Journal of Convex Analysis
Volume 22 (2015), No. 2, 569–590

On Relations between Non-Archimedean Generalized Power Series Spaces*

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Received: October 18, 2013
Revised manuscript received: January 31, 2014
Accepted: May 31, 2014

We study relations between non-Archimedean generalized power series spaces $D_f(a,r)$ and $D_g(b,s)$. We investigate when $D_f(a,r)$ has a subspace (or a quotient) isomorphic to $D_g(b,s)$.

Keywords: Non-Archimedean Köthe space, generalized power series spaces

2000 Mathematics Subject Classification: 46S10, 46A35

Introduction

In this paper all linear spaces are over a non-Archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0,\infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [10, 11, 12].

By a *Köthe space* we mean a Fréchet space with a continuous norm and with a Schauder basis. The power series spaces of finite type and infinite type, $A_1(a)$ and $A_\infty(a)$, defined in Preliminaries, are the most known and important examples of nuclear Köthe spaces. They were studied in [4, 15, 16, 17, 18].

Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant increasing odd function which is convex in $[0,\infty)$. Let $a = (a_n)$ be an increasing sequence of positive real numbers with $\lim_n a_n = \infty$. Let $r \in (-\infty, \infty]$ and let (r_k) be a strictly increasing sequence with $\lim_k r_k = r$ such that $r_k r_j > 0$ for all $k, j \in \mathbb{N}$. The linear space

$$D_f(a,r) = \{(x_n) \in \mathbb{K}^{\mathbb{N}} : \lim_n |x_n| \exp f(r_k a_n) = 0 \text{ for every } k \in \mathbb{N}\}$$

with the topology generated by the norms

$$p_k((x_n)) = \max_n |x_n| \exp f(r_k a_n), \quad k \in \mathbb{N},$$

*This research was supported by the National Center of Science, Poland, grant No. N N201 605340.

ISSN 0944-6532 / \$ 2.50 © Heldermann Verlag

is a nuclear regular Köthe space ([19, Proposition 2]). It is said to be a *generalized power series space* (see [19]). If $f(x) = x, x \in \mathbb{R}$ and $r = 0$ (respectively, $r = \infty$), then $D_f(a, r)$ is isomorphic to $A_1(a)$ (respectively, $A_\infty(a)$).

In this paper we investigate relations between generalized power series spaces.

Using [15, Propositions 9 and 19] we show that $D_f(a, 0)$ has a subspace isomorphic to $D_f(b, \infty)$, if $\sup_n(a_{2n}/a_n) < \infty$ and $\sup_n(a_n/b_n) < \infty$ (Theorem 1.1), and $D_f(a, \infty)$ has a quotient isomorphic to $D_f(b, 0)$, if $\sup(a_{2n}/a_n) < \infty$ and $\lim_n(a_n/b_n) = 0$ (Theorem 1.3). If $\sup_n(a_{n+1}/a_n) < \infty$ and $\lim_n(b_{n+1}/b_n) = \infty$, then $D_f(a, 0)$ has a subspace isomorphic to $D_f(b, \infty)$ and $D_f(a, \infty)$ has a quotient isomorphic to $D_f(b, 0)$ (Propositions 1.2 and 1.4). These results generalize [15, Propositions 11, 14, 21 and 25].

On the other hand we prove that $\sup_n(a_n/b_n) < \infty$, if $D_f(a, r)$ has a subspace (or a quotient) isomorphic to $D_f(b, s)$; if additionally $s < r = \infty$, then $\lim_n(a_n/b_n) = 0$ (Proposition 2.3). If $D_f(a, r)$ contains an isomorphic copy of $D_f(b, s)$ and $\lim_n(a_{n+1}/a_n) = \infty$, then a has a subsequence equivalent to b (Theorem 2.9). It follows that for $r \in \{0, \infty\}$ and a, b with $\lim_n(a_{n+1}/a_n) = \infty$ the space $D_f(a, r)$ contains an isomorphic copy of $D_f(b, s)$ if and only if a has a subsequence equivalent to b (Corollary 2.10). A similar result for the Archimedean power series has been proved by Dubinsky [6, Corollaries III.3.6, III.3.7]; we use and develop some ideas of his proof to get Theorem 2.9.

Assume that $\sup_n(a_{2n}/a_n) < \infty$ and $r, s \in \{0, \infty\}$ with $r \leq s$. Then $D_f(a, r)$ has a subspace isomorphic to $D_f(b, s)$ if and only if $\sup_n(a_n/b_n) < \infty$ (Corollary 2.5) and $D_f(a, \infty)$ has a quotient isomorphic to $D_f(b, 0)$ if and only if $\lim_n(a_n/b_n) = 0$ (Corollary 2.6). These results generalize [15, Corollaries 13 and 22].

The spaces $D_f(a, r)$ and $D_f(b, r)$ are isomorphic if a and b are strictly equivalent (Proposition 3.1). If the sequences a and b are not equivalent then the spaces $D_f(a, r)$ and $D_f(b, s)$ are not isomorphic (Corollary 3.5). The spaces $D_f(a, r)$ and $D_f(b, \infty)$ are not isomorphic for any $r < \infty$ (Corollary 3.4). The spaces $D_f(a, -1), D_f(b, 0)$ and $D_f(c, 1)$ are pairwise non-isomorphic if f is rapidly increasing (Proposition 3.6).

If f is strictly increasing and $f^{-1} \circ g$ is rapidly increasing, then the spaces $D_f(a, r)$ and $D_g(b, s)$ are not isomorphic (Proposition 3.7); if additionally $s > 0$, then every continuous linear map from $D_f(a, r)$ to $D_g(b, s)$ is bounded (Proposition 2.12), so $D_f(a, r)$ has no quotient isomorphic to $D_g(b, s)$ and $D_g(b, s)$ has no subspace isomorphic to $D_f(a, r)$ (Corollary 2.13).

Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant increasing odd function which is logarythmically convex in $[0, \infty)$ (i.e. the function $f(e^x)$ is convex in $\mathbb{R}$); it is easy to see that f is convex in $[0, \infty)$, so $f \in \Phi_c$. Let $a = (a_n)$ be an increasing sequence of positive real numbers with $\lim_n a_n = \infty$. Let $r \in (-\infty, \infty]$ and let (r_k) be a strictly increasing sequence with $\lim_k r_k = r$ such that $r_k r_j > 0$ for all $k, j \in \mathbb{N}$.

The linear space (over $\mathbb{C}$)

$$L_f(a,r) = \left\{ (x_n) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| \exp f(r_k a_n) < \infty \text{ for every } k \in \mathbb{N} \right\}$$

with the topology generated by the norms

$$\|(x_n)\|_k = \sum_{n=1}^{\infty} |x_n| \exp f(r_k a_n), \quad k \in \mathbb{N},$$

is a regular Köthe space. It is said to be a *Dragilev space*. These spaces were introduced by Dragilev [5] and were studied in a lot of papers, see for example [1, 2, 3, 6, 7, 8, 20].

For nuclear Dragilev spaces $L_f(a,r)$ and $L_g(b,s)$ (with some additional conditions on f and g) Dubinsky proved the following ([6, Propositions III.6.4.4–III.6.4.11]):

(1) $L_f(a,\infty)$ has a subspace isomorphic to $L_g(b,\infty)$ if and only if
   (i) there exists $M \in (0,1)$ such that for every $\rho \geq 1$ there exists $\tau > \rho$ such that $\limsup_n [f^{-1}g(\rho b_n)/f^{-1}g(\tau b_n)] \leq M$;
   (ii) for every $\rho > 0$ there exists $\tau > 0$ such that $\sup_n [f(\rho a_n)/g(\tau b_n)] < \infty$.

(2) $L_f(a,\infty)$ has a subspace isomorphic to $L_g(b,1)$ if and only if
   (i) there exists $M \in (0,1)$ such that for every $\rho \in (0,1)$ there exists $\tau \in (0,1)$ such that $\limsup_n [f^{-1}g(\rho b_n)/f^{-1}g(\tau b_n)] \leq M$;
   (ii) for every $\rho > 0$ there exists $\tau \in (0,1)$ such that $\sup_n [f(\rho a_n)/g(\tau b_n)] < \infty$.

(3) $L_f(a,\infty)$ has no subspace isomorphic to $L_g(b,r)$ for $r \in \{0,-1\}$.

(4) $L_f(a,0)$ has a subspace isomorphic to $L_g(b,\infty)$ if and only if there exist $\rho > 0$ and $\tau > 0$ such that $\sup_n [f(\rho a_n)/g(\tau b_n)] < \infty$.

(5) $L_f(a,0)$ has a subspace isomorphic to $L_g(b,1)$ if and only if there exist $\rho > 0$ and $\tau \in (0,1)$ such that $\sup_n [f(\rho a_n)/g(\tau b_n)] < \infty$.

(6) $L_f(a,0)$ has a subspace isomorphic to $L_g(b,0)$ if and only if
   (i) there exists $M > 0$ and a sequence (ρ_k) decreasing to 0 such that $\limsup_n [f^{-1}g(\rho_k b_n)/f^{-1}g(\rho_{k+1} b_n)] \leq M$ for every $k \in \mathbb{N}$;
   (ii) for every $\tau > 0$ there exists $\rho > 0$ such that $\sup_n [f(\rho a_n)/g(\tau b_n)] < \infty$.

(7) $L_f(a,0)$ has a subspace isomorphic to $L_g(b,-1)$ if and only if
   (i) there exists $M > 0$ and a sequence (ρ_k) decreasing to 1 such that $\limsup_n [f^{-1}g(\rho_k b_n)/f^{-1}g(\rho_{k+1} b_n)] \leq M$ for every $k \in \mathbb{N}$;
   (ii) for every $\tau > 0$ there exists $\rho > 0$ such that $\sup_n [f(\rho a_n)/g(\tau b_n)] < \infty$.

Generalized power series spaces in the Archimedean case were introduced by Kashirin in [7] to show that not every regular Köthe space of some type is isomorphic to some Dragilev space of infinite type; see also [8, 9].

Preliminaries

By a *seminorm* on a linear space E we mean a function $p : E \to [0,\infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x+y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$.

The linear span of a subset A of a linear space E is denoted by $\operatorname{lin} A$.

Let E, F be locally convex spaces. A map $T : E \to F$ is called a linear homeomorphism if it is linear, bijective and the maps T, T^{-1} are continuous. If there exists a linear homeomorphism $T : E \to F$, then we say that E is isomorphic to F. A linear map $T : E \to F$ is *bounded* if for some open neighbourhood U of 0 in E the set $T(U)$ is bounded in F.

A sequence (x_n) in a lcs E is a *Schauder basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $\alpha_n \in \mathbb{K}, n \in \mathbb{N}$, and the coefficient functionals $f_n : E \to \mathbb{K}, x \to \alpha_n$ $(n \in \mathbb{N})$ are continuous.

A sequence (x_n) in a lcs E is a *Schauder basic sequence* in E if it is a Schauder basis in the closed linear span of (x_n) in E.

A metrizable complete lcs is called a *Fréchet space*. A series $\sum_{n=1}^{\infty} x_n$ in a Fréchet space E is convergent if and only if $\lim_n x_n = 0$.

The set of all continuous seminorms on a Fréchet space E is denoted by $\mathcal{P}(E)$.

An increasing sequence (p_k) of continuous norms on a Fréchet space E with a continuous norm is a *base* of continuous norms on E if for any $p \in \mathcal{P}(E)$ there is $k \in \mathbb{N}$ such that $p \le k p_k$.

Let (p_k) be a base of continuous norms on a Fréchet space E. A Schauder basis (x_n) in E is

(a) *orthogonal* with respect to (p_k) if

$$p_k\left(\sum_{n=1}^{m} \alpha_n x_n\right) = \max_{1 \le n \le m} |\alpha_n| p_k(x_n)$$

for all $k, m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$;

(b) *regular* with respect to (p_k) if

$$\frac{p_k(x_n)}{p_k(x_{n+1})} \ge \frac{p_{k+1}(x_n)}{p_{k+1}(x_{n+1})}$$

for all $k, n \in \mathbb{N}$.

Let $A = (a_{n,k})$ be an infinite matrix of positive real numbers such that $a_{n,k} \le a_{n,k+1}$ for all $n, k \in \mathbb{N}$. By the Köthe space associated with the matrix A we mean the Fréchet space

$$K(A) = \{(\alpha_n) \in \mathbb{K}^{\mathbb{N}} : \lim_n |\alpha_n| a_{n,k} = 0 \text{ for any } k \in \mathbb{N}\}$$

with the base (p_k) of continuous norms: $p_k((\alpha_n)) = \max_n |\alpha_n| a_{n,k}, k \in \mathbb{N}$. The sequence (e_n) of coordinate vectors is a Schauder basis in $K(A)$ (see [4, Proposition 2.2]). It is orthogonal with respect to the base of norms (p_k).

Any infinite-dimensional Fréchet space E with a continuous norm and with a Schauder basis is isomorphic to some Köthe space $K(A)$ ([4, Proposition 2.4]).

The Köthe space $K(A)$ is nuclear if and only if for any $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$, $j > i$, with $\lim_n (a_{n,i}/a_{n,j}) = 0$ ([4, Proposition 3.5]).

Let Γ be the family of all increasing sequences $a = (a_n)$ of positive real numbers with $\lim_n a_n = \infty$.

Let $a = (a_n) \in \Gamma$. Then the following Köthe spaces are nuclear (see [4]):

(1) $A_1(a) = K(A)$ with $A = (a_{n,k}), a_{n,k} = \exp(-a_n/k)$;
(2) $A_\infty(a) = K(A)$ with $A = (a_{n,k}), a_{n,k} = \exp(ka_n)$.

$A_1(a)$ and $A_\infty(a)$ are the *power series spaces* (of *finite type* and *infinite type*, respectively).

Sequences $(a_n), (b_n) \in \Gamma$ are *equivalent* if $0 < \inf_n(a_n/b_n) \le \sup_n(a_n/b_n) < \infty$ and *strictly equivalent* if $\lim_n(a_n/b_n) = 1$.

For $r \in (-\infty, \infty]$ we denote by Λ_r the family of all strictly increasing sequences $(r_k) \subset \mathbb{R}$ with $\lim_k r_k = r$ such that $r_k r_j > 0$ for all $k, j \in \mathbb{N}$.

A map $\varphi : \mathbb{R} \to \mathbb{R}$ is *rapidly increasing* if $\lim_{t\to\infty}[\varphi(ct)/\varphi(t)] = \infty$ for any $c > 1$.

Denote by Φ_c the family of all non-constant increasing odd functions $f : \mathbb{R} \to \mathbb{R}$ such that f is convex in $[0, \infty)$.

Let $f \in \Phi_c, a \in \Gamma, r \in (-\infty, \infty]$ and $(r_k) \in \Lambda_r$. Denote by $D_f(a, r)$ the Köthe space $K(A)$, where $A = (a_{n,k}) = (\exp f(r_k a_n))$; it is clear that the space $K(A)$ does not depend on the choice of the sequence $(r_k) \in \Lambda_r$.

In [19, Lemma 1] we have shown the following

Lemma A. *Let $f \in \Phi_c$. Then (1) f is subadditive in $(-\infty, 0]$; (2) $f(u_1v_2) + f(u_2v_1) \le f(u_1v_1) + f(u_2v_2)$ for all $u_1, u_2, v_1, v_2 \in [0, \infty)$ with $u_1 \le u_2$ and $v_1 \le v_2$; (3) $f(u) + f(v) \le f(p) + f(q)$ for all $u, v, p, q \in [0, \infty)$ with $uv \le pq$ and $\max\{u, v\} \le \max\{p, q\}$; (4) $\lim_{u\to\infty} f(u) = \infty$; if additionally $f(u) > 0$ for all $u > 0$, then f is strictly increasing; (5) $f(x-y) \le f(x)-f(y)$ for all $x, y \in \mathbb{R}$ with $x \ge y$ and $xy \ge 0$; (6) $\sup_n[f(a_n) - f(b_n)] = \infty$ for all $(a_n), (b_n) \in \Gamma$ with $\limsup_n(a_n/b_n) > 1$.*

1. Some sufficient conditions

We shall need the following two propositions ([15, Propositions 9 and 19]).

Proposition I. *Let $K(A)$ and $K(B)$ be Köthe spaces. Assume that there exist $(k_n) \subset \mathbb{N}$ with $\lim_n k_n = \infty$ and $\{v(n, i) : n, i \in \mathbb{N}, 1 \le i \le k_n\} \subset \mathbb{N}$ such that*

(1) $v(n, i) = v(k, j)$ if and only if $(n, i) = (k, j)$;
(2) $(b_{n,k}/b_{n,i}) \ge (a_{v(n,i),k}/a_{v(n,i),i})$ for $n, k, i \in \mathbb{N}$ with $1 \le k \le k_n, 1 \le i \le k_n$.

Then $K(A)$ has a subspace isomorphic to $K(B)$.

Proposition II. *Let $K(A)$ and $K(B)$ be Köthe spaces. Assume that there exist $(k_n) \subset \mathbb{N}$ with $\lim_n k_n = \infty$ and $\{v(n,i) : n, i \in \mathbb{N}, 1 \leq i \leq k_n\} \subset \mathbb{N}$ such that*

(*1*) $v(n,i) = v(m,j)$ *if and only if* $(n,i) = (m,j)$;

(*2*) $(b_{n,k}/b_{n,i}) \leq (a_{v(n,i),k}/a_{v(n,i),i})$ *for all* $n, k, i \in \mathbb{N}$ *with* $1 \leq k \leq k_n, 1 \leq i \leq k_n$.

Then $K(A)$ has a quotient isomorphic to $K(B)$.

Using these propositions we get the following results.

Theorem 1.1. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$ with $\sup_n(a_n/b_n) < \infty$ and $\sup_n(a_{2n}/a_n) < \infty$. Then $D_f(a,0)$ has a subspace isomorphic to $D_f(b,\infty)$.*

Proof. Let $m > \max\{\sup_n(a_n/b_n), \sup_n(a_{2n}/a_n)\}$. Put $s_i = m^{2i+1}$ for $i \geq 0$ and $k_n = n$ for $n \in \mathbb{N}$. Put $A_{n,i} = \{k \in \mathbb{N} : s_{i-1}b_n < a_k < s_i b_n\}$ for $n, i \in \mathbb{N}$ with $1 \leq i \leq k_n$.

Let $n, i \in \mathbb{N}$ with $1 \leq i \leq k_n$. Put $s(n,i) = \max\{k \in \mathbb{N} : a_k \leq s_{i-1}b_n\}$; clearly $s(n,i) \geq n$. We have

$$\frac{a_{4s(n,i)}}{a_{s(n,i)}} < m^2 = \frac{s_i b_n}{s_{i-1}b_n}, \quad \text{so } \frac{a_{4s(n,i)}}{s_i b_n} < \frac{a_{s(n,i)}}{s_{i-1}b_n} \leq 1.$$

Hence we get

$$s_{i-1}b_n < a_{s(n,i)+1} \leq a_{4s(n,i)} < s_i b_n.$$

Thus the set $A_{n,i}$ has at least $3n$ elements.

Let $1 \leq t < n$ and $1 \leq j < k \leq k_t$ with $A_{n,i} \cap A_{t,j} \neq \emptyset \neq A_{n,i} \cap A_{t,k}$. Take $p \in A_{n,i} \cap A_{t,j}$ and $q \in A_{n,i} \cap A_{t,k}$. Then we have

$$m^2 = \frac{s_i b_n}{s_{i-1}b_n} > \frac{a_q}{a_p} > \frac{s_{k-1}b_t}{s_j b_t} = m^{2(k-1-j)},$$

so $k = j+1$. Thus the set $\{1 \leq j \leq k_t : A_{n,i} \cap A_{t,j} \neq \emptyset\}$ has at most two elements for any $1 \leq t < n$. Therefore we can choose $v(n,i) \in A_{n,i}$ for $n, i \in \mathbb{N}, 1 \leq i \leq k_n$, such that $v(n,i) = v(k,j)$ if and only if $(n,i) = (k,j)$. Clearly, $s_i > (a_{v(n,i)}/b_n) > s_{i-1}$ for $n, i \in \mathbb{N}$ with $1 \leq i \leq k_n$. Put $t_i = m^i$ and $d_i = -m^{-i}$ for $i \geq 0$.

Let $n, i, k \in \mathbb{N}$ with $1 \leq i \leq k_n$ and $1 \leq k \leq k_n$. Then

$$f(t_k b_n) + f(-d_k a_{v(n,i)}) \geq f(t_i b_n) + f(-d_i a_{v(n,i)}).$$

Indeed, put $x = t_k b_n, y = -d_k a_{v(n,i)}, z = t_i b_n, t = -d_i a_{v(n,i)}$ and $W = \{x, y, z, t\}$. Clearly $W \subset (0,\infty)$ and $xy = zt$. If $i = k$, then $x = z$ and $y = t$. If $i < k$, then $z < x, y < t, t < x$ and $y < z$, so $y = \min W$ and $x = \max W$. If $i > k$, then $z > x, y > t, y > z$ and $t > x$; so $x = \min W$ and $y = \max W$. Hence, by Lemma A (*3*), we get $f(x) + f(y) \geq f(z) + f(t)$. Thus

$$(*) \qquad f(t_k b_n) - f(t_i b_n) \geq f(d_k a_{v(n,i)}) - f(d_i a_{v(n,i)})$$

for all $n, i, k \in \mathbb{N}$ with $1 \leq i \leq k_n$ and $1 \leq k \leq k_n$.

The sequences $(t_k) \subset (0, \infty), (d_k) \subset (-\infty, 0)$ are strictly increasing and $\lim_k t_k = \infty, \lim_k d_k = 0$. Put $a_{n,k} = \exp f(d_k a_n)$ and $b_{n,k} = \exp f(t_k b_n)$ for all $n, k \in \mathbb{N}$. Let $A = (a_{n,k})$ and $B = (b_{n,k})$. Then $K(A) = D_f(a, 0)$ and $K(B) = D_f(b, \infty)$. Using $(*)$ we get $(b_{n,k}/b_{n,i}) \geq (a_{v(n,i),k}/a_{v(n,i),i})$ for all $n, k, i \in \mathbb{N}$ with $1 \leq i \leq k_n$ and $1 \leq k \leq k_n$.

By Proposition I we infer that $K(A)$ has a subspace isomorphic to $K(B)$. $\square$

Proposition 1.2. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$. Assume that $\sup_n(a_{n+1}/a_n) < \infty$ and $\lim_n(b_{n+1}/b_n) = \infty$. Then $D_f(a, 0)$ has a subspace isomorphic to $D_f(b, \infty)$.*

Proof. Let $m \in \mathbb{N}$ with $m > \sup_n(a_{n+1}/a_n)$. Passing to an equivalent sequence we can assume that $\inf_n(b_{n+1}/b_n) > m^2$ and $b_1 \geq a_1$. Then there exists a sequence $(k_n) \subset \mathbb{N}$ with $\lim_n k_n = \infty$ such that $(b_{n+1}/b_n) > m^{2k_n}$ for $n \in \mathbb{N}$. Put $s_i = m^{2i+1}$ for $i \geq 0$.

Let $n, i \in \mathbb{N}$ with $1 \leq i \leq k_n$. Since $a_1 \leq s_{i-1} b_n$ and

$$\sup_j \frac{a_{j+1}}{a_j} < \frac{s_i b_n}{s_{i-1} b_n},$$

there is $v(n, i) \in \mathbb{N}$ such that $s_{i-1} b_n < a_{v(n,i)} < s_i b_n$.

Moreover, $s_{k_n} b_n < s_0 b_{n+1}$ for $n \in \mathbb{N}$. Thus $v(n, i) = v(k, j)$ if and only if $(n, i) = (k, j)$. Clearly $s_i > (a_{v(n,i)}/b_n) > s_{i-1}$ for $n, i \in \mathbb{N}$ with $1 \leq i \leq k_n$. Then, as in the proof of Theorem 1.1, we infer that $D_f(a, 0)$ has a subspace isomorphic to $D_f(b, \infty)$. $\square$

Theorem 1.3. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$ with $\lim_n(a_n/b_n) = 0$ and $\sup_n(a_{2n}/a_n) < \infty$. Then $D_f(a, \infty)$ has a quotient isomorphic to $D_f(b, 0)$.*

Proof. Let $m \in \mathbb{N}$ with $m > \max\{\sup_n(a_{2n}/a_n), \sup_n(a_n/b_n)\}$. Then there exists a sequence $(k_n) \subset \mathbb{N}$ with $\lim_n k_n = \infty$ such that $k_n \leq n+1$ and $(a_n/b_n) < m^{3-2k_n}$ for $n \in \mathbb{N}$. Put $s_i = m^{3-2i}$ for $i \geq 0$ and $A_{n,i} = \{k \in \mathbb{N} : s_i b_n < a_k < s_{i-1} b_n\}$ for all $n, i \in \mathbb{N}$ with $1 \leq i \leq k_n$.

Let $n, i \in \mathbb{N}$ with $1 \leq i \leq k_n$. Put $s(n, i) = \max\{k \in \mathbb{N} : a_k \leq s_i b_n\}$. Then $(a_n/b_n) < s_{k_n} \leq s_i$ and $s(n, i) \geq n$. We have

$$\frac{a_{4s(n,i)}}{a_{s(n,i)}} < m^2 = \frac{s_{i-1} b_n}{s_i b_n}, \text{ so } \frac{a_{4s(n,i)}}{s_{i-1} b_n} < \frac{a_{s(n,i)}}{s_i b_n} \leq 1.$$

Hence

$$a_{s(n,i)} \leq s_i b_n < a_{s(n,i)+1} \leq a_{4s(n,i)} < s_{i-1} b_n.$$

Then the set $A_{n,i}$ has at least $3n$ elements.

Let $1 \leq t < n$ and $1 \leq j < k \leq k_t$ with $A_{n,i} \cap A_{t,j} \neq \emptyset \neq A_{n,i} \cap A_{t,k}$. Take $p \in A_{n,i} \cap A_{t,j}$ and $q \in A_{n,i} \cap A_{t,k}$. Then we have

$$m^2 = \frac{s_{i-1} b_n}{s_i b_n} > \frac{a_p}{a_q} > \frac{s_j b_t}{s_{k-1} b_t} = \frac{s_j}{s_{k-1}} = m^{2(k-j-1)},$$

so $k = j + 1$. Thus the set $\{1 \le j \le k_t : A_{n,i} \cap A_{t,j} \ne \emptyset\}$ has at most two elements for any $1 \le t < n$. Therefore we can choose $v(n,i) \in A_{n,i}$ for $n, i \in \mathbb{N}$ with $1 \le i \le k_n$ such that $v(n,i) = v(k,j)$ if and only if $(n,i) = (k,j)$. Clearly, $s_i < (a_{v(n,i)}/b_n) < s_{i-1}$ for $n, i \in \mathbb{N}$ with $1 \le i \le k_n$. Put $t_i = m^{i-2}$ and $d_i = -m^{2-i}$ for $i \ge 0$.

Let $n, k, i \in \mathbb{N}$ with $1 \le k \le k_n$ and $1 \le i \le k_n$. Then

$$f(-d_k b_n) + f(t_k a_{v(n,i)}) \ge f(-d_i b_n) + f(t_i a_{v(n,i)}).$$

Indeed, put $x = -d_k b_n, y = t_k a_{v(n,i)}, z = -d_i b_n, t = t_i a_{v(n,i)}$ and $W = \{x, y, z, t\}$. Clearly, $W \subset (0, \infty)$ and $xy = zt$.

If $i = k$, then $x = z$ and $y = t$.

If $i < k$, then $x < z, t < y, x < t$ and $z < y$, so $x = \min W$ and $y = \max W$.

If $i > k$, then $x > z, t > y, x > t$ and $z > y$, so $y = \min W$ and $x = \max W$.

Hence, by Lemma A (*3*), we get $f(x) + f(y) \ge f(z) + f(t)$. Thus

$$(*) \qquad f(d_k b_n) - f(d_i b_n) \le f(t_k a_{v(n,i)}) - f(t_i a_{v(n,i)})$$

for all $n, k, i \in \mathbb{N}$ with $1 \le k \le k_n$ and $1 \le i \le k_n$.

The sequences $(t_k) \subset (0, \infty), (d_k) \subset (-\infty, 0)$ are strictly increasing and $\lim_k t_k = \infty, \lim_k d_k = 0$. Put $a_{n,k} = \exp f(t_k a_n)$ and $b_{n,k} = \exp f(d_k b_n)$ for all $n, k \in \mathbb{N}$. Let $A = (a_{n,k})$ and $B = (b_{n,k})$. Then $K(A) = D_f(a, \infty)$ and $K(B) = D_f(b, 0)$. Using $(*)$ we get $(b_{n,k}/b_{n,i}) \le (a_{v(n,i),k}/a_{v(n,i),i})$ for all $n, i, k \in \mathbb{N}$ with $1 \le i \le k_n$ and $1 \le k \le k_n$. By Proposition II we infer that $K(A)$ has a quotient isomorphic to $K(B)$. □

Proposition 1.4. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$. Assume that $\sup_n(a_{n+1}/a_n) < \infty$ and $\lim_n(b_{n+1}/b_n) = \infty$. Then $D_f(a, \infty)$ has a quotient isomorphic to $D_f(b, 0)$.*

Proof. Let $m \in \mathbb{N}$ with $m > \sup_n(a_{n+1}/a_n)$. Passing to an equivalent sequence we can assume that $\inf_n(b_{n+1}/b_n) > m^2$ and $b_1 \ge a_1$. Then there exists a sequence $(k_n) \subset \mathbb{N}$ with $\lim_n k_n = \infty$ such that $(b_{n+1}/b_n) > m^{2k_{n+1}}$ for $n \in \mathbb{N}$; we can assume that $k_n \le n, n \in \mathbb{N}$.
Put $s_i = m^{3-2i}$ for $i \ge 0$. Let $n, i \in \mathbb{N}$ with $1 \le i \le k_n$. Since

$$a_1 = b_n \frac{a_1}{b_1} \prod_{j=1}^{n-1} \frac{b_j}{b_{j+1}} \le m^{-2(n-1)} b_n \le s_i b_n$$

and

$$\sup_j \frac{a_{j+1}}{a_j} < \frac{s_{i-1} b_n}{s_i b_n},$$

there is $v(n,i) \in \mathbb{N}$ such that $s_i b_n < a_{v(n,i)} < s_{i-1} b_n$. Moreover $s_0 b_n < s_{k_{n+1}} b_{n+1}$ for $n \in \mathbb{N}$. Thus $v(n,i) = v(k,j)$ if and only if $(n,i) = (k,j)$. Clearly $s_i < (a_{v(n,i)}/b_n) < s_{i-1}$ for $n, i \in \mathbb{N}$ with $1 \le i \le k_n$. Then, as in the proof of Theorem 1.3, we can show that $D_f(a, \infty)$ has a quotient isomorphic to $D_f(b, 0)$. □

Lemma 1.5. *For any $b \in \Gamma$ there is $a \in \Gamma$ with $\sup_n(a_{2n}/a_n) < \infty$ such that $\lim_n(a_n/b_n) = 0$.*

Proof. Let $m_k = \max\{i \in \mathbb{Z} : 2^{2i} \leq b_{2^{k-1}}\}$ for $k \in \mathbb{N}$. Clearly, $\lim_k m_k = \infty$ and $\lim_k 2^{m_k}/b_{2^{k-1}} = 0$. Let us define by induction a sequence $(n_k) \subset \mathbb{Z}$ as follows: $n_1 = m_1$; for $k > 1$ we put $n_k = n_{k-1}$ if $m_k = m_{k-1}$ and $n_k = n_{k-1} + 1$ otherwise. Let $a_n = 2^{n_k}$ for $2^{k-1} \leq n < 2^k, k \in \mathbb{N}$. Of course, $a = (a_n) \in \Gamma, \sup_n(a_{2n}/a_n) = 2$ and $\lim_n(a_n/b_n) = 0$. □

By Lemma 1.5 and Theorems 1.1 and 1.3 we get

Corollary 1.6. *Let $f \in \Phi_c$. For any $b \in \Gamma$ there is $a \in \Gamma$ such that $D_f(b, 0)$ is isomorphic to a quotient of $D_f(a, \infty)$ and $D_f(b, \infty)$ is isomorphic to a subspace of $D_f(a, 0)$.*

2. Some necessary conditions

For arbitrary subsets A, B of a linear space E and a linear subspace L of E we denote $d(A, B, L) = \inf\{|\beta| : \beta \in \mathbb{K}, A \subset \beta B + L\}$ (we put $\inf \emptyset = \infty$). Let $d_n(A, B) = \inf\{d(A, B, L) : L$ is a linear subspace of E with $\dim L < n\}$ for $n \in \mathbb{N}$.
It is easy to check the following.

Remark I. Let E and F be a linear spaces. If $A, B \subset E$ and T is a linear map from E onto F, then $d_n(A, B) \geq d_n(T(A), T(B))$ for $n \in \mathbb{N}$.

If $\alpha, \beta \in (\mathbb{K} \setminus \{0\}), A' \subset A \subset E$ and $B \subset B' \subset E$, then $d_n(A, B) \geq d_n(A', B')$ and $d_n(\alpha A, \beta B) = |\alpha\beta^{-1}| d_n(A, B)$ for $n \in \mathbb{N}$.

Lemma 2.1. *Let E be a Fréchet space with a Schauder basis (e_n) which is regular and orthogonal with respect to a base (p_k) of continuous norms on E. Let $U_k = \{x \in E : p_k(x) \leq 1\}$ for $k \in \mathbb{N}$. Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. Then for all $n, l, m \in \mathbb{N}$ with $l \geq m$ we have $|\alpha|^{-1} p_m(e_n)/p_l(e_n) \leq d_n(U_l, U_m) \leq |\alpha| p_m(e_n)/p_l(e_n)$.*

Proof. Let $l, m \in \mathbb{N}$ with $l \geq m$. Put $a_k = [p_l(e_k)]^{-1}$ and $b_k = [p_m(e_k)]^{-1}$ for $k \in \mathbb{N}$. By the regularity of (e_n) with respect to (p_k) we get $a_{k+1}a_k^{-1} \leq b_{k+1}b_k^{-1}$ for all $k \in \mathbb{N}$. Let (f_n) be the sequence of coefficient functionals associated with the Schauder basis (e_n). Then we have

$$U_l = \{x \in E : \max_k |f_k(x)| p_l(e_k) \leq 1\} = \{x \in E : |f_k(x)| \leq a_k, k \in \mathbb{N}\}$$

and

$$U_m = \{x \in E : \max_k |f_k(x)| p_m(e_k) \leq 1\} = \{x \in E : |f_k(x)| \leq b_k, k \in \mathbb{N}\}.$$

Using [14, Lemma 2] we get $|\alpha|^{-1} a_n b_n^{-1} \leq d_n(U_l, U_m) \leq |\alpha| a_n b_n^{-1}$ for any $n \in \mathbb{N}$. This completes the proof. □

Lemma 2.2. *Let $f, g \in \Phi_c, a, b \in \Gamma$ and $r, s \in (-\infty, \infty]$. Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. Assume that $D_f(a, r)$ has a subspace (or a quotient) isomorphic to $D_g(b, s)$. Then for every $l \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for every $v \in \mathbb{N}$ there is $t \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have*

$$[f(r_{k+v}a_n) - f(r_k a_n) + g(s_l b_n) - g(s_t b_n)] \leq (t - v - l + 2) \ln |\alpha|.$$

Proof. Put $X = D_f(a, r)$ and $Y = D_g(b, s)$. Let (p_k) and (q_k) be the standard bases of continuous norms on X and Y, respectively. Put $U_k = \{x \in X : p_k(x) \leq 1\}$ and $V_k = \{y \in Y : q_k(y) \leq 1\}$ for $k \in \mathbb{N}$.

Let $l \in \mathbb{N}$. Consider two cases.

Case 1: X has a subspace F isomorphic to Y. Let $T : F \to Y$ be an isomorphism. For some $k \in \mathbb{N}$ we have $T(\alpha^{-k} U_k \cap F) \subset \alpha^{-l} V_l$. Let $v \in \mathbb{N}$. Then there exists $t \in \mathbb{N}, t \geq l$, such that $\alpha^{-t} V_t \subset T(\alpha^{-k-v} U_{k+v} \cap F)$. Using [15, Proposition 4] and Remark I, we get for any $n \in \mathbb{N}$

$$\begin{aligned} d_n(U_{k+v}, U_k) &\geq d_n(U_{k+v} \cap F, U_k \cap F) \geq d_n(T(U_{k+v} \cap F), T(U_k \cap F)) \\ &\geq d_n(\alpha^{k+v-t} V_t, \alpha^{k-l} V_l) = |\alpha|^{v-t+l} d_n(V_t, V_l). \end{aligned}$$

Case 2: X has a quotient isomorphic to Y. Let $T : X \to Y$ be a continuous linear surjective map. For some $k \in \mathbb{N}$ we have $\alpha^{-l} V_l \supset T(\alpha^{-k} U_k)$. Let $v \in \mathbb{N}$. Then there exists $t \in \mathbb{N}$ such that $T(\alpha^{-k-v} U_{k+v}) \supset \alpha^{-t} V_t$. By Remark I, we get for any $n \in \mathbb{N}$

$$d_n(U_{k+v}, U_k) \geq d_n(T(U_{k+v}), T(U_k)) \geq d_n(\alpha^{k+v-t} V_t, \alpha^{k-l} V_l) = |\alpha|^{v-t+l} d_n(V_t, V_l).$$

Using Lemma 2.1, we get for all $n \in \mathbb{N}$

$$|\alpha| \exp[f(r_k a_n) - f(r_{k+v} a_n)] \geq |\alpha|^{v-t+l-1} \exp[g(s_l b_n) - g(s_t b_n)].$$

Thus for any $n \in \mathbb{N}$ we obtain

$$f(r_{k+v} a_n) - f(r_k a_n) + g(s_l b_n) - g(s_t b_n) \leq (t - v - l + 2) \ln |\alpha|. \qquad \square$$

Using Lemma 2.2 we shall prove the following proposition.

Proposition 2.3. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$ and $r, s \in (-\infty, \infty]$. Assume that $D_f(a, r)$ has a subspace (or a quotient) isomorphic to $D_f(b, s)$. Then $\sup_n(a_n/b_n) < \infty$. If additionally $s < r = \infty$, then $\lim_n(a_n/b_n) = 0$.*

Proof. Let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. By Lemma 2.2, there is $k \in \mathbb{N}$ such that for every $v \in \mathbb{N}$ there exists $t \in \mathbb{N}$ such that

$$f(s_1 b_n) - f(s_t b_n) + f(r_{k+v} a_n) - f(r_k a_n) \leq (t - v + 1) \ln |\alpha|$$

for every $n \in \mathbb{N}$. Put $s_0 = \max\{s, -s_1\}$ if $s < \infty$ and $s_0 = s_t$ if $s = \infty$. Put $r_0 = r_{k+v} - r_k$. Then $-f(s_0 b_n) \le f(s_1 b_n) - f(s_t b_n)$ and $f(r_0 a_n) \le f(r_{k+v} a_n) - f(r_k a_n)$, so $f(r_0 a_n) - f(s_0 b_n) \le (t - v + 1) \ln |\alpha|$ for $n \in \mathbb{N}$.

Suppose that $\limsup_n (a_n/b_n) > (s_0 + 1)/r_0$. Then there exists a strictly increasing sequence $(n_m) \subset \mathbb{N}$ such that $(a_{n_m}/b_{n_m}) \ge (s_0 + 1)/r_0$ for $m \in \mathbb{N}$. Hence $r_0 a_{n_m} \ge s_0 b_{n_m} + b_{n_m}$ for $m \in \mathbb{N}$. Thus

$$\exp f(b_{n_m}) \le \exp[f(r_0 a_{n_m}) - f(s_0 b_{n_m})] \le |\alpha|^{t-v+1}$$

for all $m \in \mathbb{N}$, a contradiction. It follows that $\limsup_n (a_n/b_n) \le (s_0+1)/r_0 < \infty$. If $s < r = \infty$, then $\lim_v (s_0 + 1)/(r_{k+v} - r_k) = 0$; so $\lim_n (a_n/b_n) = 0$. □

Corollary 2.4. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$ and $r, s \in (-\infty, \infty]$. Assume that $D_f(a, r)$ has a subspace (or a quotient) isomorphic to $D_f(b, s)$ and $D_f(b, s)$ has a subspace (or a quotient) isomorphic to $D_f(a, r)$. Then the sequences a and b are equivalent.*

By Theorem 1.1, Proposition 2.3 and [15, Lemma 7], we get

Corollary 2.5. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$ with $\sup_n (a_{2n}/a_n) < \infty$ and let $r, s \in \{0, \infty\}$ with $r \le s$. Then $D_f(a, r)$ has a subspace isomorphic to $D_f(b, s)$ if and only if $\sup_n (a_n/b_n) < \infty$.*

By Proposition 2.3 and Theorem 1.3 we get the following.

Corollary 2.6. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$ with $\sup_n (a_{2n}/a_n) < \infty$. Then the space $D_f(a, \infty)$ has a quotient isomorphic to $D_f(b, 0)$ if and only if $\lim_n (a_n/b_n) = 0$.*

To prove our next theorem we shall need the following two lemmas.

Lemma 2.7. *Let $u, v, x, y, c, d \in \mathbb{R}$ such that $u > v, uv > 0, x > y$ and $xy > 0$. Let $(a_n), (b_n) \subset (0, \infty)$ with $\lim_n a_n = \infty$. Assume that $f \in \Phi_c$ and*

$$f(ua_n) - f(va_n) \le c + f(xb_n) - f(yb_n)$$

for $n > d$. Then $\limsup_n (a_n/b_n) \le \max\{x, -y\}/(u - v)$.

Proof. Put $s = \max\{x, -y\}$. If $x > 0$, then $y > 0$ and $s = x$; if $x < 0$, then $y < 0$ and $s = -y$. Thus $f(xb_n) - f(yb_n) \le f(sb_n)$. Using Lemma A (*5*) we get

$$f((u - v)a_n) \le f(ua_n) - f(va_n) \le c + f(sb_n) \quad \text{for } n > d.$$

Hence, by Lemma A (*4*), we have $\lim_n b_n = \infty$. By Lemma A (*4*) and Lemma A (*1*), for every $\varepsilon > 0$ there exists $M_\varepsilon > d$ such that

$$c + f(sb_n) < f(\varepsilon b_n) + f(sb_n) \le f((\varepsilon + s)b_n) \quad \text{for } n > M_\varepsilon.$$

Thus $(u - v)a_n < (\varepsilon + s)b_n$ for $n > M_\varepsilon$, so $\limsup_n (a_n/b_n) \le [(\varepsilon + s)/(u - v)]$ for every $\varepsilon > 0$. It follows that $\limsup_n (a_n/b_n) \le [s/(u - v)]$. □

Lemma 2.8. *Let (x_n) be a Schauder basic sequence in a Fréchet space E. Let $(y_n) \subset E, s \in \mathbb{N}$ and let $(m_k) \subset \mathbb{N}$ be a strictly increasing sequence.*

(a) *If (x_n) is orthogonal with respect to a base $(|\cdot|_k)$ of continuous norms on E and $|x_n - y_n|_k < |x_n|_k$ for $n \geq m_k, k \geq s$, then $(y_n)_{n \geq m_s}$ is a Schauder basic sequence in E.*

(b) *If for some base $(\|\cdot\|_k)$ of continuous norms on E we have $\lim_n \|x_n - y_n\|_j / \|x_n\|_1 = 0$ for every $j \in \mathbb{N}$, then $(y_n)_{n \geq m}$ is a Schauder basic sequence in E for some $m \in \mathbb{N}$.*

Proof. (a) Let $x'_n = x_{n-1+m_s}$ and $y'_n = y_{n-1+m_s}$ for $n \in \mathbb{N}$. Put $\|\cdot\|_j = |\cdot|_k$ for $m_k - m_s < j \leq m_{k+1} - m_s, k \geq s$. Clearly, (x'_n) is a Schauder basic sequence in E, orthogonal with respect to the base $(\|\cdot\|_j)$ of continuous norms on E. Let $n, j \in \mathbb{N}$ with $n \geq j$. Let $k \geq s$ with $m_k - m_s < j \leq m_{k+1} - m_s$. Then

$$\|x'_n - y'_n\|_j = |x_{n-1+m_s} - y_{n-1+m_s}|_k < |x_{n-1+m_s}|_k = \|x'_n\|_j,$$

since $n - 1 + m_s \geq j - 1 + m_s \geq m_k$. Using [13, Theorem 1], we infer that (y'_n) is a Schauder basic sequence in E.

(b) Let $s \in \mathbb{N}$ with $\|\cdot\|_1 \leq s|\cdot|_s$ and let $k \geq s$. Let $l \in \mathbb{N}$ with $|\cdot|_k \leq l\|\cdot\|_l$. Then

$$\frac{|x_n - y_n|_k}{|x_n|_k} \leq \frac{kl\|x_n - y_n\|_l}{\|x_n\|_1}$$

for $n \in \mathbb{N}$. Thus for some $m_k \in \mathbb{N}$ and $n \geq m_k$ we have $|x_n - y_n|_k < |x_n|_k$. Clearly we can assume that the sequence (m_k) is strictly increasing. Using (a) we infer that $(y_n)_{n \geq m_s}$ is a Schauder basic sequence in E. □

Theorem 2.9. *Let $f \in \Phi_c$ and $r, s \in (-\infty, \infty]$. Let $a, b \in \Gamma$ with $\lim_n (a_{n+1}/a_n) = \infty$. Assume that $D_f(a, r)$ contains an isomorphic copy of $D_f(b, s)$. Then a contains a subsequence equivalent to b. If additionally $s = \infty$, then $r = \infty$.*

Proof. (S1) Let F be a closed subspace of $E = D_f(a, r)$ that is isomorphic to $D_f(b, s)$. Let $(r_k), (s_k) \subset (-\infty, \infty)$ be strictly increasing sequences with $\lim_k r_k = r$ and $\lim_k s_k = s$ such that $r_k r_l > 0$ and $s_k s_l > 0$ for all $k, l \in \mathbb{N}$. F has an orthogonal basis (f_n) with respect to some base (l_k) of continuous norms on F such that $l_k(f_n) = \exp f(s_k b_n)$ for all $k, n \in \mathbb{N}$. Clearly, (f_n) is regular with respect to (l_k). Let (p_k) be the standard base of continuous norms on E and let (e_j) be the coordinate basis in E. Then $p_k(e_j) = \exp f(r_k a_j)$ for all $k, j \in \mathbb{N}$. Put $a_{k,j} = p_k(e_j), b_{k,j} = p_k(f_j)$ and $c_{k,j} = l_k(f_j)$ for $k, j \in \mathbb{N}$. Let $f_n = (t_{n,i})$ for $n \in \mathbb{N}$. Then $b_{k,n} = \max_i |t_{n,i}| a_{k,i}$ for $k, n \in \mathbb{N}$. Put $q_{k,n} = \max\{i \in \mathbb{N} : b_{k,n} = |t_{n,i}| a_{k,i}\}$ for all $k, n \in \mathbb{N}$.

(S2) We shall prove that $q_{k,n} \leq q_{k+1,n}$ for $k, n \in \mathbb{N}$. Suppose that $q_{k+1,n} < q_{k,n}$ for some $k, n \in \mathbb{N}$. By the regularity of the basis (e_n) in E we have

$$\frac{a_{k+1,i}}{a_{k+1,i+1}} \leq \frac{a_{k,i}}{a_{k,i+1}} \quad \text{for all } i \in \mathbb{N}.$$

Hence

$$\frac{a_{k+1,q_{k+1,n}}}{a_{k+1,q_{k,n}}} \leq \frac{a_{k,q_{k+1,n}}}{a_{k,q_{k,n}}}.$$

By the definitions of $q_{k+1,n}$ and $q_{k,n}$ we have

$$|t_{n,q_{k,n}}|a_{k+1,q_{k,n}} < |t_{n,q_{k+1,n}}|a_{k+1,q_{k+1,n}}$$

and

$$|t_{n,q_{k+1,n}}|a_{k,q_{k+1,n}} \leq |t_{n,q_{k,n}}|a_{k,q_{k,n}}.$$

Hence we get

$$\frac{|t_{n,q_{k,n}}|}{|t_{n,q_{k+1,n}}|} < \frac{a_{k+1,q_{k+1,n}}}{a_{k+1,q_{k,n}}} \leq \frac{a_{k,q_{k+1,n}}}{a_{k,q_{k,n}}} \leq \frac{|t_{n,q_{k,n}}|}{|t_{n,q_{k+1,n}}|};$$

a contradiction.

(S3) Now we shall prove that there exist $v \in \mathbb{N}$ and an increasing sequence $(m_j) \subset \mathbb{N}$ such that $q_{j,n} = q_{v,n}$ for all $j, n \in \mathbb{N}$ with $j \geq v$ and $n \geq m_j$.

For all $j, k, n \in \mathbb{N}$ we have

$$\frac{a_{j,q_{k,n}}}{a_{k,q_{k,n}}} = \frac{|t_{n,q_{k,n}}|}{|t_{n,q_{k,n}}|}\frac{a_{j,q_{k,n}}}{a_{k,q_{k,n}}} \leq \frac{|t_{n,q_{j,n}}|a_{j,q_{j,n}}}{|t_{n,q_{k,n}}|a_{k,q_{k,n}}} = \frac{b_{j,n}}{b_{k,n}}$$

and

$$\frac{b_{j,n}}{b_{k,n}} = \frac{|t_{n,q_{j,n}}|}{|t_{n,q_{k,n}}|}\frac{a_{j,q_{j,n}}}{a_{k,q_{k,n}}} \leq \frac{|t_{n,q_{j,n}}|}{|t_{n,q_{j,n}}|}\frac{a_{j,q_{j,n}}}{a_{k,q_{j,n}}} = \frac{a_{j,q_{j,n}}}{a_{k,q_{j,n}}}.$$

For some strictly increasing sequences $(j_k), (i_k) \subset \mathbb{N}$ we have $(p_{j_k}|F) \leq i_k l_{i_k}$ and $l_{i_k} \leq j_{k+1}(p_{j_{k+1}}|F)$ for $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$. For $n \in \mathbb{N}$ we have

$$\frac{l_{i_{k+1}}(f_n)}{l_{i_k}(f_n)} \leq i_k j_{k+2}\frac{p_{j_{k+2}}(f_n)}{p_{j_k}(f_n)},$$

and

$$\frac{p_{j_{k+4}}(f_n)}{p_{j_{k+3}}(f_n)} \leq i_{k+4} j_{k+3}\frac{l_{i_{k+4}}(f_n)}{l_{i_{k+2}}(f_n)},$$

so

$$\frac{c_{i_{k+1},n}}{c_{i_k,n}} \leq i_k j_{k+2}\frac{b_{j_{k+2},n}}{b_{j_k,n}} \leq i_k j_{k+2}\frac{a_{j_{k+2},q_{j_{k+2},n}}}{a_{j_k,q_{j_{k+2},n}}}$$

and

$$\frac{a_{j_{k+4},q_{j_{k+3},n}}}{a_{j_{k+3},q_{j_{k+3},n}}} \leq \frac{b_{j_{k+4},n}}{b_{j_{k+3},n}} \leq i_{k+4} j_{k+3}\frac{c_{i_{k+4},n}}{c_{i_{k+2},n}}.$$

It follows that for any $n \in \mathbb{N}$ we have

$$\exp[f(s_{i_{k+1}}b_n) - f(s_{i_k}b_n)] \leq i_{k+1}j_{k+2}\exp[f(r_{j_{k+2}}a_{q_{j_{k+2},n}}) - f(r_{j_k}a_{q_{j_{k+2},n}})]$$

and

$$\exp[f(r_{j_{k+4}} a_{q_{j_{k+3},n}}) - f(r_{j_{k+3}} a_{q_{j_{k+3},n}})] \le i_{k+4} j_{k+3} \exp[f(s_{i_{k+4}} b_n) - f(s_{i_{k+2}} b_n)].$$

Using Lemma 2.7 we get $\sup_n[b_n/a_{q_{j_{k+2},n}}] < \infty$ and $\sup_n[a_{q_{j_{k+3},n}}/b_n] < \infty$. Hence $\sup_n[a_{q_{j_{k+3},n}}/a_{q_{j_{k+2},n}}] < \infty$ for $k \in \mathbb{N}$. Since $\lim_m(a_{m+1}/a_m) = \infty$ and $q_{j_{k+2},n} \le q_{j_{k+3},n}$ for $k, n \in \mathbb{N}$, for every $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ such that $q_{j_{k+2},n} = q_{j_{k+3},n}$ for $n \ge n_k$. Let $v = j_3$. Then there exists an increasing sequence $(m_j) \subset \mathbb{N}$ such that $q_{j,n} = q_{v,n}$ for all $j, n \in \mathbb{N}$ with $j \ge v$ and $n \ge m_j$.

(S4) We have $\lim_n q_{v,n} = \infty$. Indeed, let $j \ge v$. By the nuclearity of F there is $k > j$ such that $\lim_n[b_{j,n}/b_{k,n}] = 0$. For $n \ge m_k$ we have

$$\frac{b_{j,n}}{b_{k,n}} = \frac{|t_{n,q_{j,n}}|}{|t_{n,q_{k,n}}|} \frac{a_{j,q_{j,n}}}{a_{k,q_{k,n}}} = \frac{a_{j,q_{v,n}}}{a_{k,q_{v,n}}},$$

so $\lim_n[a_{j,q_{v,n}}/a_{k,q_{v,n}}] = 0$. Hence $\lim_n[f(r_j a_{q_{v,n}}) - f(r_k a_{q_{v,n}})] = -\infty$, so $\lim_n q_{v,n} = \infty$.

(S5) Next we shall prove that the sequences $(a_{q_{v,n}})$ and (b_n) are equivalent. Let $t \ge v$ with $l_1 \le t(p_t|F)$ and let $j > 1$ with $(p_{t+1}|F) \le j l_j$. Then

$$tj \frac{l_j(f_n)}{l_1(f_n)} \ge \frac{p_{t+1}(f_n)}{p_t(f_n)} = \frac{b_{t+1,n}}{b_{t,n}} = \frac{a_{t+1,q_{v,n}}}{a_{t,q_{v,n}}}$$

for $n \ge m_{t+1}$. Hence

$$tj \exp[f(s_j b_n) - f(s_1 b_n)] \ge \exp[f(r_{t+1} a_{q_{v,n}}) - f(r_t a_{q_{v,n}})]$$

for $n \ge m_{t+1}$. Using Lemma 2.7 we get $A = \sup_n(a_{q_{v,n}}/b_n) < \infty$.

Let $t \in \mathbb{N}$ with $(p_v|F) \le t l_t$ and let $j > v$ with $l_{t+1} \le j(p_j|F)$. Then

$$\frac{1}{tj} \frac{l_{t+1}(f_n)}{l_t(f_n)} \le \frac{p_j(f_n)}{p_v(f_n)} = \frac{b_{j,n}}{b_{v,n}} = \frac{a_{j,q_{v,n}}}{a_{v,q_{v,n}}}$$

for $n \ge m_j$. Hence

$$\exp[f(s_{t+1} b_n) - f(s_t b_n)] \le tj \exp[f(r_j a_{q_{v,n}}) - f(r_v a_{q_{v,n}})]$$

for $n \ge m_j$. Using Lemma 2.7 we get $B = \sup_n(b_n/a_{q_{v,n}}) < \infty$. Thus the sequences (b_n) and $(a_{q_{v,n}})$ are equivalent.

(S6) We shall prove that for some $m_0 \in \mathbb{N}$ the sequence $(q_{v,n})_{n \ge m_0}$ is strictly increasing. We have

$$\frac{a_{q_{v,n}}}{a_{q_{v,n+1}}} \le AB \frac{b_n}{b_{n+1}} \le AB.$$

Since $\lim_m(a_{m+1}/a_m) = \infty$ and $\lim_n q_{v,n} = \infty$, we infer that for some $n_0 \in \mathbb{N}$ the sequence $(q_{v,n})_{n \ge n_0}$ is increasing.

Put $h_n = t_{n,q_{v,n}} e_{q_{v,n}}$ for $n \in \mathbb{N}$. Let $j \geq v+1$. For $n \in \mathbb{N}$ we have

$$p_j(f_n - h_n) = p_j\left(\sum_{q \neq q_{v,n}} t_{n,q} e_q\right) = \max_{q \neq q_{v,n}} |t_{n,q}| p_j(e_q) = \max_{q \neq q_{v,n}} |t_{n,q}| a_{j,q}.$$

For $q \geq 1, l \geq v$ and $n \geq m_{l+1}$ we have

$$\begin{aligned} \frac{|t_{n,q}| a_{j,q}}{p_{v+1}(f_n)} &= \frac{|t_{n,q}| a_{l,q}}{b_{v+1,n}} \frac{a_{j,q}}{a_{l,q}} \leq \frac{|t_{n,q_{l,n}}| a_{l,q_{l,n}}}{b_{v+1,n}} \frac{a_{j,q}}{a_{l,q}} = \frac{b_{l,n}}{b_{v+1,n}} \frac{a_{j,q}}{a_{l,q}} = \frac{a_{l,q_{v,n}}}{a_{v+1,q_{v,n}}} \frac{a_{j,q}}{a_{l,q}} \\ &= \exp[f(r_l a_{q_{v,n}}) - f(r_{v+1} a_{q_{v,n}}) + f(r_j a_q) - f(r_l a_q)]. \end{aligned}$$

Let $z_1 = \max\{r_j, -r_v\}$ and $z_2 = \max\{r_{j+1}, -r_{v+1}\}$. If

$$\frac{a_{q_{v,n}}}{a_{q_{v,n}-1}} > \frac{2z_1}{r_{v+1} - r_v},$$

then for $1 \leq q < q_{v,n}$ we have

$$\begin{aligned} &[f(r_v a_{q_{v,n}}) - f(r_{v+1} a_{q_{v,n}}) + f(r_j a_q) - f(r_v a_q)] \\ \leq &\ -f((r_{v+1} - r_v) a_{q_{v,n}}) + f(z_1 a_q) \leq -f((r_{v+1} - r_v) a_{q_{v,n}}) + f(z_1 a_{q_{v,n}-1}) \\ \leq &\ -2f\left(\frac{r_{v+1} - r_v}{2} a_{q_{v,n}}\right) + f\left(\frac{r_{v+1} - r_v}{2} a_{q_{v,n}}\right) = -f\left(\frac{r_{v+1} - r_v}{2} a_{q_{v,n}}\right). \end{aligned}$$

If

$$\frac{a_{q_{v,n}+1}}{a_{q_{v,n}}} > \frac{2z_2 + r_{v+1} - r_v}{2(r_{j+1} - r_j)}$$

then for $q > q_{v,n}$ we have

$$\begin{aligned} &[f(r_{j+1} a_{q_{v,n}}) - f(r_{v+1} a_{q_{v,n}}) + f(r_j a_q) - f(r_{j+1} a_q)] \\ &\leq f(z_2 a_{q_{v,n}}) - f((r_{j+1} - r_j) a_q) \leq f(z_2 a_{q_{v,n}}) - f((r_{j+1} - r_j) a_{q_{v,n}+1}) \\ &\leq f(z_2 a_{q_{v,n}}) - f\left(\left(z_2 + \frac{r_{v+1} - r_v}{2}\right) a_{q_{v,n}}\right) \leq -f\left(\frac{r_{v+1} - r_v}{2} a_{q_{v,n}}\right). \end{aligned}$$

Thus for sufficiently large n we have

$$\frac{p_j(f_n - h_n)}{p_{v+1}(f_n)} = \max_{q \neq q_{v,n}} \frac{|t_{n,q}| a_{j,q}}{p_{v+1}(f_n)} \leq \exp\left(-f\left(\frac{r_{v+1} - r_v}{2} a_{q_{v,n}}\right)\right);$$

so $\lim_n [p_j(f_n - h_n)/p_{v+1}(f_n)] = 0$ for every $j \geq v+1$. Using Lemma 2.8 we infer that $(e_{q_{v,n}})_{n \geq m_0}$ is a Schauder basic sequence in E for some $m_0 \geq n_0$, so $q_{v,n} \neq q_{v,k}$ for $n > k \geq m_0$. Thus the sequence $(q_{v,n})_{n \geq m_0}$ is strictly increasing.

(S7) The sequences (b_n) and $(a_{q_{v,n}})$ are equivalent, but the sequences $(a_{q_{v,n}})$ and $(a_{q_{v,n}})_{n \geq m_0}$ are not equivalent if $m_0 > 1$ (since $\lim_n (a_{n+1}/a_n) = \infty$). Thus the sequences (b_n) and $(a_{q_{v,n}})_{n \geq m_0}$ are not equivalent if $m_0 > 1$.

To prove that the sequence (a_n) has a subsequence (a_{q_n}) equivalent to (b_n) it is enough to show that $q_{v,m} \geq m$ for some $m \geq m_0$. Indeed, putting $q_n = n$

for $1 \leq n < m$ and $q_n = q_{v,n}$ for $n \geq m$ we get a subsequence of (a_n) that is equivalent to (b_n).

(S8) We shall prove that $q_{v,m} \geq m$ for some $m \geq m_0$.

Put $U_k = \{x \in F : l_k(x) \leq 1\}$ and $V_k = \{x \in E : p_k(x) \leq 1\}$ for $k \in \mathbb{N}$. Let $t \in \mathbb{N}$ with $p_v \leq tl_t$. For some $i > t, k > t$ and some $\alpha \in \mathbb{K}$ with $0 < |\alpha| < 1$ we have

$$\alpha U_i \subset V_{k+1} \cap F \subset V_k \cap F \subset \alpha^{-1} U_t.$$

Hence, by [14, Remark 1], and [15, Lemma 3], we get

$$|\alpha|^2 d_m(U_i, U_t) = d_m(\alpha U_i, \alpha^{-1} U_t) \leq d_m(V_{k+1} \cap F, V_k \cap F) \leq d_m(V_{k+1}, V_k),$$

for $m \in \mathbb{N}$. Let $d \geq v$ with $l_i \leq dp_d$.

Using Lemma 2.1 we obtain

$$d_m(U_i, U_t) \geq |\alpha| \frac{l_t(f_m)}{l_i(f_m)} \geq \frac{|\alpha|}{td} \frac{p_v(f_m)}{p_d(f_m)} = \frac{|\alpha|}{td} \frac{b_{v,m}}{b_{d,m}}$$

and

$$d_m(V_{k+1}, V_k) \leq |\alpha|^{-1} \frac{p_k(e_m)}{p_{k+1}(e_m)} = |\alpha|^{-1} \frac{a_{k,m}}{a_{k+1,m}}$$

for $m \in \mathbb{N}$. Thus

$$\frac{b_{v,m}}{b_{d,m}} \leq C \frac{a_{k,m}}{a_{k+1,m}}$$

for $m \in \mathbb{N}$, where $C = |\alpha|^{-4} td$. Hence, for $m \geq m_d$ we have

$$\frac{a_{k+1,m}}{a_{k,m}} \leq C \frac{a_{d,q_{v,m}}}{a_{v,q_{v,m}}},$$

so

$$\exp[f(r_{k+1} a_m) - f(r_k a_m)] \leq C \exp[f(r_d a_{q_{v,m}}) - f(r_v a_{q_{v,m}})].$$

Using Lemma 2.7 we get $\sup_m (a_m / a_{q_{v,m}}) < \infty$. Since $\lim_n (a_{n+1}/a_n) = \infty$, there is an $m \geq m_0$ such that $q_{v,m} \geq m$.

(S9) Assume that $s = \infty$. Suppose that $r < \infty$. Let $m \in \mathbb{N}$. Let $t \in \mathbb{N}$ with $p_v \leq tl_t$ and $d \in \mathbb{N}$ with $s_d - s_t > m$. Let $j > v$ with $l_d \leq jp_j$. Then

$$\frac{l_d(f_n)}{l_t(f_n)} \leq jt \frac{p_j(f_n)}{p_v(f_n)} = jt \frac{b_{j,n}}{b_{v,n}}$$

for $n \in \mathbb{N}$. Hence for $n \geq m_j$ we have

$$\frac{c_{d,n}}{c_{t,n}} \leq jt \frac{a_{j,q_{v,n}}}{a_{v,q_{v,n}}},$$

so

$$\exp[f(s_d b_n) - f(s_t b_n)] \leq jt \exp[f(r_j a_{q_{v,n}}) - f(r_v a_{q_{v,n}})].$$

Using Lemma 2.7 we get

$$\limsup_n \frac{b_n}{a_{q_{v,n}}} \leq \frac{\max\{r_j, -r_v\}}{s_d - s_t} \leq \frac{\max\{r, -r_v\}}{m}.$$

Since m is arbitrary, we infer that $\lim_n(b_n/a_{q_{v,n}}) = 0$; a contradiction, because $\sup_n(a_{q_{v,n}}/b_n) < \infty$. Thus $D_f(a,r)$ contains no isomorphic copy of $D_f(b,\infty)$, if $r < \infty$; a contradiction. Thus $r = \infty$. □

Corollary 2.10. *Let $f \in \Phi_c, r \in \{0,\infty\}$ and let $a, b \in \Gamma$ with $\lim_n(a_{n+1}/a_n) = \infty$. Then $D_f(a,r)$ contains an isomorphic copy of $D_f(b,r)$ if and only if a contains a subsequence equivalent to b.*

N. De Grande-De Kimpe has proved in [4, Proposition 4.3] that every continuous linear map from $A_1(a)$ to $A_\infty(b)$ is bounded. In [19] we have generalized this result by proving that every continuous linear map from $D_f(a,0)$ to $D_g(b,\infty)$ is bounded ([19, Theorem 8]). Since generalized power series spaces are nuclear (see [19, Proposition 2]), we get the following.

Corollary 2.11. *Let $f, g \in \Phi_c$ and $a, b \in \Gamma$. Then $D_f(a,0)$ has no quotient isomorphic to $D_g(b,\infty)$ and $D_g(b,\infty)$ has no subspace isomorphic to $D_f(a,0)$.*

We have also the following.

Proposition 2.12. *Let $f, g \in \Phi_c, a, b \in \Gamma$ and $r \in (-\infty,\infty], s \in (0,\infty]$. If f is strictly increasing and $f^{-1} \circ g$ is rapidly increasing, then every continuous linear map from $D_f(a,r)$ to $D_g(b,s)$ is bounded.*

Proof. Let $(r_k) \in \Lambda_r$ and $(s_k) \in \Lambda_s$. Denote by (e_j) and (h_i) the coordinate Schauder bases in $D_f(a,r)$ and $D_g(b,s)$, respectively. Let T be a continuous linear map from $D_f(a,r)$ to $D_g(b,s)$ with $Te_j = \sum_{i=1}^\infty t_{i,j}h_i$ for $j \in \mathbb{N}$.

By the continuity of T for every $l \in \mathbb{N}$ there exist $\sigma(l) \in \mathbb{N}$ and $C_l > 0$ such that

$$(*_1) \qquad \sup_{i,j} |t_{i,j}| \exp[g(s_l b_i) - f(r_{\sigma(l)} a_j)] \leq C_l$$

(see [19, Proposition 3]). To prove that T is bounded it is enough to show that there exists $q \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ we have

$$(*_2) \qquad \sup_{i,j} |t_{i,j}| \exp[g(s_k b_i) - f(r_q a_j)] < \infty$$

(see [19, Proposition 3]).

Put $q = \sigma(1) + 1$ and $r_0 = r_q - r_{\sigma(1)}$. Let $k \in \mathbb{N}$. The function $\gamma = f^{-1} \circ g$ is rapidly increasing, so there exists i_k such that $(1 + r_k' r_0^{-1})\gamma(s_k b_i) \leq \gamma(s_{k+1} b_i)$ for all $i > i_k$ where $r_k' = |r_{\sigma(k+1)}| + |r_q|$.

Let $i, j \in \mathbb{N}$ with $g(s_k b_i) \le f(r_0 a_j)$. Using $(*_1)$ (for $l = 1$) and Lemma A (*5*) we get

$$\begin{aligned}
&|t_{i,j}| \exp[g(s_k b_i) - f(r_q a_j)] \\
&\le C_1 \exp[f(r_{\sigma(1)} a_j) - g(s_1 b_i)] \exp[g(s_k b_i) - f(r_q a_j)] \\
&\le C_1 \exp[f(r_{\sigma(1)} a_j) - f(r_q a_j) + g(s_k b_i)] \\
&\le C_1 \exp[f(-r_0 a_j) + f(r_0 a_j)] = C_1.
\end{aligned}$$

Let $i, j \in \mathbb{N}$ with $g(s_k b_i) > f(r_0 a_j)$. Then $a_j < r_0^{-1} \gamma(s_k b_i)$. If $i \le i_k$, then

$$g(s_k b_i) - g(s_{k+1} b_i) + f(r'_k a_j) \le f(r'_k r_0^{-1} \gamma(s_k b_{i_k})).$$

If $i > i_k$, then $r'_k a_j \le \gamma(s_{k+1} b_i) - \gamma(s_k b_i)$; so

$$g(s_k b_i) - g(s_{k+1} b_i) + f(r'_k a_j) = f(\gamma(s_k b_i)) - f(\gamma(s_{k+1} b_i)) + f(r'_k a_j) \le 0.$$

Thus using $(*_1)$ (for $l = k + 1$) we obtain

$$\begin{aligned}
&|t_{i,j}| \exp[g(s_k b_i) - f(r_q a_j)] \\
&\le C_{k+1} \exp[f(r_{\sigma(k+1)} a_j) - g(s_{k+1} b_i)] \exp[g(s_k b_i) - f(r_q a_j)] \\
&\le C_{k+1} \exp[g(s_k b_i) - g(s_{k+1} b_i) + f(r'_k a_j)] \\
&\le C_{k+1} \exp f(r'_k r_0^{-1} \gamma(s_k b_{i_k})).
\end{aligned}$$

It follows that

$$|t_{i,j}| \exp[g(s_k b_i) - f(r_q a_j)] \le \max\{C_1, C_{k+1} \exp f(r'_k r_0^{-1} \gamma(s_k b_{i_k}))\}$$

for all $i, j \in \mathbb{N}$. We have shown $(*_2)$. □

Corollary 2.13. *Let $f, g \in \Phi_c, a, b \in \Gamma$ and $r \in (-\infty, \infty], s \in (0, \infty]$. If f is strictly increasing and $f^{-1} \circ g$ is rapidly increasing, then $D_f(a, r)$ has no quotient isomorphic to $D_g(b, s)$ and $D_g(b, s)$ has no subspace isomorphic to $D_f(a, r)$.*

3. When $D_f(a, r)$ and $D_g(b, s)$ are isomorphic

We have the following.

Proposition 3.1. *Let $f \in \Phi_c$ and $a, b \in \Gamma$. If (1) $r \in (-\infty, \infty]$ and a, b are strictly equivalent, or (2) $r \in \{0, \infty\}$ and a, b are equivalent, then the spaces $D_f(a, r)$ and $D_f(b, r)$ are isomorphic.*

Proof. (*1*) For every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $r_k a_n < r_{k+1} b_n$ and $r_k b_n < r_{k+1} a_n$ for all $n \ge n_k$. Hence $\sup_n \exp[f(r_k a_n) - f(r_{k+1} b_n)] < \infty$ and $\sup_n \exp[f(r_k b_n) - f(r_{k+1} a_n)] < \infty$ for every $k \in \mathbb{N}$. It follows that $D_f(a, r) = D_f(b, r)$ and the identity map $I : D_f(a, r) \to D_f(b, r)$ is an isomorphism.

(*2*) For some $C > 1$ we have $C^{-1} a_n \le b_n \le C a_n$ for $n \in \mathbb{N}$. For every $k \in \mathbb{N}$ there exists $t(k) \in \mathbb{N}$ such that $r_{t(k)} \ge C r_k$ if $r = \infty$ and $r_{t(k)} \ge C^{-1} r_k$ if $r = 0$.

Hence $f(r_k b_n) \leq f(r_{t(k)} a_n)$ and $f(r_k a_n) \leq f(r_{t(k)} b_n)$ for all $n \in \mathbb{N}$.
Let $x \in D_f(a,r)$. Then $|x_n| \exp f(r_k b_n) \leq |x_n| \exp f(r_{t(k)} a_n)$ for $n \in \mathbb{N}$. Thus $\lim_n |x_n| \exp f(r_k b_n) = 0$ for every $k \in \mathbb{N}$; so $x \in D_f(b,r)$ and $q_k(x) \leq p_{t(k)}(x)$ for $k \in \mathbb{N}$, where (p_k) and (q_k) are the standard bases of continuous norms on $D_f(a,r)$ and $D_f(b,r)$, respectively. Similarly, for every $x \in D_f(b,r)$ we have $x \in D_f(a,r)$ and $p_k(x) \leq q_{t(k)}(x)$. Thus the identity map $I : D_f(a,r) \to D_f(b,r)$ is an isomorphism. $\square$

If $a, b \in \Gamma$ and $\limsup_n (a_{n+1}/a_n) < \liminf_n (b_{n+1}/b_n)$ or $\sup_n [(a_n/b_n) + (a_{2n}/a_n)] < \infty$, then the sequence a has a subsequence equivalent to b ([15, Lemma 7]. Thus, as in the proof of Proposition 3.1, we get

Corollary 3.2. *Let $f \in \Phi_c$ and $a, b \in \Gamma$. Let $r \in \{0, \infty\}$. Assume that $\sup_n [(a_n/b_n) + (a_{2n}/a_n)] < \infty$ or $\limsup_n (a_{n+1}/a_n) < \liminf_n (b_{n+1}/b_n)$. Then $D_f(a,r)$ has a complemented subspace isomorphic to $D_f(b,r)$. In particular, $D_f(a,r)$ has a quotient isomorphic to $D_f(b,r)$.*

By Proposition 2.3 we get the following.

Corollary 3.3. *Let $f \in \Phi_c$ and $a, b \in \Gamma$. Let $r \in \{0, \infty\}$. Then the spaces $D_f(a,r)$ and $D_f(b,r)$ are isomorphic if and only if the sequences a and b are equivalent.*

Corollary 3.4. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$ and $r \in \mathbb{R}$. Then the spaces $D_f(a,r)$ and $D_f(b,\infty)$ are not isomorphic.*

By Corollary 2.4 we get our next corollary.

Corollary 3.5. *Let $f \in \Phi_c$. Let $a, b \in \Gamma$ and $r, s \in (-\infty, \infty]$. If the sequences a and b are not equivalent, then the spaces $D_f(a,r)$ and $D_f(b,s)$ are not isomorphic.*

Using Lemma 2.2 and Corollaries 3.3, 3.4 and 3.5 we get

Proposition 3.6. *Let $a, b, c, d \in \Gamma$. If $f \in \Phi_c$ is rapidly increasing, then the spaces $D_f(a,-1), D_f(b,0), D_f(c,1)$ and $D_f(d,\infty)$ are pairwise non-isomorphic.*

Proof. Suppose that $D_f(a,-1)$ is isomorphic to $D_f(b,0)$. By Corollary 3.5, the sequences a and b are equivalent. Hence, by Corollary 3.3, $D_f(b,0)$ is isomorphic to $D_f(a,0)$, so $D_f(a,-1)$ is isomorphic to $D_f(a,0)$. Let $(s_k) \subset (-1,0)$ be a strictly increasing sequence with $\lim s_k = 0$. Let $(r_k) \in \Lambda_{-1}$. By Lemma 2.2, there exist $k, t \in \mathbb{N}$ and $C > 0$ such that

$$f(-r_k a_n) + f(-s_t a_n) \leq C + f(-r_{k+1} a_n) + f(-s_1 a_n)$$

for $n \in \mathbb{N}$. Hence

$$1 + \frac{f(-s_t a_n)}{f(-r_k a_n)} \leq \frac{C}{f(-r_k a_n)} + \frac{f(-r_{k+1} a_n)}{f(-r_k a_n)} + \frac{f(-s_1 a_n)}{f(-r_k a_n)} \tag{$*$}$$

for $n \in \mathbb{N}$. Since $0 < -s_t \leq -s_1 < 1 < -r_{k+1} < -r_k$, tending n to ∞ in $(*)$, we get $1 \leq 0$; a contradiction. Thus $D_f(a,-1)$ is not isomorphic to $D_f(b,0)$. By Corollary 3.4, the spaces $D_f(c,1)$ and $D_f(d,\infty)$ are not isomorphic. Now, using [19, Theorem 8] we complete the proof. □

By Corollary 2.6 the spaces $D_f(a,0)$ and $D_g(b,\infty)$ are not isomorphic. Using Lemma 2.2 we get the following.

Proposition 3.7. *Let $f, g \in \Phi_c, a, b \in \Gamma$ and $r, s \in (-\infty, \infty]$. Assume that f is strictly increasing and $f^{-1} \circ g$ is rapidly increasing. Then the spaces $D_f(a,r)$ and $D_g(b,s)$ are not isomorphic.*

Proof. It is easy to see that g is rapidly increasing. Suppose, by contradiction, that $D_f(a,r)$ is isomorphic to $D_g(b,s)$. Put $\varphi = f^{-1} \circ g$.

By Proposition 2.12 we can assume that $s \leq 0$. By Lemma 2.2, there exist integers $k, t, C \in \mathbb{N}$ and $K, T, D \in \mathbb{N}$ with $T > k+1$ such that

$$f(r_1 a_n) - f(r_t a_n) + g(s_{k+1} b_n) - g(s_k b_n) \leq C, \quad n \in \mathbb{N}$$

and

$$g(s_{k+1} b_n) - g(s_T b_n) + f(r_{K+1} a_n) - f(r_K a_n) \leq D, \quad n \in \mathbb{N}.$$

Put $r_0 = \max\{r_t, -r_1\}$ and $\widehat{r_0} = r_{K+1} - r_K$. Then we get

$$1 \leq \frac{g(-s_{k+1} b_n)}{g(-s_k b_n)} + \frac{C}{g(-s_k b_n)} + \frac{f(r_0 a_n)}{g(-s_k b_n)}$$

and

$$1 \geq \frac{g(-s_T b_n) - D}{g(-s_{k+1} b_n)} + \frac{f(\widehat{r_0} a_n)}{g(-s_{k+1} b_n)}$$

for $n \in \mathbb{N}$. Hence there exists $n_0 \in \mathbb{N}$ such that $1/2 \leq f(r_0 a_n)/g(-s_k b_n)$ and $f(\widehat{r_0} a_n)/g(-s_{k+1} b_n) < 1$ for $n \geq n_0$. Thus

$$f(2 r_0 a_n) \geq 2 f(r_0 a_n) \geq g(-s_k b_n)$$

and $f(\widehat{r_0} a_n) < g(-s_{k+1} b_n)$ for $n \geq n_0$, so $2 r_0 a_n \geq \varphi(-s_k b_n)$ and $\widehat{r_0} a_n < \varphi(-s_{k+1} b_n)$ for $n \geq n_0$. Hence

$$\frac{\varphi(-s_k b_n)}{\varphi(-s_{k+1} b_n)} < \frac{2 r_0}{\widehat{r_0}}$$

for $n \geq n_0$; a contradiction, since $\lim_n \varphi(-s_k b_n)/\varphi(-s_{k+1} b_n) = \infty$. □

Corollary 3.8. *Let $r \in (-\infty, \infty]$, $p \in \{1, \infty\}$ and $a, b \in \Gamma$. If $f \in \Phi_c$ is rapidly increasing, then $D_f(a,r)$ is not isomorphic to the power series space $A_p(b)$.*

4. Summary

In the last part of this paper we summarize our results in a table.

For Fréchet spaces X and Y we write

(1) $X \hookrightarrow Y$ if X is isomorphic to a subspace of Y;
(2) $X \rightsquigarrow Y$ if X has a quotient isomorphic to Y;
(3) $X \simeq Y$ if X is isomorphic to Y.

For sequences $a = (a_n)$, $b = (b_n)$ of positive numbers we write

(1) $a \sim b$ if a and b are equivalent;
(2) $a \approx b$ if a and b are strictly equivalent;
(3) $a \prec b$ if a is equivalent to a subsequence of b.

The abbreviations s.i. and r.i. denote *strictly incresing* and *rapidly increasing*, respectively.

For $f, g \in \Phi_c, a, b, c, d \in \Gamma$ and $r, s \in (-\infty, \infty]$ we have shown the following results.

	Assumptions	Thesis
1.1	$\sup_n(a_n/b_n) < \infty; \sup_n(a_{2n}/a_n) < \infty$	$D_f(b,\infty) \hookrightarrow D_f(a,0)$
1.2	$\sup_n(a_{n+1}/a_n) < \infty;\ \lim_n(b_{n+1}/b_n) = \infty$	$D_f(b,\infty) \hookrightarrow D_f(a,0)$
1.3	$\lim_n(a_n/b_n) = 0;\ \sup_n(a_{2n}/a_n) < \infty$	$D_f(a,\infty) \rightsquigarrow D_f(b,0)$
1.4	$\sup_n(a_{n+1}/a_n) < \infty;\ \lim_n(b_{n+1}/b_n) = \infty$	$D_f(a,\infty) \rightsquigarrow D_f(b,0)$
2.3	$D_f(b,s) \hookrightarrow D_f(a,r)$ or $D_f(a,r) \rightsquigarrow D_f(b,s)$	$\sup_n(a_n/b_n) < \infty$
2.3	$s < \infty; D_f(b,s) \hookrightarrow D_f(a,\infty)$	$\lim_n(a_n/b_n) = 0$
2.4	$D_f(a,r) \hookrightarrow D_f(b,s) \hookrightarrow D_f(a,r)$	$a \sim b$
2.4	$D_f(a,r) \rightsquigarrow D_f(b,s) \rightsquigarrow D_f(a,r)$	$a \sim b$
2.6	$\sup_n(a_{2n}/a_n) < \infty; D_f(a,\infty) \rightsquigarrow D_f(b,0)$	$\lim_n(a_n/b_n) = 0$
2.9	$\lim_n(a_{n+1}/a_n) = \infty;\ D_f(b,s) \hookrightarrow D_f(a,r)$	$b \prec a$
2.11		$D_f(a,0) \not\hookrightarrow D_g(b,\infty)$
2.11		$D_f(a,0) \not\rightsquigarrow D_g(b,\infty)$
2.13	$s > 0; f$ is s.i.; $f^{-1} \circ g$ is r.i.	$D_f(a,r) \not\hookrightarrow D_g(b,s)$
2.13	$s > 0; f$ is s.i.; $f^{-1} \circ g$ is r.i.	$D_f(a,r) \not\rightsquigarrow D_g(b,s)$
3.1	$a \approx b$	$D_f(a,r) \simeq D_f(b,r)$
3.3	$r \in \{0,\infty\};\ a \sim b$	$D_f(a,r) \simeq D_f(b,r)$
3.4	$r < \infty$	$D_f(a,r) \not\simeq D_f(b,\infty)$
3.7	f is s.i. ; $f^{-1} \circ g$ is r.i.	$D_f(a,r) \not\simeq D_g(b,s)$
3.8	f is r.i.; $p \in \{1;\infty\}$	$D_f(a,r) \not\simeq A_p(b)$

Acknowledgements. The author wishes to thank the referee for very useful remarks and suggesting improvements.

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