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Risk Measures, Convexity, and Max-Min Shortfalls

Sjur Didrik Flåm
Economics Department, University of Bergen, 5020 Norway
sjur.flaam@econ.uib.no

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Monetary risk measures are studied here in terms of acceptable outcomes, normalized non-negative prices, and resulting shortfalls. At center stage stand convex analysis, saddle functions and associated max-min formulae. The latter comply with common sense and established theory.

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1. Introduction

Various quantitative measures of economic risk have long been around, but not all are equally appealing. For good reasons, investors and financial analysts prefer that measures be sensitive to downside outcomes, and be reported in monetary terms. Accordingly, during the last two decades, mathematical finance has reconsidered and axiomatized its approach to risk measurement.[1]

The received literature invokes a non-empty set S of future states or scenarios. Any mapping $X : S \to \mathbb{R}$ – that is, each $X \in \mathbb{R}^S$ – is seen as a *financial position* or *contingent claim* to money. If state $s \in S$ comes up, the owner of X is entitled to present value $X(s)$. By assumption then, the class of admissible contingent claims constitutes a linear subspace $\mathbb{X}$ of $\mathbb{R}^S$ under pointwise addition and scaling. Also by assumption, $\mathbb{X}$ includes the constant claim $\mathbf{1}(\cdot) \equiv 1$.

Here, more generally, let $\mathbb{X}$ be any real linear space that contains a particular non-zero element denoted $\mathbf{1}$ – whence any aligned (constant) "receipt" $r = r\mathbf{1}$, $r \in \mathbb{R}$. Members of $\mathbb{X}$ are still called *claims* or *risks*, and denoted by capital letters.[2]

[1] Pioneering studies include [2], [5] and [22]. For newer, data-based risk measures, see [31].
[2] Instead of risks, $\mathbb{X}$ could comprise consumption profiles, signed measures, or lotteries on some

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Reasonable risk measures are sought only among proper functions $\rho : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$. Further, a candidate measure should translate risk-free additions ex post to equal reductions ex ante. Accordingly, each **risk measure** ρ, concerning claims, is henceforth taken to be *cash additive* in that

$$\rho(X + r) = \rho(X) - r \text{ for each } X \in \mathbb{X} \text{ and } r \in \mathbb{R}. \tag{1}$$

Put differently: adding constant reserves $r > 0$ to X, reduces risk exposure by the same amount. Most important, in economic terms, (1) divorces attitudes to sure wealth from attitudes to genuine risk; see [47].

Property (1) tells that $\rho(X)$ equals the minimal amount of money – interpreted as financial reserves r – needed up front to ensure that $\rho(X+r) \leq 0$. Emphasized is thus the central role played by the non-empty *acceptance set*

$$\mathbb{A}_\rho := \{\rho \leq 0\} := \{X \in \mathbb{X} : \rho(X) \leq 0\}.$$

Members of $\mathbb{A}_\rho$ are called *acceptable* – or "safe" – and

$$\rho(X) = \inf\{r \in \mathbb{R} : X + r \in \mathbb{A}_\rho\}. \tag{2}$$

Conversely, using the convention $\inf \emptyset = +\infty$, if a non-empty set $\mathbb{A} \subseteq \mathbb{X}$ satisfies

$$X \in \mathbb{X} \mapsto \rho_{\mathbb{A}}(X) := \inf\{r \in \mathbb{R} : X + r \in \mathbb{A}\} > -\infty,$$

the proper function $\rho_{\mathbb{A}}$, so defined, becomes a risk measure, and $\mathbb{A} \subseteq \mathbb{A}_{\rho_{\mathbb{A}}}$. If moreover,

$$X + r_n \in \mathbb{A} \ \& \ r_n \searrow r \leq 0 \Rightarrow X \in \mathbb{A},$$

then $\mathbb{A} = \mathbb{A}_{\rho_{\mathbb{A}}}$. These observations indicate that an investor – or a regulator [31] – may wish to single out the acceptance set $\mathbb{A}$ as a primitive object. This paper pursues that tack. Upon doing so, it reckons that some claims are likely to stand out as desirable. Presumably, $\mathbf{1}$ is of such sort, being totally risk-free. So, with $\mathcal{E} = \{\mathbf{1}\}$, a subset $\mathbb{D} \subset \mathbb{X}$ comprises *desirable risks* iff

$$\mathbf{1} \in \mathcal{E} \subset \mathbb{D} \quad \text{and} \quad \mathcal{E} + \mathbb{R}_+\mathbb{D} \subseteq \{\rho < 0\}. \tag{3}$$

For example, provided $\rho(D) < 0$ for each non-zero $D \in \mathbb{D}$, and $\mathbb{X} \subseteq \mathbb{R}^S$, one may posit $\mathbb{D} = \mathbb{R}^S_+ \cap \mathbb{X}$ or $\mathbb{D} = \mathbb{R}_+\mathbf{1}$.

Clearly, $\rho(\cdot)$ is a risk measure iff $\rho(\cdot) + r$ is likewise for any real r. Moreover, $\mathbb{A}_{\rho+r} = \mathbb{A}_\rho + r$. Therefore, since $\rho(\mathbf{1}) < 0$, it entails no loss of generality to *normalize* ρ – as done here – by setting $\rho(\mathbf{1}) = -1$. Hence $\rho(0) = 0$, and $0 \in \mathbb{A}$.

Apart from enjoying claims $D \in \mathbb{D}$, most investors prefer diversification – that is, combinations – of assets so as to balance or hedge various positions [43]. Accordingly, it appears reasonable that the acceptance set $\mathbb{A}$ be convex.

measure space $(S, \mathcal{F})$; see [16]. Alternatively, $\mathbb{X}$ might be the space $L^\infty(\mathbb{R}^d)$ of essentially bounded, $\mathbb{R}^d$-valued, $\mathcal{F}$-measurable functions on a probability space $(S, \mathcal{F}, \mu)$; conf. [17].

Convexity of $\mathbb{A}$ could, however, be introduced more directly. This paper explores how. It is planned as follows. To prepare the ground, Section 2 extends a robust representation of convex risk measures to reach beyond the frames of bounded claims, monotone valuations, and measurable or topological settings. That section brings out a set $\mathbb{P}$ of normalized prices such that $\mathbb{P}(\mathbb{D}) \geq 0$. Section 3 briefly comments on the appropriate price set $\mathbb{P}$. Section 4 relates to preferences which reflect risk or ambiguity aversion. Section 5 takes a couple $(\mathbb{A}, \mathbb{P})$ as primitive datum, thereby getting direct access to bilinear max-min theory. Opening up that tractable avenue is a chief objective of this paper. Section 6 concludes by briefly dealing with measures that stem from market-based pricing.

2. Convex Risk Measures

This section considers the class of convex risk measures, introduced by Föllmer and Schied [23], Frittelli and Gianin [24].[3]

Proposition 2.1 (On convexity and closure of risk measures). *A risk measure ρ is convex $\Longleftrightarrow$ its acceptance set $\{\rho \leq 0\}$ is likewise $\Longleftrightarrow$ ρ is quasi-convex. Granted convexity, one may – with no loss of generality – take the set $\mathbb{D}$ of desirable risks to be a convex cone.*
Similarly, in the topological case, a risk measure ρ is closed (meaning lower semi-continuous) $\Longleftrightarrow$ its acceptance set $\{\rho \leq 0\}$ is likewise.

Proof. The assertion about convexity is well known; see [22]. Yet, for completeness, it's proven directly here. Evidently, when ρ is convex, so is the lower level set $\mathbb{A} = \{\rho \leq 0\}$ as well. Conversely, given any two risks $X, \hat{X} \in \mathbb{X}$ and real weights w, $\hat{w} > 0$, with $w + \hat{w} = 1$, the convexity of $\mathbb{A}$ ensures that

$$X + r,\ \hat{X} + \hat{r} \in \mathbb{A} \Rightarrow wX + \hat{w}\hat{X} + wr + \hat{w}\hat{r} \in \mathbb{A}.$$

Thus, by (2),

$$\begin{aligned} w\rho(X) + \hat{w}\rho(\hat{X}) &= w \inf\{r : X + r \in \mathbb{A}\} + \hat{w} \inf\left\{\hat{r} : \hat{X} + \hat{r} \in \mathbb{A}\right\} \\ &= \inf\left\{wr + \hat{w}\hat{r} : X + r,\ \hat{X} + \hat{r} \in \mathbb{A}\right\} \\ &\geq \inf\left\{wr + \hat{w}\hat{r} : wX + \hat{w}\hat{X} + wr + \hat{w}\hat{r} \in \mathbb{A}\right\} \\ &\geq \inf\left\{r : wX + \hat{w}\hat{X} + r \in \mathbb{A}\right\} = \rho(wX + \hat{w}\hat{X}). \end{aligned}$$

The assertions about lower semicontinuity and quasi-convexity follow from $\{\rho \leq r\} = \mathbb{A} - r$. For the last assertion, let $\mathbb{C}$ denote the convex cone generated by $\mathbb{D}$. By (3), since $\mathbb{R}_+\mathbb{D} \subseteq \{\rho < 0\} - \mathcal{E}$, and the latter set is convex, it follows that

$$\mathbb{C} = \operatorname{conv}(\mathbb{R}_+\mathbb{D}) \subseteq \{\rho < 0\} - \mathcal{E},$$

from which we may conclude. □

[3] For generalizations to quasi-convex risk measures, see [16].

It deserves emphasis that assuming cash additivity (1) is far from innocuous. Reflecting on concerns with default or liquidity, Cerreia-Voglio et al. [9] argue for use of *cash subadditive* measures, satisfying

$$\rho(X - r) \le \rho(X) + r \quad \text{for each } X \in \mathbb{X} \text{ and } r \in \mathbb{R}_+. \tag{4}$$

To account then for advantages of diversification, they let ρ be quasi-convex and seek its dual representation. Broadly, quasi-convex duality resembles the one between direct and indirect utility of a non-satiated, price-taking consumer [9], [10], [37].

Attention is restricted here, however, to risk measures of convex sort. Many of these emerge as extremal convolutions. For example, given any convex function $\Phi : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, provided the sup-convolution

$$\rho(X) = \sup \{-r + \Phi(-X + r) : r \in \mathbb{R}\}$$

be proper, it's a convex risk measure [4], [5]. Such constructions create direct links to stochastic programming; see [33] and [42]. Also, noteworthy is that each proper inf-convolution

$$\rho(X) := \inf \left\{ \sum_{i \in I} \rho_i(X_i) : \sum_{i \in I} X_i = X \right\} \tag{5}$$

of finitely many convex risk measures becomes itself a measure of that type. The resulting measure could represent the over-all attitude of a *syndicate* $(\rho_i)_{i \in I}$; conf. [6], [20], [21], [46].

Anyway, a convex risk measure ρ has several appealing properties:

- The *benefit of hedging* becomes non-negative and quantifiable: $w\rho(X) + \hat{w}\rho(\hat{X}) - \rho(wX + \hat{w}\hat{X}) \ge 0$ whenever the weights $w, \hat{w} > 0$ sum to 1.
- Suppose an investor considers only *portfolios* – alias linear combinations – $\sum_{X \in \mathcal{X}} w_X X$ of "papers" X selected from a fixed finite subset $\mathcal{X} \subset \mathbb{X}$. If the weight vector $w := (w_X)$ must belong to a prescribed convex set $W \subset \mathbb{R}^{\mathcal{X}}$, the task to reduce the investor's risk fits the tractable format of convex programming [7], [23]: *minimize* $\rho(\sum_{X \in \mathcal{X}} w_X X)$ *subject to* $w \in W$.
- As seen below, many convex risk measures permit an explicit representation which emphasizes the role of two fundamental sets: $\mathbb{A}$ and $\mathbb{P}$. The first reflects acceptance; the second concerns pricing.

The rest of this section deals with the said representation. It generalizes Theorem 4.15 in [22] to allow unbounded claims and absence of measurability, order, and topology.

For this, recall that a point $X \in \mathbb{X}$ belongs to the *algebraic interior* of a subset $\mathcal{X} \subseteq \mathbb{X}$, written $X \in \operatorname{aint} \mathcal{X}$, iff $\mathbb{R}_+(\mathcal{X} - X) = \mathbb{X}$. Let $\mathbb{X}^*$ denote the *dual space* $\mathbb{X}$, consisting of *all* linear functionals $X^* : \mathbb{X} \to \mathbb{R}$. Such a functional is declared *normalized* if $X^*(\mathcal{E}) = 1$. It's *non-negative on* $\mathbb{D}$ if $X^*(D) \ge 0$ for each $D \in \mathbb{D}$.

Any $X^* \in \mathbb{X}^*$ could operate as a price regime on $\mathbb{X}$. In particular, the "*shortfall*" $A - X$ between an acceptable A and a claim X is then worth $X^*(A - X)$. The lower support function

$$X^* \mapsto \underline{\sigma}_{\mathbb{A}}(X^*) := \inf \{X^*(A) : A \in \mathbb{A}\} \tag{6}$$

indicates then the cheapest valuation $\underline{\sigma}_{\mathbb{A}}(X^*) - X^*(X)$ of such shortfall. In these economic terms, the following result on robust representation of convex risk measures extends Theorem 4.15 in [22], which deals with the instance $\mathbb{X} = L^\infty(S, \mathcal{F})$ of bounded $\mathcal{F}$-measurable risks $X : S \to \mathbb{R}$, and posits $\mathbb{D} = L^\infty_+$.

Theorem 2.2 (Representing a convex risk measure of claims). *Suppose a risk measure $\rho : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ has $\rho(0) = 0$ and a convex acceptance set $\mathbb{A} = \{\rho \leq 0\}$, such that $X + r$ belongs to its algebraic interior whenever $\rho(X)$ is finite and r is sufficiently large. Let $\mathbb{D} \subset \mathbb{X}$ denote an associated set of desirable risks* (3)*. Then there exists a non-empty convex ensemble $\mathbb{P} \subseteq \mathbb{X}^*$ of normalized prices, each $P \in \mathbb{P}$ being non-negative on $\mathbb{D}$, such that $\rho(X) < +\infty$ implies*

$$\rho(X) = \max_{P \in \mathbb{P}} \inf_{A \in \mathbb{A}} P(A - X) = \max_{P \in \mathbb{P}} \{\underline{\sigma}_{\mathbb{A}}(P) - P(X)\} \leq \sup_{P \in \mathbb{P}} P(-X) \tag{7}$$

where

$$\mathbb{P} := \{P \in \mathbb{X}^* : P \text{ normalized and } P(\mathbb{D}) \geq 0\}. \tag{8}$$

In (7) *each maximizing $P \in \mathbb{P}$ is an anti-subgradient; it belongs to $-\partial\rho(X)$. Further, $\underline{\sigma}_{\mathbb{A}}$ is a maximal penalty function; it majorizes any $p : \mathbb{X}^* \to \mathbb{R} \cup \{-\infty\}$ which satisfies*

$$p(X^*) \leq X^*(X) + \rho(X) \quad \text{for all } X^* \in \mathbb{X}^* \text{ and } X \in \mathbb{X}. \tag{9}$$

If moreover, $\mathbb{A}$ is a cone, then $\rho(X) = \max_{P \in \mathbb{P}} P(-X)$ becomes ***coherent****, being both convex and positively homogenous.*

Proof. Most arguments follow those given for Theorem 4.15 in [22], but are spelled out. Fix any risk $X \in \mathbb{X}$ for which $\rho(X)$ is finite. Then, by way of cash invariance (1), $X + \rho(X) \in \mathbb{A}$. Hence, for any normalized $P \in \mathbb{X}^*$,

$$\underline{\sigma}_{\mathbb{A}}(P) \leq P(X + \rho(X)) = P(X) + \rho(X)$$

to the effect that

$$\sup \{\underline{\sigma}_{\mathbb{A}}(P) - P(X) : P \text{ normalized}\} \leq \rho(X). \tag{10}$$

Thus – to verify the equalities in (7), and the attainment of maximum there – it suffices to find a normalized $P_X \in \mathbb{X}^*$ such that $P_X(\mathbb{D}) \geq 0$ and

$$\rho(X) \leq \underline{\sigma}_{\mathbb{A}}(P_X) - P_X(X). \tag{11}$$

With no loss of generality, assume $\rho(X) = 0$. Proposition 2.1 implies that $\mathbb{A}_0 := \{\rho < 0\}$ is convex. Further, by assumption, $\mathbb{A}_0$ has non-empty algebraic interior, to which X cannot belong. Therefore, an algebraic version of the Hahn-Banach separation theorem (see [45] Theorem 2.7) ensures existence of a non-zero $X^* \in \mathbb{X}^*$ such that

$$X^*(X) \leq \inf\{X^*(A) : A \in \mathbb{A}_0\} =: \beta. \tag{12}$$

Now, by (3),

$$X^*(X) \leq X^*(\mathcal{E}) + \mathbb{R}_+ X^*(\mathbb{D}).$$

Consequently, X^* must be non-negative on $\mathbb{D}$ whence $X^*(\mathcal{E}) \geq 0$. To show that the last inequality is strict, let r be so large that $X + r\mathcal{E} \in \operatorname{aint} \mathbb{A}_0$. Further, choose $\chi \in \mathbb{X}$ such that $X^*(\chi) < 0$ and $X + r + \chi \in \mathbb{A}_0$. Now, $X^*(\mathbf{1}) = 0$ would entail $X^*(X + r + \chi) < X^*(X)$. Since this contradicts (12), it follows that $X^*(\mathbf{1}) > 0$, and the normalized price $P_X := X^*/X^*(\mathbf{1})$ satisfies

$$\underline{\sigma}_{\mathbb{A}}(P_X) := \inf P_X(\mathbb{A}) \leq \inf P_X(\mathbb{A}_0) = \beta/X^*(\mathbf{1}).$$

Equality prevails in the last string for the simple reason that $\mathbb{A} + r\mathcal{E} \subseteq \mathbb{A}_0$ when $r > 0$; and we may let $r \searrow 0$. Inequality (11) follows from this observation because

$$\rho(X) = 0 \leq \frac{\beta - X^*(X)}{X^*(\mathbf{1})} = \underline{\sigma}_{\mathbb{A}}(P_X) - P_X(X).$$

Plainly, P_X belongs to the set $\mathbb{P}$ defined by (8). This takes care of the equalities in (7). The inequality there derives from the assumption $0 \in \mathbb{A}$ which implies that $\underline{\sigma}_{\mathbb{A}} \leq 0$.

For the differential property, let $P \in \mathbb{P}$ be maximizing in (7). Given any other risk $\hat{X}$, it holds $\rho(\hat{X}) \geq \underline{\sigma}_{\mathbb{A}}(P) - P(\hat{X})$. Subtract $\rho(X) = \underline{\sigma}_{\mathbb{A}}(P) - P(X)$ from the preceding inequality to have $\rho(\hat{X}) - \rho(X) \geq -P(\hat{X} - X)$ hence $-P \in \partial\rho(X)$.[4]

To conclude, consider any $p : \mathbb{X}^* \to \mathbb{R} \cup \{-\infty\}$ which satisfies (9). For any $X^* \in \mathbb{X}^*$ it holds

$$\begin{aligned} p(X^*) \leq\ & \inf\{X^*(X) + \rho(X) : X \in \mathbb{X}\} \\ \leq\ & \inf\{X^*(X) + \rho(X) : X \in \mathbb{A}\} \\ \leq\ & \inf\{X^*(A) : A \in \mathbb{A}\} = \underline{\sigma}_{\mathbb{A}}(X^*). \end{aligned}$$

Finally, in the coherent case, when $\mathbb{A}$ is a cone, $\underline{\sigma}_{\mathbb{A}}$ can take just two values: either 0 or $-\infty$. This completes the proof. □

Additional properties might be required of a reasonable risk measure ρ. For instance, when $(S, \mathcal{F}, \mu)$ is a probability space, and each $X \in \mathbb{X} = L^\infty(S, \mathcal{F}, \mu; \mathbb{E})$ maps S into a Euclidean space $\mathbb{E}$, one may insist that ρ be *law-invariant* [17]. Also, when $\mathbb{X} \subseteq L^0(S, \mathcal{F}; \mathbb{R})$ = the (quotient) space of all $\mathcal{F}$-measurable $X : S \to \mathbb{R}$, it appears desirable that ρ be additive over *comonotone* claims [22]. Such issues are not considered here.

[4] For more on subdifferential representations of ρ, see [40].

Theorem 2.2 makes no mention of topology. Yet the attainment of maximum in (7) gives to think about inherent compactness. Theorem 3.1 returns to this issue. It also surges in the following example:

The Sharpe Ratio. Let $(S, \mathcal{F}, \mu)$ be a probability space, $\mathbb{X} := L^2(S, \mathcal{F}, \mu) = \mathbb{X}^*$, $X^*(X) := E(X^*X)$, and $\mathbf{1}(\cdot) \equiv 1$. Choose a constant $c > 0$, and posit

$$\mathbb{A} := \{X \in \mathbb{X} : EX \geq c\sigma(X)\} =: \mathbb{D}$$

with $\sigma(X) := \|X - EX\|$. Thus a non-degenerate X becomes acceptable if its *Sharpe ratio* $E(X)/\sigma(X)$ is bounded below by c. Since the *standard deviation* $\sigma(\cdot)$ is positively homogenous and subadditive, $\mathbb{A}$ becomes a convex cone. The associated risk measure reads

$$\rho(X) := \rho_{\mathbb{A}}(X) = E(-X) + c\sigma(X).$$

Writing $\hat{X} := -(X - EX)$, and $m := EX$, notice that

$$\begin{aligned} 0 \leq \rho^*(-X^*) &:= \sup_X \{E(-X^*X) - \rho(X)\} \\ &= \sup_{\hat{X},\, E\hat{X}=0} \left\{ E[X^* - EX^*)\hat{X}] - c\left\|\hat{X}\right\| \right\} + \sup_m \{m - EX^*m\} \\ &= \begin{cases} 0 & \text{if } \|X^* - EX^*\| \leq c \text{ and } EX^* \geq 1, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

From Theorem 2.2 follows that $\rho(X) = \max_{P \in \mathbb{P}} P(-X)$ for some convex set $\mathbb{P}$ of normalized prices. With no loss, let $\mathbb{P}$ be closed. Its extended indicator $\delta_{\mathbb{P}} : \mathbb{X}^* \to \{0, +\infty\}$ takes the value 0 on $\mathbb{P}$ and $+\infty$ elsewhere. Since $\rho(X) = \delta^*_{\mathbb{P}}(-X)$, by customary convex duality, $P \in \mathbb{P} \Longleftrightarrow \delta_{\mathbb{P}}(P) = \delta^{**}_{\mathbb{P}}(P) = 0$. Equivalently, noting that $EX^* = X^*(\mathbf{1})$, it obtains

$$\begin{aligned} \mathbb{P} &= \{X^* \in \mathbb{X}^* : X^*(\mathbf{1}) = 1 \text{ and } \rho^*(-X^*) \leq 0\} \\ &= \{X^* \in \mathbb{X}^* : EX^* = 1 \text{ and } \|X^* - 1\| \leq c\}. \end{aligned}$$

This set $\mathbb{P}$ is convex and *weakly compact* whence Theorem 6.2.7 in [3] implies

$$\rho(X) = \max_{P \in \mathbb{P}} \inf_{A \in \mathbb{A}} P(A - X) = \inf_{A \in \mathbb{A}} \sup_{P \in \mathbb{P}} P(A - X).$$

Proposition 5.5 returns to this lopsided max-inf result.

For similar instances, if a convex function $C : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ satisfies $C(X + r) = C(X)$ when $r \in \mathbb{R}$, a *generalized Sharpe ratio* obtains with $X \in \mathbb{A} \Longleftrightarrow EX \geq C(X)$. Examples include $C(X) = c(X - EX)$ with $c : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ convex, or $C(X) = -K(\min\{\kappa, X - EX\})$ with $K : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ concave increasing, and κ a real constant.[5]

[5] See the discussion of *deviation functionals* in [40].

Note that if $\mathbb{A}_+ := \mathbb{A} \cap \mathbb{X}_+$ had been chosen as alternative acceptance set, and $\mathcal{F}$ isn't finitely generated, the proof of Theorem 2.2 would no longer apply. In fact, since $\mathbb{A}_+$ has empty algebraic interior, separation (12) cannot be invoked. That is, we can hardly find a set $\mathbb{P} \subset \mathbb{X}^*$ of (continuous) normalized prices such that $X \notin \mathbb{A}_+$ implies $\inf_{A \in \mathbb{A}_+} \langle P, A \rangle > \langle P, X \rangle$ for at least one $P \in \mathbb{P}$. This form of strict separation will reemerge in (22). $\Diamond$

3. On Prices and Monotonicity

This section collects some observations on prices, their continuity and their role as state-time deflators. Also noted is the role of monotonicity and the link to what is called *the Lebesgue property.*

• (*On continuous prices*). Most often, $\mathbb{X}$ is a locally convex topological vector space. Provided $X + r \in \operatorname{int} \mathbb{A}$ for $\rho(X)$ finite and r sufficiently large, then – under the other assumptions of Theorem 2.2 – one may confine $\mathbb{P}$ to *continuous* linear functionals. However, the hypothesis that the algebraic – or a fortiori, topological – interior of $\mathbb{A}$ be non-empty is stringent. It's conveniently avoided, and the assertions in Theorem 2.2 remain valid, if $\mathbb{A}$ comes closed convex. (As it does in the Sharpe ratio.) Such properties, already prominent in Proposition 2.1, are reconsidered in Section 5.

• (*On linearity of pricing and arbitrage*). Theorem 2.2 was motivated in economic terms. Still in such terms, if claims are sold and resold, added and scaled, then – by the *law of one price* [12], [34], to preclude that pure profit result from bundling and unbundling – any market pricing $P : \mathbb{X} \to \mathbb{R}$ must be linear; that is, $\mathbb{P} \subset \mathbb{X}^*$. Further, some claim should act as a common yardstick, unit of account, or *numeraire*; here $P(\mathbf{1}) = 1$. Finally, if some $D \in \mathbb{D}$ commands price $P(D) < 0$, there is *arbitrage*: the investor is paid for acquiring an item he desires. Such a situation makes no economic sense – or at least, it cannot persist.

• (*On state-time deflators*). Since no desirable claim D can reasonably be had at negative cost, $\mathbb{D}$'s *dual cone*

$$\mathbb{D}^0 := \left\{ D^0 \in \mathbb{X}^* : D^0(D) \geq 0 \ \forall D \in \mathbb{D} \right\}$$

comes naturally to the fore. When $\mathbb{X}$ is topological, and $\mathbb{X}^*$ denotes its customary continuous dual, $\mathbb{D}^0$ becomes $\sigma(\mathbb{X}^*, \mathbb{X})$-closed. Recall that if $\mathbb{A}$ is closed convex, one may take $\mathbb{D}$ (3) to be a closed convex cone. If moreover, $\mathbb{X}$ is locally convex, the bipolar theorem [1] says that $D \in \mathbb{D} \iff D^0(D) \geq 0 \ \forall D^0 \in \mathbb{D}^0$. One may then refer to

$$\mathbb{D}_{++} := \left\{ D \in \mathbb{D} : D^0(D) > 0 \text{ for each non-zero } D^0 \in \mathbb{D}^0 \right\}$$

as the set of *strictly positive desirable claims.* A closed convex cone $\mathbb{D}$ is declared *regular* if $\mathbb{D}_{++}$ is non-empty. Granted such regularity, it holds for any element $\mathbf{1} \in \mathbb{D}_{++}$ that $\mathbb{D}^0 = \mathbb{R}_+ \mathbb{P}$ with

$$\mathbb{P} := \left\{ P \in \mathbb{D}^0 : P(\mathbf{1}) = 1 \right\}.$$

In case $\mathbb{X} \subseteq \mathbb{R}^S$, each scenario $s \in S$ typically describes a particular state-time evolution; see Section 6. One may then regard any $\mathbf{1} \in \mathbb{D}_{++}$ as an *Arrow-Debreu security* [36], and each $P \in \mathbb{P}$ as a *state-time deflator* [12].

• (*On normalized prices*). Under the assumptions of Theorem 2.2 it holds that

$$\rho(X) = \inf_{A \in \mathbb{A}} \sup_{P \in \mathbb{P}} P(A - X); \tag{13}$$

see Proposition 5.2. Consequently, if the pricing ensemble $\mathbb{P} \subset \mathbb{X}^*$ *normalizes* a set $\mathcal{E} \subset \mathbb{X}$ in the sense that

$$\mathbb{P}(\mathcal{E}) \subset (0, 1], \tag{14}$$

then (13) implies:

$$\rho(X - r\mathcal{E}) \leq \rho(X) + r \quad \text{for each } r > 0.$$

The last inequality points to (4), and it may invite accommodation of a larger set $\mathcal{E}$ than the singleton $\{\mathbf{1}\}$ in (3). To indicate this, fix a function $\rho : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ and any $X \in \mathbb{X}$ such that $\rho(X) \in (0, +\infty)$, and $X + \rho(X)\mathcal{E}$ intersects $\mathbb{A}$. Then, under (14), inequality (10) holds. Specialize to a non-empty convex set $\mathbb{A} \subseteq \{\rho \leq 0\}$ such that $\mathbb{A} + r\mathcal{E} \subset \operatorname{aint} \mathbb{A}$ for $r > 0$. Further, let $\mathcal{E} \subset \mathbb{X}$ be a polytope which satisfies (3). As in the proof of Thm. 2.2, if $X + \rho(X)\mathcal{E}$ doesn't intersect $\operatorname{aint} \mathbb{A}$, the equalities in (7) hold for each $\hat{X} \in \mathbb{A} \cap \{X + \rho(X)\mathcal{E}\}$ with $\mathbb{P} = \{P \in \mathbb{X}^* : P(\mathcal{E}) \subset (0, 1] \text{ and } P(\mathbb{D}) \geq 0\}$.

• (*On monotonicity*). When $\mathbb{X} \subseteq \mathbb{R}^S$, and $\mathbb{C} = \mathbb{X} \cap \mathbb{R}^S_+$, an investor may find larger claims less risky:

$$\hat{X} - X \in \mathbb{C} \Longrightarrow \rho(\hat{X}) \leq \rho(X). \tag{15}$$

(The Sharpe ratio is, however, not monotone; see [22].) More generally, a risk measure ρ is *monotone* (decreasing) if (15) holds for some specified convex cone $\mathbb{C} \subseteq \mathbb{X}$. By Proposition 2.1, provided ρ be convex, the cone $\mathbb{C} = \operatorname{conv}(\mathbb{R}_+\mathbb{D})$ may replace $\mathbb{D}$ - in which case monotonicity (15) follows from representation (7).

• (*On pricing by non-additive measures* [15]). Suppose some state space S be endowed with a field $\mathcal{F}$, and that $\mathbb{P}$ is a family of additive measures $P : \mathcal{F} \to \mathbb{R}_+$. Let μ be the upper envelope; that is, $\mu(F) := \sup\{P(F) : P \in \mathbb{P}\}$ for each $F \in \mathcal{F}$. Let $\mathbb{X}$ equal the quotient space $L_1(S, \mathcal{F}, \mu)$ of $\mathcal{F}$-measurable functions $X : S \to \mathbb{R}$ for which the *Choquet integral* $\int |X| \, d\mu$ is finite. When valid, representation (13) reads

$$\rho(X) = \inf_{A \in \mathbb{A}} \int (A - X) d\mu. \tag{16}$$

For any class $\mathcal{C} \subset L_1(S, \mathcal{F}, \mu)$ of comonotone functions, there exists $P \in \mathbb{P}$ such that $X \in \mathcal{C} \Longrightarrow \int (A - X) d\mu = \int (A - X) dP$. Conversely, if ρ is given by (16), with μ submodular, there exists a non-empty convex set $\mathbb{P}$ of additive measures $P : \mathcal{F} \to \mathbb{R}_+$ such that (13) holds.

• (*On attainment of supremum and the Lebesgue property*). When $\mathcal{F}$ is a sigma-field in some sample set S, and μ is an associated probability measure, the classes of μ-almost surely equal, $\mathcal{F}$-measurable functions $X : S \to \mathbb{R}$ form a linear space $L^0 = L^0(S, \mathcal{F}, \mu)$, ordered in customary pointwise manner. Mathematical finance considers various subspaces $\mathbb{X} \subseteq L^0$. The two instances L^∞ and L^2 are prominent. Each of these two is *solid* in the sense that $|X| \leq \left|\hat{X}\right| \in \mathbb{X} \Longrightarrow X \in \mathbb{X}$. More generally, in the rest of this section, let $\mathbb{X}$ be any solid subspace of L^0 which contains the constants and is contained in $L^1(S, \mathcal{F}, \hat{\mu})$ for some probability measure $\hat{\mu}$ equivalent to μ. Let $X^* \in L^0$ belong to the *order-continuous dual space* $\mathbb{X}^\sim$ iff

$$X \mapsto X^*(X) := \int X^* X d\mu$$

is well defined and finite-valued on $\mathbb{X}$; see [1]. In this setting, use $\mathbb{C} = \mathbb{X} \cap \mathbb{R}^S_+$ as ordering cone (15), and let $\lambda(X) = \rho(-X)$, to see the attainment of maximum in (7) from the following alternative view-point:

Theorem 3.1 (On the Lebesgue property [39]). *Suppose $\lambda : \mathbb{X} \to \mathbb{R}$ is convex, and $\sigma(\mathbb{X}, \mathbb{X}^\sim)$-lower semicontinuous, and monotone increasing. Then the following four statements are equivalent:*

∗ *λ has the* ***Lebesgue property*** [32], [38], *meaning*

$$X_n \to X \quad \mu\text{-almost surely and} \quad |X_n| \leq \left|\hat{X}\right| \in \mathbb{X} \Longrightarrow \lim \lambda(X_n) = \lambda(X).$$

∗ *The conjugate function λ^* is $\sigma(\mathbb{X}, \mathbb{X}^\sim)$-inf-compact.*

∗ *When λ is seen as a Fenchel conjugate, the supremum is attained in that*

$$\lambda(X) = \max \{X^*(X) - \lambda^*(X^*) : X^* \in \mathbb{X}^\sim\} \quad \text{for each } X \in \mathbb{X}.$$

∗ *λ is subdifferentiable. More precisely, $\partial\lambda(X) \cap \mathbb{X}^\sim$ is non-empty for each $X \in \mathbb{X}$.* □

This theorem, proven by Owari (2014), shows that – for finite measures, operating on Riesz solid subspaces of L^0 – customary monotonicity entails properties similar to those derived from cash invariance. Links to specialized versions of Theorem 2.2 emerge by setting $\mathbb{D} = \mathbb{X}_+$ and requiring lower semicontinuity in the order sense.

4. On Ambiguity Aversion and Preferences

Since Ellsberg (1961) *ambiguity* has remained central in decision theory. On that account, this section leans on Thm. 2.2 to review a recent study which deals with preference-based representations of utility [35]. The section begins though, with a link to liabilities and insurance.

Maintaining an economic perspective, recall that $\rho(X)$ measures the risk of an *investor* who can *claim* X. His counterpart can be construed as an *insurer* who has underwritten *liability* X. Accordingly, $X \mapsto \lambda(X) := \rho(-X)$ is a risk measure of his loss. When ρ is proper convex, so is λ as well, but now $\lambda(X+r) = \lambda(X) + r$, $\mathcal{A} := \{\lambda \leq 0\} = -\mathbb{A}$, and $\lambda(\mathbf{1}) = 1$.

If $X^* \in \mathbb{X}^*$ is a price regime, the *excess* $X - A$ of liability $X \in \mathbb{X}$ over $A \in \mathcal{A}$ is worth $X^*(X - A)$. Invoking here the upper support function

$$X^* \mapsto \overline{\sigma}_{\mathcal{A}}(X^*) := \sup X^*(\mathcal{A}),$$

one may regard $\inf\{X^*(X - A) : A \in \mathcal{A}\} = X^*(X) - \overline{\sigma}_{\mathcal{A}}(X^*)$ as the *cheapest excess* of X. Risk measures of liability or loss relate easily to Fenchel conjugation. Indeed: *Under the assumptions of Theorem* 2.2, *if* $\lambda(X) = \rho(-X)$ *is finite, then*

$$\lambda(X) = \max_{P \in \mathbb{P}} \inf_{A \in \mathcal{A}} \{P(X - A)\} = \max_{P \in \mathbb{P}} \{P(X) - \overline{\sigma}_{\mathcal{A}}(P)\} \geq \sup_{P \in \mathbb{P}} P(X)$$

where $\mathbb{P} := \{P \in \mathbb{X}^* : P(\mathbf{1}) = 1,\ P(\mathbb{D}) \geq 0\}$. *Each maximizing* $P \in \mathbb{P}$ *belongs to the subdifferential* $\partial\lambda(X)$. *Further,* $\bar{\sigma}_{\mathcal{A}}$ *minorizes any penalty function* $p : \mathbb{X}^* \to \mathbb{R} \cup \{+\infty\}$ *which satisfies*

$$\lambda(X) \geq X^*(X) - p(X^*) \ \ \textit{for all}\ X^* \in \mathbb{X}^* \ \ \textit{and}\ X \in \mathbb{X}.$$

If moreover, $\mathcal{A}$ *is a cone, then* $\lambda(X) = \max_{P \in \mathbb{P}} P(X)$ *becomes coherent, meaning convex and positively homogenous.*

It's illuminating to relate risk measures to preferences and utility functions. For a backdrop, since ρ reports disutility, posit $U := -\rho$ and $\mathcal{A} := -\mathbb{A}$. Formula (7) then reads

$$U(X) = \min_{P \in \mathbb{P}} \sup_{A \in \mathcal{A}} \{P(X - A)\} = \min_{P \in \mathbb{P}} \{P(X) + c(P)\}, \tag{17}$$

featuring a convex function $c(P) = \sup P(\mathbb{A})$. Maccheroni et al. (2006) call formula (17) a *variational representation of preferences*. They see $c(\cdot)$ as an index of *ambiguity aversion*. In particular, if c is an extended indicator of some domain which contains $\mathbb{P}$, the purely *risk averse* instance $U(X) = \min_{P \in \mathbb{P}} P(X)$ emerges with *multiple priors* $P \in \mathbb{P}$, as axiomatized by Gilboa and Schmeidler (1989).

Utility function (17) is given a behavioral foundation in [35]. For a quick review, let $\mathcal{F}$ be a field on state space S, and regard each $X \in \mathbb{X}$ as an *act* – that is, as a simple $\mathcal{F}$-measurable mapping from S into a convex set $\mathbf{C}$ of *consequences*.[6] The simplex $\Delta(S)$ consist of all additive probability measures P on $\mathcal{F}$. The said foundation of (17) derives from several axioms which all pertain to an underlying preference order. These are stated next:

[6] *Simple* means that the range $X(S)$ should be a finite set.

A1: The preference order $\succsim$ on $\mathbb{X}$ is *transitive, total* and *reflexive.*

A2: There is *weak certainty independence* in that – for any two acts $X, \hat{X} \in \mathbb{X}$, consequences $c, \hat{c} \in \mathbf{C}$, and $r \in (0,1)$ –

$$rX + (1-r)c \succsim r\hat{X} + (1-r)c \Rightarrow rX + (1-r)\hat{c} \succsim r\hat{X} + (1-r)\hat{c}.$$

A3: $\succsim$ is *continuous*: For any acts $X_1, X_2, \hat{X} \in \mathbb{X}$ both sets

$$\left\{r \in [0,1] : rX_1 + (1-r)X_2 \succsim \text{ (resp. } \precsim) \ \hat{X}\right\} \text{ are closed.}$$

A4: $\succsim$ is *monotone*: When $X \in \mathbb{X}$ and $s \in S$, let $X_s \in \mathbb{X}$ denote the act with constant value $X(s)$. In these terms,

$$X_s \succsim \hat{X}_s \text{ for all } s \in S \Rightarrow X \succsim \hat{X}.$$

A5: $\succsim$ reflects *uncertainty aversion*:

$$X \sim \hat{X} \text{ and } r \in (0,1) \Rightarrow rX + (1-r)\hat{X} \succsim X.$$

A6: $\succsim$ is *non-degenerate*: $X \succ \hat{X}$ for some $X, \hat{X} \in \mathbb{X}$.

Theorem 4.1 (Robust representation of risk and ambiguity averse preferences [35]). *Let $\succsim$ be a binary order on $\mathbb{X}$. Then the following two assertions are equivalent*:

- *The relation $\succsim$ satisfies axioms* **A1**–**A6**.
- *There exists a non-constant affine function $u : \mathbf{C} \to \mathbb{R}$ and a convex, lower semicontinuous $c : \Delta(S) \to [0, +\infty]$, with $\inf c = 0$, such that $X \succsim \hat{X} \Leftrightarrow U(X) \geq U(\hat{X})$ with*

$$U(X) = \min_{P \in \Delta(S)} \left\{ \int u(X)dP + c(P) \right\}. \tag{18}$$

Further, for each appropriate u there is a unique minimal $\underline{c} : \Delta(S) \to \mathbb{R}$ that satisfies (18). *It is given by*

$$\underline{c}(P) = \sup \left\{ u(X) - \int u(X)dP : X \in \mathbb{X} \right\}. \qquad \square$$

The similarity between (17) and (18) is striking. Clearly, in (18) one may take u to be linear. Further, one may single out a desirable consequence, called $\mathbf{1} \in \mathbf{C}$, and normalize u by setting $u(\mathbf{1}) = 1$. Note that each minimizing P in (18) is a supergradient of U at $X : P \in \partial U(X)$.

To see links to markets and exchange economies, suppose agent $i \in I$ worships quasi-linear utility U_i of normalized form (17) or (18), and he holds endowment $e_i \in \mathbb{X}$. In equilibrium, demand should equal supply – that is, $\sum_{i \in I} X_i = \sum_{i \in I} e_i$ – for some market clearing price $P \in \cap_{i \in I} \partial U_i(X_i)$; see [20].

Besides being derived more directly, representation (17) has some extra appeal. It features a class of a priori acceptable claims, a restricted set of admissible prices, and it can accommodate implicit constraints by way of $U(X) = -\infty$. Further, returning to the equivalent formula (7), it opens the door to bilinear instances of received min-max theory [3], [26], [27]. This is explored next.

5. Risk Measures and Saddle Functions

In this section the non-empty sets $\mathbb{A} \subseteq \mathbb{X}$ and $\mathbb{P} \subseteq \mathbb{X}^*$ are regarded as primitive objects. Since $\mathbb{A}$ is intended as acceptance set, it should satisfy $\mathbb{A} + \mathbb{R}_+\mathbf{1} \subseteq \mathbb{A}$. Further, because $\mathbb{P}$ should be a family of normalized price functionals: $\mathbb{P}(\mathbf{1}) = 1$.

Once appropriate sets $\mathbb{A}$ and $\mathbb{P}$ are chosen, representation (7) invites a look at a *lower measure*

$$\underline{\rho}(X) := \sup_{P \in \mathbb{P}} \inf_{A \in \mathbb{A}} P(A - X) = \sup_{P \in \mathbb{P}} \{\underline{\sigma}_{\mathbb{A}}(P) - P(X)\}. \tag{19}$$

For (19) to properly qualify as as a candidate risk measure, *henceforth suppose* $\underline{\rho} > -\infty$.[7] This assumption implies that $\mathbb{P}$ must be part of $\mathbb{A}$'s *lower barrier cone*

$$\underline{b}(\mathbb{A}) := \{X^* \in \mathbb{X}^* : \underline{\sigma}_{\mathbb{A}}(X^*) > -\infty\}.$$

Associated to (19) is an *upper measure*

$$\bar{\rho}(X) := \inf_{A \in \mathbb{A}} \sup_{P \in \mathbb{P}} P(A - X). \tag{20}$$

If $\mathbb{A}$ and $\mathbb{P}$ need emphasis, one may write $\underline{\rho}_{\mathbb{A},\mathbb{P}}$ and $\overline{\rho}_{\mathbb{A},\mathbb{P}}$. Clearly,

$$\mathbb{A} \subseteq \mathcal{A},\ \mathbb{P} \supseteq \mathcal{P} \Rightarrow \begin{cases} \underline{\rho}_{\mathbb{A},\mathbb{P}} \geq \underline{\rho}_{\mathcal{A},\mathcal{P}} \\ \overline{\rho}_{\mathbb{A},\mathbb{P}} \geq \overline{\rho}_{\mathcal{A},\mathcal{P}}. \end{cases}$$

At the heart of either measure, (19) and (20), lies pricing $P(\cdot)$ of the *shortfall* $A - X$ below an acceptable A. That is, chief items are the priced deficits of X as compared to acceptable outcomes. $\underline{\rho}$ differs from $\overline{\rho}$ by the temporal order of picking P and A. Selecting $P \in \mathbb{P}$ first and $A \in \mathbb{A}$ second, gives $\underline{\rho}$. The reversed order of selection brings out $\bar{\rho}$.

Proposition 5.1 (On lower and upper measures). *When $\mathbb{P}$ is normalized, meaning $\mathbb{P}(\mathbf{1}) = 1$, either function $\underline{\rho}$, $\overline{\rho} : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ becomes cash invariant (1), and $\underline{\rho} \leq \bar{\rho}$.*
For any subset $\mathbb{D} \subset \mathbb{X}$ such that $\mathbf{1} + \mathbb{R}_+\mathbb{D} \subset \mathbb{A}$ it holds that $\mathbb{P}(\mathbb{D}) \geq 0$. If $0 \in \mathbb{A}$, then $\bar{\rho}(X) \leq \sup_{P \in \mathbb{P}} P(-X)$.
If $\mathbb{A}$ is a cone, both measures become positively homogenous. When moreover, $\mathbb{P}(\mathbb{A}) \geq 0$, the two functions equal the coherent instance $\rho(X) = \sup_{P \in \mathbb{P}} P(-X)$.

[7] It suffices for this purpose that $P \in \mathbb{P} \mapsto P(X)$ be bounded for every $X \in \mathbb{X}$, and $\sup_{P \in \mathbb{P}} \inf_{A \in \mathbb{A}} P(A) > -\infty$. In particular, suffices that $\mathbb{P}$ be pointwise bounded and $\mathbb{P}(\mathbb{A}) \geq 0$.

If $\mathbb{A}$ is convex, both measures become convex.
If both sets $\mathbb{A}$ and $\mathbb{P}$ are convex, and $\underline{\rho}(X)$ is finite, it holds for the family $\mathbb{F}$ of finite sets $F \subseteq \mathbb{P}$, and standard simplices $\Delta(F)$ over these, that

$$\underline{\rho}(X) = \sup_{F\in\mathbb{F}} \inf_{A\in\mathbb{A}} \max_{P\in F} P(A - X) = \sup_{F\in\mathbb{F}} \inf_{A\in\mathbb{A}} \max_{P\in\Delta(F)} P(A - X).$$

Proof. Only the last two assertions, invoking convexity, require proof. Note that $P(A-X)$ is jointly convex in (A, X). Hence, with $\mathbb{A}$ convex, both reduced functions

$$X \mapsto \rho_P(X) := \inf_{A\in\mathbb{A}} P(A - X) \quad \text{and} \quad (A, X) \mapsto \rho(A, X) := \sup_{P\in\mathbb{P}} P(A - X)$$

remain convex. Therefore, either measure

$$X \mapsto \underline{\rho}(X) = \sup_{P\in\mathbb{P}} \rho_P(X) \quad \text{and} \quad X \mapsto \overline{\rho}(X) = \inf_{A\in\mathbb{A}} \rho(A, X)$$

becomes convex. The last assertion follows from Theorem 6.2.2 in [3]. □

Clearly, $\mathbb{A} \subseteq \{\underline{\rho}_{\mathbb{A},\mathbb{P}} \leq 0\}$. Yet, since $\underline{\rho}_{\mathbb{A},\mathbb{P}} = \underline{\rho}_{conv\mathbb{A},\mathbb{P}}$, a primitive $\mathbb{A}$ cannot be recovered from $\underline{\rho}_{\mathbb{A},\mathbb{P}}$ unless $\mathbb{A}$ be convex. Similarly, since $\bar{\rho}_{\mathbb{A},\mathbb{P}} = \bar{\rho}_{\mathbb{A},conv\mathbb{P}}$, one cannot recover $\mathbb{P}$ from $\bar{\rho}_{\mathbb{A},\mathbb{P}}$ unless $\mathbb{P}$ be convex.

The lower and upper measures are twin values of a bilinear saddle function. As it turns out, when $\{\underline{\rho} \leq 0\} \subseteq \mathbb{A}$, risk assessment acquires the nature of a saddle value. Moreover, $\mathbb{A}$ is then identified as the common acceptance set:

Proposition 5.2 (Saddle representation of risk measures). *Suppose $\{\underline{\rho} \leq 0\} \subseteq \mathbb{A}$. Then both measures $\underline{\rho}$, $\bar{\rho}$ have $\mathbb{A}$ as acceptance set. Hence $\underline{\rho} = \bar{\rho}$.*
In particular, this happens under the assumptions of Theorem 2.2.
In general, when $\underline{\rho} = \bar{\rho}$, one may let both sets $\mathbb{A}$ and $\mathbb{P}$ be convex. Similarly, when $\mathbb{X}$ is topological and each $P \in \mathbb{P}$ is continuous, if $\underline{\rho} = \bar{\rho}$, one may take $\mathbb{A}$ and $\mathbb{P}$ to be both closed.

Proof. Since $\bar{\rho} \leq 0$ on $\mathbb{A}$, and $\underline{\rho} \leq \bar{\rho}$, the first assertions follow from the string $\mathbb{A} \subseteq \{\bar{\rho} \leq 0\} \subseteq \{\underline{\rho} \leq 0\} \subseteq \mathbb{A}$. In the special setting of Theorem 2.2, $\rho = \underline{\rho}$ and $\mathbb{A} = \{\underline{\rho} \leq 0\}$.

For the assertion about convexity, recall that $\underline{\rho}_{\mathbb{A},\mathbb{P}} = \underline{\rho}_{conv\mathbb{A},\mathbb{P}}$ and $\overline{\rho}_{\mathbb{A},\mathbb{P}} = \overline{\rho}_{\mathbb{A},conv\mathbb{P}}$. So, when $\underline{\rho} = \bar{\rho}$, it obtains

$$\underline{\rho}_{conv\mathbb{A},conv\mathbb{P}} \geq \underline{\rho}_{conv\mathbb{A},\mathbb{P}} = \underline{\rho}_{\mathbb{A},\mathbb{P}} = \bar{\rho}_{\mathbb{A},\mathbb{P}} = \bar{\rho}_{\mathbb{A},conv\mathbb{P}} \geq \bar{\rho}_{conv\mathbb{A},conv\mathbb{P}},$$

hence $\rho_{\mathbb{A},\mathbb{P}} = \rho_{conv\mathbb{A},conv\mathbb{P}}$. Finally, for the topological setting, notice that $\underline{\rho}_{\mathbb{A},\mathbb{P}} = \underline{\rho}_{cl\mathbb{A},\mathbb{P}}$ and $\overline{\rho}_{\mathbb{A},\mathbb{P}} = \overline{\rho}_{\mathbb{A},cl\mathbb{P}}$. So,

$$\underline{\rho}_{cl\mathbb{A},cl\mathbb{P}} \geq \underline{\rho}_{cl\mathbb{A},\mathbb{P}} = \underline{\rho}_{\mathbb{A},\mathbb{P}} = \bar{\rho}_{\mathbb{A},\mathbb{P}} = \bar{\rho}_{\mathbb{A},cl\mathbb{P}} \geq \bar{\rho}_{cl\mathbb{A},cl\mathbb{P}},$$

to the effect that $\rho_{\mathbb{A},\mathbb{P}} = \rho_{cl\mathbb{A},cl\mathbb{P}}$. $\square$

Example with non-convex sets. Let $S = \{\underline{s}, \bar{s}\}$ comprise merely two states (say, low and high), and posit $\mathbb{X} = \mathbb{X}^* = \mathbb{R}^S$ with the ordinary inner product. Choose non-convex primitive sets

$$\mathbb{A} = \{(-1,1),(1,-1)\} + \mathbb{R}_+\mathbf{1} \text{ and } \mathbb{P} = \{(1,0),(0,1)\}$$

to have

$$\underline{\rho}(X) = -\min_{s\in S} X(s) - 1, \qquad \bar{\rho}(X) = \min_{A\in\mathbb{A}} \max_{s\in S} [A(s) - X(s)] .$$

Here $0 \notin \mathbb{A}$, but $\underline{\rho}(0) = -1$, and $\bar{\rho}(0) = 1$. Thus, absent convexity, lower and upper measure may differ. Note that $\underline{\rho}$ is convex, but $\mathbb{A}$ is a strict subset of $\left\{\underline{\rho} \leq 0\right\} = \{X \in \mathbb{X} : \min_s X(s) \geq -1\}$. $\Diamond$

Explored next are two alternative settings which both yield the crucial inclusion $\left\{\underline{\rho} \leq 0\right\} \subseteq \mathbb{A}$ whence $\underline{\rho} = \overline{\rho}$. The first invokes a weak form of local Lipschitz continuity along lines:

Proposition 5.3 (On coincidence of the lower and upper measures). *Suppose $X \notin \mathbb{A}$ and $\underline{\rho}(X) < +\infty$ imply three things:*

1) the function $r \in \mathbb{R} \mapsto \underline{\rho}(rX)$ is locally Lipschitz continuous at $r = 1$,

2) the set

$$\{r \in [0,1] : (1-r)X + rc \in \mathbb{A}\} \text{ is closed for large enough constant } c \in \mathbb{R}. \tag{21}$$

3) $\underline{\rho}(X) \geq 0$.

Then, $\left\{\underline{\rho} \leq 0\right\} \subseteq \mathbb{A}$ to the effect that $\underline{\rho} = \overline{\rho}$, and $\mathbb{A}$ becomes the common acceptance set.

Proof. Consider any $X \notin \mathbb{A}$ such that $\underline{\rho}(X) < +\infty$. Let L be the local Lipschitz modulus of the function $r \in \mathbb{R} \mapsto \underline{\rho}(rX)$ at $r = 1$. Choose a constant $c > L$ so large that (21) holds. Then there exists $\overline{\lambda} \in (0,1)$ such that $(1-\lambda)X + \lambda c \notin \mathbb{A}$ for all $\lambda \in \left[0, \overline{\lambda}\right)$. By *3*) any of these λ gives

$$0 \leq \underline{\rho}((1-\lambda)X + \lambda c) = \underline{\rho}((1-\lambda)X) - \lambda c.$$

Consequently, when $\lambda \in (0, \overline{\lambda})$ is small enough, the local Lipschitz continuity of $\underline{\rho}(rX)$ at $r = 1$ yields

$$\underline{\rho}(X) \geq \underline{\rho}((1-\lambda)X) - \lambda L \geq \lambda(c - L) > 0.$$

Thus, $\left\{\underline{\rho} \leq 0\right\} \subseteq \mathbb{A}$, and Proposition 5.2 can be invoked to conclude. $\square$

If $\mathbb{A}$ is convex, so is $\underline{\rho}$ by Proposition 5.1. Then it suffices for the above condition *1*) that $r \in \mathbb{R} \mapsto \underline{\rho}(rX)$ be finite-valued near $r = 1$. If the vector space $\mathbb{X}$ is topological and Hausdorff, it suffices for condition *2*) that $\mathbb{A}$ be closed; its intersection with each line is then a closed set. Note that, granted such closure along lines, if $\mathbb{A}$ is convex with non-empty algebraic interior,

$$\mathbb{A} = \{X \in \mathbb{X} : \underline{\sigma}_{\mathbb{A}}(X^*) \leq X^*(X) \ \forall X^* \in \mathbb{X}^*\}.$$

To have $\mathbb{A}$ as acceptance set for $\underline{\rho}$, condition *3*) in Prop. 5.3 must, of course, be strengthened to $X \notin \mathbb{A} \Rightarrow \underline{\rho}(X) > 0$. This observation motivates the following:

Definition (Separating prices). $\mathbb{P}$ strictly separates $\mathbb{A}$ iff

$$X \notin \mathbb{A} \ \text{ implies } \ \underline{\sigma}_{\mathbb{A}}(P) := \inf_{A \in \mathbb{A}} P(A) > P(X) \ \text{ for some } P \in \mathbb{P}. \tag{22}$$

In economic terms, when X isn't acceptable, at least one price $P \in \mathbb{P}$ evaluates all $A \in \mathbb{A}$ strictly above X. In geometric terms, the prescribed set $\mathbb{P}$ of price functionals is large enough to separate any inacceptable position strictly from all acceptable counterparts. Put differently: strict separation obtains when $\mathbb{A}$ equals an intersection of "open" half-spaces $\{P > r\}$, $P \in \mathbb{P}$, $r \in \mathbb{R}$. Then clearly, $\mathbb{A}$ is convex.

Proposition 5.4 (Strictly separating prices yield coincidence of the two risk measures).

- *If $\mathbb{P}$ strictly separates $\mathbb{A}$, it does likewise with* $\operatorname{conv} \mathbb{A}$.
- *If $\underline{\rho}$, as defined in* (19), *has $\mathbb{A}$ as acceptance set, then $\mathbb{P}$ separates $\mathbb{A}$.*
- *Conversely, if $\mathbb{P}$ separates $\mathbb{A}$, then both risk measures $\underline{\rho}$, $\overline{\rho}$ have $\mathbb{A}$ as acceptance set – hence $\underline{\rho} = \overline{\rho}$.*

Proof. The first two statements are obvious. For the third, in case $X \notin \mathbb{A}$, pick $P \in \mathbb{P}$ such that (22) holds. Then

$$0 < \inf_{A \in \mathbb{A}} P(A - X) \leq \sup_{P \in \mathbb{P}} \inf_{A \in \mathbb{A}} P(A - X) = \underline{\rho}(X).$$

Hence $\{\underline{\rho} \leq 0\} \subseteq \mathbb{A}$, which implies $\underline{\rho} = \overline{\rho}$, and $\mathbb{A}$ becomes the common acceptance set. □

In fairness, a main task is *not* to derive properties of $\mathbb{A}$ and $\mathbb{P}$ from $\underline{\rho} = \overline{\rho}$. Rather, in view of min-max theory, it imports to use appropriate properties of $\mathbb{A}$ and $\mathbb{P}$ to establish that $\underline{\rho} = \overline{\rho}$:

Proposition 5.5 (Joint convexity and lopsided min-max [3]). *Let $\mathbb{X}$ be a topological vector space, its topological dual $\mathbb{X}^*$ being given the weak-star topology. When both sets $\mathbb{A} \subset \mathbb{X}$, $\mathbb{P} \subset \mathbb{X}^*$ are convex, and one is compact, it holds that $\underline{\rho} = \overline{\rho}$. In that case, $\mathbb{A}$ is the common acceptance set iff $\mathbb{P}$ separates $\mathbb{A}$* (22)*. This happens then iff $\mathbb{A}$ separates $\mathbb{P}$ in the sense that*

$$X \notin \mathbb{A} \ \textit{ implies } \ \sup_{P \in \mathbb{P}} P(A - X) > 0 \ \textit{ for some } A \in \mathbb{A}. \tag{23}$$

When $\mathbb{A}$ is compact, there exists for each $X \in \mathbb{X}$ an $A_X \in \mathbb{A}$ such that

$$\underline{\rho}(X) = \sup_{P \in \mathbb{P}} P(A_X - X) = \bar{\rho}(X).$$

If $\mathbb{P}$ is compact, there exists for each $X \in \mathbb{X}$ a $P_X \in \mathbb{P}$ such that

$$\underline{\rho}(X) = \inf_{A \in \mathbb{A}} P_X(A - X) = \bar{\rho}(X). \qquad \square$$

Example 5.6 (Restricted probabilities). Let the state set S be finite to have a Euclidean space $\mathbb{X} = \mathbb{R}^S$ of contingent claims, endowed with ordinary inner product. Then any non-empty closed convex subset $\mathbb{P}$ of the standard simplex $\Delta = \Delta(S) := \left\{P \in \mathbb{R}^S_+ : \sum_{s \in S} P_s = 1\right\}$ may serve as a tractable ensemble of normed, non-negative prices. In this case, by Proposition 5.5, employing any non-empty convex set $\mathbb{A} \subset \mathbb{X}$ yields $\underline{\rho} = \overline{\rho}$. ♢

Example 5.7 (Relative entropy). Still with S finite, and Euclidean space $\mathbb{X} = \mathbb{R}^S$, consider the *Kullback-Leibler distance*

$$K(\bar{P}, P) := \sum_{s \in S} \bar{P}_s \log \frac{\bar{P}_s}{P_s} = -\sum_{s \in S} \bar{P}_s \left\{\log P_s - \log \bar{P}_s\right\}$$

between any two probability distributions $\bar{P}, P \in \Delta$. That “distance” is jointly convex, non-negative, and vanishes only when $\bar{P} = P$; see [13]. Thus for a fixed reference distribution $\bar{P} \in \Delta$, and any $\varepsilon > 0$, letting $\mathbb{P} := \left\{P \in \Delta : K(\bar{P}, P) \leq \varepsilon\right\}$, we get a compact convex ensemble of prices [28]. Again by Proposition 5.5, $\underline{\rho} = \overline{\rho}$ for any non-empty convex $\mathbb{A} \subset \mathbb{X}$. ♢

Convexity of both sets $\mathbb{A}$, $\mathbb{P}$ – as required in Proposition 5.5 – is, however, not essential. Topological properties may partly serve as substitutes. When both spaces $\mathbb{A}$, $\mathbb{P}$ are topological, for each fixed $X \in \mathbb{X}$, let $\mathcal{C}(\mathbb{A}-X, \mathbb{P})$ denote the set of all continuous mappings from $\mathbb{A}-X$ into $\mathbb{P}$. Correspondingly, let $\mathcal{C}(\mathbb{P}, \mathbb{A})$ consist of all continuous $C : \mathbb{P} \to \mathbb{A}$. Now, Theorem 6.3.2 in [3] gives forthwith:

Proposition 5.8 (On instances where merely *one* primitive set is convex).

• *Suppose $A \mapsto P(A)$ is lower semicontinuous for each P belonging to a convex subset $\mathbb{P}$ of a topological vector space. Further suppose some $P \in \mathbb{P}$ makes*

$$\{A \in \mathbb{A} : P(A) \leq r\} \quad \textit{compact for each real } r.$$

Then it holds that

$$\sup_{C \in \mathcal{C}(\mathbb{A}-X, \mathbb{P})} \inf_{A \in \mathbb{A}} C(A - X)[A - X] = \overline{\rho}(X).$$

If moreover, $\mathbb{A}$ is convex, then $\underline{\rho} = \overline{\rho}$.

• *Let S be a separable metric space, and suppose each $P \in \mathbb{P}$ is a probability measure on S. Further, let $\mathbb{X} := BC(S, \mathbb{R})$ be the space of bounded continuous functions from S into $\mathbb{R}$. Let $\mathbb{A} \subset \mathbb{X}$ be convex, and suppose some $A \in \mathbb{A}$ makes*

$$\{P \in \mathbb{P} : P(A) \geq r\} \quad \textit{compact for each real } r. \tag{24}$$

Then

$$\inf_{C \in \mathcal{C}(\mathbb{P},\mathbb{A})} \sup_{P \in \mathbb{P}} P[C(P) - X] = \underline{\rho}(X). \tag{25}$$

If moreover, $\mathbb{P}$ is convex, we get $\underline{\rho} = \overline{\rho}$. □

Example 5.9 (Robust certainty equivalents). Let $\mathcal{U}$ be a set of quasi-concave, increasing utility functions $u : \mathbb{R} \to \mathbb{R}$. Suppose μ is a probability measure on $(S, \mathcal{F})$ and let $\mathbb{X} := L^1(S, \mathcal{F}, \mu)$ be the quotient space of $\mathcal{F}$-measurable functions $X : S \to \mathbb{R}$ such that $E_\mu |X| = \int |X(s)| \, \mu(ds)$ is finite. Fix a bounded, closed but possibly non-convex set of *probability densities* $\mathbb{P} \subseteq \mathbb{X}^* = L^\infty(S, \mathcal{F}, \mu, \mathbb{R})$. This means that each $P \in \mathbb{P}$ is non-negative on $\mathbb{X}_+$, and $P(\mathbf{1}) = \int P(s)\mu(ds) = 1$. For chosen constants $c_{P,u}$ – each reflecting a *certainty equivalent* under (P, u) – posit

$$\mathbb{A} := \cap_{P \in \mathbb{P}} \cap_{u \in \mathcal{U}} \left\{ X \in L^1(S, \mathcal{F}, \mu) : E_P u(X) \geq u(c_{P,u}) \right\}$$

to have a convex acceptance set. Then (24) holds, hence (25) applies. ♢

6. Market-based Pricing

As seen, a cash invariant risk measure ρ determines – and is determined by – its acceptance set $\mathbb{A}$. Further, as shown by Theorem 2.2, a convex instance $\rho \leftrightarrow \mathbb{A}$ may point to an ensemble $\mathbb{P}$ of associated prices. In principle, $\mathbb{A}$, $\mathbb{P}$ could derive from deliberate choice. Often though, these two sets stem from market data. This last section briefly illustrates how.

Here, for convenience, instead of a risk measure ρ, rather consider the corresponding *value measure* $\vartheta := -\rho$. Thus, $\vartheta(X + r) = \vartheta(X) + r$ for all $X \in \mathbb{X}$, $r \in \mathbb{R}$, and $\mathbb{A}_\vartheta = \mathbb{A} = \{\vartheta \geq 0\}$.

Suppose claims are made, and trades undertaken, along a *scenario tree* which has finite node set $\mathcal{N}$. Each node $n \in \mathcal{N}$, except *one*, has a unique immediate *ancestor* $n^- \in \mathcal{N}$. The exceptional *root* node, denoted 0, has none. Conversely, if n is the ancestor of c, the latter is called a *child*, belonging to $\mathcal{C}(n) \subset \mathcal{N}$. *Terminal nodes*, alias *leafs*, have no children and constitute a strict subset $\mathcal{N}_T \subset \mathcal{N}$. The tree emerges by directing an oriented branch from n to c iff $c \in \mathcal{C}(n)$.

Suppose there is a finite list J of *traded assets*. A share of asset $j \in J$ commands unit price p_{jn} (cum dividend if any) at node $n \in \mathcal{N}$. Denote by $\theta_{jn} \in \mathbb{R}$ the number of shares an investor holds in paper $j \in J$ upon leaving node n. Suppose he buys (outgoing) *portfolio* $\theta_n := (\theta_{jn}) \in \mathbb{R}^J$ at n and liquidates there the (incoming) portfolio θ_{n^-} bought at the ancestor node. Naturally, posit $\theta_{0^-} := 0$.

There prevails a price vector $p_n := (p_{jn}) \in \mathbb{R}^J$. Absent transaction costs, in terms of the standard dot inner product, those operations bring *gain* $G_n(\theta) := p_n \cdot [\theta_{n^-} - \theta_n]$ at n. If the j-price remains constant across $\mathcal{C}(n)$, paper j is declared *predictable* at n.

By assumption, the investor does not affect prices, but he might reasonably wonder: can the market be milked for money? That question motivates the following

Definition. The asset market allows **arbitrage** iff the inequality system

$$G_n(\theta) \geq 0 \text{ at all nodes } n, \quad \text{and} \quad p_n \cdot \theta_n \geq 0 \text{ for each leaf}, \tag{26}$$

admits a solution $\theta = (\theta_n)$ with at least one strict inequality. Otherwise the market is declared **arbitrage-free**.[8]

By assumption, a special paper $b \in J$, hereafter called a *bond*, trades at each node n for strictly positive price p_{bn}. That paper's price profile defines *discount factors* $\delta_n := p_{b0}/p_{bn}$ which help identifying arbitrage – if any. For the statement, note that any probability distribution $P \in \Delta(S)$ across *scenarios* $S := \mathcal{N}_T$, generates, by successive backward recursion, probabilities $P_n := \sum_{c \in \mathcal{C}(n)} P_c$ on deeper nodes $n \notin \mathcal{N}_T$. Conversely, for any such node the resulting P_n defines direct *transition probabilities*

$$P(c\,|n) := \begin{cases} P_c/P_n & \text{when } c \in \mathcal{C}(n) \text{ and } P_n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.1 (On arbitrage and martingale measures [19], [30]). *The market is arbitrage-free iff there exists a strictly positive probability measure P across leafs, inducing transition probabilities $P(c|\,n)$, $c \in \mathcal{C}(n)$, such that the discounted price process $n \in \mathcal{N} \mapsto \delta_n p_n \in \mathbb{R}^J$ becomes a vector martingale:*

$$\delta_n p_n = E_P\left[\delta_c p_c|\, n\right] = \sum_{c \in \mathcal{C}(n)} \delta_c p_c P(c\,|n) \quad \textit{for each non-terminal } n. \tag{27}$$

In particular, if the bond is predictable at some non-terminal n, and $\delta_{(n)} := p_{bn}/p_{bc} = \delta_c/\delta_n$, $c \in \mathcal{C}(n)$, denotes the local discount factor there, the corresponding equation in (27) *amounts to*

$$p_n = \delta_{(n)} E_P\left[p_c|\, n\right] = \delta_{(n)} \sum_{c \in \mathcal{C}(n)} p_c P(c\,|n).$$

Proof. Fix *any* probability distribution $\pi_n > 0$ across leafs $n \in \mathcal{N}_T$. Successively, by backward recursion, find the probability $\pi_n := \sum_{c \in \mathcal{C}(n)} \pi_c > 0$ on

[8] Other definitions of arbitrage require that θ be *self-financing* in that $G_n(\theta) = 0$ for all $n \neq 0$ (or for all n); see e.g. [30].

deeper nodes $n \notin \mathcal{N}_T$. Consider the homogeneous (conical) linear program, aimed at maximal expected present value:

$$\max_\theta \sum_n \delta_n \pi_n G_n(\theta) + \sum_{n \in \mathcal{N}_T} \delta_n \pi_n p_n \cdot \theta_n \text{ s.t. (26).} \tag{28}$$

Clearly, $\theta = 0$ is feasible for problem (28), and the market is arbitrage-free iff the optimal value equals 0. Associate a multiplier $\delta_n y_n \geq 0$ to inequality $G_n(\theta) \geq 0$, and a multiplier $\delta_n Y_n \geq 0$ to the leaf constraint $p_n \cdot \theta_n \geq 0$. Maximizing the resulting Lagrangian

$$\sum_n \delta_n(\pi_n + y_n) G_n(\theta) + \sum_{n \in \mathcal{N}_T} \delta_n(\pi_n + Y_n) p_n \cdot \theta_n \tag{29}$$

$$= \sum_{n \notin \mathcal{N}_T} \left[\sum_{c \in \mathcal{C}(n)} \delta_c(\pi_c + y_c) p_c - \delta_n(\pi_n + y_n) p_n \right] \cdot \theta_n + \sum_{n \in \mathcal{N}_T} \delta_n(Y_n - y_n) p_n \cdot \theta_n$$

with respect to θ, the resulting dual program amounts to solve the equations

$$\delta_n(\pi_n + y_n) p_n = \sum_{c \in \mathcal{C}(n)} \delta_c(\pi_c + y_c) p_c \text{ for all non-terminal } n \text{ with } y \geq 0.$$

Suppose this equation system is indeed solvable. Then, by LP duality, problem (28) has optimal value 0, whence there are no arbitrage opportunities. Component b of the last equation gives $\pi_n + y_n = \sum_{c \in \mathcal{C}(n)}(\pi_c + y_c)$. Therefore $P(c\,|n) := (\pi_c + y_c)/(\pi_n + y_n)$ defines strictly positive transition probabilities that satisfy (27).

Conversely, suppose some strictly positive measure P on the leafs suits (27). Then, in (29) let $\pi = P$ and each $y_n, Y_n = 0$ to get

$$\sum_n \delta_n \pi_n G_n(\theta) + \sum_{n \in \mathcal{N}_T} \delta_n \pi_n p_n \cdot \theta_n = \sum_{n \notin \mathcal{N}_T} \left[\sum_{c \in \mathcal{C}(n)} \delta_c \pi_c p_c - \delta_n \pi_n p_n \right] \cdot \theta_n = 0$$

for all θ. Thus arbitrage is impossible. □

Let $\mathbb{P}$ denote the bounded convex set of all strictly positive probability measures across leafs that satisfy (27). Clearly, $\mathbb{P} \subset \Delta(S)$ depends only on the price process $n \mapsto p_n$. By Proposition 6.1, $\mathbb{P}$ is non-empty iff there are no arbitrage opportunities. Any $P \in \mathbb{P}$ is called an *equivalent martingale measure.*

Asset pricing. Besides the list J of primary securities, consider another risk X which delivers monetary payout X_n in node $n \neq 0$. Then, what minimal purchase cost $p_0 \cdot \theta_0$ will a portfolio profile $n \mapsto \theta_n$ require in order to gain $G_n(\theta) \geq X_n$ at each node $n \neq 0$, and moreover, avoid terminal debt? This question amounts to look for a

superhedge: minimize $p_0 \cdot \theta_0$ s.t. $G_n(\theta) \geq X_n \; \forall n \neq 0 \;\&\; p_n \cdot \theta_n \geq 0 \; \forall n \in \mathcal{N}_T$.

Any feasible superhedge θ represents an information-adapted investment strategy that stays on the upper safe side, eliminating all downside risk.[9]

Assign a Lagrange multiplier $\delta_n P_n \geq 0$ to constraint $G_n(\theta) \geq X_n$. Similarly, couple a multiplier $\delta_n \Pi_n \geq 0$ to the terminal constraint $p_n \cdot \theta_n \geq 0$, $n \in \mathcal{N}_T$. Thereby emerges a Lagrangian

$$\mathcal{L}(\theta, P, \Pi) := \begin{cases} \sum_{n\neq 0} \delta_n P_n X_n \quad + \left[p_0 - \sum_{c\in\mathcal{C}(0)} \delta_c P_c p_c\right] \cdot \theta_0 \\ + \sum_{n\notin 0\cup\mathcal{N}_T} \left[\delta_n P_n p_n - \sum_{c\in C(n)} \delta_c P_c p_c\right] \cdot \theta_n \\ + \sum_{n\in\mathcal{N}_T} \delta_n (P_n - \Pi_n) p_n \cdot \theta_n. \end{cases}$$

As customary, minimize $\mathcal{L}(\theta, P, \Pi)$ with respect to $\theta = (\theta_n)$. That operation, when finite-valued, associates to the above problem a corresponding *dual problem*, namely:

$$\begin{array}{ll} \text{maximize} & \sum_{n\neq 0} \delta_n P_n X_n \\ \text{subject to} & \delta_n P_n p_n = \sum_{c\in\mathcal{C}(n)} \delta_c P_c p_c \ \forall n \notin \mathcal{N}_T,\ P \geq 0,\ \text{and}\ P_0 = 1. \end{array}$$

(Π disappeared because $P_n = \Pi_n$ for $n \in \mathcal{N}_T$.) Component b of the last constraint yields $P_n = \sum_{c\in\mathcal{C}(n)} P_c$ at each $n \notin \mathcal{N}_T$. Since $P_0 = 1$ and $P \geq 0$, it follows that P defines a probability measure over $\mathcal{N}_T = S$; that is, $P \in \Delta(S)$. One may regard optimal dual solutions P as risk-neutral, equivalent probability distributions that depict the expected present value of process X as favorably as possible. Clearly, when all $P_n > 0$, one recovers (27). It now follows forthwith:

Proposition 6.2 (Martingale evaluation). *Suppose the asset market is arbitrage-free, having a non-empty set $\mathbb{P}$ of equivalent martingale measures. Posit as numeraire $\sum_{n\neq 0} \delta_n := 1$. Then,*

$$\bar{\vartheta}(X) := \sup \left\{ E_P \sum_{n\neq 0} \delta_n X_n : P \in \mathbb{P} \right\}$$

is a coherent value measure, giving the same evaluation as the above superhedging. It has acceptance set $\mathbb{A} = \left\{\bar{\vartheta} \geq 0\right\}$ and price ensemble $\mathbb{P}$. □

Superhedging approximates X from above, as cheaply as possible, while avoiding intermediate expense and terminal debt. Alternatively, one could approximate X from below, as dearly as possible, exiting maybe with terminal claims. That enterprise furnishes value

$$\underline{\vartheta}(X) := \sup \left\{p_0 \cdot \theta_0 : G_n(\theta) \leq X_n \ \forall n \neq 0 \ \&\ p_n \cdot \theta_n \leq 0 \ \forall n \in \mathcal{N}_T\right\}.$$

[9] If X is demanded for price $> p_0 \cdot \theta_0$ at the root, a seller can immediately pocket the excess over $p_0 \cdot \theta_0$ and spend the latter amount on an optimal super-hedge θ. Thereafter he looks forward to net payment $G_n(\theta) - X_n \geq 0$ at node $n \neq 0$, and no terminal debt.

Absent arbitrage, arguing as above, and letting $\sum_{n\neq 0}\delta_n := 1$, the associated dual problem defines another value measure:

$$\underline{\vartheta}(X) := \inf\left\{E_P\sum_{n\neq 0}\delta_n X_n : P\in\mathbb{P}\right\}$$

Clearly, $\underline{\vartheta}\leq\bar{\vartheta}$, and – as is well known – $\underline{\vartheta}=\bar{\vartheta}$ iff $\mathbb{P}$ reduces to a singleton [41], which happens iff the market is complete, meaning that any $X\in\mathbb{R}^{\mathcal{N}\setminus 0}$ has $G_n(\theta)=X_n$ for a suitable portfolio profile $n\mapsto\theta_n$.

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