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Isoperimetric Inequalities for the Principal Eigenvalue of a Membrane and the Energy of Problems with Robin Boundary Conditions

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An inequality for the reverse Bossel-Daners inequality is derived by means of the harmonic transplantation. The first and second shape derivatives are computed for the ball. The same method is then applied to elliptic boundary value problems with inhomogeneous Neumann conditions. The first variation is computed and an isoperimetric inequality is derived for the minimal energy.

Keywords: Rayleigh-Faber Krahn inequality, Robin boundary conditions, domain variation, harmonic transplantation

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1. Introduction

Bossel [6] and Daners [7] extended the Rayleigh-Faber-Krahn inequality to the principal eigenvalue of the membrane with Robin boundary conditions. They proved that among all domains of given volume, the first eigenvalue λ of $\Delta\phi + \lambda\phi = 0$ on Ω with $\partial_\nu\phi + \beta\phi = 0$ on $\partial\Omega$, where ∂_ν is the outer normal derivative and β is a positive elasticity constant, is minimal for the ball.

Bareket [5] considered the eigenvalue of the same problem where $\beta = -\alpha$ is negative. She was able to show, that for nearly circular domains of given area the circle has the largest first eigenvalue. Recently this result was extended to higher dimensions for nearly spherical domains by Ferone, Nitsch and Trombetti [8]. The question whether or not the ball is optimal for all domains of the same volume remained open until recently when Freitas and Krejcirik [9] showed that annuli have for large α a larger eigenvalue than the ball with the same volume.

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In this note we derive an isoperimetric inequality for arbitrary domains in $\mathbb{R}^n$. The proof uses the method of harmonic transplantation which is a substitute of the conformal transplantation in higher dimensions. We then determine the first and second shape derivative for nearly spherical domains with fixed volume. It turns out that in the case of a ball the first derivative vanishes and the second is strictly negative.

All the arguments apply to a large class of variational problems related to elliptic equations with homogeneous and inhomogeneous boundary conditions. We illustrate them for some problems with inhomogeneous Neumann conditions considered by Auchmuty [2] in the context of trace inequalities.

The quantities we want to estimate in this paper are expressed by variational principles, defined for functions in $W^{1,2}(\Omega)$ and $L^2(\partial\Omega)$. For the existence of those quantities we have to require that the embedding $W^{1,2}(\Omega)$ into $L^2(\Omega)$ as well as the trace operator $\Gamma : W^{1,2}(\Omega) \to L^2(\partial\Omega)$ is compact. This is the case, cf. [1], when $\Omega \subset \mathbb{R}^n$ is a bounded domain such that its boundary consists of a finite number of closed Lipschitz surfaces of finite surface area. *Throughout this paper we shall assume that this condition is satisfied.*

2. Eigenvalue problem

2.1. Existence

Consider the eigenvalue defined by

$$\lambda(\Omega) = \inf_{W^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla u|^2 \, dx - \alpha \oint_{\partial\Omega} u^2 \, dS \right\} \text{ with } \int_\Omega u^2 \, dx = 1 \text{ and } \alpha > 0 \, . \tag{1}$$

Under our assumptions on Ω the infimum is finite. This is a consequence of the following trace inequality: for all $\epsilon > 0$ there exists a positive constant $c = c(\epsilon)$ such that

$$\oint_{\partial\Omega} u^2 \, dS \le \epsilon \int_\Omega |\nabla u|^2 \, dx + c(\epsilon) \int_\Omega u^2 \, dx.$$

By the direct method of the calculus of variations and the compactness properties of $W^{1,2}(\Omega)$ the minimum is achieved. The minimizer u is a solution of

$$\Delta u + \lambda(\Omega) u = 0 \text{ in } \Omega, \qquad \partial_\nu u = \alpha u \text{ on } \partial\Omega. \tag{2}$$

If we choose a constant as a trial function in (1), we see immediately that $\lambda(\Omega)$ is negative. The minimizer is of constant sign and the eigenspace corresponding to the smallest eigenvalue is one-dimensional.

According to M. Bareket [5] this problem appears in acoustics.

2.2. Harmonic Transplantation and Isoperimetric Inequality

In this section we recall the method of harmonic transplantation which has been deviced by Hersch [10], (cf. also [3]) to construct trial functions for variational problems of the type (1). To this end we need the Green's function with Dirichlet boundary condition

$$G_\Omega(x,y) = \gamma(S(|x-y|) - H(x,y)), \tag{3}$$

where

$$\gamma = \begin{cases} \frac{1}{2\pi} & \text{if } n = 2 \\ \frac{1}{(n-2)|\partial B_1|} & \text{if } n > 2 \end{cases} \qquad \text{and} \qquad S(t) = \begin{cases} -\ln(t) & \text{if } n = 2 \\ t^{2-n} & \text{if } n > 2. \end{cases} \tag{4}$$

For fixed $y \in \Omega$ the function $H(\cdot, y)$ is harmonic. Set $B_R = \{x : |x| < R\}$.

Definition 2.1. The harmonic radius at a point $y \in \Omega$ is given by

$$r(y) = \begin{cases} e^{-H(y,y)} & \text{if } n = 2, \\ H(y,y)^{-\frac{1}{n-2}} & \text{if } n > 2. \end{cases}$$

The harmonic radius vanishes on the boundary $\partial\Omega$ and takes its maximum r_Ω at the harmonic center y_h. It satisfies the isoperimetric inequality [10], [3]

$$|B_{r_\Omega}| \le |\Omega|. \tag{5}$$

Note that $G_{B_R}(x,0)$ is a monotone function in $r = |x|$. Consider any radial function $\phi : B_{r_\Omega} \to \mathbb{R}$ thus $\phi(x) = \phi(r)$. Then there exists a function $\omega : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) = \omega(G_{B_{r_\Omega}}(x,0)).$$

To $\phi(x)$ we associate the transplanted function $U : \Omega \to \mathbb{R}$ defined by $U(x) = \omega(G_\Omega(x,y_h))$. Then for any positive function $f(s)$, cf. [10] or [3], the following inequalities hold true

$$\int_\Omega |\nabla U|^2 \, dx = \int_{B_{r_\Omega}} |\nabla \phi|^2 \, dx \tag{6}$$

$$\int_\Omega f(U) \, dx \ge \int_{B_{r_\Omega}} f(\phi) \, dx. \tag{7}$$

For our purpose we need an estimate of $\int_\Omega f(U)\, dx$ from above. For this some auxiliary lemmata are needed. The following notation will be used.

$$\Omega^t := \{x \in \Omega : G_\Omega(x,y_h) > t\}, \qquad B^t := \{x \in B_{r_\Omega} : G_{B_{r_\Omega}}(x,0) > t\},$$
$$m_\Omega(t) = |\Omega^t|, \qquad m_B(t) := |B^t|.$$

Recall that the Green's function $G_\Omega(x, y_h)$ is harmonic in the domain $\Omega \setminus \Omega^t$ and constant on the boundary and that $G_{B_{r_\Omega}}(x, 0)$ has analogous properties. Furthermore the capacity of the two sets is given by

$$\mathrm{cap}(\Omega \setminus \Omega^t) = \frac{1}{t^2} \int\limits_{\Omega\setminus\Omega^t} |\nabla G_\Omega(x, y_h)|^2 \, dx$$

$$\text{and} \quad \mathrm{cap}(B_{r_\Omega} \setminus B^t) = \frac{1}{t^2} \int\limits_{B_{r_\Omega}\setminus B^t} |\nabla G_{B_{r_\Omega}}(x, 0)|^2 \, dx.$$

By (4) we have

$$\oint_{\partial(\Omega\setminus\Omega^t)} \partial_\nu G_\Omega(x, y_h) \, dS = \oint_{\partial\left(B_{r_\Omega}\setminus B^t\right)} \partial_\nu G_{B_{r_\Omega}}(x, 0) \, dS = t.$$

A simple computation shows that the capacities of $\Omega \setminus \Omega^t$ and $B_{r_\Omega} \setminus B^t$ are equal. Let r_t be the radius of B^t, then

$$\mathrm{cap}(\Omega \setminus \Omega^t) = \mathrm{cap}(B_{r_\Omega} \setminus B^t) = \begin{cases} |\partial B_1| \, \frac{n-2}{r_t^{2-n} - r_\Omega^{2-n}} & \text{if } n > 2, \\ |\partial B_1| \, (\ln(r_t) - \ln(r_\Omega)) & \text{if } n = 2. \end{cases} \tag{8}$$

The following lemma is crucial for our optimization result.

Lemma 2.2. *Let*

$$\gamma := \left(\frac{|\Omega|}{|B_{r_\Omega}|}\right)^{\frac{1}{n}}.$$

Then $m_\Omega(t) \le \gamma^n \, m_B(t)$ for all $t \in (0, \infty)$.

Proof. By a rearrangement argument

$$\mathrm{cap}(\Omega \setminus \Omega^t) \ge \mathrm{cap}(B_R \setminus B_{\rho_t}),$$

where B_R is the ball with the same volume as $|\Omega|$ and B_{ρ_t} is the ball with the same volume as $|\Omega^t|$. From (8) we deduce that

$$\frac{1}{r_t^{2-n} - r_\Omega^{2-n}} \ge \frac{1}{\rho_t^{2-n} - R^{2-n}} \quad \text{if } n > 2,$$
$$\ln(r_t) - \ln(r_\Omega) \ge \ln(\rho_t) - \ln(R) \quad \text{if } n = 2.$$

Hence

$$\rho_t^{2-n} - R^{2-n} \ge r_t^{2-n} - r_\Omega^{2-n}. \tag{9}$$

By the definitions of R and r_Ω

$$\gamma = \frac{R}{r_\Omega} \; > \; 1,$$

hence $R = \gamma\, r_\Omega$. Introducing this expression into (9) we find

$$\rho_t^{2-n} \geq r_t^{2-n} - r_\Omega^{2-n}\left(1 - \frac{1}{\gamma^{n-2}}\right) \geq (\gamma\, r_t)^{2-n} \quad \text{for } n \geq 3,$$

and

$$\ln(r_t) \geq \ln(\rho_t) - \ln(\gamma) \quad \text{for } n = 2.$$

Consequently $\rho_t \leq \gamma\, r_t$ which completes the proof. □

This lemma enables us to construct an upper bound for $\int_\Omega f(U)\, dx$.

Lemma 2.3. *Suppose that $f(s)$ is positive and monotone increasing. Let $\phi(x) : B_{r_\Omega} \to \mathbb{R}$ be radial and monotone increasing. Then*

$$\int_\Omega f(U)\, dx \leq \gamma^n \int_{B_{r_\Omega}} f(\phi)\, dx. \tag{10}$$

Proof. Integration along level surfaces implies

$$\begin{aligned}\int_\Omega f(U)\, dx &= -\int_0^\infty f(\omega(t))\, dm_\Omega(t) \\ &= -f(\omega(t))\, m_\Omega(t)\Big|_0^\infty + \int_0^\infty f'(\omega(t))\, \omega'(t)\, m_\Omega(t)\, dt.\end{aligned}$$

If $f(\omega(0))$ is bounded

$$\int_\Omega f(U)\, dx = f(\omega(0))|\Omega| + \int_0^\infty f'(\omega(t))\, \omega'(t)\, m_\Omega(t)\, dt.$$

The assertion now follows from Lemma 2.2. □

We are now in position to prove

Theorem 2.4. *If $\Omega \subset \mathbb{R}^n$ is any domain with maximal harmonic radius r_Ω then*

$$|\Omega|\lambda(\Omega) \leq |B_{r_\Omega}|\lambda(B_{r_\Omega}).$$

Equality holds if and only if Ω is the ball B_{r_Ω}.

Proof. Let u be the eigenfunction corresponding to $\lambda(B_{r_\Omega})$. As mentioned before it is of constant sign, simple and therefore radial. Let U be its transplantation into Ω. Then by (1)

$$\lambda(\Omega) \leq \frac{\int_\Omega |\nabla U|^2\, dx - \alpha \oint_{\partial\Omega} U^2\, dS}{\int_\Omega U^2\, dx}.$$

In view of the equality (6) and since $\partial\Omega$ is a level set of U the numerator becomes

$$\int_{B_{r_\Omega}} |\nabla u|^2 \, dx - \alpha u^2(r_\Omega) \oint_{\partial\Omega} dS.$$

The isoperimetric inequality together with (5) implies

$$|\partial\Omega| \geq c_n |\Omega|^{\frac{n-1}{n}} \geq c_n |B_{r_\Omega}|^{\frac{n-1}{n}} = |\partial B_{r_\Omega}|.$$

From these estimates and the fact that $\lambda(B_{r_\Omega}) < 0$ it follows that

$$\int_{B_{r_\Omega}} |\nabla u|^2 \, dx - \alpha u^2(r_\Omega) \oint_{\partial\Omega} dS < 0.$$

Since $u(|x|)$ is a positive radial increasing function (10) applies and thus

$$\lambda(\Omega) \leq \frac{\int_{B_{r_\Omega}} |\nabla u|^2 \, dx - \alpha \oint_{\partial B_{r_\Omega}} u^2 \, dS}{\gamma^n \int_{B_{r_\Omega}} u^2 \, dx} = \gamma^{-n} \lambda(B_{r_\Omega})$$

which completes the proof. $\square$

3. Nearly spherical domains

3.1. Eigenfunction and eigenvalues for the ball

By standard arguments the first eigenfunction is radial and of constant sign. The eigenvalue problem (2) then reads as

$$u''(r) + \frac{n-1}{r} u'(r) + \lambda(B_R) \, u(r) = 0, \quad u'(R) = \alpha u(R). \tag{11}$$

The solution is expressed by means of the modified Bessel functions I_ν of order $\nu = \frac{n-2}{2}$, namely

$$u(r) = r^{\frac{2-n}{2}} \, I_{\frac{n-2}{2}}(\sqrt{\lambda(B_R)} \, r).$$

The first eigenvalue $\lambda(B_R)$ is determined implicitly by

$$\sqrt{|\lambda(B_R)|} = \left(\alpha + \frac{n-2}{2R}\right) \frac{I_\nu(\sqrt{|\lambda(B_R)|} R)}{I'_\nu(\sqrt{|\lambda(B_R|} R)}. \tag{12}$$

In the next subsection we discuss a monotonicity property of $\lambda(B_r)$ with respect to r.

3.2. Domain variation and first variation

In this section we follow closely the paper [4]. Let Ω_t be a family of perturbations of the domain $\Omega \subset \mathbb{R}^n$ of the form

$$\Omega_t := \{y = x + tv(x) + o(t) : x \in \Omega,\ t \text{ small}\}, \tag{13}$$

where $v = (v_1(x), v_2(x), \dots, v_n(x))$ is a smooth vector field and where $o(t)$ collects all terms such that $\frac{o(t)}{t} \to 0$ as $t \to 0$. Note that with this notation we have

$$|\Omega_t| = |\Omega| + t \int_\Omega \text{div } v\, dx + o(t), \tag{14}$$

where $|\Omega|$ denotes the n–dimensional Lebesgue measure of Ω.

We say that $y : \Omega_t \to \Omega$ is volume preserving of the first order if

$$\int_\Omega \text{div}\, v\, dx = \oint_{\partial\Omega} (v \cdot \nu) = 0. \tag{15}$$

Let $\lambda(\Omega_t)$ be the eigenvalue of a perturbed domain Ω_t (as described in (13)). Let $u(t)$ be the corresponding eigenfunction. Thus $u(t)$ and $\lambda(\Omega_t)$ solve

$$\Delta u(t) + \lambda(\Omega_t)u(t) = 0 \ \text{ in } \Omega_t, \qquad \partial_{\nu_t} u(t) = \alpha\, u(t) \ \text{ on } \partial\Omega_t,$$

where ν_t is the outer normal of Ω_t. We will use the notation $\lambda = \lambda(0) = \lambda(\Omega)$.

The first variation of $\frac{d}{dt}\lambda(\Omega_t)\big|_{t=0} =: \dot\lambda(0)$ has been computed by different authors and assumes the form

$$\dot\lambda(0) = \oint_{\partial\Omega} (|\nabla u|^2 - \lambda(0)u^2 - 2\alpha^2 u^2 - \alpha(n-1)Hu^2)(v \cdot \nu)\, dS, \tag{16}$$

where H is the mean curvature of $\partial\Omega$. In this formula u is normalized by $\int_\Omega u^2\, dx = 1$. From this formula we get immediately the

Lemma 3.1. *Let $\Omega = B_R$ be the ball of radius R centered at the origin. Suppose that $|\Omega_t| > |B_R|$ for small $|t|$. Then*

$$\dot\lambda(0) > 0.$$

In particular $\lambda(B_{R_1}) > \lambda(B_{R_0})$ if $R_1 > R_0$.

Proof. For the integrand in (16) we have

$$\begin{aligned} &|\nabla u|^2 - \lambda\, u^2 - 2\alpha^2 u^2 - \alpha(n-1)Hu^2 \\ = &\ |u'(R)|^2 - \lambda\, u^2(R) - 2\alpha^2\, u^2(R) - \alpha\, \frac{n-1}{R} u^2(R) \\ = &\ -\left(\lambda + \alpha^2 + \alpha\, \frac{n-1}{R}\right) u^2(R) \end{aligned}$$

since $u'(R) = \alpha u(R)$. Thus

$$\dot\lambda(0) = -\left(\lambda + \alpha^2 + \alpha\,\frac{n-1}{R}\right) u^2(R) \oint_{\partial B_R} (v\cdot\nu)\, dS.$$

Since $|\Omega_t| > |B_R|$ for small t, formula (14) and then (15) imply

$$\oint_{\partial B_R} (v\cdot\nu)\, dS > 0.$$

Thus

$$\dot\lambda(0) > 0 \ \text{ iff } \left(\lambda + \alpha^2 + \alpha\,\frac{n-1}{R}\right) < 0.$$

This will be proved with the help of (11). We set $z = \frac{u_r}{u}$ and observe that

$$\frac{dz}{dr} + z^2 + \frac{n-1}{r} z + \lambda = 0 \ \text{ in } (0,R).$$

At the endpoint

$$\frac{dz}{dr}(R) + \alpha^2 + \frac{n-1}{R}\alpha + \lambda = 0.$$

We know that $z(0) = 0$ and $z(R) = \alpha > 0$. Note that

$$z_r(0) = -\lambda > 0, \tag{17}$$

thus $z(r)$ increases near 0. Let us now determine the sign of $z_r(R)$. If $z_r(R) \le 0$ then because of (17) there exists a number $\rho \in (0,R)$ such that $z_r(\rho) = 0$, $z(\rho) > 0$ and $z_{rr}(\rho) \le 0$. From the equation we get $z_{rr}(\rho) = \frac{n-1}{\rho^2} z(\rho) > 0$ which leads to a contradiction. Consequently

$$z_r(R) = -\left(\alpha^2 + \frac{\alpha(n-1)}{R} + \lambda\right) > 0. \tag{18}$$

Hence

$$\dot\lambda(0) > 0$$

for all volume increasing perturbations $\oint_{\partial B_R} v\cdot\nu\, dS > 0$. This completes the proof of the lemma. □

This monotonicity is opposite to the usual case where α is negative and it will be crucial for the upper bounds derived in the next section.

From Lemma 3.1 it follows that $\sqrt{|\lambda(B_r)|}$ decreases as r increases. The asymptotic behavior of $I_\nu(z)$, namely $I_\nu(z) \sim \frac{e^z}{\sqrt{\pi z}}$ as $z \to \infty$, gives

$$\lim_{r\to\infty} \sqrt{|\lambda(B_r)|} = \alpha.$$

Theorem 2.4 is weaker than the estimate $\lambda(\Omega) \le \lambda(B_R)$. This follows from the next lemma.

Lemma 3.2. *$r^n\,\lambda(B_r)$ is monotonically decreasing in r.*

Proof. This is a consequence of (12). In fact if we set $r^n|\lambda(B_r)| =: y^2$ then (12) reads as

$$y = r^{n/2}\left(\alpha + \frac{n-2}{2r}\right)\frac{I_\nu(yr^{-\nu})}{I'_\nu(yr^{-\nu})}. \tag{19}$$

Since I_ν/I'_ν is decreasing straightforward differentiation shows that y is increasing. In fact if we set

$$g(z) := \frac{I_\nu(z)}{I'_\nu(z)}$$

then from (19) we get

$$\begin{aligned} y'(r) = {} & \frac{n}{2}r^{\frac{n-2}{2}}\left(\alpha + \frac{n-2}{2r}\right)g(yr^{-\nu}) - r^{n/2}\frac{n-2}{2r^2}g(yr^{-\nu}) \\ & + r^{n/2}\left(\alpha + \frac{n-2}{2r}\right)g'(yr^{-\nu})y'(r)r^{-\nu} \\ & - \nu\, r^{n/2}\left(\alpha + \frac{n-2}{2r}\right)g'(yr^{-\nu})y(r)r^{-\nu-1}. \end{aligned}$$

From this it follows $y'(r) > 0$. □

Remark 3.3. It is always possible to find a domain with a large boundary surface such that $\lambda(\Omega) < \lambda(B_R)$. In fact, if we introduce a constant in (1) then

$$\lambda(\Omega) < -\frac{\alpha|\partial\Omega|}{|\Omega|}.$$

The expression at the right-hand side can be made arbitrarily small whereas $\lambda(B_R)$ is fixed for given $|\Omega|$.

3.3. Nearly spherical domains with prescribed volume

In this section we consider perturbations of the following type. Let $y := \Phi_t(x) \in \overline{\Omega}_t$, then

$$\overline{\Omega}_t = \left\{ y : y = x + tv(x) + \frac{t^2}{2}w(x) + o(t^2) \;:\; x \in \overline{\Omega} \right\}. \tag{20}$$

For the vector fields v and w we assume

$$\mathcal{L}^n(\Omega_t) = \mathcal{L}^n(\Omega) + o(t^2)$$

Thus in addition to (15) we have to require

$$\int_{\partial\Omega} (v\cdot\nu)\ \operatorname{div} v - v\cdot Dv\cdot\nu + w\cdot\nu\, dS = 0.$$

Here $v \cdot Dv \cdot \nu = \sum_{i,j=1}^{n} v_i \partial_i v_j \nu_j$.

Let u' satisfy the following equation

$$\Delta u' + \lambda(0) u' = 0 \quad \text{in } B_R$$
$$\partial_\nu u' + \alpha\, u' = k(u(R)\, (v \cdot \nu) \quad \text{in } \partial B_R,$$

where

$$k(u(R)) := \big(\alpha^2 R - \alpha(n-1) + \lambda\, R\big)\, \frac{u(R)}{R}$$

Note that u' only depends on the normal component of v. Moreover let

$$S(t) = \int_{\partial\Omega} m(t)\, dS$$

denote the surface area of $\partial\Omega_t$.

From [4] we obtain the following lemma.

Lemma 3.4. *Let*

$$Q(u') := \int_{B_R} |\nabla u'|^2\, dx - \lambda(B_R) \int_{B_R} u'^2\, dx - \alpha \int_{\partial B_R} u'^2\, dS.$$

Then

$$\ddot{\lambda}(0) = -2Q(u') - 2\alpha\, u(R) k(u(R)) \int_{\partial B_R} (v \cdot \nu)^2\, dS - \alpha\, u^2(R)\, \ddot{S}(0).$$

Following the computations in Chapter 7 in [4] we get

$$\ddot{\lambda}(0) \leq -\alpha\, u^2(R)\, \ddot{S}(0) < 0$$

where u is the normalized eigenfunction for the ball B_R.

4. Steklov type problems

In this section we study problems with a variable weight on the boundary. Let $\rho(x)$ be a continuous function defined in $D \supset \Omega_t$ for $|t| \leq \epsilon$. Consider boundary value problems of the type

$$\Delta u + G'(u) = 0 \quad \text{in } \Omega, \qquad \partial_\nu u = \mu\, \rho(x) \quad \text{in } \partial\Omega. \tag{21}$$

This equation is understood in the weak sense

$$\int_{\Omega} \nabla w \cdot \nabla \varphi\, dx - \int_{\Omega} G'(u) \varphi\, dx - \mu \int_{\partial\Omega} \varphi\, \rho(x)\, dS = 0 \tag{22}$$

for all $\varphi \in H^{1,2}(\Omega)$. It is the Euler–Lagrange equation corresponding to the energy

$$\mathcal{E}(v,\Omega) := \int_{\Omega} |\nabla v|^2 \, dx \, dx - 2\int_{\Omega} G(v)\, dx - 2\mu \int_{\partial\Omega} v\, \rho(x)\, dS \tag{23}$$

for $u \in H^{1,2}(\Omega)$. A special case where $G'(u)$ is constant appears in [2].

We consider problem (21) in the perturbed domains Ω_t described in (13). We assume that there exists a unique solution. The corresponding energy (23) will be denoted by $\mathcal{E}(t)$. Following [4] we compute the first variation $\dot{\mathcal{E}}(0)$ formally. For the calculation we refer to [4]. There it has been carried out in detail for a more general case. In particular we use Section 4.1 and Formula (2.18).

Let us decompose v in its normal and tangential components $v = v^\tau + (v\cdot\nu)\nu$. Let $\operatorname{div}_{\partial\Omega} v = \partial_i v_i - \nu_j \partial_j v_i \nu_i$ be the tangential divergence. Here we use the Einstein convention. Then

$$v\cdot\nabla u = v^\tau \cdot \nabla^\tau u + \mu(v\cdot\nu)\rho(x).$$

It then follows that

$$\begin{aligned}\dot{\mathcal{E}}(0) = & \oint_{\partial\Omega} \{|\nabla u|^2 - 2G(u)\}(v\cdot\nu)\, dS - 2\oint_{\partial\Omega} (v\cdot\nabla u)\, \partial_\nu u\, dS \\ & - 2\mu \oint_{\partial\Omega} u\, v\cdot\nabla\rho(x)\, dS - 2\mu\int_{\partial\Omega} u\,\rho(x) \operatorname{div}_{\partial\Omega} v^\tau \, dS \\ & + 2(n-1)\mu \int_{\partial\Omega} (v\cdot\nu) H\, dS.\end{aligned} \tag{24}$$

By (21)

$$\begin{aligned}-2\oint_{\partial\Omega} (v\cdot\nabla u)\,\partial_\nu u\, dS = & -2\oint_{\partial\Omega} (v^\tau\cdot\nabla^\tau u)\,\partial_\nu u\, dS - 2\mu\oint_{\partial\Omega} (v\cdot\nu)\rho(x)\,\partial_\nu u\, dS \\ = & -2\mu\oint_{\partial\Omega} (v^\tau\cdot\nabla^\tau u)\,\rho(x)\, dS - 2\mu^2\oint_{\partial\Omega} (v\cdot\nu)\rho(x)^2\, dS \\ = & \, 2\mu\oint_{\partial\Omega} \operatorname{div}_{\partial\Omega} v^\tau\, u\,\rho(x)\, dS + 2\mu\oint_{\partial\Omega} (v^\tau\cdot\nabla^\tau\rho(x))\, u\, dS \\ & - 2\mu^2\oint_{\partial\Omega} (v\cdot\nu)\rho(x)^2\, dS.\end{aligned}$$

We compare this with (24) and finally get

$$\dot{\mathcal{E}}(0) = \oint_{\partial\Omega} \{|\nabla u|^2 - 2G(u) - 2\mu^2\rho(x)^2 - 2\mu u\partial_\nu\rho(x) + 2(n-1)\mu H\}(v\cdot\nu) dS. \tag{25}$$

Definition. A domain Ω is said to be *critical in the class of* Ω_t if $\dot{\mathcal{E}}(0) = 0$.

Observe that (25) gives a necessary condition for the solution u of (21) in a critical domain and in particular for an extremal domain. For specific perturbation such as volume preserving perturbation the discussion of the overdetermined boundary problem is still open.

Example. 1. B_R is a critical domain if the solution u of (21) and ρ are radial and if the perburtation is volume preserving in the sense of (15).

2. Let $\Omega = B_R$ and $G(u) = ku$. Then (21) becomes

$$\Delta u + k = 0 \text{ in } B_R, \qquad \partial_\nu u = \mu\rho \text{ on } \partial B_R.$$

Note that this problem has a solution if and only if $k|B_R| = \mu \oint_{\partial B_R} \rho \, dS =: \mu M$. The solution is not unique and the energy is not a minimizer.

In the next section we will investigate a relaxed formulation of a related optimization problem.

4.1. Isoperimetric inequalities

In this section we reconsider the energy given in (23). In particular we assume that

$$\mathcal{E}(\Omega) = \inf_{W^{1,2}(\Omega)} \int_\Omega |\nabla v|^2 \, dx \, dx - 2 \int_\Omega G(v) \, dx - 2\mu \int_{\partial\Omega} v \, \rho(x) \, dS$$

attains its minimum and that there is a unique minimizer u which solves (21). The aim is to derive an upper bound by means of harmonic transplantation. We shall distinguish between two cases.

(i) $G(s) > 0$. Consider the comparison problem

$$\Delta\phi + G'(\phi) = 0 \text{ in } B_{r_\Omega}, \qquad \partial_\nu \phi = \frac{\mu M}{|\partial B_{r_\Omega}|} \text{ on } \partial B_{r_\Omega} \text{ where } M := \oint_{\partial B_{r_\Omega}} \rho \, dS. \tag{26}$$

Because of the extremal property of the corresponding energy, ϕ is radially symmetric. The arguments of Section 2.2, in particular the inequalities (6) and (7) apply. Let $\phi(x) = \omega(G_{B_{r_\Omega}}(x, 0))$ and set $U(x) = \omega(G_\Omega(x, y_h))$. Then

$$\mathcal{E}(\Omega) \leq \int_\Omega |\nabla U|^2 \, dx - 2 \int_\Omega G(U) \, dx - 2\mu \int_{\partial\Omega} U \, \rho(x) \, dS.$$

Since $U = \text{const.}$ on $\partial\Omega$,

$$\mathcal{E}(\Omega, \rho) \leq \mathcal{E}\left(B_{r_\Omega}, \frac{M}{|\partial B_{r_\Omega}|}\right). \tag{27}$$

(ii) $G(s) = -H(s)$, where $H : \mathbb{R}^+ \to \mathbb{R}^+$ and $H'(s) = h(s) > 0$. The comparison problem will be in this case

$$\Delta\phi = \gamma^n h(\phi) \text{ in } B_{r_\Omega}, \qquad \partial_\nu \phi = \frac{\mu M}{|\partial B_{r_\Omega}|} \text{ on } \partial B_{r_\Omega}.$$

If $\mu M > 0$ then ϕ is increasing. Moreover we assume that h is such that ϕ is positive. Under these assumptions Lemma 2.3 yields for the transplanted function U of ϕ

$$\int_\Omega H(U)\, dx \le \gamma^n \int_{B_{r_\Omega}} H(\phi)\, dx.$$

Consequently

$$\mathcal{E}(\Omega) \le \int_{B_{r_\Omega}} |\nabla\phi|^2\, dx + 2\gamma^n \int_{B_{r_\Omega}} H(\phi)\, dx - 2\phi(r_\Omega)\mu M.$$

The right-hand side is the energy of the comparison problem. An example is $h(s) = c^2 s$.

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