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Maharam-Types and Lyapunov's Theorem for Vector Measures on Locally Convex Spaces with Control Measures*

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This paper presents an equivalence between (i) the Lyapunov property under which a vector measure with values in a sequentially complete, separable locally convex Hausdorff space (lcHs) has a weakly compact and convex range, (ii) the *thinness* property of subsets of Bochner integrable functions due to Kingman-Robertson (1968) and (iii) the saturation property due to Maharam (1942) and Hoover-Keisler (1984). It also considers the case of a non-separable range space, and presents versions of the Lyapunov theorem for a quasicomplete lcHs based either on the Egorov property or the notion of Maharam-types. The results are applied to two canonical objects in convex analysis: the integral and the distribution of a multifunction.

Keywords: Saturation property, Lyapunov's theorem, locally convex space, thin sets, integral, distribution, multifunction, Radon-Nikodym property, control measure, Maharam-type

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Contents

1. Introduction

The need and the motivation for a version of the Lyapunov theorem on the range of a nonatomic vector measure taking values in a locally convex Hausdorff space (lcHs) originally came from the study of optimal controls; see the paper of Knowles [35] that relied on the earlier work [32, 33] and found consolidation in the 1975 monograph [34]. Indeed, the importance of the theorem goes beyond optimal control theory to other areas of the mathematical sciences including the study of statistical decisions, economies with many agents and non-atomic games; see [4, 5, 31, 39] and their references. And even though the predominance of the theorem was loosened as the subject gained maturity and its development embraced different areas of mathematics, it continued to be called upon in one of two alternative versions: an *exact* version for measures with values in a finite-dimensional range space, and an *approximate* version for those with values in an infinite-dimensional one; see [11, 50]. To be sure, sufficient conditions that relied on the non-injectivity of the Lindenstrauss-Rudin operator for an exact infinite-dimensional version were offered in the control theory literature – a necessary and sufficient result is presented in [35, Theorems 1 and 2] – but there was a certain lack of closure and elusiveness to the general versions that were obtained. This stemmed from the fact that there was no translation of these sufficient conditions to the measure-space itself on which the vector measures were defined; see the monographs [11, 34] for discussion and references.

Such a measure space was offered by Loeb [37], and it is interesting that its discovery is essentially contemporaneous with the monographs of Kluvánek and Knowles [34] and Diestel and Uhl [11]. Apart from its attractive properties relating to asymptotic implementation and permutational invariance, a Loeb measure space furnished an additional, concretely-interpretable condition on an abstract measure space, and has been an object of active investigation in all branches of mathematical sciences, including mathematical physics; see [1, 31, 43] and their references. However, as far as an exact Lyapunov theorem was concerned, its development had to await a detour involving its straightforward consequence concerning the convexity of the integral of a multifunction (synonymously, correspondence, random set, set-valued mapping) defined on a nonatomic probability space; see [4, 5, 46] for the original result. Within mathematical economics, it too, just as its parent, found employment in *exact* and *approximate* forms depending on the dimensionality of the range space. The breakthrough towards an exact result for a multifunction with values in separable Banach space came a good two decades after Loeb's paper, and through work that ironically privileged the classic marriage lemma rather than a reliance on the Lyapunov theorem as the driving engine. Sun considered the distribution of a multifunction on a nonatomic Loeb space with values in a Polish space; derived its convexity through the Hall-Halmos lemma; and used this convexity property to obtain an exact theorem on the set-valued integral; see [54, Lemma 3.4], [55, Section 4] and the reference to the lemma. This work, however, left open the question as to what it was about a non-atomic Loeb space that delivered the exact form of the result. This is to say that it left the necessity question open.

What then was the specific property of a nonatomic Loeb space that was "exclusively" responsible for delivering the convexity of the distribution of a multifunction, and through it, exclusively responsible for delivering the convexity of the corresponding integral of a multifunction? and was this property necessary? It took another five years to finally, and definitively, pin this down. In work circulated in 2002, Keisler-Sun identified the saturation property, formalized in Hoover and Keisler [25], as *the* property of Loeb spaces that not only delivers a convexity result, but a fully comprehensive theory for the distribution of a multifunction; see [26, Section 3]. Given that the connection of the saturation property to Maharam's [38] classification of measure algebras was already indicated in [25], and further explicated in [16], and that the distribution of a multifunction already identified as the vehicle for a straightforward translation of its properties to those of the corresponding integral, Keisler-Sun did not address integration. Such a derivation was given an explicit exposition in [56], and also undergirded by an *ab initio* derivation from the Maharam definition itself without utilizing the saturation property at all; see [45].

At this point in the delineation of the trajectory of our subject, and the execution of this introduction, it is also important to give explicit recognition to an approach to exact integration that is independent of the work involving the Loeb construction. In a paper not engaged and discussed in either [45] or [56], Podczeck also emphasized the need for an exact integral of a multifunction, and offered a condition on the underlying measure space of agents under which it holds; see in particular Assumption A2, Theorem 3.1 in [44]. But what about the necessity and sufficiency theory for the conclusion of the Lyapunov theorem itself? In this welcome advance concerning the convexity of the set-valued integral, motivated by and within mathematical economics, it was left by the wayside. To be sure, the fact that the integration result could be translated to an exact Lyapunov theorem for a saturated vector measure with the Radon-Nikodym property (RNP), and thus for any vector measure with bounded variation taking values in a Banach space with the RNP, was well-understood in earlier work; see [53, Proposition 4.5], in addition to [55], for the requisite argumentation. But the issue at hand was a Lyapunov theorem for a Banach space *without* the RNP, and for an arbitrary vector measure taking its values in such a space? This is our primary concern here.

A closing of this door has taken another five years, but has now been pinned down to yield a most satisfactory theory. Relying on the operator-theoretic methods of Lindenstrauss-Rudin, as further utilized in [45, 49], the authors show an equivalence between the saturation and Lyapunov properties for the case when the vector measure takes values in a separable Banach space; see [28, Theorem 4.2]. And with this characterization in hand, they also proceed to the case of measures with values in a Banach space that is not necessarily separable. Relying on the notion of higher-order Maharam-types, they show that corresponding to any Banach space, there exists a measure space of high-enough determinable Maharam-type that yields the Lyapunov property and is implied by it; see [28, Theorem and Corollary 5.1]. As such, a saturated measure space represents a terminus of some sort for the Lyapunov property, and for the integration result, only if the range space is a separable Banach space; and that one has to proceed to higher-order Maharam types if one desires to proceed beyond this terminus. These two important claims, adduced and validated in [28], and extended in [20] as discussed in the sequel, form the backdrop to the work reported here.

One can then turn to the results delineated in this paper, and specifically to their marginal contribution as regards the Lyapunov theorem, and thereby to convex analysis. Briefly and succinctly put, we present generalizations of the results in [28] from an infinite-dimensional Banach space to an infinite-dimensional lcHs.

In particular, we offer:

(i) a characterization of the saturation property of [25] through (a) the Lyapunov property requiring vector measures with values in a sequentially complete separable lcHs to have a weakly compact and convex range, (b) the *thinness* property of subsets of integrable functions due to [32] (see Theorem 3.6 below);

(ii) a characterization of the Lyapunov property of a sequentially complete lcHs through the use of higher-order Maharam types; and remaining with the non-separable case but with the RNP, the establishment of such higher-order types as a substitutable alternative to the Egorov property (see Proposition 4.4 and Theorem 4.5 below for the former and Theorem 4.10 and Corollary 4.11 for the latter);

(iii) two applications to applied convex analysis that furnish: (a) a new result on the convexity of the set-valued integral in a quasicomplete lcHs, (b) an alternative proof of the result in [26] on the convexity of the distribution of a multifunction with values in a compact metric space (see Corollary 5.2 and Theorem 5.4 below).

Two points need to be addressed here. First, why the need for an extension to a lcHs setting? Second, granting this need, what are the principal analytical hurdles that need to be overcome? The first question can be answered through the register either of the theory or of its applications. For both, a lcHs dispenses with the need to derive the convexity of the set-valued integral, and other results and applications, in a veritable variety of topologies: one result suffices for the norm, weak, weak*, Mackey and strong topologies, and any important dual topology lying in between; see for example the catalogue of variety of results in [45, 53, 55, 56] pertaining to integration. Thus, dual spaces can be directly and effortlessly covered in the lcHs setting; see Subsection 5.2 below for an exemplification of this point of view. This is after all the *raison d'etre* of the subject itself, and it is a little surprising that there have not been earlier efforts in this direction. Furthermore, by emphasizing the viewpoint of integration, the results reverses the downplaying of Lyapunov's theorem, and the movement, as discussed above, from the distribution of a multifunction to its corresponding integral. Given that an induced distribution of a multifunction is its integral when it is lifted to the space of probability measures, such a lifting is a natural and expected move. In this connection, it is indeed satisfying that one can obtain the convexity of the set-valued integral in a quasicomplete lcHs that we do by essentially following the outline of the proof that Blackwell initially gave – a simple consequence of the relevant form of the Lyapunov theorem that we develop. Indeed, the point of view being espoused here is explicit in privileging Lyapunov's theorem and the convexity of the integral of a multifunction as a somewhat neglected door for systematic investigations, albeit in the future, of Bochner and Gelfand integration in the context of both vector measures and multifunctions, and towards other results such as Fatou's lemma; see [29] for preliminary work. On the applications side, leaving aside the applications in optimal control theory as discussed in [9, 34], we can refer the reader to [44] for an appreciation of how lcHs arise naturally in questions pertaining to general equilibrium theory.

Finally, we turn to the analytical hurdles for the extensions that we undertake. To say that we build, and carry forward, arguments in [28] from a Banach space to a suitable lcHs, obscures the specific technical decisions that had to be made, and shown in what follows, to bear fruit. The basic point is that unlike an infinite-dimensional Banach space, the existence of control measures cannot necessarily be taken for granted, and this serves as an analytical point of departure for this work. It leads us to introduce the countable chain condition (ccc), countable summable condition (csc) and the Bartle-Dunford-Schwartz (BDS) property as structural properties of measures and vector spaces to guarantee the existence of a control measure for vector measures in lcHs. The six-part preliminary section of the paper already brings out how the results draw on a variety of concepts that coalesce in the Banach space case. Furthermore, in terms of (i) above, we need, even in the case of a sequentially-complete separable lcHs, the assured existence of a subspace isomorphic to a infinite-dimensional Banach space. Again, while we do not want to overemphasize Theorem 4.7 that generalizes Uhl's approximate Lyapunov theorem [61] to a lcHs with the RNP, it does give a benchmark result relative to which the advance in the non-separable case can to be measured. And surely the fact that the Egorov property serves as a substitute for a higher-order Maharam type is also of some surprise and interest. Finally, it bears emphasis that for the convexity of distributions, we identify the space of Radon measures with a Fréchet space in a way that our Lyapunov theorem in lcHs can be applied. Thus, in addition to reversing the devaluation of the Lyapunov theorem in recent work, we show it to have interest in its own right.

The plan of the paper is relatively straightforward. After the multi-facetted preliminary section, the paper consists of two theoretical sections divided between the separable and non-separable case, and a final section on two applications. Readers interested primarily in the applications may want to go to Section 6 first.

2. Preliminaries

2.1. Maharam Types and Cellularity in Boolean Algebras

Let $\mathcal{F}$ be a Boolean algebra with binary operations $\vee$ and $\wedge$, and a unary operation ^{c}, furnished with the order $\le$ given by $A \le B \Longleftrightarrow A = A \wedge B$, where $\emptyset = \Omega^c$ and $\Omega = \emptyset^c$ are the smallest and largest elements in $\mathcal{F}$ respectively. A subset $\mathcal{I}$ of $\mathcal{F}$ is an *ideal* if $\emptyset \in \mathcal{I}$, $A \vee B \in \mathcal{I}$ for every $A, B \in \mathcal{I}$ and $B \le A$ with $A \in \mathcal{I}$ implies $B \in \mathcal{I}$. The *principal ideal* $\mathcal{F}_E$ generated by $E \in \mathcal{F}$ is an ideal of $\mathcal{F}$ given by $\mathcal{F}_E = \{A \in \mathcal{F} \mid A \le E\}$, which is a Boolean algebra with unit E.

A *subalgebra* of $\mathcal{F}$ is a subset of $\mathcal{F}$ that contains Ω and is closed under the Boolean operations $\vee$, $\wedge$ and ^{c}. A subalgebra $\mathcal{U}$ of $\mathcal{F}$ is *order-closed* with respect to the order $\leq$ if any nonempty upwards directed subsets of $\mathcal{U}$ with its supremum in $\mathcal{F}$ has the supremum in $\mathcal{U}$. A subset $\mathcal{U} \subset \mathcal{F}$ *completely generates* $\mathcal{F}$ if the smallest order closed subalgebra in $\mathcal{F}$ containing $\mathcal{U}$ is $\mathcal{F}$ itself. The *Maharam type* of a Boolean algebra $\mathcal{F}$ is the smallest cardinal of any subset $\mathcal{U} \subset \mathcal{F}$ which completely generates $\mathcal{F}$. By $\tau(\mathcal{F})$ we denote the Maharam type of $\mathcal{F}$.

The *cellularity* of $\mathcal{F}$ is a cardinal $c(\mathcal{F})$ defined by

$$c(\mathcal{F}) = \sup\{\#\mathcal{U} \mid \mathcal{U} \subset \mathcal{F} \setminus \{\emptyset\} \text{ and } A \wedge B = \emptyset \ \forall A, B \in \mathcal{U}\},$$

where $\#\mathcal{U}$ denotes the cardinality of $\mathcal{U}$. (If $\mathcal{F} = \{\emptyset\}$, then take $c(\mathcal{F}) = 0$.) A Boolean algebra $\mathcal{F}$ is said to satisfy the *countable chain condition* (*ccc*) if $c(\mathcal{F}) \leq \aleph_0$. An element $A \in \mathcal{F}$ is an *atom* if $A \neq \emptyset$ and $E \leq A$ with $E \in \mathcal{F}$ implies either $E = \emptyset$ or $E = A$. A Boolean algebra $\mathcal{F}$ is *purely atomic* if for every nonnull element $E \in \mathcal{F}$ there is an atom $A \in \mathcal{F}$ such that $A \leq E$; $\mathcal{F}$ is *nonatomic* if it has no atom. If $c(\mathcal{F})$ is finite, then $\mathcal{F}$ is purely atomic (see [18, 332X(b)]).

Proposition 2.1. *Let $\mathcal{F}$ be a Boolean algebra and $\mathcal{F}_E$ be the principal ideal generated by $E \in \mathcal{F}$. Then the following conditions are equivalent:*

(*i*) *$\mathcal{F}$ is nonatomic.*
(*ii*) *$\tau(\mathcal{F}_E)$ is infinite for every $E \in \mathcal{F}$ with $E \neq \emptyset$.*
(*iii*) *$c(\mathcal{F}_E)$ is infinite for every $E \in \mathcal{F}$ with $E \neq \emptyset$.*

Proof. (*i*) $\Rightarrow$ (*iii*): If $c(\mathcal{F}_E)$ is finite for some $E \in \mathcal{F}$ with $E \neq \emptyset$, then $\mathcal{F}_E$ is purely atomic. This implies that there is an atom $A \in \mathcal{F}$ such that $A \leq E$.

(*iii*) $\Rightarrow$ (*ii*): It follows from [18, Proposition 331H] that $\tau(\mathcal{F}_E)$ is finite if and only if $\mathcal{F}_E$ is a finite Boolean algebra. Thus, if $\tau(\mathcal{F}_E)$ is finite for some $E \in \mathcal{F}$ with $E \neq \emptyset$, then $c(\mathcal{F}_E)$ is finite.

(*ii*) $\Rightarrow$ (*i*): If $E \in \mathcal{F}$ is an atom, then $\mathcal{F}_E$ contains no proper subalgebra, so $\tau(\mathcal{F}_E) = 0$. $\square$

2.2. Vector Measures in LcHs

Throughout this paper, $(\Omega, \mathcal{F})$ is a measurable space and X is a lcHs. A set function $m : \mathcal{F} \to X$ is *countably additive* if for every pairwise disjoint sequence $\{A_n\}$ in $\mathcal{F}$, we have $m(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty m(A_n)$, where the series is unconditionally convergent with respect to the locally convex topology on X. It is well known that if m is countably additive with respect to "some" locally convex topology on X that is consistent with the dual pair $\langle X, X^* \rangle$, then it is countably additive with respect to "any" locally consistent convex topology on X (see [60, Proposition 4]). Therefore, the countable additivity of vector measures is independent of the particular topologies lying between the weak and Mackey topologies of X.

A countably additive set function $m : \mathcal{F} \to X$ is called a *vector measure*. For a vector measure m, a set $N \in \mathcal{F}$ is *m-null* if $m(A \cap N) = \mathbf{0}$ for every $A \in \mathcal{F}$. An equivalence relation $\sim$ on $\mathcal{F}$ is given by $A \sim B$ if and only if $A \triangle B$ is m-null, where $A \triangle B$ is the symmetric difference of A and B in $\mathcal{F}$. The collection of equivalence classes is denoted by $\widehat{\mathcal{F}} = \mathcal{F}/\sim$ and its generic element $\widehat{A}$ is the equivalence class of $A \in \mathcal{F}$. The lattice operations $\vee$ and $\wedge$ in $\widehat{\mathcal{F}}$ are given in a usual way by $\widehat{A} \vee \widehat{B} = \widehat{A \cup B}$ and $\widehat{A} \wedge \widehat{B} = \widehat{A \cap B}$. The unary operation ^{c} in $\widehat{\mathcal{F}}$ is obtained for taking complements in $\widehat{\mathcal{F}}$ by $\widehat{A}^c = \widehat{(A^c)}$. Under these operations $\widehat{\mathcal{F}}$ is a Boolean σ-algebra. Let $\hat{m} : \widehat{\mathcal{F}} \to X$ be an X-valued vector measure on $\widehat{\mathcal{F}}$ defined by $\hat{m}(\widehat{A}) = m(A)$ for $\widehat{A} \in \widehat{\mathcal{F}}$. Then the pair $(\widehat{\mathcal{F}}, \hat{m})$ is called a *vector measure algebra* induced by m. The Maharam type of a vector measure $m : \mathcal{F} \to X$ is defined to be $\tau(\widehat{\mathcal{F}})$. Denote by $\mathcal{F}_E = \{A \cap E \mid A \in \mathcal{F}\}$ a σ-algebra of $E \in \mathcal{F}$ inherited from $\mathcal{F}$ and $(\widehat{\mathcal{F}_E}, \widehat{m_E})$ a vector measure algebra induced by the restriction m_E of m to $\mathcal{F}_E$. Then $\widehat{\mathcal{F}_E}$ is the principal ideal of $\widehat{\mathcal{F}}$ generated by the element $\widehat{E} \in \widehat{\mathcal{F}}$. A set $A \in \mathcal{F}$ is an *atom* of m if $m(A) \neq \mathbf{0}$ and $\widehat{A}$ is an atom of $\widehat{\mathcal{F}}$. If m has no atom, it is said to be *nonatomic*.

Definition 2.2. A vector measure $m : \mathcal{F} \to X$ is said to satisfy (*ccc*) if its associated vector measure algebra $\widehat{\mathcal{F}}$ satisfies (ccc).

Corollary 2.3. *Let $(\widehat{\mathcal{F}}, \hat{m})$ be a vector measure algebra of a vector measure $m : \mathcal{F} \to X$. Then the following conditions are equivalent:*

(*i*) *m is nonatomic.*

(*ii*) *$\tau(\widehat{\mathcal{F}_E})$ is infinite for every m-nonnull $E \in \mathcal{F}$.*

(*iii*) *$c(\widehat{\mathcal{F}_E})$ is infinite for every m-nonnull $E \in \mathcal{F}$.*

2.3. Control Measures and the BDS Property

A vector measure $m : \mathcal{F} \to X$ is *absolutely continuous* (or *μ-continuous*) with respect to a scalar measure μ if $\mu(A) = 0$ implies that $m(A \cap E) = \mathbf{0}$ for every $E \in \mathcal{F}$. A finite measure μ is a *control measure* of m whenever $\mu(A) = 0$ if and only if $m(A \cap E) = \mathbf{0}$ for every $E \in \mathcal{F}$. Thanks to the classical result by [47, Theorem 3.1], μ is a control measure of m if and only if

$$\lim_{m(A) \to \mathbf{0}} \mu(A) = 0 \quad \text{and} \quad \lim_{\mu(A) \to 0} m(A) = \mathbf{0}. \tag{1}$$

Denote by $ca(\mathcal{F}, X)$ the space of all X-valued vector measures on $\mathcal{F}$. It is well known that if X is a Banach space, then every vector measure in $ca(\mathcal{F}, X)$ has a control measure by the Bartle-Dunford-Schwartz theorem (see [3, Corollary 2.4], [11, Corollary I.2.6] or [14, Lemma IV.10.5]). More generally, if X is a metrizable locally convex space, then every X-valued vector measure defined on a σ-ring has a control measure (see [13, p. 210] or [40, Corollary 2]). This result is further extended to exhaustive, finitely additive vector measures on rings (see [62, Corollary 7.5]).

However, when X is a metrizable topological vector space, the existence of a control measure is by no means a trivial problem. In particular, there exists a metrizable topological vector space with a translation invariant metric and an exhaustive, finitely additive vector measure on a Boolean algebra that does not have a control measure (see [58, Theorem 1.4]). This counterexample clarifies the role of local convexity for the existence of control measures. This observation motivates us to present the following definition.

Definition 2.4. A lcHs X has the *Bartle-Dunford-Schwartz* (*BDS*) *property* if every vector measure in $ca(\mathcal{F}, X)$ has a control measure.

As mentioned in the above, metrizable locally convex spaces, especially, Fréchet spaces have the BDS property. While the BDS property is defined in [41] for a given vector measure in $ca(\mathcal{F}, X)$ as having a control measure, we impose it instead on the range space X.

Proposition 2.5. *A vector measure $m : \mathcal{F} \to X$ satisfies* (ccc) *if and only if it has a control measure.*

This characterization of (ccc) for vector measures is due to [13, Theorem 2.3] and [40, Theorem 2]. It is an intrinsic equivalent condition on the underlying vector measure algebra for the existence of a control measure.

Corollary 2.6. *A lcHs X has the BDS property if and only if every vector measure in $ca(\mathcal{F}, X)$ satisfies* (ccc).

Definition 2.7. A lcHs X satisfies the *countable summability condition* (*csc*) if the following condition is satisfied: If $\{x_i\}_{i\in I}$ is a family of nonzero elements in X such that every countable subfamily $\{x_i\}_{i\in J}$ with $J \subset I$ is summable, then $\{x_i\}_{i\in I}$ is at most countable.

(csc) was introduced by [33] under the name of "property (Σ)". It is equivalent to the following condition: If $\mathcal{R}_\sigma(I)$ is the σ-ring of all countable subsets of a set I and $m : \mathcal{R}_\sigma(I) \to X$ is a vector measure, then the set $\{i \in I \mid m(\{i\}) \neq \mathbf{0}\}$ belongs to $\mathcal{R}_\sigma(I)$.

Note that all locally convex topologies lying between the weak and Mackey topologies of X satisfy (csc) or none of them does. This shows in particular that there exist nonmetrizable locally convex topologies which satisfy (csc).

Theorem 2.8. *A lcHs has the BDS property if it satisfies* (csc).

Proof. A lcHs X satisfies (csc) if and only if for every σ-ring $\mathcal{R}$, any X-valued vector measure defined on $\mathcal{R}$ possesses a control measure (see [13, Theorem 2.5]). Therefore, if (csc) holds, then every vector measure in $ca(\mathcal{F}, X)$ has a control measure. □

Example 2.9. We illustrate some examples of lcHs with (csc): (i) metrizable locally convex spaces; (ii) lcHs whose origin is a G_δ set (see [42, Proposition 2.2]); (iii) lcHs whose dual is the union of a sequence of weakly* compact sets (see [36, Lemma 2.1]). Here we clarify that the lcHs spaces listed under (i), (ii) and (iii) form an increasingly general classes of such spaces. A lcHs of type (i) is obviously a lcHs of type (ii). Suppose that X is a lcHs of type (ii). Let U_n be a sequence of neighborhoods of the origin $\mathbf{0}$ in X such that $\{\mathbf{0}\} = \bigcap_{n=1}^\infty U_n$. Denote by U_n° the (absolute) polar of U_n defined by

$$U_n^\circ = \{x^* \in X^* \mid |\langle x^*, x\rangle| \leq 1 \ \forall x \in U_n\}, \quad n = 1, 2, \dots.$$

By the Banach-Alaoglu theorem (see [2, Theorem 5.105]), each U_n° is weakly* compact. By the bipolar theorem (see [2, Theorem 5.103]), we have $X^* = (\bigcap_{n=1}^\infty U_n)^\circ = \bigcup_{n=1}^\infty U_n^\circ$. Thus, X is of type (iii).

Locally convex Suslin spaces are of type (ii) (see [52, Lemmas II.1.7 and II.1.8 of Part I]). Type (iii) spaces are used also by [60] and X is of type (iii) if X^* is weakly* separable. Locally convex Suslin spaces possess a weakly* separable dual (see [52, Lemma I.4.4 of Part II]). In particular, the weak* separability is satisfied for lcHs with a separable dual space. A simpler example for type (iii) is a normed space; if X is a normed space and B^* is the closed unit ball in X^*, then the sequence $\{rB^*\}_{r\in\mathbb{Q}_+}$ is a desired one, where $\mathbb{Q}_+$ is the set of positive rationals.

2.4. Bochner Integrals in LcHs

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and X be a lcHs with its topology generated by a separating family of seminorms $\mathcal{P}$, for which each seminorm in $\mathcal{P}$ is continuous on X. Denote by $S_\mathcal{F}(X)$ the space of all X-valued simple functions; that is, functions of the form $g(\omega) = \sum_{i=1}^n \chi_{A_i}(\omega)x_i$ with $A_i \cap A_j = \emptyset$, where χ_{A_i} are the characteristic functions of the set $A_i \in \mathcal{F}$ and x_i are vectors in X. It is clear that $S_\mathcal{F}(X)$ is seminormed via the relation $\gamma_p(g) = \int p \circ g d\mu$ for $g \in S_\mathcal{F}(X)$ and $p \in \mathcal{P}$, which makes the quotient space $S_\mathcal{F}(X)/\bigcap_{p\in\mathcal{P}} \gamma_p^{-1}(0)$ a lcHs. Denote by $L^1(\mu, X)$ the completion of $S_\mathcal{F}(X)/\bigcap_{p\in\mathcal{P}} \gamma_p^{-1}(0)$ with respect to the locally convex Hausdorff topology generated by the family of seminorms $\{\gamma_p \mid p \in \mathcal{P}\}$ (see [51, III.6.4]).

We define Bochner integrals in quasicomplete lcHs along the lines of [51, Ch. III §6]. Recall that a lcHs X is *quasicomplete* if every bounded, closed subset of X is complete. If $g : \Omega \to X$ is an X-valued simple function of the form $g = \sum_{i=1}^n \chi_{A_i} x_i$, define as usual the integral of g with respect to μ by $\int g d\mu = \sum_{i=1}^n \mu(A_i)x_i$. We then have

$$\begin{aligned}\gamma_p(g) &= \int p\left(\sum_{i=1}^n \chi_{A_i} x_i\right) d\mu = \sum_{i=1}^n \int_{A_i} p(x_i) d\mu \\ &= \sum_{i=1}^n \mu(A_i)p(x_i) \geq p\left(\sum_{i=1}^n \mu(A_i)x_i\right) = p\left(\int g d\mu\right)\end{aligned}$$

for every $p \in \mathcal{P}$. If g is in $L^1(\mu, X)$, take a net $\{g_\alpha\}$ of X-valued simple functions such that $\gamma_p(g_\alpha - g) \to 0$ for every $p \in \mathcal{P}$. It follows from the above inequality and the quasicompleteness of X that the *Bochner integral* of g can be defined by $\int g d\mu = \lim_\alpha \int g_\alpha d\mu$. The integral $\int_A g d\mu$ is defined to be $\int g\chi_A d\mu$. An element in $L^1(\mu, X)$ is called a *Bochner integrable function*. When X is a Banach space, Bochner integrability and integrals coincide with the usual definition (see [11, Definition II.1.1]). The definition of Bochner integrable functions in lcHs introduced here is broader than the definitions adopted by [6, 10, 15, 17, 19, 22, 50].

When $X = \mathbb{R}$, instead of $L^1(\mu, \mathbb{R})$, denote by $L^1(\mu)$ the space of integrable functions on Ω and by $L^\infty(\mu)$ the space of essentially bounded functions on Ω. Denote by $L^1_E(\mu) = \{f\chi_E \mid f \in L^1(\mu)\}$ the vector subspace of $L^1(\mu)$ consisting of μ-integrable functions restricted to E and similarly, by $L^\infty_E(\mu) = \{f\chi_E \mid f \in L^\infty(\mu)\}$ the vector subspace of $L^\infty(\mu)$ consisting of essentially bounded functions on Ω restricted to E.

The following approximation of a Bochner integrable function in terms of its mean values is an extension of the well-known result [11, Lemma III.2.1] to the case where X is a lcHs and its proof is found in [22, Lemma 4].

Lemma 2.10. *Let X be a quasicomplete lcHs and Π be the family of all finite measurable partitions of Ω directed by refinement and define the continuous linear operator $E_\pi : L^1(\mu, X) \to L^1(\mu, X)$ by*

$$E_\pi g = \sum_{A \in \pi} \frac{\int_A g d\mu}{\mu(A)} \chi_A, \quad \pi \in \Pi,\ g \in L^1(\mu, X)$$

where the usual convention $\mathbf{0}/0 = \mathbf{0}$ is in force. Then $E_\pi g \to g$ in $L^1(\mu, X)$.

2.5. Integration with Respect to Vector Measures

A generic element $\pi = \{E_1 \dots, E_n\}$ denotes a finite (measurable) partition of $A \in \mathcal{F}$. Given a seminorm $p \in \mathcal{P}$, the *p-variation* of a vector measure $m : \mathcal{F} \to X$ is defined by

$$|m|_p(A) = \sup \left\{ \sum_{E \in \pi} p(m(E)) \mid \pi \text{ is a finite partition of } A \right\}.$$

We say that m is of *bounded variation* if $|m|_p(\Omega) < \infty$ for every $p \in \mathcal{P}$. Let $U_p = \{x \in X \mid p(x) \le 1\}$ and U_p° be a polar of U_p. The *p-semivariation* of m is defined by

$$\begin{aligned} \|m\|_p(A) &= \sup_{x^* \in U_p^\circ} |x^* m|(A) \\ &= \sup \left\{ p\left(\sum_{E \in \pi} \alpha_E m(E) \right) \;\middle|\; \begin{array}{l} \pi \text{ is a finite partition of } A \\ |\alpha_E| \le 1\ \forall E \in \pi \end{array} \right\} \end{aligned}$$

where $|x^*m|$ is the variation of the scalar measure $x^*m := \langle x^*, m\rangle : \mathcal{F} \to \mathbb{R}$ (see for the second equality [34, Lemma II.1.1]). We say that m is *bounded* if $\|m\|_p(\Omega) < \infty$ for every $p \in \mathcal{P}$. Every vector measure is bounded (see [34, Lemma II.1.2]). If a vector measure is of bounded variation, then it is bounded, but not vice versa, that is, there is a bounded vector measure of unbounded variation (see [11, Example I.1.6]).

We define the integral with respect to a vector measure along the lines of [24, 57] when X is a sequentially complete lcHs. Recall that X is said to be *sequentially complete* if every Cauchy sequence in X converges. Every quasicomplete lcHs is sequentially complete. If $f : \Omega \to \mathbb{R}$ is a simple function of the form $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ with $\alpha_i \in \mathbb{R}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, define as usual the integral of f with respect to m by $\int f dm = \sum_{i=1}^n \alpha_i m(E_i)$. We then have

$$p\left(\int f dm\right) = p\left(\sum_{i=1}^n \frac{\alpha_i m(E_i)}{\|f\|_\infty}\right)\|f\|_\infty \leq \|m\|_p(\Omega)\|f\|_\infty$$

for every $p \in \mathcal{P}$, where $\|f\|_\infty = \max_{1\leq i\leq n}|\alpha_i|$. If m is absolutely continuous with respect to a scalar measure μ, we may define the *integration operator* $T_m f = \int f dm$ of m by extending the integral to any $f \in L^\infty(\mu)$ and still retaining the above inequality. Therefore, the integration operator $T_m : L^\infty(\mu) \to X$ is well defined and it is a continuous linear operator (see [51, III.1.1]). This definition allows us to define $\int f dm = \lim_n \int f_n dm$, where $\{f_n\}$ is a sequence of simple functions converging to $f \in L^\infty(\mu)$ in the essential sup norm, using the fact that X is sequentially complete. The integral $\int_A f dm$ is defined to be $\int f\chi_A dm$.

The following continuity property of the integration operator is well known and its proof is found in [11, Corollary I.2.7].

Lemma 2.11. *Let X be a sequentially complete lcHs. If $m : \mathcal{F} \to X$ is a μ-continuous vector measure, then the integration operator $T_m : L^\infty(\mu) \to X$ is continuous for the weak* topology of $L^\infty(\mu)$ and the weak topology of X.*

Let $ba(\mathcal{F}, \mu)$ be the space of μ-continuous, bounded, finitely additive set functions on $\mathcal{F}$. A linear operator $T : L^\infty(\mu) \to X$ is of *finite rank* if it is of the form $Tf = \sum_{i=1}^n \alpha_i \nu_i(f) x_i$ for every $f \in L^\infty(\mu)$ with $\alpha_i \in \mathbb{R}$, $\nu_i \in ba(\mathcal{F}, \mu)$ and $x_i \in X$, where $\nu_i(f) = \int f d\nu_i$. A linear operator T is *compact* if it maps bounded sets in $L^\infty(\mu)$ into relatively compact sets in X. Finite-rank operators are compact. A linear operator T is *weakly compact* if it maps bounded sets in $L^\infty(\mu)$ into relatively weakly compact sets in X. Lemma 2.11 implies that the integration operator T_m is a weakly compact operator.

2.6. Lyapunov Measures and Lyapunov Operators

Definition 2.12. The integration operator $T_m : L^\infty(\mu) \to X$ is a *nonatomic operator* if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ and every neighborhood U of the origin in X there exists $f \in L_E^\infty(\mu) \setminus \{0\}$ with signed values $\{-1, 0, 1\}$ such that $T_m f \in U$.

Theorem 2.13. *Let X be a sequentially complete lcHs and $m : \mathcal{F} \to X$ be a μ-continuous vector measure. Then m is nonatomic if and only if $T_m : L^\infty(\mu) \to X$ is a nonatomic operator.*

Proof. Suppose that T_m is a nonatomic operator. Then for every neighborhood U of $\mathbf{0}$ there exists $f \in L^\infty_E(\mu) \setminus \{0\}$ with signed values $\{-1, 0, 1\}$ such that $T_m f \in U \cap (-U)$. If m has an atom $E \in \mathcal{F}$, then $m(E) \neq \mathbf{0}$ and E is an atom of μ by the μ-continuity of m. Since measurable functions are constant on atoms of μ, either $f = \chi_E$ or $f = -\chi_E$. We thus obtain $m(E) \in \{\pm T_m f\} \subset U \cap (-U)$ for every neighborhood U of $\mathbf{0}$, and hence, $m(E) = \mathbf{0}$, a contradiction.

Conversely, suppose that T_m is not a nonatomic operator. Then there exists $E \in \mathcal{F}$ with $\mu(E) > 0$ and a neighborhood U of $\mathbf{0}$ such that $T_m f \notin U$ for every $f \in L^\infty_E(\mu) \setminus \{0\}$ with signed values $\{-1, 0, 1\}$. Thus, for every $A \in \mathcal{F}_E$ with $\mu(A) > 0$, we have $m(A) = T_m \chi_A \notin U$. Thus, there exist $p \in \mathcal{P}$ and $\varepsilon > 0$ such that $\|m\|_p(A) \geq \varepsilon$ for every $A \in \mathcal{F}_E$. If E is not an atom of m, then there exists $A \in \mathcal{F}_E$ such that $m(A) \neq m(E)$ and $m(A) \neq \mathbf{0}$. By the μ-continuity of m, we have $\mu(A) > 0$. Hence, there exists $\delta > 0$ such that $\mu(A \cap B) < \delta$ with $B \in \mathcal{F}$ implies $\|m\|_p(A \cap B) < \varepsilon$, a contradiction. Therefore, E is an atom of m. $\square$

Definition 2.14.

(i) A vector measure $m : \mathcal{F} \to X$ is a *Lyapunov measure* if for every $E \in \mathcal{F}$ the set $m(\mathcal{F}_E)$ is weakly compact and convex in X.

(ii) The integration operator $T_m : L^\infty(\mu) \to X$ is a *Lyapunov operator* of m if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the restriction $T_m : L^\infty_E(\mu) \to X$ is not injective.

The range of a Lyapunov measure m is weakly compact and convex in X in view of $m(\mathcal{F}) = m(\mathcal{F}_\Omega)$ with $\Omega \in \mathcal{F}$. If m has an atom $E \in \mathcal{F}$, then evidently, $m(\mathcal{F}_E)$ is not convex in X. Therefore, every Lyapunov measure is nonatomic. As the next result demonstrates, the nonatomicity of vector measures is reinforced as well by the notion of Lyapunov operators. (See [28, Theorem 3.2].)

Theorem 2.15. *Let X be a sequentially complete lcHs and $m : \mathcal{F} \to X$ be a μ-continuous vector measure. If $T_m : L^\infty(\mu) \to X$ is a Lyapunov operator, then it is a nonatomic operator.*

Instead of using $T_m f$, denote by $m(f)$ the integral of f with respect to m, that is, $m(f) = \int f dm$. We next characterize Lyapunov measures in terms of Lyapunov operators. While the proof of the following proposition is provided for the case where X is a Banach space (see [11, Theorem IX.1.4]), the proof is valid as it stands for the case where X is a sequentially complete lcHs.

Proposition 2.16. *Let X be a sequentially complete lcHs and $m : \mathcal{F} \to X$ be a μ-continuous vector measure. Then m is a Lyapunov measure if and only if $T_m : L^\infty(\mu) \to X$ is a Lyapunov operator of m. The range of a Lyapunov* *measure m is given by*

$$m(\mathcal{F}) = \{m(f) \in X \mid 0 \le f \le 1,\ f \in L^\infty(\mu)\}.$$

Denote by $ca(\mathcal{F}, \mu, X)$ be the space of μ-continuous, X-valued vector measures on $\mathcal{F}$ and by $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ the space of linear operators from $L^\infty(\mu)$ to X which are continuous for the weak* topology of $L^\infty(\mu)$ and the weak topology of X.

Definition 2.17. A linear operator $T : L^\infty(\mu) \to X$ is a *local injection* if there exist $E \in \mathcal{F}$ with $\mu(E) > 0$ such that the restriction $T : L^\infty_E(\mu) \to X$ is an injection.

The following result is due to [28, Corollary 3.1].

Proposition 2.18. *Let X be a sequentially complete lcHs. Then every vector measure in $ca(\mathcal{F}, \mu, X)$ is a Lyapunov measure if and only if every linear operator in $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is not a local injection.*

3. Lyapunov Measures in Separable LcHs

3.1. Saturation: A Sufficiency Theorem

We introduce here the notion of saturation for vector measures. This is a natural analogue of saturation for scalar measures developed in [25, 26, 38].

Definition 3.1. Let $(\widehat{\mathcal{F}}, \hat{m})$ be a vector measure algebra induced by a vector measure $m : \mathcal{F} \to X$. Then m is *saturated* if $\tau(\widehat{\mathcal{F}_E})$ is uncountable for every m-nonnull $E \in \mathcal{F}$.

If μ and ν are finite measures, μ is saturated, and ν is absolutely continuous with respect to μ, then ν is also saturated. This is because the measure algebra induced by ν contains the measure algebra induced by μ. Therefore, for every lcHs X with the BDS property, a vector measure $m : \mathcal{F} \to X$ is saturated if and only if it is absolutely continuous with respect to a saturated finite measure.

We turn to a canonical situation in the literature where a finite measure space $(\Omega, \mathcal{F}, \mu)$ is saturated and X is sequentially complete and separable.

Proposition 3.2. *If $(\Omega, \mathcal{F}, \mu)$ is saturated and X is a sequentially complete, separable lcHs, then every linear operator in $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is not a local injection.*

Proof. The proof employs the same argument as in [28, Proposition 3.2]. Here, it suffices to note that [45, Lemma 1] is valid as it stands even if X is a sequentially complete, separable lcHs instead of a separable Banach space. □

The next theorem states that the saturation of m is a sufficient condition for m to be a Lyapunov measure, or equivalently, for T_m to be a Lyapunov operator, whenever X is a sequentially complete, separable lcHs.

Theorem 3.3. *Let X be a sequentially complete, separable lcHs and $m : \mathcal{F} \to X$ be a μ-continuous vector measure. If $(\Omega, \mathcal{F}, \mu)$ is saturated, then m is a Lyapunov measure with its range $m(\mathcal{F})$ given by*

$$m(\mathcal{F}) = \{m(f) \in X \mid 0 \leq f \leq 1,\ f \in L^\infty(\mu)\}.$$

Proof. By the sequential completeness of X, the integration operator $T_m : L^\infty(\mu) \to X$ is well defined. Since T_m is not a local injection by Lemma 2.11 and Proposition 3.2, it is a Lyapunov operator of m and the result follows from Proposition 2.16. □

Remark 3.4. When a lcHs has the BDS property or a vector measure satisfies (ccc), the above definition of saturation coincides with the definition introduced in [28]. We develop here the definition of saturated vector measures that is independent of the existence of control measures. [30] provides a characterization of the saturation and Lyapunov property of vector measures in lcHs without control measures, based on the notion of integrals with respect to vector measures investigated at length in [34].

3.2. Saturation: A Characterization

Let $\mathcal{M}$ be a subset of $L^1(\mu)$ and denote by $\mathcal{M}^\perp$ the *annihilator* of $\mathcal{M}$ in $L^\infty(\mu)$, i.e.,

$$\mathcal{M}^\perp = \left\{ f \in L^\infty(\mu) \mid \int f\varphi d\mu = 0\ \forall \varphi \in \mathcal{M} \right\}$$

and let the restriction of $\mathcal{M}^\perp$ to $E \in \mathcal{F}$ be defined by

$$\mathcal{M}^\perp_E = \{f \in \mathcal{M}^\perp \mid f(\omega) = 0 \text{ a.e. } \omega \in \Omega \setminus E\}.$$

It is clear that $\mathcal{M}^\perp$ and $\mathcal{M}^\perp_E$ are weakly* closed vector subspaces of $L^\infty(\mu)$. For any Bochner integrable function $g \in L^1(\mu, X)$ define the subset of $L^1(\mu)$ by $\mathcal{M}_g = \{x^*g \mid x^* \in X^*\}$ and let $m_g \in ca(\mathcal{F}, \mu, X)$ be a vector measure defined by a Bochner indefinite integral of g with $m_g(A) = \int_A g d\mu$ for $A \in \mathcal{F}$.

The following notion of "thin sets" in $L^1(\mu)$ was introduced in [32] to investigate the validity of a version of the Lyapunov theorem for infinite-dimensional vector spaces.

Definition 3.5. A subset $\mathcal{M}$ of $L^1(\mu)$ is *thin* if $\mathcal{M}^\perp_E$ is a nontrivial vector subspace of $L^\infty_E(\mu)$ for every $E \in \mathcal{F}$ with $\mu(E) > 0$.

We present here a necessity result on saturation that provides a complete characterization of saturated measure spaces in terms of Lyapunov measures, Lyapunov operators and thin sets.

Theorem 3.6. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and X be a sequentially complete, separable lcHs with a vector subspace that is isomorphic to an infinite-dimensional Banach space. Then the following conditions are equivalent:*

(*i*) $(\Omega, \mathcal{F}, \mu)$ *is saturated.*

(*ii*) *Every vector measure in* $ca(\mathcal{F}, \mu, X)$ *is a Lyapunov measure.*

(*iii*) $m_g \in ca(\mathcal{F}, \mu, X)$ *is a Lyapunov measure for every* $g \in L^1(\mu, X)$.

(*iv*) *The integration operator* $T_m : L^\infty(\mu) \to X$ *is a Lyapunov operator for every* $m \in ca(\mathcal{F}, \mu, X)$.

(*v*) *The integration operator* $T_{m_g} : L^\infty(\mu) \to X$ *is a Lyapunov operator for every* $g \in L^1(\mu, X)$.

(*vi*) $\mathcal{M}_g$ *is thin in* $L^1(\mu)$ *for every* $g \in L^1(\mu, X)$.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$: See Theorem 3.3.

$(ii) \Leftrightarrow (iv)$ and $(iii) \Leftrightarrow (v)$: See Proposition 2.16.

$(iv) \Rightarrow (v)$: Trivial.

$(v) \Rightarrow (vi)$: Let $\mathcal{N}$ be the kernel of the T_{m_g}, that is, $\mathcal{N} = \{f \in L^\infty(\mu) \mid T_{m_g} f = \mathbf{0}\}$ and let $\mathcal{N}_E = \{f \in \mathcal{N} \mid f(\omega) = 0 \text{ a.e. } \omega \in \Omega \setminus E\}$. Since T_{m_g} is not a local injection, $\mathcal{N}_E$ is a nontrivial vector subspace of $L^\infty_E(\mu)$ for every $E \in \mathcal{F}$ with $\mu(E) > 0$. Note that for every $x^* \in X^*$ and $f \in \mathcal{N}$ we have

$$\int (x^* g) f d\mu = \left\langle x^*, \int f g d\mu \right\rangle = \langle x^*, T_{m_g} f \rangle = 0.$$

Hence, $\mathcal{M}_g \subset {}^\perp\mathcal{N}$, where ${}^\perp\mathcal{N}$ is the annihilator of $\mathcal{N}$ in $L^1(\mu)$, i.e., ${}^\perp\mathcal{N} = \{\varphi \in L^1(\mu) \mid \int \varphi f d\mu = 0 \ \forall f \in \mathcal{N}\}$. We thus obtain $\mathcal{M}_g^\perp \supset ({}^\perp\mathcal{N})^\perp = \mathcal{N}$ (see [48, Theorem 4.7]). Therefore, $(\mathcal{M}_g^\perp)_E \supset \mathcal{N}_E$ is a nontrivial vector subspace of $L^\infty_E(\mu)$ for every $E \in \mathcal{F}$ with $\mu(E) > 0$.

$(vi) \Rightarrow (ii)$: Let $\chi_\mathcal{F}$ be the set of characteristic functions of sets in $\mathcal{F}$ and $\mathcal{I}_E = \{f \in L^\infty_E(\mu) \mid 0 \le f \le 1\}$. Since $\mathcal{M}_g$ is thin in $L^1(\mu)$, by [32, Lemma 1], we have $\mathcal{I}_\Omega \subset \chi_\mathcal{F} + \mathcal{M}_g^\perp$. Hence, $\mathcal{I}_E \subset \chi_{\mathcal{F}_E} + (\mathcal{M}_g^\perp)_E$ for every $E \in \mathcal{F}$ with $\mu(E) > 0$. Since $\mathcal{I}_E$ is weakly* compact and convex in $L^\infty(\mu)$, and T_{m_g} is continuous for the weak* topology of $L^\infty(\mu)$ and the weak topology of X by Lemma 2.11, the set $m_g(\mathcal{I}_E)$ is weakly compact and convex in X. It follows from the thinness of $\mathcal{M}_g$ that $m_g((\mathcal{M}_g^\perp)_E) = \mathbf{0}$. Therefore, we obtain

$$m_g(\mathcal{I}_E) \subset m_g(\mathcal{F}_E) + m_g((\mathcal{M}_g^\perp)_E) = m_g(\mathcal{F}_E) \subset m_g(\mathcal{I}_E),$$

and hence, $m_g(\mathcal{F}_E)$ is weakly compact and convex in X for every $E \in \mathcal{F}$.

$(iii) \Rightarrow (i)$: Let Y be an infinite-dimensional Banach space that is isomorphic to a vector subspace of X. If μ is not saturated, then by [28, Lemma 4.1], there exists $E \in \mathcal{F}$ with $\mu(E) > 0$ and $g \in L^1(\mu, Y)$ such that $\tilde{m}_g \in ca(\mathcal{F}_E, \mu_E, Y)$ and the range $\tilde{m}_g(\mathcal{F}_E)$ is not convex in Y, where μ_E is the restriction of μ to $\mathcal{F}_E$. Extend $\tilde{m}_g$ from $\mathcal{F}_E$ to $\mathcal{F}$ by $m_g(A) = \tilde{m}_g(A \cap E)$ for $A \in \mathcal{F}$. Then m_g is a non-Lyapunov measure in $ca(\mathcal{F}, \mu, X)$, a contradiction. □

The requirement in Theorem 3.6 that X possesses a vector subspace isomorphic to an infinite-dimensional Banach space is employed only to demonstrate the implication $(iii) \Rightarrow (i)$. For the implication $(vi) \Rightarrow (ii)$, the separability of X is unnecessary. Note that [32] proved the implication $(vi) \Rightarrow (iii)$ without the separability assumption. The equivalence of (i), (ii) and (iv) was proved in [28] for the case where X is a Banach space.

Example 3.7. Let $(X_i, \|\cdot\|_i)$ with $i \in \mathbb{N}$ be a sequence of separable Banach spaces and let $X = \prod_{i\in\mathbb{N}} X_i$ be the product space with its generic element denoted by $x = (x_1, x_2, \dots)$, $x_i \in X_i$ for each $i \in \mathbb{N}$. Under the conventional vector operations, X is a vector space and its product topology τ is metrizable by the metric

$$\rho(x,y) = \sum_{i\in\mathbb{N}} \frac{1}{2^i} \frac{\|x_i - y_i\|_i}{1 + \|x_i - y_i\|_i},$$

which makes X a separable Fréchet space. If some X_j is infinite-dimensional, then the inclusion mapping from $(X_j, \|\cdot\|_j)$ to (X, τ) furnishes an isomorphism and X_j is a vector subspace of X.

4. Lyapunov Measures in Nonseparable LcHs

4.1. Maharam-Types and the Lyapunov Property

Definition 4.1. A lcHs X has the *Lyapunov property* with respect to a finite measure space $(\Omega, \mathcal{F}, \mu)$ if every vector measure in $ca(\mathcal{F}, \mu, X)$ is a Lyapunov measure.

The next result reveals an ambivalence of the Lyapunov property in infinite-dimensional vector spaces, which follows from the proof of [28, Proposition 5.1].

Proposition 4.2. *No sequentially complete lcHs with a vector subspace that is isomorphic to an infinite-dimensional Banach space has the Lyapunov property with respect to any countably-generated, nonatomic probability space. However, for every sequentially complete lcHs there exists a nonatomic probability space with respect to which it has the Lyapunov property.*

Definition 4.3. A finite measure space $(\Omega, \mathcal{F}, \mu)$ is (*Maharam-type-*)*homogeneous* if for every $E \in \mathcal{F}$ with $\mu(E) > 0$ the Maharam type of the measure algebra induced by $(E, \mathcal{F}_E, \mu)$ is equal to that of $(\Omega, \mathcal{F}, \mu)$.

Denote by $(\Omega_\mathfrak{m}, \mathcal{F}_\mathfrak{m}, \mu_\mathfrak{m})$ a finite measure space that induces a homogeneous measure algebra of Maharam type $\mathfrak{m}$. Then $\mu_\mathfrak{m}$ is nonatomic whenever $\mathfrak{m} \geq \aleph_0$ and moreover $\mathfrak{m} = \aleph_0$ if and only if $\mathcal{F}_\mathfrak{m}$ is countably generated modulo the null sets. By definition, $(\Omega_\mathfrak{m}, \mathcal{F}_\mathfrak{m}, \mu_\mathfrak{m})$ is saturated whenever $\mathfrak{m} \geq \aleph_1$. By Theorem 3.3, if $\mathfrak{m} \geq \aleph_1$, then sequentially complete, separable lcHs spaces have the Lyapunov property with respect to $(\Omega_\mathfrak{m}, \mathcal{F}_\mathfrak{m}, \mu_\mathfrak{m})$. Thus, the smallest cardinal satisfying the Lyapunov property for sequentially complete, separable lcHs with a vector subspace that is isomorphic to an infinite-dimensional Banach space is $\mathfrak{m} = \aleph_1$ in view of Proposition 4.2.

When X is sequentially complete, but non-separable, it is proved in [20, Corollary 1] that the Lyapunov property for X is satisfied if $\mathfrak{m} > \dim_{\mathrm{top}} X$, where $\dim_{\mathrm{top}} X$ is the *topological dimension* of X, the smallest cardinal corresponding to a subset of X with linear span dense in the locally convex topology. This is an improvement of the corresponding result in [28, Theorem 5.1], one based on the algebraic dimension of X. In any case, since the class of all cardinals is well-ordered with respect to cardinality, the smallest cardinal satisfying the Lyapunov property for X indeed exists.

For any sequentially complete lcHs X, let $\mathfrak{m}(X)$ be the smallest cardinal $\mathfrak{m}$ such that X has the Lyapunov property with respect to $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$. Then $\mathfrak{m}(X) = \aleph_0$ if X is finite dimensional; $\mathfrak{m}(X) \geq \aleph_1$ if X is infinite dimensional.

We summarize the above observations through the following characterization of the Lyapunov property for lcHs.

Proposition 4.4. *If $(\Omega, \mathcal{F}, \mu)$ is Maharam-type-homogeneous and X is a sequentially complete lcHs, then the following conditions are equivalent.*

(*i*) *X has the Lyapunov property with respect to $(\Omega, \mathcal{F}, \mu)$.*

(*ii*) *Every linear operator in $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is not a local injection.*

(*iii*) *The Maharam type of $(\Omega, \mathcal{F}, \mu)$ is greater than or equal to $\mathfrak{m}(X)$.*

Denote by $\mathfrak{m}^+$ the smallest cardinal strictly greater than $\mathfrak{m}$. Under the generalized continuum hypothesis, $\mathfrak{m}^+ = 2^{\mathfrak{m}}$, in particular, $\mathfrak{c} = \aleph_1$. (However, we do not assume it here.) For non-separable lcHs, we have the following estimate of $\mathfrak{m}(X)$. (See [28, Theorem 5.1] and [20, Corollary 1]).

Theorem 4.5. *If X is a sequentially complete, non-separable lcHs, then $\aleph_1 \leq \mathfrak{m}(X) \leq (\dim_{\mathrm{top}} X)^+$.*

As demonstrated in [28, Example 4.1] and [20, Remark 4], the upper bound $(\dim_{\mathrm{top}} X)^+$ for $\mathfrak{m}(X)$ is the "best possible" cardinal in the sense that there exists a nonseparable Banach space X and a Maharam-type-homogeneous finite measure space $(\Omega_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}, \mu_{\mathfrak{m}})$ with $\mathfrak{m} = \dim_{\mathrm{top}} X$ for which X does not possesses the Lyapunov property.

4.2. Nonatomicity and the Radon-Nikodym Property

Definition 4.6. A quasicomplete lcHs X has the *RNP* if for every finite measure space $(\Omega, \mathcal{F}, \mu)$ and μ-continuous vector measure $m : \mathcal{F} \to X$ of bounded variation, there exists a Bochner integrable function $g \in L^1(\mu, X)$ such that $m = m_g$.

We can now present a generalization of Uhl's theorem (see [61] and [11, Theorem IV.1.10]) to the case where X is a quasicomplete lcHs, which is an "approximate" version of the Lyapunov theorem for the strong topology. This is also a variant of [50, Theorem 3.5].

Theorem 4.7. *Let X be a quasicomplete lcHs with the RNP. If $m : \mathcal{F} \to X$ is a μ-continuous, nonatomic vector measure of bounded variation, then the closure of the range of m is compact and convex in X.*

Proof. Let $m : \mathcal{F} \to X$ be a μ-continuous vector measure of bounded variation. Denote by $\mathcal{L}_b(L^\infty(\mu), X)$ the space of all linear operators from $L^\infty(\mu)$ to X, endowed with the topology of bounded convergence. Since X has the RNP, there is a function $g \in L^1(\mu, X)$ such that $m(A) = \int_A g d\mu$ for $A \in \mathcal{F}$. For each partition $\pi = \{E_1, \dots, E_n\}$ of Ω, define the continuous linear operator $T_\pi : L^\infty(\mu) \to X$ by $T_\pi f = \int f E_\pi g d\mu$ for $f \in L^\infty(\mu)$. Define the μ-continuous measures ν_i by $\nu_i(A) = \mu(A \cap E_i)$ and the vectors in X by $x_i = \int_{E_i} g d\mu / \mu(E_i)$ whenever $\mu(E_i) > 0$, otherwise $x_i = \mathbf{0}$. We then have $T_\pi f = \sum_{i=1}^n \nu_i(f) x_i$ for every $f \in L^\infty(\mu)$. Hence, T_π is of finite rank, and consequently, it is also a compact operator. If the linear operator $T : L^\infty(\mu) \to X$ is defined by $Tf = \int f g d\mu$, then by Lemma 2.10, we have

$$p(Tf - T_\pi f) \le \int p(f(g - E_\pi g)) d\mu \le \|f\|_\infty \gamma_p(g - E_\pi g) \to 0$$

for every $p \in \mathcal{P}$. Therefore, T is the uniform limit of the net of finite-rank operators $\{T_\pi \mid \pi \in \Pi\}$ in $\mathcal{L}_b(L^\infty(\mu), X)$. Since the space of all compact operators of $L^\infty(\mu)$ to X is a closed vector subspace of $\mathcal{L}_b(L^\infty(\mu), X)$ (see [51, III.9.3]), T is a compact operator. Thus, $m(\mathcal{F}) = \{T(\chi_A) \mid A \in \mathcal{F}\}$ is relatively compact in X.

To prove that the closure of $m(\mathcal{F})$ is convex, let x_1 and x_2 be in the closure of $m(\mathcal{F})$. Let $\varepsilon_1, \dots, \varepsilon_n > 0$ and $p_1, \dots, p_n \in \mathcal{P}$ be arbitrary and choose A_1 and A_2 in $\mathcal{F}$ such that $p_k(x_i - m(A_i)) < \varepsilon_k/2$ for each $i = 1, 2$ and $k = 1, \dots, n$. Choose a partition π that refines the trivial partitions $\{A_1, \Omega \setminus A_1\}$ and $\{A_2, \Omega \setminus A_2\}$ and satisfies $\gamma_{p_k}(g - E_\pi g) < \varepsilon_k/2$ for each $k = 1, \dots, n$ by virtue of Lemma 2.10. Then $\int_{A_i} E_\pi g d\mu = m(A_i)$ for $i = 1, 2$. Moreover, the measure $A \mapsto \int_A E_\pi g d\mu$ has a convex range since it has a finite dimensional range and μ is nonatomic. Thus, if $\alpha \in [0, 1]$, then there is a set $A_0 \in \mathcal{F}$ such that $\int_{A_0} E_\pi g d\mu = \alpha m(A_1) + (1 - \alpha) m(A_2)$. Accordingly, we have

$$\begin{aligned}
&p_k(\alpha x_1 + (1-\alpha) x_2 - m(A_0)) \\
&\le \alpha p_k(x_1 - m(A_1)) + (1-\alpha) p_k(x_2 - m(A_2)) + p_k\left(\int_{A_0} E_\pi g d\mu - \int_{A_0} g d\mu\right) \\
&\le \frac{\alpha \varepsilon_k}{2} + \frac{(1-\alpha)\varepsilon_k}{2} + \frac{\varepsilon_k}{2} = \varepsilon_k
\end{aligned}$$

for each $k = 1, \dots, n$. This shows that the convex combination $\alpha x_1 + (1-\alpha) x_2 \in X$ is in the closure of $m(\mathcal{F})$. □

Remark 4.8. For a characterization of the RNP in terms of the geometric and measure-theoretic aspects of X such as "dentability" and the "martingale convergence property", the usual definition of the RNP involves the (local) boundedness of the μ-average range of m (see, for instance, [15, 19, 50]), a somewhat stronger condition than Definition 4.6, implying the μ-absolute continuity and bounded variation of m. If X is a Banach space, then the definition in terms of the average range coincides with Definition 4.6 due to the existence of the variation of m (see [8, p. 16]). Since our interest is limited to the original range of a vector measure, we adopt the above familiar definition (see [11, Definition III.1.3]).

4.3. Saturation and the Radon-Nikodym Property

When the underlying vector measure is saturated, we can strengthen Theorem 4.7 and derive an "exact" version of the Lyapunov theorem for quasicomplete lcHs with the RNP, a version in which the closure operation is removed. To this end, we need another condition on lcHs, called the Egorov property. The strong measurability of Bochner integrable functions imposed in [10, 19, 22] is synonymous with the Egorov property in this paper, which is imposed as a condition on lcHs along the lines of [21].

Definition 4.9. A lcHs X has the *Egorov property* if for every $g \in L^1(\mu, X)$ there is a sequence of X-valued simple functions $\{g_n\}$ such that for every $\varepsilon > 0$ there exists $A \in \mathcal{F}$ with $\mu(\Omega \setminus A) < \varepsilon$ satisfying $g_n \to g$ uniformly on A, i.e.,

$$\lim_{n\to\infty} \sup_{\omega \in A} p(g_n(\omega) - g(\omega)) = 0 \quad \text{for every } p \in \mathcal{P}.$$

The Egorov property implies that the sequence $\{g_n\}$ of X-valued simple functions converges to g uniformly on some set $A_k \in \mathcal{F}$ with $\mu(\Omega \setminus A_k) < 1/k$ for each k, and hence, $g_n \to g$ pointwise on the set $\bigcup_{k=1}^{\infty} A_k$ with $\mu(\Omega \setminus \bigcup_{k=1}^{\infty} A_k) = 0$. If X is a Fréchet space, then the almost everywhere convergence $g_n \to g$ is equivalent to the local uniform convergence in Definition 4.9 by the classical Egorov's theorem (see [12, Theorem II.6.1]). It is easy to see that under the Egorov property the range of a Bochner integrable function in $L^1(\mu, X)$ is essentially separable (see [12, Corollary II.6.1]).

Theorem 4.10. *Let X be a quasicomplete lcHs with the RNP and the Egorov property, and $m : \mathcal{F} \to X$ be a μ-continuous vector measure of bounded variation. If $(\Omega, \mathcal{F}, \mu)$ is saturated, then the range of m is compact and convex in X.*

Proof. By the RNP of X, there exists $g \in L^1(\mu, X)$ such that $m(A) = \int_A g d\mu$ for every $A \in \mathcal{F}$. By the Egorov property, g is essentially separably valued. Thus, we may assume without loss of generality that g takes values in a separable lcHs X (consider the linear span generated by the essential range of g). Since quasicompleteness implies sequentially completeness, by Theorem 3.3, the range $m(\mathcal{F})$ is weakly compact and convex in X. Since m is nonatomic by Theorems 2.13 and 2.15, the closure of $m(\mathcal{F})$ is compact by Theorem 4.7. Therefore, $m(\mathcal{F})$ is compact and convex in X because on convex sets in X (strong) closedness is equivalent to weak closedness (see [51, Corollary II.9.2]). $\square$

As is evident from the proof of Theorem 4.10, the RNP replaces the separability of X very partially and still relies crucially on the separability of the range of Bochner integrable functions with the aid of the Egorov property. On the contrary, if the Maharam type of the finite measure space is large enough, then the Egorov property can be removed from Theorem 4.10 and the separability argument is dispensed with completely. The following result brings this out.

Corollary 4.11. *Let X be a quasicomplete lcHs with the RNP and $m : \mathcal{F} \to X$ be a μ-continuous vector measure of bounded variation. If $(\Omega, \mathcal{F}, \mu)$ is Maharam-type-homogeneous of $\mathfrak{m} \geq \mathfrak{m}(X)$, then the range of m is compact and convex in X.*

Proof. Let $m \in ca(\mathcal{F}, \mu, X)$ be a vector measure defined in the proof of Theorem 4.10. It follows from the definition of $\mathfrak{m}(X)$ that m is obviously a Lyapunov measure in view of Proposition 4.4. The rest of the proof is totally same as that of Theorem 4.10. $\square$

Remark 4.12. Because the Egorov property is automatically satisfied for Fréchet spaces, the assumption on higher-order Maharam types is unnecessary for Theorem 4.10 and Corollary 4.11 to be true in nonseparable Fréchet spaces. This means that the Egorov property and the higher-order Maharam types never arise in Fréchet spaces with the RNP for the compactness and convexity of the range of a vector measure. The need for the Egorov property comes to the surface when one goes beyond Fréchet spaces and that for the higher-order Maharam types appears when one surpasses the Egorov property. In any case, these requirements are peculiar to nonseparable lcHs with the RNP.

5. Convex Analysis: Two Specific Applications

5.1. Integration of Multifunctions

Let $\mathcal{B}_X$ be the Borel σ-algebra of a lcHs X. A set-valued mapping from Ω to X with nonempty values is called a *multifunction*. A multifunction $\Gamma : \Omega \twoheadrightarrow X$ is *measurable* if its graph is $\mathcal{F} \otimes \mathcal{B}_X$-measurable. It is *integrably bounded* if for every $p \in \mathcal{P}$ there exists $\varphi_p \in L^1(\mu)$ such that $\Gamma(\omega) \subset \varphi_p(\omega)U_p$ a.e. $\omega \in \Omega$. If Γ is a measurable multifunction from a complete finite measure space $(\Omega, \mathcal{F}, \mu)$ to a locally convex Suslin space X, then by the measurable selection theorem (see [9, Theorem III.22]), there exists a measurable function $g : \Omega \to X$ such that $g(\omega) \in \Gamma(\omega)$ for every $\omega \in \Omega$. Denote by $\mathcal{S}^1_\Gamma$ the set of all Bochner integrable selections of Γ, i.e.,

$$\mathcal{S}^1_\Gamma = \{g \in L^1(\mu, X) \mid g(\omega) \in \Gamma(\omega) \text{ a.e. } \omega \in \Omega\}.$$

If $\Gamma : \Omega \twoheadrightarrow X$ is measurable and integrably bounded, then $\mathcal{S}^1_\Gamma$ is nonempty whenever $(\Omega, \mathcal{F}, \mu)$ is a complete finite measure space and X is a locally convex Suslin space. When X is quasicomplete, the *integral* of Γ with respect to μ is defined by

$$\int \Gamma d\mu = \left\{ \int g d\mu \in X \mid g \in \mathcal{S}^1_\Gamma \right\}.$$

Theorem 5.1. *If* $(\Omega, \mathcal{F}, \mu)$ *is saturated and* X *is a quasicomplete lcHs with the Egorov property, then the integral* $\int \Gamma d\mu$ *is convex in* X *for every multifunction* $\Gamma : \Omega \twoheadrightarrow X$.

Proof. Choose any $g_0, g_1 \in \mathcal{S}^1_\Gamma$ and $\alpha \in [0,1]$. By the Egorov property, g_0 and g_1 have an essentially separable range. Thus, we may assume without loss of generality that g_0 and g_1 take values in a quasicomplete separable lcHs X. It suffices to show that there exists $g \in \mathcal{S}^1_\Gamma$ such that $\int g d\mu = \int (\alpha g_0 + (1-\alpha) g_1) d\mu$. To this end, define the vector measure $m : \mathcal{F} \to X \times X$ by

$$m(A) = \left(\int_A g_0 d\mu, \int_A g_1 d\mu \right), \quad A \in \mathcal{F}.$$

Since m is saturated by its μ-continuity, the range of m is convex by Theorem 3.3. Then there exists $A \in \mathcal{F}$ such that $m(A) = \alpha m(\Omega)$, which is equivalent to $\int_A g_i d\mu = \alpha \int g_i d\mu$ for each $i = 0, 1$. Define $g = g_0 \chi_A + g_1 \chi_{\Omega \setminus A}$. We then have $g \in \mathcal{S}^1_\Gamma$ and

$$\int g d\mu = \int_A g_0 d\mu + \int_{\Omega \setminus A} g_1 d\mu = \alpha \int g_0 d\mu + (1-\alpha) \int g_1 d\mu. \qquad \square$$

Similar to Corollary 4.4, the separability and Egorov property of X can be removed from Theorem 5.1 if the Maharam type of μ is large enough. Specifically, we have the following result.

Corollary 5.2. *If* $(\Omega, \mathcal{F}, \mu)$ *is Maharam-type-homogeneous of* $\mathfrak{m} \geq \mathfrak{m}(X \times X)$ *for a quasicomplete lcHs* X*, then the integral* $\int \Gamma d\mu$ *is convex in* X *for every multifunction* $\Gamma : \Omega \twoheadrightarrow X$.

Proof. Let $m : \mathcal{F} \to X \times X$ be the vector measure given in the proof of Theorem 5.1. It follows from the definition of $\mathfrak{m}(X \times X)$ that m is a Lyapunov measure by Proposition 4.4. The rest of the proof is same with that of the proof of Theorem 5.1. $\square$

As easily anticipated, $\mathfrak{m}(X \times X) \geq \mathfrak{m}(X)$ whenever X is a sequentially complete lcHs. For, if $(\Omega, \mathcal{F}, \mu)$ is Maharam-type-homogeneous of $\mathfrak{m}(X \times X)$, then by Proposition 2.18, every linear operator in $\mathcal{L}(L^\infty_{w^*}(\mu), X_w \times X_w)$ is not a local injection. Since $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is a vector subspace of $\mathcal{L}(L^\infty_{w^*}(\mu), X_w \times X_w)$, every linear operator in $\mathcal{L}(L^\infty_{w^*}(\mu), X_w)$ is not a local injection. Again by Proposition 2.18, this implies that X satisfies the Lyapunov property for $(\Omega, \mathcal{F}, \mu)$. Hence, the result follows.

Remark 5.3. If X is a quasicomplete lcHs with a vector subspace that is isomorphic to an infinite-dimensional Banach space, then the converse of Theorem 5.1 is also true, i.e., $(\Omega, \mathcal{F}, \mu)$ is saturated whenever $\int \Gamma d\mu$ is convex in X for every multifunction $\Gamma : \Omega \twoheadrightarrow X$. This can be proved similar to the Banach space case as in [45, Theorem 1] and [56, Remark 1(2)(3)].

5.2. Distribution of Multifunctions

Let S be a compact metric space with the Borel σ-algebra $\mathcal{B}_S$ and $C(S)$ be the space of continuous functions on S furnished with the sup norm. Denote by $ca(\mathcal{B}_S)$ the space of finite signed Borel measures (Radon measures) on S. Every signed measure in $ca(\mathcal{B}_S)$ is regular (see [7, Theorem 7.1.7]). By the Riesz representation theorem $(C(S))^* = ca(\mathcal{B}_S)$ (see [7, Theorem 7.10.4]) and the duality is given by $\langle u, v\rangle = \int_S udv$ for $u \in C(S)$ and $v \in ca(\mathcal{B}_S)$. The weak* topology of $ca(\mathcal{B}_S)$ is metrizable, in particular, $ca(\mathcal{B}_S)$ is a separable metric space (see [7, Theorem 8.9.4 and Proposition 7.2.2]). Since $ca(\mathcal{B}_S)$ is weakly complete (see [14, Theorem IV.9.4]) and its weak* topology is coarser than its weak topology, it is also weakly* complete. Hence, $ca(\mathcal{B}_S)$ is a Fréchet space. Denote by $ca_1(\mathcal{B}_S)$ the family of probability measures on S. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. The *distribution* of a multifunction $\Phi : \Omega \twoheadrightarrow S$ is a subset of $ca_1(\mathcal{B}_S)$ defined by $\mathcal{D}_\Phi = \{\mu \circ \varphi^{-1} \mid \varphi \in \mathcal{S}_\Phi\}$.

Theorem 5.4. *If $(\Omega, \mathcal{F}, \mu)$ is a saturated probability space, then the distribution $\mathcal{D}_\Phi$ is convex in $ca_1(\mathcal{B}_S)$ for every multifunction $\Phi : \Omega \twoheadrightarrow S$.*

Proof. Let δ_s be the point mass at $s \in S$. For every measurable function $\varphi : \Omega \to S$, we have $\langle u, \delta_{\varphi(\omega)}\rangle = u(\varphi(\omega))$ for every $u \in C(S)$ and $\omega \in \Omega$. Since the $ca(\mathcal{B}_S)$-valued function $\omega \mapsto \delta_{\varphi(\omega)}$ is measurable if and only if it is *scalarly measurable* in the sense that the scalar function $\omega \mapsto \langle u, \delta_{\varphi(\omega)}\rangle$ is measurable on Ω for every $u \in C(S)$ (see [59, Theorem 1]), the Fréchet space valued function $\omega \mapsto \delta_{\varphi(\omega)}$ is measurable and Bochner integrable, and it is an element in $L^1(\mu, ca(\mathcal{B}_S))$. We thus obtain

$$\left\langle u, \int_\Omega \delta_{\varphi(\cdot)} d\mu \right\rangle = \int_\Omega \langle u, \delta_{\varphi(\cdot)}\rangle d\mu = \int_\Omega u \circ \varphi d\mu = \langle u, \mu \circ \varphi^{-1}\rangle$$

for every $u \in C(S)$, where the last equality follows from the change-of-variable formula. Therefore, $\mu \circ \varphi^{-1} = \int_\Omega \delta_{\varphi(\cdot)} d\mu$. Define the multifunction $\Gamma_\Phi : \Omega \twoheadrightarrow ca_1(\mathcal{B}_S) \subset ca(\mathcal{B}_S)$ by $\Gamma_\Phi(\omega) = \{\delta_{\varphi(\omega)} \mid \varphi \in \mathcal{S}_\Phi\}$. By Theorem 5.1, $\int \Gamma_\Phi d\mu = \mathcal{D}_\Phi$ is convex in $ca_1(\mathcal{B}_S)$. $\square$

Remark 5.5. For the proof of Theorem 5.4, we exploit the connection between the distribution and integral of the function:

$$\mu \circ \varphi^{-1} = \int_\Omega \delta_{\varphi(\cdot)} d\mu \quad \text{for every } \varphi \in \mathcal{S}_\Phi.$$

This connection was elaborated in [27, Lemma 1] for the case with $S = [-1, 1]$. As opposed to the treatment of $\int_\Omega \delta_{\varphi(\cdot)} d\mu$ as a Gelfand integral in the dual space of $C(S)$, as in [27], we identify it with a Bochner integral in Fréchet space $ca(\mathcal{B}_S)$: this is done to apply Theorem 5.1, a direct consequence of the Lyapunov theorem in lcHs (Theorem 3.3).

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