Journal of Convex Analysis
Volume 22 (2015), No. 3, 673–685

Ball Proximinal and Strongly Ball Proximinal Spaces

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Received: May 7, 2014
Accepted: July 6, 2014

Let Y be an E-proximinal (respectively, a strongly proximinal) subspace of X. We prove that Y is (strongly) ball proximinal in X if and only if for any $x \in X$ with $(x+Y) \cap B_X \neq \emptyset$, $(x+Y) \cap B_X$ is (strongly) proximinal in $x+Y$. Using this characterization and a smart construction, we obtain three Banach spaces $Z \subset Y \subset X$ such that Z is ball proximinal in X and Y/Z is ball proximinal in X/Z, but Y is not ball proximinal in X. This solves a problem raised by Bandyopadhyay, Lin, and Rao [1] in 2007.

Keywords: Ball proximinal, strongly ball proximinal, E-proximinal

1991 Mathematics Subject Classification: 46B20, 41A50

1. Introduction

Let Y be a subspace of a real (or a complex) Banach space X. By subspaces, we always mean closed linear subspaces. Let $x \in X$ and let A be a nonempty subset of X. The Y-distance $d_Y(x,A)$ from x to A is defined by

$$d_Y(x,A) = \begin{cases} \inf\{\|x-c\| : c \in (x+Y) \cap A\} & \text{if } (x+Y) \cap A \neq \emptyset, \\ \infty & \text{if } (x+Y) \cap A = \emptyset. \end{cases}$$

*Corresponding author. The research of Bentuo Zheng is supported in part by NSF grant DMS-1200370.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

For any $r > 0$, let

$$B_Y[x, r] = \{x + y : y \in Y \text{ and } \|y\| \leq r\} = x + B_Y[0, r].$$

$B_Y[x, r]$ is called the Y-ball of radius r and center x. For simplicity, we use B_Y to denote $B_Y[x, 1]$. An element y in A is called a best approximation of x from A if

$$d_X(x, A) = \|x - y\|.$$

A is said to be *proximinal* in X if for any $x \in X$, there is a best approximation of x from A. A proximinal subset A of X is said to be *strongly proximinal* in X if for any $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x - a\| < d_X(x, A) + \delta$, there is $a' \in A$ such that a' is a best approximation of x from A and $\|a - a'\| < \epsilon$. In [6], Saidi showed that for any non-reflexive Banach space Y, there is a Banach space X such that Y is isometrically isomorphic to a hyperplane Z of X such that Z is proximinal in X but the unit ball of Z is not proximinal in X. Bandyopadhyay, Lin, and Rao ([1, Example 3.3]; also see [3]) showed that the space Z is strongly proximinal in X. Motivated by this example, Bandyopadhyay, Lin, and Rao [1] introduced the following notion:

Definition 1.1. A subspace Y of X is said to be (strongly) *ball proximinal* in X if $B_Y[0, 1]$ is (strongly) proximinal in X.

To characterize (strongly) ball proximinality, Indumathi and Prakash [5] introduced the so called E-proximinality. Recall that a proximinal subspace Y of X is said to be E-proximinal [5] if for any $x \in X \setminus Y$, and $\gamma, \epsilon > 0$, $d_X(x, B_Y[0, \gamma]) = d_X(x, Y)$ implies there is $y \in Y$ such that $\|x - y\| = d_X(x, Y)$ and $\|y\| < \gamma + \epsilon$. It is easy to see that if Y is either ball proximinal or strongly proximinal in X, then Y is E-proximinal in X [3, 4, 5]. But there exists a Banach space X such that X is E-proximinal but not strongly proximinal [5, Example 4.1]. So E-proximinality does not imply ball proximinality or strong proximinality.

In [5], Indumathi and Prakash proved that Y is a (strongly) ball proximinal hyperplane if and only if Y is E-proximinal (respectively, strongly proximinal) in X and for any $x \in X$ with $(x + Y) \cap B_X \neq \emptyset$, $(x + Y) \cap B_X$ is (strongly) proximinal in $x + Y$. However there are two imprecisions in their paper. The first imprecision is the last sentence in the statement of [5, Proposition 2.2]. The following is a simple counterexample.

Example 1.2. Example. Let $X = R^2$, $H = \{(0, y) : y \in R\}$, and $x = (1, 1/2)$. Then

$$x + H = \{(1, y) : y \in R\} \quad \text{and} \quad d_X(\mathbf{0}, x + H) = 1.$$

Note that $B_X[\mathbf{0}, 1] \cap (x + H) = \{(1, 0)\}$. Denote by α the distance from x to H, and β is the distance from x to B_H. It is easy to see that $\alpha = \beta$. Without loss of generality, we assume $\alpha = \beta = 1$ and hence $\lambda = 1$. Now a simple computation shows that

$$d(x, H_\lambda \cap B_X) = 1/2 < 1/\beta.$$

The second imprecision is that the statements of both [5, Theoem 2.2 and 2.3] must include the value of $\lambda = 0$. The proof of [5, Theorem 2.2] should also consider the case when $d(x, H) = 0 < d(x, B_H)$. In Section 2, we correct these mistakes and generalize their results from E-proximinal hyperplanes to E-proximinal subspaces. To be more precise, we show that if Y is an E-proximinal (respectively, a strongly proximinal) subspace of X, then Y is (strongly) ball proximinal in X if and only if for any $x \in X$ with $(x+Y)\cap B_X \neq \emptyset$, $(x+Y)\cap B_X$ is (strongly) proximinal in $x + Y$.

In [1], Bandyopadhyay, Lin, and Rao proved that for Banach spaces $Z \subset Y \subset X$, if Y/Z is proximinal in X/Z and Z is proximinal in X, then Y is proximinal in X. They asked in Question 4.3 that whether a similar result holds for ball proximinality. In Section 3, by using the aforementioned characterization of ball proximinality and a pretty construction, we are able to construct Banach spaces $Z \subset Y \subset X$ such that $\dim(Z) = 1$, Y/Z is ball proximinal in X/Z, but Y is not ball proximinal in X. This gives an answer to a question of Bandypadyay, Lin, and Rao [1, Question 4.3].

2. Main results

To prove our main results, we need the following two lemmas which are formulated differently but are essentially generalizations of [5, Proposition 2.1 and 2.2] respectively. For completeness, we give the proofs here.

Lemma 2.1. *Let Y be a subspace of a Banach space X, $\beta > 0$, and $x \in X \setminus B_X[0,\beta]$. Suppose that $(x+Y)\cap B_X[0,\beta]$ is nonempty. Let*

$$\begin{aligned}\gamma = &\inf\{\|x - z\| : z \in (x+Y)\cap B_X[0,\beta]\}\\ &= d_X(x, (x+Y)\cap B_X[0,\beta]) = d_Y(x, B_X[0,\beta]).\end{aligned}$$

Then $d_X(0, B_Y[x,\gamma]) = \beta$.

Proof. Suppose that $d_X(0, B_Y[x,\gamma]) < \beta$. Then there is $y \in Y$ such that $\|y\| \leq \gamma$ and $\|x + y\| < \beta$. Note that $y \neq 0$ (because $x \notin B_X[0,\beta]$). Let $\delta = \frac{1}{2}(\beta - \|x + y\|)$. Then

$$\left\| x + y - \frac{\delta y}{\|y\|} \right\| \leq \|x + y\| + \delta < \beta;$$

$$\left\| y - \frac{\delta y}{\|y\|} \right\| < \gamma.$$

We get a contradiction. So $d_X(0, x + B_Y[0,\gamma]) \geq \beta$.

By the definition of γ, for any $\epsilon > 0$ there is $y \in B_Y[0,1]$ such that

$$\|x - (\gamma + \epsilon)y\| \leq \beta.$$

This implies that

$$d_X(x, B_Y[0,\gamma]) \leq \beta + \epsilon.$$

Since ϵ is arbitrary positive real, we have proved that

$$\beta = d_X(x, B_Y[0,\gamma]) = d_X(0, B_Y[x,\gamma]). \qquad \square$$

Lemma 2.2. *Let Y be an E-proximinal subspace of X, $\gamma > 0$ and $x \in X \setminus B_Y[0,\gamma]$. If $\beta = d(0, B_Y[x,\gamma])$ and $\alpha = d(x, Y)$, then*

$$\gamma \geq \inf\{\|y\| : y \in Y,\ x + y \in B_X[0,\beta]\} = d_Y(x, B_X[0,\beta]).$$

The equality holds if $\alpha < \beta$.

Proof. Let x be any element in $X \setminus B_Y[0,\gamma]$, $\alpha = d(x, Y)$, and $\beta = d(0, B_Y[x,\gamma])$. Since Y is proximinal in X, there is $z \in x + Y$ such that

$$\|z\| = \alpha = d_X(0, x + Y).$$

It is clear that $\beta \geq \alpha$.

Case 1. $\alpha = \beta$. Since Y is E-proximinal, for any $\epsilon > 0$, there is $z' \in x + Y$ such that $\|z'\| = \beta$ and $\|x - z'\| \leq \gamma + \epsilon$. This implies that

$$\gamma \geq d_Y(x, B_X[0,\beta]).$$

Case 2. $\alpha < \beta$. Note that $\|z\| = \alpha < \beta$. For any $\epsilon > 0$, there is $1 > \delta > 0$ such that

$$\delta \max\{\beta - \alpha, \|x - z\|\} < \epsilon.$$

By the definition of β, there is $y \in B_Y[0,\gamma]$ such that

$$\|x + y\| \leq \beta + \delta(\beta - \alpha).$$

Then $(1-\delta)(x+y) + \delta z \in x + Y$,

$$\begin{aligned}\|(1-\delta)(x+y) + \delta z\| &\leq (1-\delta)(\beta + \delta(\beta - \alpha)) + \delta\alpha \\ &\leq (1-\delta)\beta + \delta(\beta - \alpha) + \delta\alpha = \beta; \\ \|[(1-\delta)(y+x) + \delta z] - x\| &= \|(1-\delta)y + \delta(z - x)\| \\ &\leq \|y\| + \|\delta(z - x)\| \leq \gamma + \epsilon.\end{aligned}$$

This implies that

$$\gamma \geq d_Y(x, B_X[0,\beta]).$$

Suppose that there is $y' \in Y$ such that $\|y'\| < \gamma$ and $\|x + y'\| \leq \beta$. Select $\delta > 0$ such that $\delta\|x - z\| < \gamma - \|y'\|$. Then $(1-\delta)y' + \delta(z - x) \in Y$, and

$$\|(1-\delta)y' + \delta(z - x)\| \leq \|y'\| + \delta\|z - x\| < \gamma,$$

$$\|(1-\delta)(x + y') + \delta z\| \leq (1-\delta)\beta + \delta\alpha < \beta.$$

This contradicts to the definition of β. So $\gamma = d_Y(x, B_X[0,\beta])$. The proof is complete. $\square$

Theorem 2.3. *Let Y be an E-proximinal subspace of a Banach space X. Then the following are equivalent.*

(*1*) *Y is ball-proximinal in X.*
(*2*) *For any $\gamma > 0$ and any $x \in X$, the Y-ball $B_Y[x, \gamma]$ is proximinal in X.*
(*3*) *For any $x \in X \setminus Y$ and $\beta > 0$ with $\beta \geq d_X(x, Y) = \alpha > 0$, the set $(x + Y) \cap B_X[0, \beta]$ is proximinal in $x + Y$.*
(*4*) *For any $x \in X \setminus Y$ with $1 \geq d_X(x, Y) > 0$, the set $(x + Y) \cap B_X[0, 1]$ is proximinal in $x + Y$.*

Proof. Clearly, (*1*) $\Rightarrow$ (*2*) and (*3*) $\Rightarrow$ (*4*). We need only to prove that (*2*) $\Rightarrow$ (*3*) $\Rightarrow$ (*1*) and (*4*) $\Rightarrow$ (*3*).

(*4*) $\Rightarrow$ (*3*). Let x be any element in $X \setminus Y$ (i.e., $d_X(x, Y) > 0$), $\alpha = d_X(x, Y)$, and β a positive real such that $\beta \geq \alpha$. Set $x' = \frac{x}{\beta}$. Then

$$0 < d_X(x', Y) = \frac{\alpha}{\beta} \leq 1.$$

By the assumption (of (*4*)), there is a best approximation y' of x' from $(x' + Y) \cap B_X[0, 1]$. Note that

$$(x + Y) \cap B_X[0, \beta] = \beta \left[(x' + Y) \cap B_X[0, 1]\right].$$

So $\beta y'$ is a best approximation of x from $(x + Y) \cap B_X[0, \beta]$. We have proved (*4*) $\Rightarrow$ (*3*).

(*2*) $\Rightarrow$ (*3*). Suppose that Y is a ball proximinal subspace of X, $x \in X \setminus Y$ and $\beta > 0$ such that $\beta \geq d_X(x, Y) = \alpha$. Then $(x + Y) \cap B_X[0, \beta]$ is nonempty. Let

$$\gamma = \inf\{\|x - z\| : z \in (x + Y) \cap B_X[0, \beta]\} = d_Y(x, B_X[0, \beta]).$$

By Lemma 2.1, $d_X(0, B_Y[x, \gamma]) = \beta$. Since Y is ball proximinal in X, $B_Y[x, \gamma]$ is proximinal in X. Let z be a best approximation of 0 from $B_Y[x, \gamma]$. Then $z \in x + Y$, $\|z\| = \beta$, and $\|z - x\| \leq \gamma$. So z is a best approximation of x from $(x + Y) \cap B_X[0, \beta]$.

(*3*) $\Rightarrow$ (*1*). Assume that Y is E-proximinal in X. Let x be any element in $X \setminus B_Y[0, 1]$. If $x \in Y$, then $x/\|x\|$ is the best approximation of x from $B_Y[0, 1]$. So we may assume that $x \notin Y$, i.e., $d_X(x, Y) > 0$. It is known that the following two statements are equivalent.

- There is a best approximation (say z) of x from $B_Y[0, 1]$.
- There is a best approximation $(x - z)$ of 0 from $B_Y[x, 1]$.

So we need only to show that for any $x \in X$ with $d_X(x, Y) > 0$, there is a best approximation of 0 from the Y-ball $B_Y[x, 1]$. Let

$$\beta = d_X(x, B_Y[0, 1]) = d_X(0, B_Y[x, 1]) > 0;$$
$$\alpha = d_X(x, Y).$$

Clearly $\alpha \leq \beta$. Since Y is proximinal, there is $z \in x + Y$ such that $\|z\| = \alpha$. Since $(x+Y) \cap B_X[0,\beta]$ is nonempty, by Lemma 2.2,

$$d_Y(x, B_X[0,\beta]) \leq 1.$$

Case 1. $d_Y(x, B_X[0,\beta]) < 1$. Then by Lemma 2.2, $\alpha = \beta$. So there is $y \in Y$ with $\|y\| < 1$ and $\|x+y\| = \beta$. Therefore $u = x+y$ is a best approximation of 0 from $B_Y[x,1]$.

Case 2. $d_Y(x, B_X[0,\beta]) = 1$. Since $(x+Y) \cap B_X[0,\beta]$ is proximinal in $x+Y$, there is $u \in (x+Y) \cap B_X[0,\beta]$ such that

$$\|x-u\| = 1 \quad \text{and} \quad \|u\| \leq \beta.$$

Then u is the best approximation of 0 from $B_Y[x,1]$. We have proved that $B_Y[x,1]$ is proximinal in X. So Y is ball proximinal. □

Let Y be a subspace of a Banach space X. It is easy to see the following are equivalent.

- $B_Y[0,1]$ is strongly promininal in Y.
- For any $\beta > 0$, $B_Y[0,\beta]$ is strongly promininal in Y.

As mentioned in the introduction, the proofs of Theorem 3.2 and 3.3 of [5] need one additional condition that B_H (the unit ball of the hyperplane of H) is strongly proximinal in H. So we add a similar condition to the assumptions of our next theorem.

Theorem 2.4. *Let Y be a strongly proximinal subspace of a Banach space X. The following statements are equivalent.*

(*1*) *Y is strongly ball proximinal in X.*
(*2*) *For any $x \in X$ and $\gamma > 0$, the Y-ball $B_Y[x,\gamma]$ is strongly proximinal in X.*
(*3*) *For any $x \in X \setminus Y$ and any $\beta > 0$ with $d(x,Y) \leq \beta$, $(x+Y) \cap B_X[0,\beta]$ is strongly proximinal in $x+Y$.*
(*4*) *For any $x \in X \setminus Y$ with $d(x,Y) \leq 1$, $(x+Y) \cap B_X[0,1]$ is strongly proximinal in $x+Y$.*

Proof. Clearly, (*1*) $\Rightarrow$ (*2*). (*3*) $\Leftrightarrow$ (*4*) follows from a proof similar to that of (*4*) $\Rightarrow$ (*3*) in Theorem 2.3. We need only to prove that (*2*) $\Rightarrow$ (*3*) and (*3*) $\Rightarrow$ (*1*).

(*2*) $\Rightarrow$ (*3*). Suppose that Y is strongly ball proximinal in X, $x \in X \setminus Y$ and $\beta \geq d_X(x,Y)$. Then $(x+Y) \cap B_X[0,\beta] \neq \emptyset$. Let

$$\gamma = \inf\{\|x-z\| : z \in (x+Y) \cap B_X[0,\beta]\} = d_Y(x, B_X[0,\beta]).$$

By Lemma 2.1, $d_X(0, B_Y[x,\gamma]) = \beta$. Let $\epsilon > 0$ be any positive real number. Since $B_Y[0,1]$ is strongly proximinal in X, there is $\delta > 0$ such that for any

$w \in B_Y[x,\gamma]$ with $\|w\| \le \beta + \delta$, there is $z \in B_Y[x,\gamma] \cap B_X[0,\beta]$ such that $\|z - w\| < \frac{\epsilon}{2}$. Give any element w' in $x + Y$ such that $\|w'\| \le \beta$ and

$$\gamma \le \|w' - x\| \le \gamma + \min(\epsilon/2, \delta).$$

Let $w = x + \gamma(w' - x)/\|w' - x\|$. Then $w \in B_Y[x,\gamma]$ and

$$\|w - w'\| = \|w' - x\| - \gamma < \min\left(\delta, \frac{\epsilon}{2}\right);$$

$$\|w\| \le \|w'\| + \|w - w'\| < \beta + \delta.$$

So there is $z \in B_Y[x,\gamma] \cap B_X[0,\beta]$ such that $\|z - w\| < \frac{\epsilon}{2}$. We have shown that there is a best approximation z of x from $(x + Y) \cap B_X[0,\beta]$ such that

$$\|w' - z\| < \|w' - w\| + \|w - z\| < \epsilon.$$

(3) $\Rightarrow$ *(1)*. Assume that Y is strongly proximinal in X and $(x + Y) \cap B_X[0,\beta]$ is strongly proximinal in $x + Y$ whenever $(x + Y) \cap B_X[0,\beta] \ne \emptyset$. Let x be any point in X. If $x \in B_Y[0,1]$ (this is equivalent to $0 \in B_Y[x,1]$), then x is the best approximation of x from $B_Y[0,1]$. We may assume that $x \in X \setminus B_Y[0,1]$. Let

$$\beta = d_X(x, B_Y[0,1]) = d_X(0, B_Y[x,1]) > 0;$$

$$\alpha = d_X(x, Y).$$

We need to show that for any $\epsilon > 0$, there is $\delta > 0$ such that for any $z \in B_Y[x,1]$ with $\|z\| \le \beta + \delta$, there is a best approximation u of 0 from $B_Y[x,1]$ such that $\|u - z\| < \epsilon$.

By Lemma 2.2,

$$1 \ge \inf\{\|y\| : y \in Y,\ x + y \in B_X[0,\beta]\} = d_Y(x, B_X[0,\beta]).$$

Case 1. $\alpha = \beta$ and $1 = \inf\{\|y\| : y \in Y,\ x + y \in B_X[0,\beta]\}$. Let ϵ be any positive real number. Since $(x + Y) \cap B_X[0,\beta]$ is strongly proximinal in $x + Y$, there is $\delta_1 > 0$ such that for any $z' \in (x + Y) \cap B_X[0,\beta]$ with $d(x, z') < 1 + \delta_1$, there is $u \in (x + Y) \cap B_X[0,\beta]$ such that

$$\|u - z'\| < \frac{\epsilon}{2} \quad \text{and} \quad \|u - x\| = 1.$$

Since Y is strongly proximinal in X, there is $\delta_2 > 0$ such that if $z \in x + Y$ and $\|z\| \le \beta + \delta_2$, then there is $z' \in x + Y$ such that

$$\|z'\| = \beta \quad \text{and} \quad \|z - z'\| \le \min\left\{\delta_1, \frac{\epsilon}{2}\right\}.$$

Therefore, if $z \in B_Y[x,1]$ with $\|z\| \le \beta + \delta_2$, then there is $z' \in (x + Y) \cap B_X[0,\beta]$ such that

$$\|z'\| = \beta;$$

$$\|z - z'\| \le \min\left\{\delta_1, \frac{\epsilon}{2}\right\};$$

$$\|z' - x\| \le \|z' - z\| + \|z - x\| < 1 + \delta_1.$$

Then there is $u \in (x+Y) \cap B_X[0,\beta]$ such that

$$\|u - z'\| < \frac{\epsilon}{2};$$
$$\|u - x\| = 1;$$
$$\|u - z\| \le \|u - z'\| + \|z' - z\| < \epsilon.$$

So u is a best approximation of 0 from $B_Y[x,1]$ such that $\|u - z\| < \epsilon$.

Case 2. $\alpha = \beta$ and $1 > d_Y(x, B_X[0,\beta])$. Then there is $w \in x+Y$ such that $\|w\| = \beta$ and $\|w - x\| < 1$. Since Y is strongly proximinal in X, for any $n \in \mathbb{N}$, there is $1 > \delta > 0$ such that if $z \in B_Y[x,1]$ and $\|z\| \le \beta + \delta$, then there is $z' \in x+Y$ such that

$$\|z'\| = \beta \quad \text{and} \quad \|z - z'\| \le \frac{1 - \|w - x\|}{4n}.$$

Then

$$\beta = \alpha \le \|(1-\lambda)z' + \lambda z\| \le \beta \;\text{ for any } 0 < \lambda < 1;$$
$$\|z' - x\| \le \|z' - z\| + \|z - x\| \le 1 + \frac{1 - \|w - x\|}{4n}.$$

Let $u = (1 - \frac{1}{n})z' + \frac{w}{n}$. Then

$$\begin{aligned} \|u - x\| &= \left\| \left(1 - \frac{1}{n}\right)(z' - x) + \frac{w - x}{n} \right\| \\ &\le \frac{n-1}{n} + \frac{1 - \|w - x\|}{4n} + \frac{\|w - x\|}{n} < 1, \end{aligned}$$

and

$$\begin{aligned} \|z' - u\| &= \left\| z' - \left(\left(1 - \frac{1}{n}\right) z' + \frac{w}{n} \right) \right\| \\ &= \frac{1}{n}\|z' - w\| \le \frac{\|z'\| + \|w\|}{n} \le \frac{2\beta}{n}. \end{aligned}$$

We have proved that u is a best approximation of 0 from $B_Y[x,1]$ such that

$$\|z - u\| \le \|z - z'\| + \|z' - u\| \le \frac{1}{4n} + \frac{2\beta}{n}.$$

Case 3. $\alpha < \beta$. Then there is $w \in x+Y$ such that $\|w\| = \alpha < \beta$. Since w is an interior point of $B_X[0,\beta]$, we have

$$\|w - x\| > 1.$$

Let ϵ be a positive real. Since $(x+Y)\cap B_X[0,\beta]$ is strongly proximinal in $x+Y$. There is $\delta_1>0$ such that for any $z\in B_Y[x,1]$ and $d_Y(z,B_X[0,\beta])<1+\delta_1$, there is $u\in(x+Y)\cap B_X[0,\beta]$ with $\|u-x\|\le 1$ and $\|u-z\|<\frac{\epsilon}{2}$. Select $1>\lambda>0$ and $\delta_2>0$ such that

$$\lambda(\|x\|+\|w\|+1)<\min(\epsilon/2,\delta_1);$$
$$(1-\lambda)(\beta+\delta_2)+\lambda\alpha\le\beta.$$

Give any element z in $B_Y[x,1]$ such that $\|z\|<\beta+\delta_2$. Let

$$z'=(1-\lambda)z+\lambda w\in x+Y.$$

Then

$$\begin{aligned}\|z'\| &\le (1-\lambda)\|z\|+\lambda\alpha\le\beta;\\ \|z'-z\| &\le \lambda\|z-w\|\le\lambda(\|z\|+\|w\|)\\ &\le \lambda(\|x\|+1+\|w\|)\le\min(\epsilon/2,\delta_1);\\ \|z'-x\| &\le \|x-z\|+\|z-z'\|\le 1+\delta_1.\end{aligned}$$

Then there is $u\in(x+Y)\cap B_X[0,\beta]$ with $\|u-x\|\le 1$ and $\|u-z'\|<\frac{\epsilon}{2}$. So u is a best approximation of 0 from $x+B_Y[0,1]$ such that

$$\|u-z\|\le\|u-z'\|+\|z'-z\|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

We have proved that $B_Y[x,1]$ is proximinal in X. The proof is complete. □

3. Some applications and Remarks

In this section, we consider the Banach spaces over $\mathbb{R}$. Let Y be a real Banach space and C a closed convex subset C of $B_Y[0,1]$ that contains 0. Let $X=\mathbb{R}\oplus Y$ with the norm

$$\|(r,y)\|_X=|r|+d_Y(y,rC).$$

This renorming was considered in [6]. We will give a necessary and sufficient condition for Y to be (strongly) ball proximinal in X. First, we prove the following lemma.

Lemma 3.1. *Let Y be a Banach space and C a closed convex subset of the unit ball of Y that contains 0. Let $X=\mathbb{R}\oplus Y$ with the norm*

$$\|(r,y)\|_X=|r|+d_Y(y,rC).$$

Then Y is strongly proximinal in X.

Proof. Let (r,y) be an element in X. Then $(0,y)$ is a best approximation of (r,Y) from Y. For any $\epsilon>0$ let y' be an element in Y such that

$$|r|+\epsilon>\|(r,y)-(0,y')\|_X=|r|+d_Y(y-y',rC).$$

So $d_Y(y-y', rC) < \epsilon$ and there is $c \in C$ such that $\|y-y'-rc\|_Y < \epsilon$. Then

$$\|(r,y)-(0,y-rc)\|_X = |r| + d_Y(rc, rC) = |r|,$$

and

$$\|(0,y-rc)-(0,y')\|_X = \|y-rc-y'\|_Y < \epsilon.$$

The proof is complete. □

It is easy to see that for any $|r| \leq 1$,

$$((r,0)+Y) \cap B_X[0,1] = \{(r, rc+y) : c \in C \text{ and } y \in B_Y[0, 1-|r|]\}.$$

By Theorem 2.3 and Theorem 2.4, we have the following theorem.

Theorem 3.2. *Let Y be a Banach space and C a closed convex subset of the unit ball of Y that contains 0. Let $X = \mathbb{R} \oplus Y$ with the norm*

$$\|(r,y)\|_X = |r| + d_Y(y, rC).$$

Then Y is (strongly) ball proximinal in X if and only if $rB_Y[0,1]+C$ is (strongly) proximinal in Y for all $r \in \mathbb{R}$.

We need the following theorem that is proved by Barbu and Precupanu [2, Theorem 3.99].

Theorem 3.3. *For any nonempty subset C of a linear normed space Y, C is proximinal in Y if and only if $C + B_Y[0,r]$ is closed for all $r \geq 0$.*

It is known that for any $r, r' \geq 0$,

$$C + B_Y[0,r] + B_Y[0,r'] = C + B[0, r+r'].$$

We have the following theorem.

Theorem 3.4. *Let Y be a Banach space and C a closed convex subset of the unit ball of Y that contains 0. Let $X = \mathbb{R} \oplus Y$ with the norm*

$$\|(r,y)\|_X = |r| + d_Y(y, rC).$$

Then Y is ball proximinal in X if and only if C is proximinal in Y.

We do not know whether Y is strongly ball proximinal in X if C is strongly proximinal in Y.

Example 3.5 (Saidi). Let Y be a non-reflexive Banach space. There are subspaces $W \subset Z \subset Y$ such that Z is a hyperplane of Y and W is a non-proximinal hyperplane of Z (this is possible since Z is not reflexive). Let z be an element of Z such that $\|z\| \leq \frac{5}{8}$ and $d(z,W) = \frac{1}{2}$. Let

$$C = \overline{\mathrm{co}}\left((B_Z[z, 1/4] \cap (4z/5 + W)) \cup \{0\}\right).$$

Then C is not a proximinal subset of Z. Let $X = \mathbb{R} \oplus Z$ with the norm

$$\|(r, z')\| = |r| + d_Z(z', rC).$$

Then

(1) X is isomorphic to Y;
(2) Z is strongly proximinal in X;
(3) Z is not ball proximinal in X.

It is known that for Banach spaces $Z \subset Y \subset X$, if Y is proximinal in X, then Y/Z is proximinal in X/Z. It is natural to ask the following question:

Question 3.6. Let $Z \subset Y \subset X$ be three Banach spaces. Suppose that Y is ball proximinal in X. Is Y/Z ball proximinal in X/Z?

The following theorem shows the answer of above question is affirmative if Z is reflexive.

Theorem 3.7. *Let X be a Banach space and let Z be a reflexive subspace of X. If Y is a ball proximinal subspace of X such that $Z \subseteq Y$, then Y/Z is ball proximinal in X/Z.*

Proof. First, we note that Z is proximinal in X (so in Y). For any $y \in Y$, there is $z \in Z$ such that $d_X(y, Z) = \|y - z\|$. Let x be any vector in X. Then

$$d_{X/Z}\left(x, B_{Y/Z}[0,1]\right) = d_X(x, B_Y[0,1] + Z) = d_X(x, (B_Y[0,1] + Z) \cap 2\|x\|B_X).$$

Define a function $H : Z \to \mathbb{R}$ by

$$H(z) = d_X(x, z + B_Y[0,1]).$$

Since Z is reflexive and H is convex, H attains its minimum at some point $z_1 \in Z$. Let y' be a best approximate of x from $z_1 + B_Y$. Then

$$\begin{aligned}\|y' - x\| &= d_X(x, z_1 + B_Y[0,1]) = \inf_{z \in Z} d_X(x, z + B_Y[0,1]) \\ &= d_X(x, B_Y[0,1] + Z) = dX/Z(x, B_{Y/Z}[0,1].\end{aligned}$$

This implies that $y' + Z$ is a best of $x + Z$ from Y/Z. □

Let $Z \subseteq Y \subseteq X$ be three Banach spaces. In [1, Question 4.3], Bandyopadhyay, Lin, and Rao asked whether Y is ball proximinal in X if Z is ball proximinal in X and Y/Z is ball proximinal in X/Z. The following result gives a negative answer to the question.

Theorem 3.8. *There exist three Banach spaces $Z \subseteq Y \subseteq X$ so that Z is ball proximinal in X and Y/Z is ball proximinal in X/Z, but Y is not ball proximinal in X.*

Proof. Let $Y = \ell_1$ and let (e_i) be the natural basis of Y. Let C_1, C_2 be two convex subsets of Y defined by

$$C_1 = \overline{\mathrm{co}}\left(\{0\} \cup \left\{\frac{e_1}{2} + \left(\frac{1}{4} + \frac{1}{4k}\right) e_k : k \geq 2\right\}\right);$$
$$C_2 = \overline{\mathrm{co}}\left(\{0\} \cup \left\{\left(\frac{1}{4} + \frac{1}{4(k+1)}\right) e_{k+1} : k \geq 1\right\}\right).$$

It is easy to check that $d(e_1, C_1) = \frac{3}{4}$ and for any $c \in C_1$,

$$\|e_1 - c\| > \frac{3}{4}.$$

So C_1 is not proximinal in Y. We claim that C_2 is proximinal in Y. Let $y = \sum_{k=1}^{\infty} a_k e_k$ be any fixed element in Y and let

$$A = \{k : a_{k+1} > 0\},$$
$$m = \inf\left\{n : \sum_{k \in A, k \leq n} a_{k+1} / \left(\frac{1}{4} + \frac{1}{4(k+1)}\right) \geq 1\right\}.$$

(1) If $m = \infty$, then $c = \sum_{k \in A} a_{k+1} e_{k+1}$ is a best approximation of y from C_2.
(2) $m < \infty$. Let

$$b_{m+1} = 1 - \sum_{k \in A, k < m} a_{k+1} / \left(\frac{1}{4} + \frac{1}{4(k+1)}\right),$$
$$c = \sum_{k \in A, k < m} a_{k+1} e_{k+1} + b_{m+1}\left(\frac{1}{4} + \frac{1}{4(k+1)}\right) e_{m+1}.$$

Then c is a best approximation of y from C_2.

Let X_1 (respectively, X_2) be the space $\mathbb{R} \oplus Y$ with the norm

$$\|(r, y)\|_{X_1} = |r| + d_Y(y, rC_1);$$
$$(\text{respectively, } \|r, y\|_{X_2} = |r| + d_Y(y, C_2)).$$

By Theorem 3.4, Y (as a subspace of X_1) is not ball proximinal in X_1 and Y (as a subspace of X_2) is ball proximinal in X_2. Let $Z = \mathrm{span}\{e_1\}$. Then $Z \subseteq Y \subseteq X_1$ and Z is ball proximinal in X_1. It is easy to see that $X_1/Z = X_2/Z$. Since Y is ball proximinal in X_2, by Theorem 3.7, we have Y/Z is ball proximinal in X_2/Z and hence Y/Z is ball proximinal in X_1/Z. However Y is not ball proximinal in X_1. $\square$

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