Journal of Convex Analysis
Volume 22 (2015), No. 2, 591–601

On the Equivalence of the Theorems of Helly, Radon, and Carathéodory via Convex Analysis

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Received: May 31, 2014
Revised manuscript received: June 13, 2014
Accepted: June 14, 2014

Helly's theorem is an important result from Convexity and Combinatorial Geometry. It gives sufficient conditions for a family of convex sets to have a nonempty intersection. A large variety of proofs as well as applications are known. Helly's theorem has close connections to two other well-known theorems: Radon's theorem and Carathéodory's theorem. In this paper we study Helly's theorem and its relations to Radon's theorem and Carathéodory's theorem by using tools of Convex Analysis and Optimization. More precisely, we will give a new proof of Helly's theorem, and in addition we show in a complete way that these three theorems are equivalent in the sense that using one of them allows us to derive the others.

Keywords: Carathéodory's theorem, convex function, convex hull, distance function, Helly's theorem, Radon's theorem, subdifferential, subgradient

2000 Mathematics Subject Classification: 52A05, 52A20, 52A35, 52A37

1. Introduction

In *Convexity* geometric properties of convex sets and functions are investigated. The foundations of this field were developed by many accomplished mathematicians such as Hermann Brunn, Constantin Carathéodory, Werner Fenchel, Eduard Helly, and Hermann Minkowski; see [8]. At the beginning of the 1960's *Convex Analysis* was grown out of Convexity, and this new field was systemat-

*The research of Nguyen Mau Nam was partially supported by the Simons Foundation under grant #208785.

ISSN 0944-6532 / $ 2.50

ically developed by the works of Jean-Jacques Moreau, R. Tyrrell Rockafellar, and others; see, e.g., [11, 12, 13, 14, 15, 16, 17]. Convex Analysis is more concerned with the generalized differentiation theory of convex functions and sets rather than only with their geometric properties. The presence of convexity makes it possible to develop calculus rules for a generalized derivative concept called the *subdifferential*, which can be used to deal with convex functions that are not necessarily differentiable. Convex Analysis then becomes the mathematical fundament for Convex Optimization, a fast growing field with numerous applications.

In this paper we use Convex Analysis and Optimization to study some basic results of Convexity and Combinatorial Geometry. We mainly focus on a theorem introduced by Eduard Helly in 1913 which gives sufficient conditions for a family of convex sets to have a nonempty intersection. In particular, we will give a new proof, via Convex Analysis, of Helly's theorem and its connection to Radon's theorem and Carathéodory's theorem, which are also important results from Combinatorial Geometry.

It has been mentioned in several references that the above-mentioned theorems of Helly, Radon, and Carathéodory are equivalent in the sense that using one of them allows us to derive the others. However, in [9, p. 47] P. M. Gruber says that "*we were not able to locate in the literature a complete proof in the context of* $\mathbb{R}^n$"; see also the excellent surveys [5] and [6], as well as the monograph [3]. In this paper we provide a complete treatment for the so meant equivalence of these theorems. The analysis involves the usage of generalized differentiation properties of distance functions. This class of functions, which reflects the connection between convex functions and sets, allows us to give a simple, self-contained, complete proof of the described threefold equivalence.

2. Notions from Convex Analysis and Optimization

In this section we introduce some important concepts and notions from Convex Analysis and Optimization that will be used in the subsequent sections. The material presented here can be found in many books on Convex Analysis; see, e.g., [10, 17] and the references therein.

Throughout the paper we consider the Euclidean space $\mathbb{R}^n$ equipped with the Euclidean norm $\|\cdot\|$ and the usual inner product $\langle\cdot,\cdot\rangle$. By $\mathbb{B}(x;t)$ we denote the *closed Euclidean ball* with center x and radius $t>0$.

A subset Ω of $\mathbb{R}^n$ is said to be *convex* if

$$\lambda x + (1-\lambda)y \in \Omega$$

whenever $x, y \in \Omega$ and $\lambda \in (0,1)$. Geometrically, a set Ω is convex if for any two points $x, y \in \Omega$ the line segment connecting x and y belongs to the set. Let

$\lambda_i \geq 0$ and suppose $\lambda_1 + \cdots + \lambda_m = 1$. Then

$$y := \sum_{i=1}^{m} \lambda_i x_i$$

is called a *convex combination* of the points $x_1, \ldots, x_m$.

From these definitions it is obvious that if $\{\Omega_i\}_{i \in I}$ is a collection of convex sets in $\mathbb{R}^n$, then the intersection $\bigcap_{i \in I} \Omega_i$ is also a (possibly empty) convex set. This property motivates the definition of the convex hull of an arbitrary subset of $\mathbb{R}^n$. Given a subset $\Omega \subset \mathbb{R}^n$, the *convex hull* of Ω is defined by

$$\operatorname{co}\Omega := \bigcap \{C \mid C \text{ is convex and } \Omega \subset C\}.$$

Equivalently, the convex hull of a set Ω is the smallest (regarding inclusion) convex set containing Ω.

The following important result is a direct consequence of the definition of convex hull; see, e.g., [10, Proposition 1.20].

Proposition 2.1. *For any subset Ω of $\mathbb{R}^n$, its convex hull consists precisely of all convex combinations of elements of Ω.*

A function f defined on $\Omega \subset \mathbb{R}^n$ is called a *convex function* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all $x, y \in \Omega$ and $\lambda \in (0, 1)$. If this inequality becomes strict for any $x \neq y$, $x, y \in \Omega$, we say that f is *strictly convex* on Ω.

The concept of distance functions presented now plays a crucial role throughout our paper. Given a set $\Omega \subset \mathbb{R}^n$ and some $x \in \mathbb{R}^n$, the *distance function* associated with Ω is defined by

$$d(x; \Omega) := \inf \big\{ \|x - \omega\| \;\big|\; \omega \in \Omega \big\}.$$

For each $x \in \mathbb{R}^n$, the *Euclidean projection* from x to Ω is defined by

$$\Pi(x; \Omega) := \big\{ \omega \in \Omega \;\big|\; \|x - \omega\| = d(x; \Omega) \big\}.$$

It is well-known that if $\Omega \subset \mathbb{R}^n$ is a nonempty closed convex set, then $\Pi(x; \Omega)$ is a singleton for every $x \in \mathbb{R}^n$.

The convexity of the distance function follows directly from the convexity of the associated set as in the proposition below; see, e.g., [10, Proposition 1.77].

Proposition 2.2. *If Ω is a nonempty convex set, then the distance function $d(\cdot; \Omega)$ is a convex function.*

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\bar{x} \in \mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is called a *subgradient* of f at $\bar{x}$ if

$$\langle v, x - \bar{x} \rangle \le f(x) - f(\bar{x}) \text{ for all } x \in \mathbb{R}^n.$$

The set of all subgradients of f at $\bar{x}$ is called the *subdifferential* of the function at this point and denoted by $\partial f(\bar{x})$.

Given a nonempty convex set $\Omega \subset \mathbb{R}^n$ and a point $\bar{x} \in \Omega$, the *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x}; \Omega) := \{v \in \mathbb{R}^n \mid \langle v, x - \bar{x} \rangle \le 0 \text{ for all } x \in \Omega\}.$$

In what follows we present subdifferential formulas for distance functions associated with convex sets. The formulas depend on whether the reference point belongs to the set; see, e.g., [10, Proposition 2.39].

Proposition 2.3. *Let Ω be a nonempty closed convex set. If $\bar{x} \in \Omega$, then*

$$\partial d(\bar{x}; \Omega) = N(\bar{x}; \Omega) \cap \mathbb{B},$$

where $\mathbb{B}$ is a closed unit ball of $\mathbb{R}^n$.

If $\bar{x} \notin \Omega$, then

$$\partial d(\bar{x}; \Omega) = \left\{ \frac{\bar{x} - \Pi(\bar{x}; \Omega)}{d(\bar{x}; \Omega)} \right\}.$$

The proposition below gives a subdifferential formula for the maximum function of a finite number of convex functions; see, e.g., [10, Proposition 2.54].

Theorem 2.4. *Suppose that $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are convex functions. Then*

$$\partial(\max f_i)(\bar{x}) = \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial f_i(\bar{x}),$$

where $I(\bar{x}) := \{i = 1, \ldots, m \mid f_i(\bar{x}) = f(\bar{x})\}$.

3. The Theorems of Carathéodory and Helly

In this section we study the equivalence of the theorems of Carathéodory and Helly based on subdifferential properties of distance functions. Let us start with two useful lemmas.

Lemma 3.1. *Let Ω_i, $i = 1, \ldots, m$, be nonempty closed convex sets. If at least one of the sets Ω_i is bounded, then the maximum function*

$$f(x) := \max\{d(x; \Omega_i) \mid i = 1, \ldots, m\}, \quad x \in \mathbb{R}^n, \tag{1}$$

has an absolute minimum on $\mathbb{R}^n$.

Proof. Let $\gamma := \inf\{f(x) \mid x \in \mathbb{R}^n\}$. Since $f(x) \geq 0$ for all $x \in \mathbb{R}^n$, it is obvious that γ is a real number. Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$ such that $\lim_{k\to\infty} f(x_k) = \gamma$. Without loss of generality, assume that Ω_1 is bounded. By the definition of limit, one finds $k_0 \in \mathbb{N}$ such that

$$0 \leq d(x_k; \Omega_1) \leq f(x_k) < \gamma + 1 \quad \text{for all } k \geq k_0.$$

Choosing $w_k \in \Omega_1$ with $\|x_k - w_k\| = d(x_k; \Omega)$, we get $\|x_k - w_k\| \leq \gamma + 1$, and hence $\|x_k\| \leq \|w_k\| + 1$ for all $k \geq k_0$. Since Ω_1 is bounded, the sequence $\{x_k\}$ is bounded as well, and so we can assume that it has a subsequence $\{x_{k_\ell}\}$ that converges to $\bar{x}$. By the continuity of f,

$$\gamma = \lim_{\ell\to\infty} f(x_{k_\ell}) = f(\bar{x}) \leq f(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Therefore, f has an absolute minimum at $\bar{x}$. □

Given any $u \in \mathbb{R}^n$, define the active index set at u, associated with the function f given in (1), by

$$I(u) := \{i = 1, \ldots, m \mid d(u; \Omega_i) = f(u)\}.$$

Lemma 3.2. *Let Ω_i, $i = 1, \ldots, m$, be nonempty closed convex sets with $\bigcap_{i=1}^m \Omega_i = \emptyset$. Consider the function f defined in* (1)*. Then f has an absolute minimum at $\bar{x}$ if and only if*

$$\bar{x} \in \operatorname{co}\{w_i \mid i \in I(\bar{x})\},$$

where $w_i := \Pi(\bar{x}; \Omega_i)$.

Proof. The assumption $\bigcap_{i=1}^m \Omega_i = \emptyset$ implies that $f(x) = \max\{d(x; \Omega_i) \mid i = 1, \ldots, m\} > 0$ for all $x \in \mathbb{R}^n$ and $x \notin \Omega_i$ for every $i \in I(x)$. It follows from Theorem 2.4 that the function f has an absolute minimum at $\bar{x}$ if and only if

$$0 \in \partial f(\bar{x}) = \operatorname{co}\{\partial d(\bar{x}; \Omega_i) \mid i \in I(\bar{x})\} = \operatorname{co}\left\{\frac{\bar{x} - w_i}{d(\bar{x}; \Omega_i)} \mid i \in I(\bar{x})\right\}.$$

Since $f(\bar{x}) = d(\bar{x}; \Omega_i) > 0$ for all $i \in I(\bar{x})$, we can denote this common value by r. By Proposition 2.1, there exist $\lambda_i \geq 0$ for $i \in I(\bar{x})$ with $\sum_{i\in I(\bar{x})} \lambda_i = 1$ such that

$$0 = \sum_{i\in I(\bar{x})} \lambda_i \frac{\bar{x} - w_i}{r},$$

which is equivalent to $\bar{x} \in \operatorname{co}\{w_i \mid i \in I(\bar{x})\}$. □

Now we are ready to prove the main theorem of this section, namely showing that Carathéodory's theorem and Helly's theorem are equivalent in the described sense.

Theorem 3.3. *Consider the following statements:*

(*i*) *Carathéodory's theorem: For any nonempty set* $\Omega \subset \mathbb{R}^n$ *with* $\bar{y} \in \operatorname{co}\Omega$*, there exist* $\lambda_i \geq 0$ *and* $w_i \in \Omega$ *for* $i = 1, \ldots, n+1$ *such that*

$$\bar{y} = \sum_{i=1}^{n+1} \lambda_i \omega_i \,.$$

(*ii*) *Helly's theorem: For any collection of nonempty closed convex sets* $\{\Omega_1, \ldots, \Omega_m\}$*,* $m \geq n+2$*, in* $\mathbb{R}^n$ *with the property that the intersection of any* $n+1$ *sets from this collection is nonempty, one has*

$$\bigcap_{i=1}^{m} \Omega_i \neq \emptyset \,.$$

Then statement (*i*) *implies statement* (*ii*)*, and vice versa.*

Proof. Let us first prove (ii) assuming that (i) holds. Consider the case where Ω_i, for $i = 1, \ldots, m$, are nonempty closed convex sets such that at least one of them is bounded. The assumption implies that the intersection of any collection of k sets from the whole collection, where $k \leq n+1$, is nonempty. Suppose that $\bigcap_{i=1}^{m} \Omega_i = \emptyset$. Consider the function f defined by (1) and let $\bar{y}$ be a point in $\mathbb{R}^n$ at which f has an absolute minimum. Note that this point exists by Lemma 3.1. From Lemma 3.2 one has

$$\bar{y} \in \operatorname{co}\{w_i \mid i \in I(\bar{y})\} \,,$$

where $w_i := \Pi(\bar{y}; \Omega_i)$. Applying Carathéodory's theorem, there exist $J \subset I(\bar{y})$, $|J| \leq n+1$, and $\lambda_i \geq 0$ for $i \in J$ with $\sum_{j \in J} \lambda_j = 1$ such that

$$\bar{y} = \sum_{j \in J} \lambda_j w_j.$$

Defining the function

$$g(x) := \max\{d(x; \Omega_j) \mid j \in J\},$$

by the given assumption, $\bigcap_{j \in J} \Omega_j \neq \emptyset$, there exists a point u in this intersection with $g(u) = 0$. Since $d(\bar{y}; \Omega_j) = r > 0$ for all $j \in J$, where $r := f(\bar{y}) = g(\bar{y}) > 0$, the active index set at $\bar{y}$ associated with the function g is J. By Lemma 3.2, the function g has an absolute minimum at $\bar{y}$, and hence $0 < r = g(\bar{y}) \leq g(u) = 0$. This contradiction implies the statement (ii).

In the case where we do not assume that at least one of the sets is bounded, we choose an element in each intersection of $n+1$ sets. Let $t > 0$ be sufficiently large such that the closed ball $\mathbb{B}(0; t)$ covers all such points. Then we only need to apply the previous case for the collection $\{\Theta_i \mid i = 1, \ldots, m\}$, where $\Theta_i := \Omega_i \cap \mathbb{B}(0; t)$.

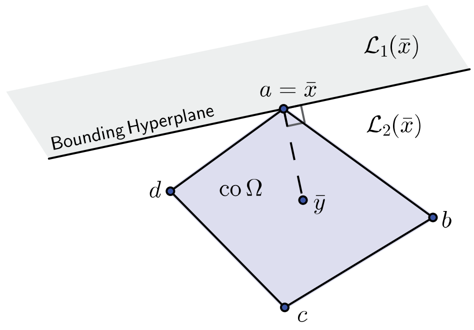

Figure 3.1: Proof illustration: Helly's theorem implies Carathéodory's theorem

Now we show how to derive (i) from (ii). Fix any element $\bar{y} \in \operatorname{co}\Omega$. If $\bar{y} \in \Omega$, then it is obviously a convex combination of itself, so the conclusion is obvious. Suppose that $\bar{y} \notin \Omega$.

We follow and simplify significantly the proof in [7], pp. 40–41. In particular, we do not assume that $\Omega \subset \mathbb{R}^n$ is closed and bounded as in [7], pp. 40–41. For each point $\bar{x} \in \Omega$, denote by $\mathcal{L}_1(\bar{x})$ and $\mathcal{L}_2(\bar{x})$ the closed half-spaces with bounding hyperplane passing through $\bar{x}$ and being perpendicular to the line connecting $\bar{x}$ and $\bar{y}$, where $\mathcal{L}_2(\bar{x})$ contains $\bar{y}$, i.e.,

$$\mathcal{L}_1(\bar{x}) := \{w \in \mathbb{R}^n \mid \langle \bar{x} - \bar{y}, w - \bar{x}\rangle \geq 0\},$$
$$\mathcal{L}_2(\bar{x}) := \{w \in \mathbb{R}^n \mid \langle \bar{x} - \bar{y}, w - \bar{x}\rangle \leq 0\}.$$

Let us now show that

$$\bigcap_{\bar{x}\in\Omega} \mathcal{L}_1(\bar{x}) = \emptyset.$$

Indeed, suppose that this set is nonempty. Then there exists $\bar{z} \in \bigcap_{\bar{x}\in\Omega} \mathcal{L}_1(\bar{x})$, implying that

$$\langle x - \bar{y}, \bar{z} - x\rangle \geq 0 \ \text{ for all } x \in \Omega\,.$$

It follows that

$$\begin{aligned}\langle \bar{z} - \bar{y}, \bar{y} - x\rangle &= \langle \bar{z} - x + x - \bar{y}, \bar{y} - x\rangle = \langle \bar{z} - x, \bar{y} - x\rangle + \langle x - \bar{y}, \bar{y} - x\rangle \\ &= \langle \bar{z} - x, \bar{y} - x\rangle - \|x - \bar{y}\|^2 < 0\end{aligned}$$

for all $x \in \Omega$ (note that $\|x - \bar{y}\|^2 > 0$, since $\bar{y} \notin \Omega$). Since $\bar{y} \in \operatorname{co}\Omega$, there exist $x_i \in \Omega$ and $\lambda_i \geq 0$ for $i = 1, \ldots, m$ such that

$$\bar{y} = \sum_{i=1}^{m} \lambda_i x_i \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = 1.$$

Then

$$0 = \langle \bar{z} - \bar{y}, \bar{y} - \bar{y}\rangle = \left\langle \bar{z} - \bar{y}, \bar{y} - \sum_{i=1}^{m} \lambda_i x_i \right\rangle = \sum_{i=1}^{m} \lambda_i \langle \bar{z} - \bar{y}, \bar{y} - x_i\rangle < 0\,.$$

This contradiction shows that

$$\bigcap_{\bar{x}\in\Omega} \mathcal{L}_1(\bar{x}) = \emptyset.$$

Since $\mathcal{L}_1(x)$ is a nonempty closed convex set for every x, by (ii) there exist $x_1, \ldots, x_{n+1} \in \Omega$ such that

$$\bigcap_{i=1}^{n+1} \mathcal{L}_1(x_i) = \emptyset.$$

However, this implies $\bar{y} \in \operatorname{co}\{x_1, \ldots, x_{n+1}\}$. Let us prove this by contradiction. Assuming that this is not the case, there exist $a, b \in \mathbb{R}^n$ such that

$$\langle a, \bar{y}\rangle > b \quad \text{and} \quad \langle a, x_i\rangle \leq b \quad \text{for all } i = 1, \ldots, n+1.$$

Then

$$\langle x_i - \bar{y}, \bar{y} - ta - x_i\rangle = \langle x_i - \bar{y}, \bar{y} - x_i\rangle - t\langle a, x_i - \bar{y}\rangle\,.$$

Thus we can find a value t_0 such that this expression is positive for all $t > t_0$ and for all i, but this implies $\bigcap_{i=1}^{n+1} \mathcal{L}_1(x_i) \neq \emptyset$. This contradiction shows that $\bar{y} \in \operatorname{co}\{x_1, \ldots, x_{n+1}\}$. □

4. The Theorems of Carathéodory and Radon

In this section we study the equivalence of the theorems of Carathéodory and Radon, thus establishing the equivalence of all three theorems.

Theorem 4.1. *Consider the following statements:*

(i) *Carathéodory's theorem: For any nonempty set $\Omega \subset \mathbb{R}^n$ with $\bar{y} \in \operatorname{co}\Omega$, there exist $\lambda_i \geq 0$ and $w_i \in \Omega$ for $i = 1, \ldots, n+1$ such that*

$$\bar{y} = \sum_{i=1}^{n+1} \lambda_i \omega_i\,.$$

(ii) *Radon's theorem: Given any set $\Omega := \{\omega_1, \ldots, \omega_m\}$, $m \geq n+2$, in $\mathbb{R}^n$, there exist two nonempty, disjoint subsets $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ such that*

$$\Omega_1 \cup \Omega_2 = \Omega \quad \text{and} \quad \operatorname{co}\Omega_1 \cap \operatorname{co}\Omega_2 \neq \emptyset\,.$$

The statements (i) and (ii) are equivalent.

Proof. $(i) \Longrightarrow (ii)$: Consider the set $\Omega \subset \mathbb{R}^n$ defined by $\Omega = \{w_1, \ldots, w_m\}$ with $m \geq n+2$. Then

$$\frac{w_1 + \cdots + w_m}{m} \in \operatorname{co}\{w_1, \ldots, w_m\}\,.$$

By Carathéodory's theorem, with rearranging the indices if necessary, we have the representation

$$\frac{w_1 + \cdots + w_m}{m} = \sum_{i=1}^{n+1} \lambda_i w_i,$$

with $\lambda_i \geq 0$ and $\sum_{i=1}^{n+1} \lambda_i = 1$. It follows that

$$\sum_{i=1}^{m} \beta_i w_i = \sum_{i=1}^{m} \lambda_i w_i\,,$$

where $\beta_i = \frac{1}{m}, i = 1, \ldots, m$, $\lambda_i \geq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, and $\lambda_i = 0$ for all $i = n+2, \ldots, m$. So

$$\sum_{i=1}^{m} (\beta_i - \lambda_i) w_i = 0\,.$$

We rewrite the above as

$$\sum_{i=1}^{m} \gamma_i w_i = 0\,,$$

with $\gamma_i = \beta_i - \lambda_i$, $\sum_{i=1}^{m} \gamma_i = 0$, where not all $\gamma_i's$ are zeros. Then

$$\sum_{i \in I} \gamma_i w_i + \sum_{j \in J} \gamma_j w_j = 0\,,$$

where $I := \{i \in \{1, \ldots, m\} \mid \gamma_i > 0\}$ and $J := \{i \in \{1, \ldots, m\} \mid \gamma_i \leq 0\}$. Observe that I and J are both nonempty. Define $\gamma := \sum_{i \in I} \gamma_i$, $\Omega_1 := \{w_i \mid i \in I\}$, and $\Omega_2 := \{w_j \mid j \in J\}$. Then $\gamma = -\sum_{j \in J} \gamma_j$ and

$$\frac{1}{\gamma} \sum_{i \in I} \gamma_i w_i = -\frac{1}{\gamma} \sum_{j \in J} \gamma_j w_j \in \operatorname{co} \Omega_1 \cap \operatorname{co} \Omega_2.$$

At this point, we can see that Ω_1 and Ω_2 satisfy the requirements of Radon's theorem.

To prove that Carathéodory's theorem follows from Radon's theorem, we complete the approach to [9, Theorem 3.2].

(ii) $\Longrightarrow$ *(i)*: Fix any $\bar{x} \in \operatorname{co} \Omega$. Then there exist $\lambda_i > 0$ for $i = 1, \ldots, m$, with $\sum_{i=1}^{m} \lambda_i = 1$ and

$$\bar{x} = \sum_{i=1}^{m} \lambda_i w_i\,,$$

where $w_i \in \Omega$ for all $i = 1, \ldots, m$ and the representation is chosen such that m is minimal. We will show that $m \leq n + 1$. Assume the contrary, i.e., that

$m > n+1$. By Radon's theorem applied to the set $\Omega := \{w_1, \ldots, w_m\}$ we can assume that, without loss of generality,

$$\sum_{i=1}^{k} \gamma_i w_i = \sum_{i=k+1}^{m} \gamma_i w_i \,,$$

where $\gamma_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{k} \gamma_i = \sum_{i=k+1}^{m} \gamma_i = 1$, $1 \leq k < m$. Then we have the representation

$$\sum_{i=1}^{m} \beta_i w_i = 0 \,,$$

where $\beta_i := \gamma_i$ for $i = 1, \ldots, k$, and $\beta_i = -\gamma_i$ for $i = k+1, \ldots, m$. Observe that $\sum_{i=1}^{m} \beta_i = 0$. Choose an index $i_0 \in \{1, \ldots, k\}$ such that

$$\epsilon := \frac{\lambda_{i_0}}{\beta_{i_0}} = \min \left\{ \frac{\lambda_i}{\beta_i} \mid i = 1, \ldots, k \text{ with } \beta_i > 0 \right\} .$$

Define $\alpha_i := \lambda_i - \epsilon \beta_i$ for $i = 1, \ldots, m$. Then $\alpha_i \geq 0$ for $i = 1, \ldots, m$, $\alpha_{i_0} = 0$, $\sum_{i=1}^{m} \alpha_i = 1$, and

$$\bar{x} = \sum_{i=1}^{m} \alpha_i w_i \,.$$

This contradicts the minimal property of m. The proof is complete. □

Remark 4.2. (i) Note that the closedness property assumed in Radon's theorem presented in Theorem 3.3(ii) can be relaxed, i.e., the statements in Theorem 3.3 and Theorem 4.1 are equivalent to the following statement:

For any collection of nonempty convex sets $\{\Omega_1, \ldots, \Omega_m\}$, $m \geq n+2$, in $\mathbb{R}^n$ with the property that the intersection of any $n+1$ sets from this collection is nonempty, one has

$$\bigcap_{i=1}^{m} \Omega_i \neq \emptyset \,.$$

Indeed, this statement obviously implies Theorem 3.3(ii), and it follows from Theorem 4.1(ii); see, e.g., [10, Theorem 3.13].

(ii) As a next step, it would be natural to extend the present investigations to *generalized convexity notions*, like for example H-convexity (see, e.g., [1] and [2]) or d-convexity (cf. [3, Chapter II]).

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