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# Some aspects of the representation of $c$-monotone operators by $C$-convex functions

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Several takes on generalizing the theory of representation of monotone operators by convex functions to the full generality of $c$-convexity have been proposed in recent years. In particular, given a monotone operator, a new family of convex antiderivatives is now associated with it, both in classical convex analysis as well as in the generality of $c$-convexity. In the present paper we take the generalization of the theory to $c$-convexity a few steps farther. In particular, we study the $C$-convex separable representation in detail, construct the sequence of Fitzpatrick functions of higher orders and present its basic properties in the generality of $c$-convexity, and, finally, we present a new example that demonstrates why the associated family of antiderivatives is a more natural environment for the Fitzpatrick function in an even more dramatic manner than in the classical case: the Fitzpatrick function turns out to be the maximal(!) member of the Fitzpatrick family, although it is still a minimal convex antiderivative.

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## 1. Introduction and Preliminaries

The theory of $c$-convexity is a main branch of *abstract convex analysis* (or *generalized convexity*) and has attracted more and more attention in recent years. Now this topic is studied continually, both theoretically and from the point of view of applications. A unifying and very detailed treatment of abstract convex analysis and, in particular, of $c$-convexity can be found in [19]. We now recall the basic definitions and notations of $c$-convexity theory; unless otherwise specified, throughout the paper $X$ and $Y$ are arbitrary sets and $c : X \times Y \to \mathbb{R}$ is an arbitrary function. We say that a function $f : X \to (-\infty, +\infty]$ is proper if $\mathrm{dom}(f) := \{x \in X \mid f(x) < \infty\}$ is not empty.

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**Definition 1.1 (c-transform).** Let $X$ and $Y$ be nonempty sets and let $c : X \times Y \to \mathbb{R}$ be a function. Given a function $f : X \to [-\infty, \infty]$, its $c$-transform $f^c : Y \to [-\infty, \infty]$ is defined by

$$f^c(y) := \sup_{x \in X} c(x, y) - f(x), \quad y \in Y. \tag{1}$$

Similarly, the $c$-transform of a function $g : Y \to [-\infty, \infty]$ is the function $g^c : X \to [-\infty, \infty]$ defined by

$$g^c(x) := \sup_{y \in Y} c(x, y) - g(y), \quad x \in X. \tag{2}$$

**Definition 1.2 (c-convexity).** A proper function $f : X \to (-\infty, +\infty]$ is said to be $c$-convex if there exists a (necessarily proper) function $g : Y \to (-\infty, +\infty]$ such that $f = g^c$. The set of all $c$-convex functions defined on $X$ is denoted by $\Gamma_c(X)$. The function $(f^c)^c$ is the $c$-convexification of $f$ and is denoted by $f^{cc}$.

The $c$-transform of a function is also known as its *c-conjugate* function. This generalization of Fenchel's conjugate function from classical convex analysis was introduced and studied by Moreau in [12]. In more detailed discussions of $c$-convexity one distinguishes between the function $c$ and the function $c' : Y \times X$ defined by $c'(y, x) = c(x, y)$ for $x \in X$ and $y \in Y$. Furthermore, sometimes the function $c$ is allowed to take the values $\pm\infty$; however, in our discussion we focus our attention on the settings in the above definitions. When referring to the theoretical study of $c$-convexity, the function $c$ is called a *coupling function* between $X$ and $Y$, while in the particular application to the study of optimal transport it is the *cost function*. As is often standard in convex analysis, the indicator function of a subset $S$ of $X$ is the function $\iota_S : X \to (-\infty, +\infty]$ defined by

$$\iota_S(x) := \begin{cases} 0 & x \in S \\ \infty & x \notin S. \end{cases}$$

For example, for every $y \in Y$, the function $c(\cdot, y) : X \to \mathbb{R}$ is $c$-convex since $c(\cdot, y) = \iota^c_{\{y\}}$. (In fact, the functions $c(\cdot, y)$, $y \in Y$, are the ones playing the role linear functionals play in classical convex analysis.)

Clearly, for any function $f : X \to (-\infty, +\infty]$,

$$c(x, y) \leq f(x) + f^c(y) \text{ for all } x \in X \text{ and } y \in Y, \tag{3}$$

which is a generalization of the Young-Fenchel inequality from classical convex analysis. The case of equality is captured in the following definition of the $c$-subdifferential. We denote the graph of a multivalued mapping $M : X \rightrightarrows Y$ by $G(M) := \{(x, y)|\ y \in M(x)\}$. The mapping $M$ is called proper if $\mathrm{dom}(M) := \{x \in X|\ M(x) \neq \emptyset\}$ is not empty. The image of the mapping $M$ is the subset of $Y$ given by $\mathrm{Im}(M) := \cup_{x \in X} M(x)$.

**Definition 1.3 (c-subdifferential and c-antiderivative).** Let $f : X \to (-\infty, +\infty]$ be a proper function. The $c$-subdifferential of $f$ is the mapping $\partial_c f : X \rightrightarrows Y$ defined by

$$\partial_c f(x) := \{y \in Y \mid f(x) + c(x', y) \leq f(x') + c(x, y) \ \forall x' \in X\} \tag{4}$$

$$= \{y \in Y \mid f(x) + f^c(y) = c(x, y)\}. \tag{5}$$

When $\partial_c f(x) \neq \emptyset$, we say that $f$ is $c$-subdifferentiable at $x$. When $M : X \rightrightarrows Y$ and $G(M) \subset G(\partial_c f)$, we say that $f$ is a $c$-antiderivative of $M$.

When $X$ is a topological vector space, $Y = X^*$ is its dual and $c(x, y^*) = \langle y^*, x\rangle = y^*(x)$ for $x \in X$ and $y^* \in Y$, then $f^* = f^c$ and $\partial f = \partial_c f$. We will refer to these particular settings as the classical case. Many features from classical convex analysis hold in the generality of $c$-convexity. For example, upper envelopes of $c$-convex functions are $c$-convex functions (when proper), the function $f^{cc} := (f^c)^c$ is the $c$-convexification of $f$, that is, it is the greatest $c$-convex function majorized by $f$, in particular, $f^{cc} = f$ if and only if $f$ is $c$-convex. However, we cannot view $c$-convexity as a generalization of classical convex analysis. One reason is that classical convex analysis has an extension beyond the case of reflexive spaces, which then departs from the theory of $c$-convexity. Another, more fundamental, reason is that, roughly speaking, in the classical case $c$-convexity tells only one side of the story: classically, a function is defined to be convex via line segments inside its epigraph. This can be viewed as "inner convexity". On the other hand, a function is $c$-convex if its graph is the upper envelope of $c$-affine functions. This can be viewed as "outer convexity". In the classical case these two sides of the story are unified by the Hahn-Banach separation theorem.

Still discussing the classical case, we recall that the mapping $T : X \rightrightarrows X^*$ is monotone if

$$(x, x^*), (y, y^*) \in G(T) \quad \Rightarrow \quad \langle y^* - x^*, y - x\rangle \geq 0.$$

The mapping $T$ is said to be maximal monotone if it has no proper monotone extension (in $X \times X^*$). Monotone operators capture subdifferentials, positive linear operators, the $p$-Laplacian and many other nonlinear operators. In the past half-century, many range and calculus results for maximal monotone operators have been established, but the validity of several desired properties still remains an open problem. Much success has been achieved in getting such results when the operators at hand were subdifferentials. These were mainly considered a particular nice case of maximal monotone operators. However, many indications pointed to closer relations between classical convexity and monotonicity. In [11] Fitzpatrick was the first to associate a convex function defined on the product space $X \times X^*$ with a monotone operator. Given a monotone mapping $T : X \rightrightarrows X^*$, this function is now known as the Fitzpatrick function $F_T : X \times X^* \to (-\infty, +\infty]$ and is defined by

$$F_T(x, x^*) := \sup_{(s,s^*) \in G(T)} \langle s^*, x\rangle + \langle x^*, s\rangle - \langle s^*, s\rangle.$$

The Fitzpatrick family of functions is defined by

$$\mathcal{F}_T := \left\{ f : X \times X^* \to (-\infty, +\infty] \;\middle|\; \begin{array}{c} f \text{ is lower semicontinuous and convex,} \\ \langle \cdot, \cdot \rangle \le f \quad \text{and} \quad f|_{G(T)} = \langle \cdot, \cdot \rangle|_{G(T)} \end{array} \right\}.$$

Fitzpatrick proved that when $T$ is maximal monotone, the function $F_T$ is the lower envelope of $\mathcal{F}_T$ and every function $h \in \mathcal{F}_T$ satisfies the equality $h = \langle \cdot, \cdot \rangle$ exactly at the points of $G(T)$. Therefore members of $\mathcal{F}_T$ are called representative functions of $T$. In the past decade results in monotone operator theory were sharpened and given convex analytic proofs using the Fitzpatrick function by many different authors and it has become a common belief that a path in the direction of embedding monotone operator theory in convex analysis has been paved. Currently, an intense study is being conducted in this direction and hundreds of papers regarding this evolving theory of representation of monotone operators by convex functions have been published. A tour of five decades of monotone operator theory can be found in [6], a most detailed current account of many of the aspects of the theory is presented in [18] and a recent detailed presentation of the theory in Hilbert spaces is provided in [5].

The aim of our paper is twofold. On the one hand, we generalize some aspects of the theory of representation of monotone operators by convex functions to the general settings of $c$-convexity. On the other hand, we point out critical differences between the classical settings and the general $c$-convexity settings. Furthermore, in [2] a family of classical antiderivatives was associated with any monotone mapping and it was implied that this family of functions is a more natural environment for the Fitzpatrick function than the original Fitzpatrick family. This program was then carried out in the settings of $c$-convexity in [3]. In the present paper we shed some new light on this matter by pointing out Example 6.3, where the underlying mapping is maximal $c$-monotone, a $c$-subdifferential and the Fitzpatrick function is maximal(!) in the Fitzpatrick family, although it is still a minimal $C$-convex $C$-antiderivative. (Given a coupling function $c : X \times Y \to (-\infty, +\infty]$, we define a natural coupling function $C : (X \times Y) \times (Y \times X) \to (-\infty, +\infty]$.) Thus, this example strongly supports the more natural features of the associated family of antiderivatives which are lost in the Fitzpatrick family.

The rest of this paper is organized as follows: In Section 2 we recall the family of $c$-convex $c$-antiderivatives associated with a mapping, given some initial values of a $c$-antiderivative. We also recall in Section 2 some basic and natural properties of such families, as well as the notions of $c$-cyclic monotonicity, $c$-monotonicity, maximal $c$-monotonicity and Rockafellar's antiderivative. In Section 3 we start the discussion of the representation of $c$-monotone mappings by $C$-convex functions. In Section 4 we discuss the separable $C$-convex representation, in Section 5 we study the sequence of higher order Fitzpatrick functions and in Section 6 we present examples and some concluding remarks.

## 2. The Family $\mathcal{A}_{[c,f|_s,M]}$ of $c$-Convex $c$-Antiderivatives and Its Envelopes

In [3] families of $c$-convex $c$-antiderivatives were introduced and studied. They are defined as families of solutions to the following problem:

**Definition 2.1.** Given a mapping $M : X \rightrightarrows Y$, a $c$-antiderivative $f$ of $M$ and a subset $S$ of $\mathrm{dom}(M)$, we denote the set of all $c$-convex functions $h : X \to (-\infty, +\infty]$ which satisfy

$$G(M) \subset G(\partial_c h) \quad \text{and} \quad h|_S = f|_S \tag{6}$$

by $\mathcal{A}_{[c,f|_S,M]}$.

Given a mapping $M : X \rightrightarrows Y$, recall that its inverse mapping $M^{-1} : Y \rightrightarrows X$ is defined by $M^{-1}(y) = \{x \in X |\ y \in M(x)\}$, $y \in Y$. In the above setting, since $M(S)$ is a subset of $\mathrm{dom}(M^{-1})$, it is also possible to consider the $c$-dual problem: $f^c$ is a $c$-antiderivative of $M^{-1}$. We therefore denote the set of $c$-convex solutions $h : Y \to (-\infty, +\infty]$ to the problem

$$G(M^{-1}) \subset G(\partial_c h) \quad \text{and} \quad h|_{M(S)} = f^c|_{M(S)} \tag{7}$$

by $\mathcal{A}_{[c,f^c|_{M(S)},M^{-1}]}$.

Besides the existence of solutions, the two main $c$-convexity theoretical results from [3], Theorem 2.2 and Theorem 2.6 below, imply rather natural structure and duality relations between the families and, in particular, the existence of optimal solutions and duality relations between them. In all applications thus far, the fact that we do not assume that $f$ is $c$-convex, just a $c$-antiderivative, is crucial. This means that we are given a function $f$ with good $c$-convexity properties only on $\mathrm{dom}(M)$, which can be arbitrary, and then $f|_S$ is extensible to a "good" solution on all of $X$. The nonemptiness, duality relations, and existence and duality relations for the envelopes are concentrated in the following result, the first one of the two:

**Theorem 2.2.** *Suppose that $f : X \to (-\infty, +\infty]$ is a $c$-antiderivative of the mapping $M : X \rightrightarrows Y$. Suppose further that $\emptyset \neq S \subset \mathrm{dom}(M)$. Then $\mathcal{A}_{[c,f|_S,M]}$ is nonempty and contains both its upper envelope, that is, the function $\gamma_{[c,f|_S,M]} : X \to (-\infty, +\infty]$ defined by*

$$\gamma_{[c,f|_S,M]}(x) := \sup\{h(x) \mid h \in \mathcal{A}_{[c,f|_S,M]}\},$$

*as well as its lower envelope, that is, the function $\alpha_{[c,f|_S,M]} : X \to (-\infty, +\infty]$ defined by*

$$\alpha_{[c,f|_S,M]}(x) := \inf\{h(x) \mid h \in \mathcal{A}_{[c,f|_S,M]}\}.$$

*In fact, if $h : X \to (-\infty, +\infty]$ is any function such that*

$$G(M) \subset G(\partial_c h) \quad \textit{and} \quad h|_S = f|_S, \tag{8}$$

*then* $\alpha_{[c,f|_S,M]} \le h$ *and* $h^c \in \mathcal{A}_{[c,f^c|_{M(S)},M^{-1}]}$. *If* $h$ *is* $c$*-convex, then*

$$h \in \mathcal{A}_{[c,f|_S,M]} \quad \Leftrightarrow \quad h^c \in \mathcal{A}_{[c,f^c|_{M(S)},M^{-1}]}. \tag{9}$$

*Furthermore,*

$$\alpha^c_{[c,f|_S,M]} = \gamma_{[c,f^c|_{M(S)},M^{-1}]} \quad \text{and} \quad \gamma^c_{[c,f|_S,M]} = \alpha_{[c,f^c|_{M(S)},M^{-1}]}. \tag{10}$$

*In the case where* $S = \mathrm{dom}(M)$*, we have*

$$\gamma_{[c,f|_{\mathrm{dom}(M)},M]} = (f + \iota_{\mathrm{dom}(M)})^{cc} \tag{11}$$

*and*

$$\alpha_{[c,f|_{\mathrm{dom}(M)},M]}(x) = (f^c + \iota_{\mathrm{Im}(M)})^c(x) \tag{12}$$

$$= \sup_{(s,t)\in G(M)} [f(s) + c(x,t) - c(s,t)], \quad x \in X. \tag{13}$$

*In this case, if* $h : X \to (-\infty, +\infty]$ *is* $c$*-convex, then*

$$h \in \mathcal{A}_{[c,f|_{\mathrm{dom}(M)},M]} \quad \Leftrightarrow \quad \alpha_{[c,f|_{\mathrm{dom}(M)},M]} \le h \le \gamma_{[c,f|_{\mathrm{dom}(M)},M]}. \tag{14}$$

We see that besides existence and duality relations between optimal $c$-convex $c$-antiderivatives of the two dual families, an explicit construction of these optimal functions is provided in (11) and (13) in the case where the set where we keep the values of $f$ fixed is $S = \mathrm{dom}(M)$, which is the case of our interest in the present paper. An explicit construction of the optimal $c$-convex $c$-antiderivatives under the more general assumptions of Theorem 2.2 was also presented in [3]. It is the second of the two main results mentioned above and is given in Theorem 2.6 below. To this end, we will need to recall the concept of $c$-cyclic monotonicity:

**Definition 2.3 (c-cyclic monotonicity, c-monotonicity and maximality).** A mapping $M : X \rightrightarrows Y$ is said to be cyclically monotone of order $n$ with respect to $c$, $n$-$c$-monotone for short, when given any set of $n$ ordered pairs $\{(x_i, y_i)\}_{i=1}^n \subset G(M)$, if we set $x_{n+1} = x_1$, then

$$0 \le \sum_{i=1}^{n} [c(x_i, y_i) - c(x_{i+1}, y_i)]. \tag{15}$$

In this case we say that $G(M)$ is an $n$-$c$-monotone set. A mapping $M$ is said to be cyclically monotone with respect to $c$, $c$-cyclically monotone for short, if it is $n$-$c$-monotone for all $n \in \mathbb{N}$. A 2-$c$-monotone mapping is simply called a $c$-monotone mapping. Explicitly, the mapping $M$ is $c$-monotone if for all $(x_1, y_1), (x_2, y_2) \in G(M)$, we have

$$0 \le c(x_1, y_1) - c(x_1, y_2) - c(x_2, y_1) + c(x_2, y_2). \tag{16}$$

The mapping $M$ is said to be maximal $n$-$c$-cyclically monotone if $G(M)$ has no proper $n$-$c$-cyclically monotone extension in $X \times Y$.

The minimal $c$-convex $c$-antiderivative $\alpha_{[c,f|_S,M]}$ is constructed in Theorem 2.6 below by also making use of the following well-known construction of an antiderivative due to Rockafellar. This fact from classical convex analysis also holds in the generality of $c$-convexity:

**Definition 2.4 (Rockafellar's antiderivative).** With the function $c$, the mapping $M : X \rightrightarrows Y$ and the point $s \in \mathrm{dom}(M)$, we associate Rockafellar's function $R_{[c,M,s]} : X \to (-\infty, +\infty]$, defined by

$$R_{[c,M,s]}(x) := \sup_{\substack{n\in\mathbb{N},\\ x_1=s,\ x_{n+1}=x,\\ \{(x_i,y_i)\}_{i=1}^n\subset G(M)}} \sum_{i=1}^{n}[c(x_{i+1}, y_i) - c(x_i, y_i)]. \tag{17}$$

**Theorem 2.5.** *A proper mapping $M : X \rightrightarrows Y$ is $c$-cyclically monotone if and only if it has a proper $c$-antiderivative. In this case, in particular, for any $s \in \mathrm{dom}(M)$, Rockafellar's function $R_{[c,M,s]}$ is a proper $c$-convex $c$-antiderivative of $M$ which satisfies $R_{[c,M,s]}(s) = 0$. In fact, $R_{[c,M,s]}$ is proper if and only if $M$ is proper and $c$-cyclically monotone.*

Employing Rockafellar's antiderivative, we reestablish the nonemptiness of the family $\mathcal{A}_{[c,f|_S,M]}$ by explicitly constructing the function $\alpha_{[c,f|_S,M]}$. Then duality relations from Theorem 2.2 also yield an explicit construction of $\gamma_{[c,f|_S,M]}$.

**Theorem 2.6.** *Suppose that $f : X \to (-\infty, +\infty]$ is a $c$-antiderivative of the mapping $M : X \rightrightarrows Y$ and suppose that $\emptyset \neq S \subset \mathrm{dom}(M)$. Then the minimal $c$-antiderivative of $M$ that equals $f$ at the points of $S$, the function $\alpha_{[c,f|_S,M]} \in \mathcal{A}_{[c,f|_S,M]}$, is given by*

$$\alpha_{[c,f|_S,M]}(x) = \sup_{s\in S}\, [f(s) + R_{[c,M,s]}(x)] \quad \forall x \in X. \tag{18}$$

Thus far the family $\mathcal{A}$ and its envelopes $\alpha$ and $\gamma$ have been employed in several applications in diverse fields of analysis: [2] contains an application to the classical representation of monotone operators by convex functions, [3] contains an application to the representation of $c$-monotone mappings by $C$-convex functions (which we recall in Section 3 below) as well as an application to the construction of optimal solutions to a constrained Lipschitz extension problem and, finally, [4] contains an application to the construction of optimal pricing strategies for the dual Kantorovich problem in the theory of optimal transport.

## 3. Optimal Antiderivatives in the Representation of $c$-Monotone Mappings by $C$-Convex Functions

In [14] and [9], as two examples, the discussion regarding the representation of monotone mappings by convex functions is extended to the settings of $c$-convexity and $c$-monotonicity. In particular, the definition of the Fitzpatrick function and a part of its basic properties are extended to hold for $c$-monotone mappings. This discussion requires a type of the following convention: given $c : X \times Y \to \mathbb{R}$, we let

$$C : (X \times Y) \times (Y \times X) \to \mathbb{R}$$

be the function defined by

$$C((x, y), (t, s)) := c(x, t) + c(s, y), \quad ((x, y), (t, s)) \in (X \times Y) \times (Y \times X). \tag{19}$$

**Definition 3.1 (Fitzpatrick function and family).** With a mapping $T : X \rightrightarrows Y$ (and the function $c$) we associate the Fitzpatrick function $F_{[c,T]} : X \times Y \to (-\infty, +\infty]$, defined by

$$F_{[c,T]}(x, y) := \sup_{(s,t)\in G(T)} c(x, t) + c(s, y) - c(s, t) \tag{20}$$

$$= \sup_{(s,t)\in G(T)} C((x, y), (t, s)) - c(s, t), \quad (x, y) \in X \times Y \tag{21}$$

as well as the Fitzpatrick family of functions $\mathcal{F}_{[c,T]}$, defined by

$$\mathcal{F}_{[c,T]} := \left\{ f \in \Gamma_C(X \times Y) \mid c \leq f,\ f|_{G(T)} = c|_{G(T)} \right\}. \tag{22}$$

When $T$ is proper and $c$-monotone, $F_{[c,T]}(x, y) \leq c(x, y)$ for each $(x, y)$ which is in $c$-monotone relations with the points of $G(T)$; in particular, $F_{[c,T]}|_{G(T)} = c|_{G(T)}$. In this case $F_{[c,T]}$ is proper and hence also $C$-convex as a proper upper envelope of $C$-convex functions. When $T$ is maximal $c$-monotone, $c \leq F_{[c,T]}$ with equality only at the points of $G(T)$. Summing up, we get the following result.

**Theorem 3.2.** *Let $T : X \rightrightarrows Y$ be a mapping. Then the following assertions are equivalent:*

1) *$T$ is $c$-monotone;*
2) *$F_{[c,T]}|_G \leq c|_G$ for any $c$-monotone extension $G$ of $G(T)$;*
3) *$F_{[c,T]}|_{G(T)} = c|_{G(T)}$.*

*If, in addition, $T$ is maximal $c$-monotone, then*

$$F_{[c,T]} \in \mathcal{F}_{[c,T]}. \tag{23}$$

*In this case $T$ is characterized as follows:*

$$G(T) = \left\{ (x, y) \in X \times Y \mid F_{[c,T]}(x, y) = c(x, y) \right\}. \tag{24}$$

Another way to see that $F_{[c,T]}$ is $C$-convex is to observe that

$$F_{[c,T]}(x, y) = (c + \iota_{G(T)})^C(y, x) \quad \forall (x, y) \in X \times Y. \tag{25}$$

It follows, after permuting coordinates, that $F^C_{[c,T]}$ is the $C$-convexification of $c + \iota_{G(T)}$. When $T$ is $c$-monotone, then for any $(x, y) \in X \times Y$, we have

$$F^C_{[c,T]}(y, x) = \sup_{(s,t)\in X\times Y} C((x, y), (t, s)) - F_{[c,T]}(s, t) \tag{26}$$

$$\geq \sup_{(s,t)\in G(T)} C((x, y), (t, s)) - F_{[c,T]}(s, t) \tag{27}$$

$$= \sup_{(s,t)\in G(T)} C((x, y), (t, s)) - c(s, t) = F_{[c,T]}(x, y). \tag{28}$$

It now follows that when $T$ is maximal $c$-monotone, we have

$$c(x,y) \le F_{[c,T]}(x,y) \le F^{C}_{[c,T]}(y,x) \le (c+\iota_{G(T)})(x,y) \quad \forall (x,y) \in X \times Y, \tag{29}$$

and each one of the inequalities becomes an equality for $(x,y) \in G(T)$.

In the classical case, it is by now a well-known fact (first discovered by Fitzpatrick [11]) that when $T$ is maximal monotone, $F_{[\langle\cdot,\cdot\rangle,T]}$ is the minimal function in $\mathcal{F}_{[\langle\cdot,\cdot\rangle,T]}$. For a general function $c$, we see that when $T$ is maximal monotone, then $F_{[c,T]} \in \mathcal{F}_{[c,T]}$ and, after permuting coordinates, $F^{C}_{[c,T]} \in \mathcal{F}_{[c,T]}$. However, for a while, no minimality property of $F_{[c,T]}$ was available in the generality of $c$-monotone mappings. The framework of optimal antiderivatives gave rise to a different minimality property: $F_{[c,T]}$ is minimal in a certain family of $C$-convex $C$-antiderivatives which is determined by $T$. This idea was first implemented in the classical case in [2] and then in the generality of $c$-convexity in [3]. In order to recall these results, we will make use of the following notation for a mapping the $C$-antiderivatives of which we study:

**Definition 3.3.** The subset $\{((x,y),(y,x)) \mid (x,y) \in X \times Y\}$ of $(X\times Y)\times(Y \times X)$ is denoted by $\Delta$. With a mapping $T : X \rightrightarrows Y$, we associate the mapping $\Delta_T : X \times Y \rightrightarrows Y \times X$, defined by

$$G(\Delta_T) = \{((x,y),(y,x)) \mid (x,y) \in T\}. \tag{30}$$

**Theorem 3.4.** ($A$) *For a mapping $T : X \rightrightarrows Y$, the following assertions are equivalent:*

1. *$T$ is c-monotone;*
2. *$\Delta_T$ is $C$-monotone;*
3. *$\Delta_T$ is $C$-cyclically monotone;*
4. *$G(\Delta_T) \subset G(\partial_C(c+\iota_{G(T)}))$, that is, $c+\iota_{G(T)}$ is a $C$-antiderivative of $\Delta_T$.*

*Consequently, the following assertions are equivalent:*

1′. *$T$ is maximal c-monotone;*
2′. *$\Delta_T$ is maximal $C$-monotone in $\Delta$;*
3′. *$\Delta_T$ is maximal $C$-cyclically monotone in $\Delta$;*
4′. *$T$ is a maximal subset in $X \times Y$ such that $G(\Delta_T) \subset G(\partial_C(c+\iota_{G(T)}))$.*

($B$) *If $T$ is monotone, then*

$$\alpha_{[C,c|_{G(T)},\Delta_T]} = F_{[c,T]}. \tag{31}$$

*If $T$ is maximal monotone, then*

$$\mathcal{A}_{[C,c|_{G(T)},\Delta_T]} \subset \mathcal{F}_{[c,T]}. \tag{32}$$

In the classical case some more structure is available:

**Theorem 3.5.** *Let $X$ be a topological vector space. If the mapping $T : X \rightrightarrows X^*$ is monotone, then*

$$\alpha_{[\langle\cdot,\cdot\rangle|_{G(T)},\Delta_T]} = F_T \tag{33}$$

*and*

$$\mathcal{A}_{[\langle\cdot,\cdot\rangle|_{G(T)},\Delta_T]} \supseteq \mathcal{F}_T. \tag{34}$$

*If $T$ is maximal monotone, then*

$$\mathcal{A}_{[\langle\cdot,\cdot\rangle|_{G(T)},\Delta_T]} = \mathcal{F}_T. \tag{35}$$

We see that even in the classical case, without the notion of minimal antiderivatives, when $T$ is monotone, but not maximal monotone, there was a problem discussing the minimality of $F_{[c,T]}$. In this case, it may happen that $c \not\leq F_{[c,T]}$, that is, $F_{[c,T]}$ is not a member of $\mathcal{F}_{[c,T]}$. This gap has been filled in [2], where the extended family $\mathcal{A}_{[C,c|_{G(T)},\Delta_T]}$ appeared for the first time together with Theorem 3.5, which asserts that $\mathcal{A}_{[C,c|_{G(T)},\Delta_T]}$ contains both the Fitzpatrick function and the Fitzpatrick family. A more detailed discussion of the classical case is also available in [2]. Thus, in the classical case, it appears that the family of associated antiderivatives is a more natural and suitable environment for the Fitzpatrick function than the original Fitzpatrick family. Outside the scope of the classical case, Theorem 3.4 asserts that the associated family of $C$-antiderivatives is still a natural environment for the Fitzpatrick function which is the minimal $C$-antiderivative. Thus far, more precise relations with the Fitzpatrick family were missing. In Section 6 we shed some more light on these relations and present Example 6.3, where the Fitzpatrick function is not minimal in the Fitzpatrick family although it does belong to the Fitzpatrick family. Thus, also outside the classical case, the family of associated $C$-antiderivatives captures the Fitzpatrick function better than the Fitzpatrick family.

## 4. The Separable $C$-Convex Representation

In this section we study representing functions of a special form for which we make use of the following notation:

**Definition 4.1.** Given functions $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$, we let the function $f \oplus g : X \times Y \to (-\infty, +\infty]$ be defined by

$$(f \oplus g)(x, y) = f(x) + g(y), \quad (x, y) \in X \times Y.$$

A function of the form $f \oplus g : X \times Y \to (-\infty, +\infty]$ is said to be a separable function on $X \times Y$.

That is, we are interested in separable representing functions of $c$-monotone mappings. A clear example of this situation is already at hand: if the underlying $c$-monotone mapping $T : X \rightrightarrows Y$ has a $c$-convex $c$-antiderivative $f : X \to (-\infty, +\infty]$, then we know that

$$c \leq f \oplus f^c \quad \text{and} \quad y \in T(x) \quad \Rightarrow \quad c(x, y) = (f \oplus f^c)(x, y),$$

that is, $f \oplus f^c \in \mathcal{F}_{[c,T]}$. (We will prove the $C$-convexity of $f \oplus f^c$ shortly.) The motivation for our discussion stems also from the converse direction, that is, if $c \leq f \oplus g$, then $c \leq f \oplus f^c \leq f \oplus g$, which means that at the points of contact, $c = f \oplus g$, we have $c$-subdifferentiability. Such an observation, made by Burachik and Svaiter in [10], where they also rediscovered the Fitzpatrick family and function, in the framework of classical convex analysis and monotonicity, is that the existence of a separable representing function is, in fact, a characterization of the case where the underlying maximal monotone mapping is a subdiffrential:

**Theorem 4.2.** *Let $X$ be a Banach space and let $T : X \rightrightarrows X^*$ be maximal monotone. If there exists a separable function $f \oplus g \in \mathcal{F}_T$, then*

$$T = \partial f \quad \textit{and} \quad g = f^*. \tag{36}$$

The above result departs from the context of $c$-convexity because it holds outside the reflexive case and, also, its proof required special features of the classical subdifferential. More precisely, we will see in this section that the equality $f^*|_{\mathrm{Im}(T)} = g|_{\mathrm{Im}(T)}$ does follow from our general discussion in $c$-convexity; however, in order to obtain the equality $f^* = g$, the special features of the classical case are needed.

We would also like to relate this discussion to the more recent family of $C$-antiderivatives $\mathcal{A}_{[C,c|_{G(T)},\Delta_T]}$. To this end, we pave our way through necessary basic connections and relations between the $c$-convexity of functions $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$, and the $C$-convexity of $f \oplus g$, and between the $c$-subdifferentials of $f$ and $g$, and the $C$-subdifferential of $f \oplus g$.

**Proposition 4.3.** *Let $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ be functions. Then*

$$(f \oplus g)^C = f^c \oplus g^c. \tag{37}$$

**Proof.** Let $(y, x) \in Y \times X$. Then

$$\begin{aligned}
&(f \oplus g)^C(y, x) \\
&= \sup_{(x',y') \in X \times Y} [C((x', y'), (y, x)) - (f \oplus g)(x', y')] \\
&= \sup_{(x',y') \in X \times Y} [c(x', y) - f(x') + c(x, y') - g(y')] \\
&= \sup_{x' \in X} [c(x', y) - f(x')] + \sup_{y' \in Y} [c(x, y') - g(y')] = (f^c \oplus g^c)(y, x). \qquad \square
\end{aligned}$$

Using the classical definition of a convex function, when $X$ is a topological vector space and $Y = X^*$, we find that the functions $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ are convex and lower semicontinuous if and only if $f \oplus g$ is convex and lower semicontinuous. A similar property holds when we consider $c$-convexity on $X$ and $Y$ and $C$-convexity on $X \times Y$:

**Proposition 4.4.** *Let $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ be functions. Then $f$ and $g$ are $c$-convex if and only if $f \oplus g$ is $C$-convex.*

**Proof.** Applying (37) twice, we get the following equality:

$$((f \oplus g)^C)^C = f^{cc} \oplus g^{cc}.$$

Now, suppose that $f$ and $g$ are $c$-convex. Since $f = f^{cc}$ and $g = g^{cc}$, it follows that $(f \oplus g)^{CC} = f \oplus g$, that is, $f \oplus g$ is $C$-convex.

Conversely, suppose that $f \oplus g$ is $C$-convex. Then

$$f \oplus g = (f \oplus g)^{CC} = f^{cc} \oplus g^{cc}.$$

This implies that $f$ and $f^{cc}$ differ by an additive constant, say $f = f^{cc} + a$ (the properness assumption for $c$-convex functions implies that $a \in \mathbb{R}$). Thus,

$$f^{cc} = (f^{cc} + a)^{cc} = f^{cc} + a,$$

which implies that $a = 0$ and consequently that $f = f^{cc}$. The same argument also implies that $g = g^{cc}$. $\square$

We now relate the separability of a function with the "separability" of its $C$-subdifferential:

**Proposition 4.5.** *Let $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ be functions. Then*

$$y \in \partial_c f(x) \quad \textit{and} \quad u \in \partial_c g(v) \qquad \Leftrightarrow \qquad (y, u) \in \partial_C (f \oplus g)(x, v).$$

**Proof.** If $y \in \partial_c f(x)$ and $u \in \partial_c g(v)$, then summing up the equalities

$$c(x, y) = f(x) + f^c(y) \quad \text{and} \quad c(u, v) = g(v) + g^c(u),$$

we get

$$\begin{aligned} C((x, v), (y, u)) = c(x, y) + c(u, v) &= f(x) + g(v) + f^c(y) + g^c(u) \\ &= (f \oplus g)(x, v) + (f \oplus g)^C(y, u), \end{aligned}$$

which implies that $(y, u) \in \partial_C (f \oplus g)(x, v)$.

Conversely, suppose that $(y, u) \in \partial_C (f \oplus g)(x, v)$. Then

$$\begin{aligned} c(x, y) + c(u, v) &\leq f(x) + f^c(y) + g(v) + g^c(u) \\ &= (f \oplus g)(x, v) + (f \oplus g)^C(y, u) = C((x, v), (y, u)) \\ &= c(x, y) + c(u, v), \end{aligned} \tag{38}$$

that is, equality must hold in both of the inequalities

$$c(x, y) \leq f(x) + f^c(y) \quad \text{and} \quad c(u, v) \leq g(v) + g^c(u),$$

a fact which implies that $y \in \partial_c f(x)$ and $u \in \partial_c g(v)$. $\square$

Having established the necessary and natural relation between $c$ and $C$ convexities, we are now ready to deal with the representation of $c$-monotone mappings by separable functions. Firstly, if we restrict our attention to the case where $(y, x) = \Delta(x, y) \in \partial_C(f \oplus g)(x, y)$ (see Definition 3.3 of the mapping $\Delta$), we get the following corollary:

**Corollary 4.6.** *Let $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ be functions. Then*

$$y \in \partial_c f(x) \quad \text{and} \quad x \in \partial_c g(y) \qquad \Leftrightarrow \qquad (y, x) \in \partial_C(f \oplus g)(x, y).$$

*In this case, if $(f \oplus g)(x, y) = c(x, y)$, then $f^c(y) = g(y)$ and $g^c(x) = f(x)$.*

**Proof.** The equivalence follows immediately from Proposition 4.5 when we let $u = x$ and $y = v$. Now, if $y \in \partial_c f(x)$ and $(f \oplus g)(x, y) = c(x, y)$, then $(f \oplus g)(x, y) = (f \oplus f^c)(x, y)$, that is, $g(y) = f^c(y)$. (In this case all three values $g(y)$, $f^c(y)$ and $f(x)$ are finite.) In the same manner, if $x \in \partial_c g(y)$ and $(f \oplus g)(x, y) = c(x, y)$, then $g^c(x) = f(x)$. □

The property of the $c$ transform that lies at the heart of matter in the context of this section is given in the following proposition:

**Proposition 4.7.** *Suppose that $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ are functions such that $c \leq f \oplus g$. Then $f^c \leq g$. Consequently,*

$$c(x, y) = f(x) + g(y) \;\; \textit{if and only if } y \in \partial_c f(x) \textit{ and } f^c(y) = g(y).$$

**Proof.** It is clear that if $y \in \partial_c f(x)$ and $f^c(y) = g(y)$, then

$$c(x, y) = f(x) + f^c(y) = f(x) + g(y).$$

Conversely, suppose that $c \leq f \oplus g$ and let $y \in Y$. Since for every $x \in X$ we have $c(x, y) - f(x) \leq g(y)$, it follows that $f^c(y) \leq g(y)$. Since this inequality holds for any $y \in Y$, we get $f^c \leq g$. We conclude that

$$c \leq f \oplus f^c \leq f \oplus g.$$

Now, if we suppose that $c(x, y) = f(x) + g(y)$, then equality holds in both inequalities and therefore both $y \in \partial_c f(x)$ and $f^c(y) = g(y)$ follow. □

Clearly, under the assumptions of Proposition 4.7, replacing the roles of $f$ and $g$, we obtain

$$c(x, y) = f(x) + g(y) \;\; \text{if and only if } x \in \partial_c f(y) \text{ and } g^c(x) = f(x).$$

Thus, combining Proposition 4.7 with Corollary 4.6, we arrive at the following corollary regarding a separable representing function and the mapping $\Delta$:

**Corollary 4.8.** *Let $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ be functions such that $c \le f \oplus g$. Then*

$$c(x, y) = f(x) + g(y) \quad \Leftrightarrow \quad (y, x) \in \partial_C(f \oplus g)(x, y).$$

We sum up and employ the above properties in the main result of this section:

**Theorem 4.9.** *Let $T : X \rightrightarrows Y$ be a mapping. For a separable function $f \oplus g : X \times Y \to (-\infty, +\infty]$, the following implications hold:*

$$f \oplus g \ \in \mathcal{F}_{[c,T]} \quad \Rightarrow \quad f \oplus g \ \in \mathcal{A}_{[C, c|_{G(T)}, \Delta_T]} \tag{39}$$

$$\Rightarrow \quad G(T) \subset G(\partial_c f) \quad \text{and} \quad f^c|_{\mathrm{Im}(T)} = g|_{\mathrm{Im}(T)}. \tag{40}$$

*Consequently, if $T$ is maximal $c$-monotone, then $T = \partial_c f$ and*

$$f \oplus g \in \mathcal{F}_{[c,T]} \quad \Leftrightarrow \quad f \oplus g \in \mathcal{A}_{[C, c|_{G(T)}, \Delta_T]}. \tag{41}$$

**Proof.** Suppose that the separable function $f \oplus g : X \times Y \to (-\infty, +\infty]$ belongs to $\mathcal{F}_{[c,T]}$ and let $(x, y) \in G(T)$. Then $c \le f \oplus g$ and $c(x, y) = f(x) + g(y)$. It now follows from Corollary 4.8 that

$$(y, x) \in \partial_C(f \oplus g)(x, y).$$

Since this inclusion holds for every $(x, y) \in G(T)$, we see that $f \oplus g$ is a $C$-antiderivative of $\Delta_T$. Recalling that $(f \oplus g)|_{G(T)} = c|_{G(T)}$ and that $f \oplus g$ is $C$-convex, we obtain that $f \oplus g \ \in \mathcal{A}_{[C, c|_{G(T)}, \Delta_T]}$, as claimed in implication (39). Now, suppose that $f \oplus g \ \in \mathcal{A}_{[C, c|_{G(T)}, \Delta_T]}$ and let $(x, y) \in G(T)$. Then

$$(y, x) \in \partial_C(f \oplus g)(x, y) \quad \text{and} \quad c(x, y) = f(x) + g(y).$$

According to Corollary 4.6, this means that $y \in \partial_c f(x)$ and that $f^c(y) = g(y)$. Since this holds for every $(x, y) \in G(T)$, we conclude that the implication in (40) holds.

Now, if we assume that $T$ is maximal $c$-monotone, then Theorem 3.4 guarantees that $\mathcal{A}_{[C, c|_{G(T)}, \Delta_T]} \subset \mathcal{F}_{[c,T]}$, which implies that

$$f \oplus g \ \in \mathcal{F}_{[c,T]} \quad \Leftarrow \quad f \oplus g \ \in \mathcal{A}_{[C, c|_{G(T)}, \Delta_T]}$$

and completes the proof of (41). Finally, since $\partial_c f$ is $c$-monotone and since $T$ is maximal $c$-monotone, the inclusion $G(T) \subset G(\partial_c f)$ must be an equality, which completes the proof. □

## 5. The Representation of $c$-Cyclically Monotone Operators of Order $n$ by $C$-Convex Functions

As we have seen, the Fitzpatrick function captures the $n$-$c$-monotonicity of the underlying mapping in the case where the order is $n = 2$. Given an $n$-$c$-monotone mapping of order $n$ higher than 2, the corresponding Fitzpatrick function of order $n$ was first introduced and studied in the setting of classical convex analysis in [1]. Since the appearance of the Fitzpatrick functions of order $n$ about seven years ago, they became the subject matter of about forty papers and even made their way to the solid mechanics literature; see, for example, [20]. In this section we extend their construction and their basic properties to the generality of $c$-monotonicity and $c$-convexity. Furthermore, we will relate these Fitzpatrick functions to the newly associated family of antiderivatives. We now extend the definition of higher order Fitzpatrick functions to the settings of $c$-convexity and $c$-monotonicity as follows:

**Definition 5.1 (Fitzpatrick function of order $n$).** Let $X$ and $Y$ be sets and let $c : X \times Y \to \mathbb{R}$ be a function. With the mapping $M : X \rightrightarrows Y$ (and with the function $c$), we associate, for each $n \in \{2, 3, \cdots\}$, the Fitzpatrick function of order $n$, $F_{[c,M,n]} : X \times Y \to (-\infty, +\infty]$, defined by

$$F_{[c,M,n]}(x, y) := \sup_{\substack{x_n = x,\\ \{(x_i,y_i)\}_{i=1}^{n-1} \subset G(M)}} c(x_1, y) + \sum_{i=1}^{n-1} [c(x_{i+1}, y_i) - c(x_i, y_i)]. \tag{42}$$

The Fitzpatrick function of order 2 is simply the Fitzpatrick function:

$$F_{[c,M,2]}(x, y) = \sup_{(x_1,y_1) \in G(M)} c(x_1, y) + c(x, y_1) - c(x_1, y_1) = F_{[c,M]}.$$

The Fitzpatrick function of infinite order is the function $F_{[c,M,\infty]} : X \times Y \to (-\infty, +\infty]$, defined by

$$F_{[c,M,\infty]} := \sup_{n \in \{2,3,\cdots\}} F_{[c,M,n]}. \tag{43}$$

We now aim at attaining the basic properties of the Fitzpatrick function for the higher order Fitzpatrick functions in the generality of $n$-$c$-monotonicity and understanding their relationship with the associated family of antiderivatives. Due to these properties, the Fitzpatrick functions of higher order play the role for $n$-$c$-monotone mappings that the Fitzpatrick function plays for 2-$c$-monotone mappings. First, we note that $F_{[c,M,n]}$ is $C$-convex , when proper, as the upper envelope of $C$-affine functions. Other basic properties are gathered in the following theorem.

**Theorem 5.2.** *Let $M : X \rightrightarrows Y$ be a mapping and let $n \in \{2, 3, \ldots, \infty\}$. Then the following assertions are equivalent:*

1) *$M$ is $n$-$c$-monotone;*
2) *$F_{[c,M,n]}|_G \le c|_G$ for any $n$-$c$-monotone extension $G$ of $G(M)$;*
3) *$F_{[c,M,n]}|_{G(M)} = c|_{G(M)}$.*

*In this case,*

$$F_{[c,M,n]} \in \mathcal{A}_{[C,c|_{G(M)},\Delta_M]}. \tag{44}$$

*If, in addition, $M$ is maximal $n$-$c$-monotone, then*

$$F_{[c,M,n]} \in \mathcal{F}_{[c,M]}. \tag{45}$$

*In this case, $M$ is characterized as follows:*

$$G(M) = \left\{(x,y) \in X \times Y \mid F_{[c,M,n]}(x,y) = c(x,y)\right\}. \tag{46}$$

**Proof.** Suppose that $n \in \{2,3,\ldots\}$, that $M$ is $n$-$c$-monotone and let $(x,y) \in G$, where $G$ is an $n$-$c$-monotone extension of $G(M)$. Given $n-1$ pairs $\{(x_i,y_i)\}_{i=1}^{n-1} \subset G(M)$, we let $(x_n,y_n) = (x,y)$. Since $\{(x_i,y_i)\}_{i=1}^{n}$ is now an $n$-$c$-monotone set (see Definition 2.3), we have

$$\begin{aligned} 0 \le & \sum_{i=1}^{n-1}[c(x_i,y_i) - c(x_{i+1},y_i)] + c(x,y) - c(x_1,y) \\ \Rightarrow \quad & c(x_1,y) + \sum_{i=1}^{n-1}[c(x_{i+1},y_i) - c(x_i,y_i)] \le c(x,y). \end{aligned} \tag{47}$$

Since this holds for any set of $n-1$ pairs $\{(x_i,y_i)\}_{i=1}^{n-1} \subset G(M)$, when we let $x_n = x$, it follows from the definition of $F_{[c,M,n]}$ that $F_{[c,M,n]}(x,y) \le c(x,y)$. Since this inequality holds for any pair $(x,y) \in G$, we conclude that *1*) $\Rightarrow$ *2*). Suppose that *2*) holds and let $(x,y) \in G(M)$. Then $F_{[c,M,n]}(x,y) \le c(x,y)$. Letting $(x_1,y_1) = (x_2,y_2) = \ldots = (x_{n-1},y_{n-1}) = (x,y)$ in (42), we see that, in fact, $F_{[c,M,n]}(x,y) = c(x,y)$. We conclude that *2*) $\Rightarrow$ *3*). Now, suppose that *3*) holds. Then given any set of $n-1$ pairs $\{(x_i,y_i)\}_{i=1}^{n-1} \subset G(M)$ and given any $(x,y) \in G(M)$, we let $(x_n,y_n) = (x,y)$. Then we have (47). Altogether, given any set of $n$ pairs $\{(x_i,y_i)\}_{i=1}^{n} \subset G(M)$, we have

$$0 \le \sum_{i=1}^{n}[c(x_i,y_i) - c(x_{i+1},y_i)],$$

that is, $M$ is $n$-$c$-monotone. This completes the proof of *3*) $\Rightarrow$ *1*). Since $F_{[c,M,\infty]} = \sup_{n\in\{2,3,\ldots\}} F_{[c,M,n]}$, we see that *1*) $\Leftrightarrow$ *2*) $\Leftrightarrow$ *3*) also holds for $F_{[c,M,\infty]}$.

In Definition (42), letting $(x_i,y_i) = (x_1,y_1)$ for each $1 \le i \le n-1$, we see that

$$F_{[c,M]} = F_{[c,M,2]} \le F_{[c,M,n]} \quad \forall n \in \{2,3,\ldots,\infty\}.$$

Now, if $M$ is $n$-$c$-monotone, where $n \in \{2,3,\ldots,\infty\}$, then $M$ is $c$-monotone and according to Theorem 3.4,

$$\alpha_{[C,c|_{G(M)},\Delta_M]} = F_{[c,M]} \le F_{[c,M,n]}.$$

Since we already have $F_{[c,M,n]} \le (c + \iota_{G(M)}) = (c + \iota_{\mathrm{dom}(\Delta_M)})$ and since $F_{[c,M,n]}$ is $C$-convex, taking the $C$-convexification and employing (11), we arrive at

$$\alpha_{[C,c|_{G(M)},\Delta_M]} \le F_{[c,M,n]} \le (c + \iota_{\mathrm{dom}(\Delta_M)})^{CC} = \gamma_{[C,c|_{G(M)},\Delta_M]}.$$

Thus (14) holds and this completes the proof of (44). If, in addition, $M$ is maximal $n$-$c$-monotone and $(x,y) \notin G(M)$, then $(x,y)$ is not in $n$-$c$-monotone relations with $G(M)$, that is, there is a set of $n-1$ pairs $\{(x_i,y_i)\}_{i=1}^{n-1} \subset G(M)$ such that when we let $x_n = x$, we obtain

$$\sum_{i=1}^{n-1}[c(x_i,y_i) - c(x_{i+1},y_i)] + c(x,y) - c(x_1,y) < 0$$
$$\Rightarrow \quad c(x_1,y) + \sum_{i=1}^{n-1}[c(x_{i+1},y_i) - c(x_i,y_i)] > c(x,y), \tag{48}$$

which implies that $F_{[c,M,n]}(x,y) > c(x,y)$. Since this holds for any $(x,y) \notin G(M)$ and since $F_{[c,M,n]}|_{G(M)} = c|_{G(M)}$, we conclude that both (45) and (46) hold. $\square$

When the underlying mapping $M$ is not $n$-$c$-monotone for some $n$, then the Fitzpatrick functions of order $k$, where $k \geq n$, no longer have the features of Theorem 5.2. In particular, $F_{[c,M,\infty]}$ is not proper in this case.

In the classical setting, given a convex, lower semicontinuous and proper function $f : X \to (-\infty,+\infty]$, a main result from [1] asserts that the sequence of Fitzpatrick functions corresponding to $\partial f$ converges to the unique separable representing function, that is,

$$F_{[\partial f,n]} \nearrow F_{[\partial f,\infty]} = f \oplus f^*. \tag{49}$$

Already in the classical setting, this equality no longer holds in cases where the underlying operator is not maximal cyclically monotone. However, even in the generality of $c$-monotonicity, we now prove that any separable representation is an upper bound for the sequence as follows:

**Proposition 5.3.** *Let $M : X \rightrightarrows Y$ be proper and $c$-cyclically monotone and let $f : X \to (-\infty,+\infty]$ be any $c$-antiderivative of $M$. Then for any $(x,y) \in X \times Y$, we have*

$$F_{[c,M,2]}(x,y) \leq F_{[c,M,3]}(x,y) \leq \cdots \leq F_{[c,M,n]}(x,y) \longrightarrow F_{[c,M,\infty]}(x,y) \tag{50}$$
$$\leq f(x) + f^c(y). \tag{51}$$

**Proof.** For each $n \in \{2,3,\cdots\}$, letting $(x_n,y_n) = (x_{n-1},y_{n-1})$ in the definition of $F_{[c,M,n+1]}$, we see that $F_{[c,M,n]} \leq F_{[c,M,n+1]}$. This verifies (50). Now, suppose that $f : X \to (-\infty,+\infty]$ is a $c$-antiderivative of $M$. It follows that at a point

$(x, y) \in X \times Y$, we have the following estimate:

$$
\begin{aligned}
& F_{[c,M,\infty]}(x, y) \\
= & \sup_{\substack{n\in\{2,3,\cdots\},\ x_n=x,\\ \{(x_i,y_i)\}_{i=1}^{n-1}\subset G(M)}} c(x_1, y) + \sum_{i=1}^{n-1}[c(x_{i+1}, y_i) - c(x_i, y_i)] \\
\leq & \sup_{\substack{n\in\{2,3,\cdots\},\ x_n=x,\\ \{(x_i,y_i)\}_{i=1}^{n-1}\subset G(M)}} c(x_1, y) + \sum_{i=1}^{n-1}[f(x_{i+1}) - f(x_i)] \\
= & \sup_{x_1\in\mathrm{dom}(M)} c(x_1, y) + f(x) - f(x_1) \\
\leq & f(x) + \sup_{x_1\in X} c(x_1, y) - f(x_1) = f(x) + f^c(y).
\end{aligned}
$$ □

In order to prove (49) in [1], the fact that when $f$ is convex, lower semicontinuous and proper, then the subdifferential $\partial f$ determines $f$ and $f^*$ uniquely, up to an additive constant, was crucial. In particular, the following useful formula was employed:

$$
f^*(x^*) = \sup_{x\in\mathrm{dom}(\partial f)} [\langle x^*, x\rangle - f(x)] = (f + \iota_{\mathrm{dom}(\partial f)})^*(x^*) \quad \forall x^* \in X^*. \tag{52}
$$

Thus, in the generality of $c$-convexity, it is natural to ask if a similar, sharper result regarding the sequence of Fitzpatrick functions holds when $f$ is uniquely defined (up to an additive constant) by $\partial_c f$. To this end, we aim at generalizing (52). (Although strictly speaking, calling this a generalization is inaccurate because (52) also holds outside the reflexive case.)

**Proposition 5.4.** *Suppose that the function $f : X \to (-\infty, +\infty]$ is the unique $c$-antiderivative (up to an additive constant) of the mapping $M : X \rightrightarrows Y$. Then $f$ is $c$-convex,*

$$
f(x) - f(s) = R_{[c,M,s]}(x) \quad \forall (s, x) \in \mathrm{dom}(M) \times X \tag{53}
$$

*and*

$$
f^c = (f + \iota_{\mathrm{dom}(M)})^c. \tag{54}
$$

**Proof.** Since $f$ is the unique antiderivative of $M$, from the nonemptiness of $\mathcal{A}|_{[c,f|_{\mathrm{dom}(M)},M]}$ in the case where $S = \mathrm{dom}(M)$ in Theorem 2.2, it is clear that

$$
\mathcal{A}_{[c,f|_{\mathrm{dom}(M)},M]} = \{f\} = \{\gamma_{[c,f|_{\mathrm{dom}(M)},M]}\} = \{(f + \iota_{\mathrm{dom}(M)})^{cc}\}. \tag{55}
$$

Thus, $f$ is $c$-convex and we get that

$$
f^c = (f + \iota_{\mathrm{dom}(M)})^{ccc} = (f + \iota_{\mathrm{dom}(M)})^c,
$$

that is, (54) holds. Now, according to Theorem 2.5, since $M$ has a $c$-antiderivative $f$, $M$ is $c$-cyclically monotone. In this case, given any $s \in \mathrm{dom}(M)$, Rockafellar's $c$-antidrivative $R_{[c,M,s]}$ vanishes at $s$. Since the $c$-antiderivative of $M$ is unique, up to an additive constant, and since $f(\cdot)-f(s)$ is a $c$-antiderivative of $M$ that vanishes at $s$, we arrive at (53). □

Finally, we are now able to generalize (49) as follows:

**Theorem 5.5.** *Let $M : X \rightrightarrows Y$ be a mapping. If $M$ admits a unique (up to an additive constant) c-antiderivative $f : X \to (-\infty, +\infty]$, then the pointwise limit of the sequence of the Fitzpatrick functions $F_{[c,M,n]}$ is*

$$F_{[c,M,\infty]} \;=\; f \oplus f^c. \tag{56}$$

**Proof.** Let $(x, y) \in X \times Y$. Then for every $s \in \operatorname{dom}(M)$, by the definition of $F_{[c,M,\infty]}$, we have

$$\begin{aligned}
&F_{[c,M,\infty]}(x, y)\\
\geq &\sup_{\substack{[n\in\{2,3,\cdots\},\\ x_1=s,\ x_n=x,\\ \{(x_i,y_i)\}_{i=1}^{n-1}\subset G(M)}} c(s, y) + \sum_{i=1}^{n-1}[c(x_{i+1}, y_i) - c(x_i, y_i)]\\
&= c(s, y) + R_{[c,M,s]}(x) = c(s, y) + f(x) - f(s),
\end{aligned}$$

where the first equality above follows from the definition of Rockafellar's antiderivative while the second equality above follows from (53). Since this holds for every $s \in \operatorname{dom}(M)$, we may employ (54) and arrive at

$$\begin{aligned}
F_{[c,M,\infty]}(x, y) &\geq f(x) + \sup_{s_1\in\operatorname{dom}(M)} [c(s, y) - f(s)]\\
&= f(x) + (f + \iota_{\operatorname{dom}(M)})^c(y) = f(x) + f^c(y).
\end{aligned}$$

When combined with the reverse inequality in (51), this inequality completes the proof. □

## 6. Examples, Conclusions and Other Remarks

In order to present examples outside the scope of the classical case, we focus our attention on the case where $(X, d)$ is a metric space, $Y = X$ and $c = -d$. In this case, the $c$-convexity structure can be described as follows:

**Proposition 6.1 ($-d$-convexity).** *Let $(X, d)$ be a metric space. For any proper function $f : X \to (-\infty, +\infty]$, the following assertions are equivalent:*

1. *$f$ is 1-Lipschitz;*
2. *$f^{-d} = -f$;*
3. *$f$ is $-d$-convex;*
4. *$f$ is a $-d$-antiderivative of the identity $I : X \to X$. That is, $G(I) \subset G(\partial_{-d} f)$.*

*In this case, it follows that $f : X \to \mathbb{R}$. The graph of the $-d$-subdifferential of $f$ is the set of ordered pairs $(x, y) \in X \times X$ such that $f$ preserves the distance $d(x, y)$ in the sense that $f(y) - f(x) = d(x, y)$.*

A proof of Proposition 6.1 can be found in [3]. We see that in the metric case considered above, the identity mapping is the most elementary $-d$-subdifferential. The following example from [3] deals with the identity's $-d$-monotonicity features, the associated Fitzpatrick function and the two associated families of functions. We complement it here by pointing out the Fitzpatrick functions of higher orders.

**Example 6.2.** The identity mapping $I$ on a metric space is $-d$-cyclically monotone. $G(I)$ is contained in the graph of the $-d$-subdifferential of any $-d$-convex function. As a consequence, any $-d$-cyclically monotone set $G(T) \subset X \times X$ extends to the $-d$-cyclically monotone set $G(T) \cup G(I)$. In particular, $G(I)$ is extensible by one point to a $-d$-cyclically monotone set, using any point in $X \times X$. Therefore, if $X$ is not a singleton, $I$ is not maximal $-d$-cyclically monotone and it is not maximal $-d$-monotone. However, it is the (whole) $-d$-subdifferential of any function $f : X \to \mathbb{R}$ such that $f(y) - f(x) < d(x, y)$ for all $x \neq y$. In particular, it is the $-d$-subdifferential of any $K$-Lipschitz function, where $0 \leq K < 1$. Therefore, $I$ does not determine its $-d$-convex $-d$-antiderivative up to an additive constant.

Let the function $D : (X \times X) \times (X \times X) \to \mathbb{R}$ be defined by $D = -C$. Then $D$ is a metric and

$$\mathcal{F}_{[-d,I]} = \mathcal{A}_{[-D,-d|_{G(I)},\Delta_I]}. \tag{57}$$

The associated Fitzpatrick function and its $-D$-transform are given by

$$F_{[-d,I]}(x, y) = -d(x, y) \quad \text{and} \quad F^{-D}_{[-d,I]}(y, x) = d(y, x), \quad (x, y) \in X \times X, \tag{58}$$

If $T$ is either maximal $-d$-monotone or the $-d$-subdifferential of a $-d$-convex function, then

$$\mathcal{A}_{[-D,-d|_{G(T)},\Delta_T]} \subset \mathcal{F}_{[-d,T]} \subset \mathcal{A}_{[-D,-d|_{G(I)},\Delta_I]}. \tag{59}$$

In particular, in this case,

$$F_{[-d,I]}(x, y) \leq F_{[-d,T]}(x, y) \leq F^{-D}_{[-d,T]}(y, x) = -F_{[-d,T]}(y, x) \leq F^{-D}_{[-d,I]}(y, x) \tag{60}$$

for every $x, y \in X$. Finally, the sequence of Fitzpatick functions associated with the identity is constant, that is,

$$F_{[-d,I,n]} = F_{[-d,I,\infty]} = -d \ \ \forall n \in \{2, 3, \cdots\}. \tag{61}$$

**Proof.** We prove (61). The rest of the proof can be found in [3]. Let $(x, y) \in X \times X$. By the definition of $F_{[c,M,n]}$, letting $M = I$ and $c = -d$, we get

$$\begin{aligned} & F_{[c,M,n]}(x, y) \\ = & \sup_{\substack{x_n = x, \\ \{(x_i,x_i)\}_{i=1}^{n-1} \subset G(M)}} -d(x_1, y) + \sum_{i=1}^{n-1} [-d(x_{i+1}, x_i) + d(x_i, x_i)] \\ = & \sup_{\{x_i\}_{i=1}^{n-1} \subset X} -d(x_1, y) - d(x_2, x_1) - d(x_3, x_2) - \ldots - d(x, x_{n-1}) \leq -d(x, y), \end{aligned}$$

where the last inequality is the triangle inequality. Letting $x_1 = x_2 = \ldots = x_{n-1} = x$ (or $= y$), we get equality. $\square$

The following example clarifies some very basic issues which have not been addressed so far. We first recall that so far we know that the minimality of the Fitzpatrick function holds in the associated family of antiderivatives, but not necessarily in the Fitzpatrick family. This is already evident in the classical case when the underlying mapping is not maximal monotone and the Fitzpatrick function no longer necessarily belongs to the Fitzpatrick family. This implies that the family of associated antiderivatives is a more natural environment for the Fitzpatrick function than the Fitzpatrick family. The following example shows that in the generality of $c$-monotonicity, the Fitzpatrick function loses its minimality in the Fitzpatrick family, even when the Fitzpatrick function does belong to it, and even when the underlying mapping is maximal $c$-monotone. In fact, this time the Fitzpatrick function turns out to be the maximal(!) function in the Fitzpatrick family. Since the minimality of the Fitzpatrick function always holds inside the associated family of antiderivatives, it is now even clearer that the family of antiderivatives is more natural and suitable for the Fitzpatrick function than the Fitzpatrick family. Second, the following example implies that the generalization to $c$-convexity provided by Theorem 5.5 makes a point also outside the classical setting. Namely, this is a non-classical situation where the $c$-antiderivative is determined uniquely, up to an additive constant:

**Example 6.3.** Let $X = Y = \mathbb{R}$ and let $c = -d$, where $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. We consider the case where $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x$ for $x \in X$ and the mapping $M = \partial_{-d} f$. Then $M$ is given by

$$M(x) = \{y \in \mathbb{R} \mid x \leq y\}, \quad x \in \mathbb{R}. \tag{62}$$

The mapping $M$ determines its $-d$-antiderivative uniquely (up to an additive constant) and is maximal $-d$-monotone. The sequence of associated Fitzpatrick functions in this case is constant, that is, for $(x, y) \in X \times X$ we have

$$F_{[c,M,n]}(x, y) = F_{[c,M,\infty]}(x, y) = f \oplus f^c(x, y) = x - y \quad \forall n \in \{2, 3, \cdots\}. \tag{63}$$

In fact, $\mathcal{A}|_{[-D,-d|_{G(M)},\Delta_M]}$ is a singleton, that is,

$$\mathcal{A}|_{[-D,-d|_{G(M)},\Delta_M]} = \{x - y\}. \tag{64}$$

We see that $F_{[-d,M]}$ is not the minimal function in $\mathcal{F}_{[-d,M]}$ since

$$\min \mathcal{F}_{[-d,M]} = -d \quad \text{and, in fact, } F_{[-d,M]} = \max \mathcal{F}_{[-d,M]}. \tag{65}$$

**Proof.** Since $f$ is 1-Lipschitz,

$$G(\partial_{-d} f) = \{(x, y) \in X \times X \mid f(y) - f(x) = d(x, y)\}$$

(see Proposition 6.1). In our case, we get (62). Suppose that $(x,y) \notin G(M)$. Then $y < x$ and it follows that $(y,x) \in G(M)$. Since

$$-d(x,y) + d(x,x) + d(y,y) - d(y,x) < 0,$$

we see that $(x,y)$ is not in a $-d$-monotone relation with the point $(y,x) \in G(M)$, that is, $G(M) \cup \{(x,y)\}$ is not a $-d$-monotone extension of $G(M)$. Since $(x,y)$ was an arbitrary point outside $G(M)$, we see that $M$ is maximal $-d$-monotone. Now, suppose that $h : \mathbb{R} \to (-\infty, +\infty]$ is a $-d$-antiderivative of $M$. Then $h$ is a $-d$-antiderivative of $I$ (since $G(I) \subset G(M)$) and therefore $h$ is 1-Lipschitz and satisfies

$$h(y) - h(x) = |y - x| \;\; \forall (x,y) \in G(M) = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

If we assume that $h(0) = 0$, then for $y \geq 0$, we get $(0,y) \in G(M)$ and therefore

$$h(y) = h(y) - h(0) = |y - 0| = y = f(y).$$

For $x \leq 0$, we get $(x,0) \in G(M)$ and therefore

$$h(x) = -(h(0) - h(x)) = -|0 - x| = x = f(x).$$

Summing up, we see that $h = f$, that is, $M$ determines its $-d$-antiderivative which vanishes at 0 uniquely. We now compute the associated Fitzpatrick function. Let $(x,y) \in \mathbb{R} \times \mathbb{R}$. If $x \leq y$, then $(x,y) \in G(M)$ and consequently $F_{[-d,M]}(x,y) = -d(x,y) = -|x-y| = x - y$. If $x > y$, then in the supremum in the definition of $F_{[-d,M]}(x,y)$ in this case,

$$F_{[-d,M]}(x,y) = \sup_{\substack{(s,t) \in \mathbb{R} \times \mathbb{R}, \\ s \leq t}} -|x-t| - |s-y| + |s-t|,$$

we may let $s = y$ and $t = x$ in order to get $F_{[-d,M]}(x,y) \geq |x-y| = x-y$. Since, according to (50) and (51), the sequence of Fitzpatrick functions $F_{[c,M,n]}(x,y)$ is increasing and bounded from above by $(f \oplus f^{-d})(x,y) = x - y$, we arrive at (63). We now recall that any 1-Lipschitz function on $X \times X$ (with respect to the metric $D$, in our case, the $\ell_1$ metric on $\mathbb{R}^2$) that vanishes at the points of $G(I)$ is bounded from above by the metric $d$ and from below by $-d$. Indeed, let $h : X \times X \to \mathbb{R}$ be 1-Lipschitz (with respect to $D$) such that $h(x,x) = 0$ for every $x \in X$ and let $(x,y) \in X \times Y$. Then

$$|h(x,y)| = |h(x,y) - h(x,x)| \leq D((x,y),(x,x)) = d(x,y).$$

This means that when $h \in \mathcal{A}|_{[-D,-d|_{G(M)},\Delta_M]}$, then if $(x,y) \in G(M)$, that is, if $x \leq y$, then $h(x,y) = -|x-y| = x - y$. If $(x,y) \notin G(M)$, that is, if $x > y$, then

$$x - y = F_{[-d,M]}(x,y) \leq h(x,y) \leq |x-y| = x - y.$$

Summing up, we see that $h(x,y) = x-y$ for every $(x,y) \in \mathbb{R} \times \mathbb{R}$. Since $h$ was an arbitrary function in $\mathcal{A}|_{[-D,d|_{G(M)},\Delta_M]}$, we obtain (64). Finally, we already know that $-d$ is $-D$-convex and, since it is the coupling function, $-d \in \mathcal{F}_{[-d,M]}$. Since $-d$ is a lower bound for the functions in $\mathcal{F}_{[-d,M]}$, it is the minimal function there. Since the functions in $\mathcal{F}_{[-d,M]}$ coincide with $-d$ on $\{(x, y) \in \mathbb{R}\times\mathbb{R}|\ x \leq y\}$ and are bounded from above by $d$ elsewhere, we infer (65) and complete the proof. $\square$

**Remark 6.4.** In the classical case, let $h : X \to (-\infty, +\infty]$ be convex and let $g : X \to (-\infty, +\infty]$ be such that $g \leq h$. If $g$ is Gâteaux differentiable at the point $x \in X$ and $g(x) = h(x)$, then $h$ is subdifferentiable at $x$ and

$$\nabla g(x) \in \partial h(x).$$

If we apply this fact to a function $h \in \mathcal{F}_{[c,T]}$ and to the function $g = c$, then at a point $(x, y^*) \in G(T)$, that is, a point where $c = h$, we get

$$(y^*, x) = \nabla g(x, y^*) \in \partial_C h(x, y^*).$$

This is why Theorem 3.5 holds. Outside the classical case, in general $c$-convexity we lose the notion of a differential. However, for particular coupling functions $c$, where for a $C$-convex function $h$, $c \leq h$, the equality $c(x, y) = h(x, y)$ implies that $(y, x) \in \partial_C h(x, y)$, the same conclusions hold. Since $\mathcal{A} \subsetneq \mathcal{F}$ in Example 6.3, this is not true in general.

**Remark 6.5.** Given a mapping $T : X \rightrightarrows Y$ which admits more than one $c$-antiderivative, a natural question arises: does the equality

$$F_{[c,T,\infty]} = \inf_{G(T)\subset G(\partial_c f)} (f \oplus f^c)$$

hold?

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