Journal of Convex Analysis
Volume 22 (2015), No. 3, 711–731

Regularity Conditions for Strong Duality in Evenly Convex Optimization Problems. An Application to Fenchel Duality.

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Received: May 11, 2013
Revised manuscript received: July 8, 2014
Accepted: July 10, 2014

In this work, via perturbational approach, we give an alternative dual problem for a general infinite dimensional optimization primal one, by means of a conjugation scheme based on generalized convex conjugation theory, rather than the classical Fenchel conjugation. Two sufficient regularity conditions for strong duality are stated, where the evenly convexity of the perturbation function plays a fundamental role. One of these conditions has been obtained due to the relationship between evenly convexity and cs-closedness, allowing us also to derive new properties for evenly convex sets and functions. The regularity conditions in the general setting are applied to the Fenchel primal-dual problem, and a comparison between them and an already existing closedness-type regularity condition for Fenchel duality is studied.

Keywords: Evenly convex function, generalized convex conjugation, Fenchel dual problem

2010 Mathematics Subject Classification: 52A20, 26B25, 90C25

1. Introduction

Let X be a separated locally convex space and $F : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ a proper function. Consider the general optimization problem

$$\begin{array}{lll} (GP) & \text{Inf} & F(x) \\ & \text{s.t.} & x \in X. \end{array}$$

There exists a well developed theory on pertubational approach to duality for convex optimization problems, as one can find for instance in [1], [2], [4], [11], [14] and [16]. The main idea is to consider a perturbation function $\Phi : X \times \Theta \to \overline{\mathbb{R}}$, where Θ is also a separated locally convex space, named the *perturbation variable space*, and Φ verifies the relation

$$\Phi\left(x, 0\right) = F\left(x\right),$$

ISSN 0944-6532 / $ 2.50

for all $x \in X$. A dual problem for (GP) can be defined in the following way:

$$\begin{array}{lll} (GD) & \text{Sup} & -\Phi^* (0, z^*) \\ & \text{s.t.} & z^* \in \Theta^*, \end{array}$$

where Θ^* is the dual space of Θ and Φ^* the Fenchel conjugate of Φ. Denoting by $v(GP)$ and $v(GD)$ the optimal values of the primal and the dual problems, respectively, it holds $-\infty \leq v(GD) \leq v(GP) \leq +\infty$, and this inequality is called *weak duality*, whereas *strong duality* is $v(GD) = v(GP)$ and the dual problem has an optimal solution.

Assuming that Φ is a proper convex function on $X \times \Theta$, the property of being lower semicontinuous plays an important role in some regularity conditions, which are those conditions guaranteeing strong duality for (GP) and (GD). See for instance [1] as a presentation of the state of art in this field.

In this paper we consider a generalized concept of lower semicontinuous convex function, called evenly convex function, and a suitable conjugation squeme for this class of convex functions, instead of the classical Fenchel conjugation pattern. It will allow us to build a new dual problem for (GP), denoted by (GD_c), and state some sufficient regularity conditions for strong duality. Fenchel duality will be derived as a particular case, and regularity conditions will be compared with another one obtained in [6].

The concept of evenly convexity (e-convexity, in brief) is due to Fenchel [5]. E-convex sets are defined as the intersection of an arbitrary family (possibly empty) of open halfspaces. They have been applied in linear inequality systems containing stritc inequalities ([7], [8]) and basic properties of this class of sets by means of their sections and projections are given in [9]. Furthermore, a non-separational geometrical characterization of e-convex sets is given in [3].

E-convex sets lead to the so-called e-convex functions, which were introduced in [15]. The survey by Martínez-Legaz [10], where generalized convex duality theory based on Fenchel-Moreau conjugation is applied to quasiconvex programming, motivated in [12] a conjugation pattern for evenly convex functions.

In a recent paper [6] we have extended some well-known results for lower semicontinuous convex functions to the more general framework of e-convex functions, obtaining a closedness-type regularity condition for the fulfilment of strong Fenchel duality for optimization problems of the form

$$\begin{array}{lll} (P) & \text{Inf} & f(x) \\ & \text{s.t.} & x \in A, \end{array} \tag{1}$$

where both feasible set A and objective function f are e-convex.

The organization is as follows. In Section 2 we summarize the basic properties for e-convex sets and functions, as well as all the necessary tools and results, in order to make the paper self-contained. In particular, the conjugation scheme for e-convex functions will be reminded, as well as its most important properties. In Section 3 we shall relate the e-convexity with other closedness-type notions, which will be applied in Section 4 in order to obtain an interior point regularity condition for strong duality for (GP) and (GD_c). In the same framework, Section 5 presents a closedness-type regularity condition. Finally, Section 6 will be devoted to the comparison between these two conditions and an already existing condition for the fulfilment of strong Fenchel duality for the problem (1).

2. Preliminaries

Let X be a separated locally convex space and X^* its topological dual space endowed with the weak* topology. For the set $D \subset X$, the *closure* of D is denoted by $\operatorname{cl} D$. If $A \subset X^*$, then $\operatorname{cl} A$ stands for the weak* closure of A. As a consequence of the Hahn-Banach theorem, every open or closed convex set is e-convex. The *e-convex hull* of $C \subset X$, denoted by $\operatorname{eco} C$, is the smallest e-convex set that contains C. This operator is well defined because X is e-convex and the class of e-convex sets is closed under intersection. Moreover, if C is convex, then $C \subset \operatorname{eco} C \subset \operatorname{cl} C$. The duality product will be denoted by $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$, i.e. $\langle x, x^* \rangle = x^*(x)$ for all $(x, x^*) \in X \times X^*$.

The following characterization of e-convex sets can be found in [7, Prop. 3.1], in the case $X = \mathbb{R}^n$, but it can be extended easily to the infinite dimensional case we are dealing with.

Proposition 2.1. *A set $C \subset X$ is e-convex if and only if for every $x \notin C$ there exists a closed hyperplane H such that $x \in H$ and $C \cap H = \emptyset$.*

For any function $f : X \to \overline{\mathbb{R}}$, the effective domain and the epigraph of f are denoted in the usual way,

$$\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\} \quad \text{and} \quad \operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\},$$

respectively. The *lower semicontinuous* (*lsc,* in short) *hull* of f, $\operatorname{cl} f : X \to \overline{\mathbb{R}}$, is defined such that $\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f)$, and f is said to be *lsc at* $x \in X$ if $f(x) = (\operatorname{cl} f)(x)$. Furthermore, f is said to be *proper* if f does not take on the value $-\infty$ and $\operatorname{dom} f \neq \emptyset$. On the other hand, according to [15], we will say that f is *e-convex* if its epigraph is an e-convex set in $X \times \mathbb{R}$. Obviously, any lsc convex function is e-convex, but the reverse statement is not true (see an example in [15]). The *e-convex hull* of f, $\operatorname{eco} f : X \to \overline{\mathbb{R}}$, is defined as the largest e-convex minorant of f, that is,

$$\operatorname{eco} f := \sup\{g \mid g \text{ is e-convex and } g \leq f\}.$$

Due to the fact that Fenchel conjugation is not suitable for e-convex functions, a new conjugation scheme is provided for this class of convex functions in [12], based on the generalized convex conjugation theory introduced by Moreau [13].

We will consider the set $W := X^* \times X^* \times \mathbb{R}$ with the coupling function $c : X \times W \to \overline{\mathbb{R}}$ given by

$$c(x,(x^*,y^*,\alpha)) := \begin{cases} \langle x, x^* \rangle & \text{if } \langle x, y^* \rangle < \alpha, \\ +\infty & \text{otherwise.} \end{cases}$$

For any $f : X \to \overline{\mathbb{R}}$, its *c-conjugate* $f^c : W \to \overline{\mathbb{R}}$ is defined by

$$f^c((x^*,y^*,\alpha)) := \sup_{x \in X} \{c(x,(x^*,y^*,\alpha)) - f(x)\}.$$

Similarly, the *c'-conjugate* of $g : W \to \overline{\mathbb{R}}$ is the function $g^{c'} : X \to \overline{\mathbb{R}}$ defined by

$$g^{c'}(x) := \sup_{(x^*,y^*,\alpha) \in W} \{c'((x^*,y^*,\alpha),x) - g(x^*,y^*,\alpha)\},$$

where $c' : W \times X \to \overline{\mathbb{R}}$ is given by $c'((x^*,y^*,\alpha),x) = c(x,(x^*,y^*,\alpha))$. The following conventions are used: $+\infty + (-\infty) = -\infty + (+\infty) = +\infty - (+\infty) = -\infty - (-\infty) = -\infty$.

Functions of the form $x \in X \to c(x,(x^*,y^*,\alpha)) - \beta \in \overline{\mathbb{R}}$, with $(x^*,y^*,\alpha) \in W$ and $\beta \in \mathbb{R}$ are called *c-elementary*; in the same way, *c'-elementary* functions are those of the form $(x^*,y^*,\alpha) \in W \to c(x,(x^*,y^*,\alpha)) - \beta \in \overline{\mathbb{R}}$, with $x \in X$ and $\beta \in \mathbb{R}$.

Actually, according to [12], the family of the proper e-convex functions from X into $\overline{\mathbb{R}}$ along with the function identically $-\infty$ is the family of pointwise suprema of a set of c-elementary functions. Using an analogous terminology, a function $g : W \to \overline{\mathbb{R}}$ is said *e'-convex* if it is the pointwise supremum of a set of c$'$-elementary functions. Also, the *e'-convex hull* of any function $k : W \to \overline{\mathbb{R}}$ is the largest e$'$-convex minorant of k, and it is denoted by $\mathrm{e'co}\, k$. The following proposition is given in [6].

Proposition 2.2. *Let $f : X \to \overline{\mathbb{R}}$ and $g : W \to \overline{\mathbb{R}}$. Then*

(*i*) *f^c is e'-convex; $g^{c'}$ is e-convex.*

(*ii*) *If f has a proper e-convex minorant,* $\mathrm{eco}\, f = f^{cc'}$; $\mathrm{e'co}\, g = g^{c'c}$.

(*iii*) *If f does not take on the value $-\infty$, then f is e-convex if and only if $f = f^{cc'}$; g is e'-convex if and only if $g = g^{c'c}$.*

(*iv*) *$f^{cc'} \leq f$; $g^{c'c} \leq g$.*

Remark 2.3. According to (*iii*) in the previous proposition, if f does not take on the value $-\infty$, f is said to be *e-convex at* $x \in X$ if $f(x) = f^{cc'}(x)$, and g is said to be *e'-convex at* $(x^*,y^*,\alpha) \in W$ if $g(x^*,y^*,\alpha) = g^{c'c}(x^*,y^*,\alpha)$.

The following definitions and results from [12] and [6] will be used throughout the paper.

Definition 2.4. A function $a : X \to \overline{\mathbb{R}}$ is said to be *e-affine* if there exist $x^*, y^* \in X^*$ and $\alpha, \beta \in \mathbb{R}$ such that

$$a(x) = \begin{cases} \langle x, x^* \rangle - \beta & \text{if } \langle x, y^* \rangle < \alpha, \\ +\infty & \text{otherwise.} \end{cases}$$

For any $f : X \to \overline{\mathbb{R}}$, $\mathcal{E}_f$ denote the set of all e-affine functions minorizing f, that is,

$$\mathcal{E}_f := \left\{ a : X \to \overline{\mathbb{R}} \mid a \text{ is e-affine and } a \le f \right\}.$$

Theorem 2.5. *Let $f : X \to \overline{\mathbb{R}}$, f not identically $+\infty$ or $-\infty$. Then f is a proper e-convex function if and only if*

$$f = \sup \left\{ a \mid a \in \mathcal{E}_f \right\}. \tag{2}$$

Definition 2.6. A set $D \subset W \times \mathbb{R}$ is *e′-convex* if there exists an e′-convex function $k : W \to \overline{\mathbb{R}}$ such that $D = \operatorname{epi} k$. The *e′-convex hull* of an arbitrary set $D \subset W \times \mathbb{R}$ is defined as the smallest e′-convex set containing D, and it will be denoted by $\mathrm{e}'\mathrm{co}\, D$.

Remark 2.7. Actually $\mathrm{e}'\mathrm{co}\, D$ is the epigraph of the e′-convex hull of the function $f_D : W \to \overline{\mathbb{R}}$ defined by $f_D(x^*, y, \alpha) := \inf\{a \in \mathbb{R} \mid (x^*, y^*, \alpha, a) \in D\}$. Hence,

$$\mathrm{e}'\mathrm{co}\, D = \operatorname{epi}(\mathrm{e}'\mathrm{co}\, f_D) = \operatorname{epi} f_D^{c'c}.$$

Definition 2.8. Consider two functions $f, g : X \to \overline{\mathbb{R}}$. A function $a : X \to \overline{\mathbb{R}}$ belongs to the set $\widetilde{\mathcal{E}}_{f,g}$ if there exist $a_1 \in \mathcal{E}_f$, $a_2 \in \mathcal{E}_g$ such that, if

$$a_1(\cdot) = \begin{cases} \langle \cdot, x_1^* \rangle - \beta_1 & \text{if } \langle \cdot, y_1^* \rangle < \alpha_1, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\text{and} \quad a_2(\cdot) = \begin{cases} \langle \cdot, x_2^* \rangle - \beta_2 & \text{if } \langle \cdot, y_2^* \rangle < \alpha_2, \\ +\infty & \text{otherwise,} \end{cases}$$

then

$$a(\cdot) = \begin{cases} \langle \cdot, x_1^* + x_2^* \rangle - (\beta_1 + \beta_2) & \text{if } \langle \cdot, y_1^* + y_2^* \rangle < \alpha_1 + \alpha_2, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 2.9. *Let $f, g : X \to \overline{\mathbb{R}}$ be proper e-convex functions such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, *and let* $h := \sup \left\{ a \mid a \in \widetilde{\mathcal{E}}_{f,g} \right\}$. *Then*

$$\mathrm{e}'\mathrm{co}\,(\operatorname{epi} f^c + \operatorname{epi} g^c) = \operatorname{epi} h^c.$$

Finally, we recall from [12] the subdifferentiability notion associated with the conjugation pattern described in this paper.

Definition 2.10. Let $f : X \to \overline{\mathbb{R}}$ be a function. We will say that the vector $(u^*, v^*, \omega) \in W$ is an c-*subgradient* of f at $x_0 \in X$ if $f(x_0) \in \mathbb{R}$, $\langle x_0, v^* \rangle < \omega$ and

$$f(x) - f(x_0) \geq c\,(x, (u^*, v^*, \omega)) - c\,(x_0, (u^*, v^*, \omega))\,, \quad \forall x \in X.$$

We denote by $\partial_c f\,(x_0)$ the set of all the c-subgradients of f at x_0, and it will be called the c-*subdifferential* of f at x_0.

3. New properties of e-convex sets and functions

In [16] the notions of closed convex set and lsc convex function are generalized in Sections 1.2 and 2.2. Again for the sake of completness, we give the most important concepts and their relations for our purposes. All of them are valid in a general topological vector space, named X.

Let $A \subset X$, $A \neq \emptyset$. A *convex series with elements of* A is a series of the form $\Sigma_{k\geq 1}\lambda_k x_k$ with $\{\lambda_k\} \subset \mathbb{R}_+$, $\{x_k\} \subset A$ and $\Sigma_{k\geq 1}\lambda_k = 1$. If the sequence $\{x_k\}$ is bounded, we deal with *b-convex series*. The set A is *cs-closed* (*ideally convex*, respectively) if any convergent convex (b-convex, respectively) series with elements of A has its sum in A. It is clear that every cs-closed set is ideally convex and every ideally convex set is convex, but the converse statements are not true. We also assume that the empty set is cs-closed.

The class of cs-closed sets contains the open and the closed convex sets. Since the latter sets are also e-convex, we ask about the relationship between cs-closed sets and e-convex sets. The proof of the following proposition is similar to the proof in [16, Prop. 1.2.1] when the set A is convex and open.

Proposition 3.1. *If $A \subset X$ is e-convex then it is cs-closed.*

The converse statement is not true, as the following example shows.

Example 3.2. Let $A = \{(x, y)\,|x > 0, y \geq 0\} \cup \{(0, 0)\} \subset \mathbb{R}^2$. This set is not e-convex but it is cs-closed. Actually, any finite-dimensional convex set in a separated topological vector space is cs-closed, as it is shown in [16, Prop. 1.2.1].

The classification for convex sets which we focus is as follows:

$$\begin{aligned} &A \text{ is closed and convex} \Longrightarrow A \text{ is e-convex} \\ &\Longrightarrow A \text{ is cs-closed} \Longrightarrow A \text{ is ideally convex,} \end{aligned}$$

and the converse statements are not true.

This convex sets classification leads to a convex functions classification in the usual way: a function $f : X \to \overline{\mathbb{R}}$ is *cs-closed* (*ideally convex*, respectively) if the set $\operatorname{epi} f$ is cs-closed (ideally convex, respectively). Moreover, f is *cs-convex* if for any convergent convex series $\Sigma_{k\geq 1}\lambda_k x_k$ with sum $\overline{x} \in X$ it holds

$$f\,(\overline{x}) \leq \liminf \Sigma_{k\geq 1}\lambda_k f\,(x_k)\,.$$

Any lsc convex function is cs-convex, and any cs-convex function is cs-closed, which is ideally convex. The converse statements are not true. Where are the e-convex functions placed?

Proposition 3.3. *If $f : X \to \overline{\mathbb{R}}$ is proper and e-convex then it is cs-convex.*

Proof. We will use [16, Prop. 2.2.17], where it is shown that any cs-closed function which is minorized by a continuous affine one is cs-convex. According to Proposition 3.1, this result is also valid for e-convex functions.

From (2), we can take any $a \in \mathcal{E}_f$ and $a(x) \leq f(x)$, for all $x \in X$, where

$$a(x) = \begin{cases} \langle x, x^* \rangle - \beta & \text{if } \langle x, y^* \rangle < \alpha, \\ +\infty & \text{otherwise.} \end{cases}$$

Hence $f(x) \geq \langle x, x^* \rangle - \beta$, for all $x \in X$, and f is cs-convex. □

It is necessary for an e-convex function to be proper in order to get cs-convexity.

Example 3.4. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be

$$f(x) = \begin{cases} -\infty & \text{if } x < 0, \\ +\infty & \text{otherwise.} \end{cases}$$

We have $\operatorname{epi} f =]-\infty, 0[\times \mathbb{R}$ which is an e-convex set. Now, taking $\overline{x} = 0$ and $\lambda_k = \left(\frac{1}{2}\right)^k$, $x_k = (-1)^k (3k - 2)$ for all $k \in \mathbb{N}$, we get that the convex series $\Sigma_{k \geq 1} \lambda_k x_k$ has sum $\overline{x}$, but $f(\overline{x}) = +\infty$ and $\liminf \Sigma_{k \geq 1} \lambda_k f(x_k) = -\infty$. Hence f is not cs-convex.

The converse statement of the above proposition is not true.

Example 3.5. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1, \\ 1 & \text{if } x = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\operatorname{epi} f = ([0, 1] \times [0, +\infty[) \setminus \{0\} \times [0, 1[$ which is not an e-convex set. On the other hand, f is cs-closed, in virtue of [16, Prop. 1.2.1]. We can also find a continuous affine minorant for f, for instance, $\varphi(x) = -x$. Hence it is cs-convex.

Therefore, if $f : X \to \overline{\mathbb{R}}$ is a proper function, we get

$$\begin{aligned} f \text{ is lsc and convex} &\Longrightarrow f \text{ is e-convex} \Longrightarrow f \text{ is cs-convex} \\ &\Rightarrow f \text{ is cs-closed} \Longrightarrow f \text{ is ideally convex} \Rightarrow f \text{ is convex,} \end{aligned}$$

and the converse statements are not true.

4. Interior point regularity conditions

Let $F : X \to \overline{\mathbb{R}}$ be a proper function and $\Phi : X \times \Theta \to \overline{\mathbb{R}}$ a perturbation function. Denoting by $Z = X \times \Theta$, the c-conjugate of the perturbation function Φ, $\Phi^c : Z^* \times Z^* \times \mathbb{R} \to \overline{\mathbb{R}}$, is defined by

$$\Phi^c\left(\left(x^*, u^*\right), \left(y^*, v^*\right), \alpha\right) = \sup_{(x,u)\in Z} \left\{\bar{c}\left((x,u), \left(\left(x^*, u^*\right), \left(y^*, v^*\right), \alpha\right)\right) - \Phi\left(x,u\right)\right\},$$

where $\bar{c} : Z \times Z^* \times Z^* \times \mathbb{R} \to \overline{\mathbb{R}}$ is

$$\bar{c}\left((x,u), \left(\left(x^*, u^*\right), \left(y^*, v^*\right), \alpha\right)\right) = \begin{cases} \langle x, x^*\rangle + \langle u, u^*\rangle & \text{if } \langle x, y^*\rangle + \langle u, v^*\rangle < \alpha, \\ +\infty & \text{otherwise.} \end{cases}$$

Following the same steps than in [6], we associate to the general problem

$$\begin{array}{lll} (GP) & \text{Inf} & F(x) \\ & \text{s.t.} & x \in X, \end{array}$$

the following dual problem

$$\begin{array}{lll} (GD_c) & \text{Sup} & -\Phi^c\left(\left(0, u^*\right), \left(0, v^*\right), \alpha\right) \\ & \text{s.t.} & u^*, v^* \in \Theta^*, \\ & & \alpha > 0, \end{array}$$

verifying $v(GD_c) \leq v(GP)$.

The dual problem can also be expressed by means of the *infimum value function,* $p : \Theta \to \overline{\mathbb{R}}$,

$$p\left(u\right) := \operatorname*{Inf}_{x\in X} \Phi\left(x,u\right),$$

and $p\left(0\right) = v(GP)$ whereas $\Phi^c\left(\left(0, u^*\right), \left(0, v^*\right), \alpha\right) = p^c\left(u^*, v^*, \alpha\right)$, for all $u^*, v^* \in \Theta^*$, $\alpha > 0$. Then (GD_c) can be rewritten as

$$\begin{array}{lll} (GD_c) & \text{Sup} & -p^c\left(u^*, v^*, \alpha\right) \\ & \text{s.t.} & u^*, v^* \in \Theta^*, \\ & & \alpha > 0. \end{array}$$

We focus in obtaining interior point regularity conditions for strong duality for (GP) and (GD_c). It is evident that strong duality holds if $v(GP) = -\infty$, hence we deal with the case $v(GP) \in \mathbb{R}$.

Proposition 4.1. *If $v(GP) \in \mathbb{R}$ then $v(GD_c) = p^{cc'}\left(0\right)$.*

Proof. Since, for all $\alpha \leq 0$ and $u^*, v^* \in \Theta^*$ it holds $c\left(0, \left(u^*, v^*, \alpha\right)\right) = +\infty$, and moreover $p\left(0\right) \in \mathbb{R}$, we have

$$p^c\left(u^*, v^*, \alpha\right) \geq c\left(0, \left(u^*, v^*, \alpha\right)\right) - p\left(0\right) = +\infty,$$

and, for all $\alpha > 0$ and $u^*, v^* \in \Theta^*$ one has $c\left(0, (u^*, v^*, \alpha)\right) = 0$, hence

$$\begin{aligned} v(GD_c) &= \sup_{\substack{u^*, v^* \in \Theta^* \\ \alpha > 0}} -p^c\left(u^*, v^*, \alpha\right) \\ &= \sup_{\substack{u^*, v^* \in \Theta^* \\ \alpha \in \mathbb{R}}} \left\{c\left(0, (u^*, v^*, \alpha)\right) - p^c\left(u^*, v^*, \alpha\right)\right\} = p^{cc'}\left(0\right). \end{aligned} \qquad \square$$

This proposition allows us to get the following result.

Proposition 4.2. *Let us assume that $v(GP) \in \mathbb{R}$. Then $v(GD_c) = v(GP)$ if and only if p is e-convex at 0.*

Proof. Assume first that $v(GD_c) = v(GP)$. Because $v(GD_c) = p^{cc'}(0)$, by Proposition 4.1, and $v(GP) = p\left(0\right)$, $v(GD_c) = v(GP)$ is equivalent to $p\left(0\right) = p^{cc'}\left(0\right)$. Taking into account Remark 2.3, it is also necessary for p to be proper in order to get the e-convexity at 0. Now, according to Proposition 2.2, $p^{cc'}$ is an e-convex function. Moreover, it is proper, due to the fact that $p^{cc'}\left(0\right) \in \mathbb{R}$ and taking into account [15, Lemma 2.5][1]. Then, since

$$p^{cc'}\left(u\right) \leq p\left(u\right),$$

for all $u \in \Theta$, p is also proper.

The converse statement comes directly from the definition of e-convexity of a function at a point and Proposition 4.1. $\square$

Proposition 4.3. *Let us assume that $v(GP) \in \mathbb{R}$. Then (GP) and (GD_c) verifies strong duality if and only if $\partial_c p\left(0\right) \neq \emptyset$; in this case, $\partial_c p\left(0\right)$ is the solution set of (GD_c).*

Proof. Assuming that $v(GD_c) = v(GP) \in \mathbb{R}$ and that $(\overline{u}^*, \overline{v}^*, \overline{\alpha})$ is a solution of (GD_c), where $\overline{u}^*, \overline{v}^* \in \Theta^*$ and $\overline{\alpha} > 0$, we have

$$v(GD_c) = p^{cc'}\left(0\right) = -p^c\left(\overline{u}^*, \overline{v}^*, \overline{\alpha}\right) = v(GP) = p\left(0\right).$$

Hence

$$p\left(0\right) = -\sup_{u \in \Theta}\left\{c\left(u, (\overline{u}^*, \overline{v}^*, \overline{\alpha})\right) - p\left(u\right)\right\} = \inf_{u \in \Theta}\left\{p\left(u\right) - c\left(u, (\overline{u}^*, \overline{v}^*, \overline{\alpha})\right)\right\}.$$

Thus $p\left(0\right) \leq p\left(u\right) - c\left(u, (\overline{u}^*, \overline{v}^*, \overline{\alpha})\right)$, for all $u \in \Theta$. Moreover $\overline{\alpha} > 0$, which means that $(\overline{u}^*, \overline{v}^*, \overline{\alpha}) \in \partial_c p\left(0\right)$ and this set is not empty.

On the other hand, if we assume that $\partial_c p\left(0\right) \neq \emptyset$, let us take $(\overline{u}^*, \overline{v}^*, \overline{\alpha}) \in \partial_c p\left(0\right)$. Then $\overline{\alpha} > 0$ and

$$p\left(0\right) \leq p\left(u\right) - c\left(u, (\overline{u}^*, \overline{v}^*, \overline{\alpha})\right),$$

[1]This lemma is stated in the case $X = \mathbb{R}^n$ but it can be easily extended to the framework of locally convex spaces.

for all $u \in \Theta$. Hence

$$p(0) \leq \inf_{u \in \Theta} \{p(u) - c(u, (\overline{u}^*, \overline{v}^*, \overline{\alpha}))\} = -p^c(\overline{u}^*, \overline{v}^*, \overline{\alpha}).$$

We obtain $v(GP) = p(0) \leq -p^c(\overline{u}^*, \overline{v}^*, \overline{\alpha}) \leq v(GD_c)$. Thus, $v(GD_c) = v(GP)$ and $(\overline{u}^*, \overline{v}^*, \overline{\alpha})$ is a solution of (GD_c). □

Remark 4.4. The obtained results in the three propositions above can be derived as a particular case in [11], from Proposition 6.4, Theorem 6.7 and Corollary 6.2, because the dual problem with $u_0 = 0$ suggested by Martínez-Legaz is equivalent to (GD_c), as it is shown in [6]. We have decided to include them in this work for the sake of completness.

According to Proposition 4.3, strong duality for (GP) and (GD_c) will hold if and only if $\partial_c p(0) \neq \emptyset$, which is also equivalent to $\partial p(0) \neq \emptyset$, taking into account Corollary 46 in [12]. Hence any regularity condition which guarantees that $\partial p(0) \neq \emptyset$ is also valid in our setting. In [16], we can find nine conditions of this kind when the perturbation function Φ is proper and convex, and $0 \in \mathrm{Pr}_\Theta(\mathrm{dom}\,\Phi)$, where $\mathrm{Pr}_\Theta : X \times \Theta \to \Theta$ is the projection operator on Θ. Some of these conditions are expressed in terms of different extensions of the classical interior of a set.

Let X be a topological vector space and $U \subset X$ any subset of X. We denote by $\mathrm{cone}\,U$ the conical hull of U. If $A \subset X$ is a nonempty convex subset in X, the *algebraic relative interior* of A is the set of all the points $a \in A$ verifying $\mathrm{cone}(A - a)$ is a linear subspace of X. It is denoted by ${}^iA$. Moreover, a point $a \in {}^{ic}A$ if $\mathrm{cone}(A - a)$ is a closed linear subspace of X, and $a \in {}^{ib}A$ if $\mathrm{cone}(A - a)$ is a barreled linear subspace of X.

Let us consider the following regularity conditions from [16]:

(C1) there exist $\lambda_0 \in \mathbb{R}$ and $x_0 \in X$ such that for any neighbourhood U in X, the set

$$\{u \in \Theta : \exists x \in x_0 + U, \Phi(x, u) \leq \lambda_0\}$$

is a neighbourhood of 0 in the lineality space generated by $\mathrm{Pr}_\Theta(\mathrm{dom}\,\Phi)$.

(C2) X and Θ are metrizable, $0 \in {}^{ib}(\mathrm{Pr}_\Theta(\mathrm{dom}\,\Phi))$ and $\mathrm{epi}\,\Phi$ verifies the following condition: for any sequences $\{(x_k, u_k, \gamma_k)\}_{k\geq 1} \subset \mathrm{epi}\,\Phi$ and $\{\lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ such that $\{(x_k, u_k, \gamma_k)\}_{k\geq 1}$ is bounded, $\Sigma_{k\geq 1}\lambda_k = 1$, $\Sigma_{k\geq 1}\lambda_k(u_k, \gamma_k) = (u, \gamma)$ and $\Sigma_{k\geq 1}\lambda_k x_k$ is Cauchy, it holds that $\Sigma_{k\geq 1}\lambda_k x_k$ is convergent and its sum $x \in X$ verifies $(x, u, \gamma) \in \mathrm{epi}\,\Phi$.

(C3) X and Θ are Fréchet spaces, Φ is lsc and $0 \in {}^{ic}(\mathrm{Pr}_\Theta(\mathrm{dom}\,\Phi))$.

We have (C3) $\Longrightarrow$ (C2) $\Rightarrow$ (C1) $\Longrightarrow$ $\partial p(0) \neq \emptyset$ if $p(0) \in \mathbb{R}$.

We formulate the following new condition:

(C4) X and Θ are Fréchet spaces, Φ is e-convex and $0 \in {}^{ic}(\mathrm{Pr}_\Theta(\mathrm{dom}\,\Phi))$.

Proposition 4.5. *Condition* (C4) *implies* (C2) *and is implied by* (C3)*. Hence under* (C4)*, $\partial p(0) \neq \emptyset$ if $p(0) \in \mathbb{R}$ and strong duality for (GP) and (GD_c) holds. Moreover, $\overline{x} \in X$ is a minimum of $\Phi(\cdot, 0)$ if and only if there exist $\overline{u}^*, \overline{v}^* \in \Theta^*$ and $\overline{\alpha} > 0$ such that $((0, \overline{u}^*), (0, \overline{v}^*), \overline{\alpha}) \in \partial_c \Phi(\overline{x}, 0)$.*

Proof. It is evident that (C4) is implied by (C3). On the other hand, if (C4) holds, since X and Θ are Fréchet spaces, they are metrizable. Now, because $\operatorname{epi} \Phi \subset X \times \Theta \times \mathbb{R}$ is e-convex, it is ideally convex and the condition for it in (C2) is satisfied also due to the completness of X. The inclusion $0 \in {}^{ib}(\operatorname{Pr}_\Theta(\operatorname{dom} \Phi))$ in (C2) is also true since any closed linear subspace in a Fréchet space is barreled. Hence the fulfilment of condition (C4) guarantees strong duality for (GP) and (GD_c).

Let $(\overline{u}^*, \overline{v}^*, \overline{\alpha})$ be a solution of (GD_c), where $\overline{u}^*, \overline{v}^* \in \Theta^*$ and $\overline{\alpha} > 0$. We have that $\overline{x} \in X$ is a minimum of $\Phi(\cdot, 0)$ if and only if $v(GP) = p(0) = \Phi(\overline{x}, 0) \in \mathbb{R}$. Since

$$v(GD_c) = -\Phi^c((0, \overline{u}^*), (0, \overline{v}^*), \overline{\alpha}),$$

we get that $v(GP) = \Phi(\overline{x}, 0)$ if and only if $-\Phi^c((0, \overline{u}^*), (0, \overline{v}^*), \overline{\alpha}) = \Phi(\overline{x}, 0)$, which is equivalent to

$$\bar{c}((x, u), ((0, \overline{u}^*), (0, \overline{v}^*), \overline{\alpha})) - \Phi(x, u) \leq -\Phi(\overline{x}, 0),$$

for all $(x, u) \in X \times \Theta$. Since $\overline{\alpha} > 0$, this means that $((0, \overline{u}^*), (0, \overline{v}^*), \overline{\alpha}) \in \partial_c \Phi(\overline{x}, 0)$. □

5. Closedness-type regularity conditions

Let us consider the same framework than in the previous section. Recalling the notation $W = X^* \times X^* \times \mathbb{R}$, we consider also the following functions:

$$\Phi(\cdot, 0) : X \to \overline{\mathbb{R}},$$
$$\Phi(\cdot, 0)^c : W \to \overline{\mathbb{R}},$$

and

$$\inf_{u^*, v^* \in \Theta^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot) : W \to \overline{\mathbb{R}}.$$

Lemma 5.1. *It holds*

$$\Phi(\cdot, 0)^c \leq \inf_{u^*, v^* \in \Theta^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot).$$

Proof. Let $(x^*, y^*, \alpha) \in W$. Then for all $u^*, v^* \in \Theta^*$ and $(x, u) \in X \times \Theta$, we have

$$\bar{c}((x, u), (x^*, u^*), (y^*, v^*), \alpha) - \Phi(x, u) \leq \Phi^c((x^*, u^*), (y^*, v^*), \alpha).$$

Taking $u = 0$, for all $x \in X$,

$$c(x, (x^*, y^*, \alpha)) - \Phi(x, 0) \leq \Phi^c((x^*, u^*), (y^*, v^*), \alpha).$$

Hence

$$\begin{aligned}\Phi(\cdot,0)^c(x^*,y^*,\alpha) &= \sup_{x\in X}\{c(x,(x^*,y^*,\alpha))-\Phi(x,0)\}\\ &\leq \inf_{u^*,v^*\in\Theta^*}\Phi^c((x^*,u^*),(y^*,v^*),\alpha). \qquad \square\end{aligned}$$

Lemma 5.2. *If Φ is a proper e-convex function then $\Phi(\cdot,0)^c$ is the e'-convex hull of the function $\inf_{u^*,v^*\in\Theta^*}\Phi^c((\cdot,u^*),(\cdot,v^*),\cdot)$.*

Proof. For all $x\in X$, it holds

$$\begin{aligned}&\left(\inf_{u^*,v^*\in\Theta^*}\Phi^c((\cdot,u^*),(\cdot,v^*),\cdot)\right)^{c'}(x)\\ &= \sup_{(x^*,y^*,\alpha)\in W}\left\{c(x,(x^*,y^*,\alpha))-\inf_{u^*,v^*\in\Theta^*}\Phi^c((x^*,u^*),(y^*,v^*),\alpha)\right\}\\ &= \sup_{\substack{(x^*,y^*,\alpha)\in W\\ u^*,v^*\in\Theta^*}}\{c(x,(x^*,y^*,\alpha))-\Phi^c((x^*,u^*),(y^*,v^*),\alpha)\}.\end{aligned}$$

Taking into account that $c(x,(x^*,y^*,\alpha))=\bar{c}((x,0),(x^*,u^*),(y^*,v^*),\alpha)$, for all $(x^*,y^*,\alpha)\in W$ and $u^*,v^*\in\Theta^*$, we get

$$\left(\inf_{u^*,v^*\in\Theta^*}\Phi^c((\cdot,u^*),(\cdot,v^*),\cdot)\right)^{c'}(x)=(\Phi^c)^{c'}(x,0).$$

Since Φ is proper and e-convex, we have $(\Phi^c)^{c'}=\Phi$, hence $(\inf_{u^*,v^*\in\Theta^*}\Phi^c((\cdot,u^*),(\cdot,v^*),\cdot))^{c'}=\Phi(\cdot,0)$. Now, according to Proposition 2.2 (ii), we obtain

$$\begin{aligned}&\mathrm{e}'\mathrm{co}\left(\inf_{u^*,v^*\in\Theta^*}\Phi^c((\cdot,u^*),(\cdot,v^*),\cdot)\right)\\ &= \left(\inf_{u^*,v^*\in\Theta^*}\Phi^c((\cdot,u^*),(\cdot,v^*),\cdot)\right)^{c'c}=\Phi(\cdot,0)^c. \qquad \square\end{aligned}$$

Lemma 5.3. *If Φ is a proper e-convex function then $\operatorname{epi}\Phi(\cdot,0)^c$ is the e'-convex hull of $\operatorname{Pr}_{W\times\mathbb{R}}(\operatorname{epi}\Phi^c)$.*

Proof. First, we shall prove that $\operatorname{Pr}_{W\times\mathbb{R}}(\operatorname{epi}\Phi^c)\subset\operatorname{epi}\Phi(\cdot,0)^c$.

Take any point $(x^*,y^*,\alpha,\beta)\in\operatorname{Pr}_{W\times\mathbb{R}}(\operatorname{epi}\Phi^c)$. Then there exist $u^*,v^*\in\Theta^*$ such that, for all $(x,u)\in X\times\Theta$,

$$\bar{c}((x,u),(x^*,u^*),(y^*,v^*),\alpha)-\Phi(x,u)\leq\beta.$$

Setting $u=0$, we get $\Phi(\cdot,0)^c(x^*,y^*,\alpha)\leq\beta$ and $(x^*,y^*,\alpha,\beta)\in\operatorname{epi}\Phi(\cdot,0)^c$.

Now, according to Remark 2.7, we have $\mathrm{e}'\mathrm{co}\left(\mathrm{Pr}_{W\times\mathbb{R}}\left(\mathrm{epi}\,\Phi^c\right)\right) = \mathrm{epi}\,g^{c'c}$, where $g : W \to \overline{\mathbb{R}}$ is defined by

$$g\left(x^*, y^*, \alpha\right) = \inf\left\{\beta \in \mathbb{R} \,|\, \exists u^*, v^* \in \Theta^*, \Phi^c\left(\left(x^*, u^*\right), \left(y^*, v^*\right), \alpha\right) \leq \beta\right\}.$$

It is clear that in the case $\inf_{u^*,v^*\in\Theta^*} \Phi^c\left(\left(x^*, u^*\right), \left(y^*, v^*\right), \alpha\right) = -\infty$, it holds $g\left(x^*, y^*, \alpha\right) = -\infty$. Assuming that $\inf_{u^*,v^*\in\Theta^*} \Phi^c\left(\left(x^*, u^*\right), \left(y^*, v^*\right), \alpha\right) = \gamma \in \mathbb{R}$, we can take a sequence $\{(u_k^*, v_k^*)\} \subset \Theta^* \times \Theta^*$ such that for all $k \in \mathbb{N}$,

$$\Phi^c\left(\left(x^*, u_k^*\right), \left(y^*, v_k^*\right), \alpha\right) \leq \gamma + \frac{1}{k},$$

and it means that $g\left(x^*, y^*, \alpha\right) \leq \gamma + \frac{1}{k}$, for all $k \in \mathbb{N}$, or, equivalently, $g\left(x^*, y^*, \alpha\right) \leq \gamma$.

In any case, for all $(x^*, y^*, \alpha) \in W$,

$$g\left(x^*, y^*, \alpha\right) \leq \inf_{u^*,v^*\in\Theta^*} \Phi^c\left(\left(x^*, u^*\right), \left(y^*, v^*\right), \alpha\right).$$

We obtain, for all $x \in X$,

$$\begin{aligned} g^{c'}(x) &= \sup_{(x^*,y^*,\alpha)\in W} \left\{c\left(x, \left(x^*, y^*, \alpha\right)\right) - g\left(x^*, y^*, \alpha\right)\right\} \\ &\geq \sup_{\substack{(x^*,y^*,\alpha)\in W \\ u^*,v^*\in\Theta^*}} \left\{\bar{c}\left((x, 0), \left(x^*, u^*\right), \left(y^*, v^*\right), \alpha\right) - \Phi^c\left(\left(x^*, u^*\right), \left(y^*, v^*\right), \alpha\right)\right\} \\ &= \left(\Phi^c\right)^{c'}(x, 0) = \Phi(x, 0). \end{aligned}$$

It implies that $g^{c'c} \leq \Phi\left(\cdot, 0\right)^c$ and

$$\mathrm{e}'\mathrm{co}\left(\mathop{\mathrm{Pr}}_{W\times\mathbb{R}}\left(\mathrm{epi}\,\Phi^c\right)\right) \subset \mathrm{epi}\,\Phi\left(\cdot, 0\right)^c \subset \mathrm{epi}\,g^{c'c},$$

and the proof is completed. □

Now we are able to state the following closedness-type regularity condition:

(C5) Φ is a proper e-convex function and $\mathrm{Pr}_{W\times\mathbb{R}}\left(\mathrm{epi}\,\Phi^c\right)$ is e$'$-convex.

The following proposition presents an alternative formulation of (C5).

Proposition 5.4. *If Φ is a proper e-convex function then $\mathrm{Pr}_{W\times\mathbb{R}}\left(\mathrm{epi}\,\Phi^c\right)$ is e$'$-convex if and only if*

$$\Phi\left(\cdot, 0\right)^c = \min_{u^*,v^*\in\Theta^*} \Phi^c\left(\left(\cdot, u^*\right), \left(\cdot, v^*\right), \cdot\right).$$

Proof. We have $\mathrm{Pr}_{W\times\mathbb{R}}\left(\mathrm{epi}\,\Phi^c\right)$ is e$'$-convex if and only if $\mathrm{Pr}_{W\times\mathbb{R}}\left(\mathrm{epi}\,\Phi^c\right) = \mathrm{epi}\,\Phi\left(\cdot, 0\right)^c$, according to Lemma 5.3. This is equivalent to say that, for all

$(x^*, y^*, \alpha) \in \operatorname{dom} \Phi(\cdot, 0)^c$, there exist $\overline{u}^*, \overline{v}^* \in \Theta^*$ such that $\Phi^c((x^*, \overline{u}^*), (y^*, \overline{v}^*), \alpha) \leq \Phi(\cdot, 0)^c (x^*, y^*, \alpha)$, which means that

$$\inf_{u^*, v^* \in \Theta^*} \Phi^c((x^*, u^*), (y^*, v^*), \alpha) \leq \Phi^c((x^*, \overline{u}^*), (y^*, \overline{v}^*), \alpha) \leq \Phi(\cdot, 0)^c (x^*, y^*, \alpha).$$

Applying Lemma 5.1 we get the result.

In the case $(x^*, y^*, \alpha) \notin \operatorname{dom} \Phi(\cdot, 0)^c$, $\sup_{x \in X} \{c(x, (x^*, y^*, \alpha)) - \Phi(x, 0)\} = +\infty$ and the conclusion is trivial. □

Proposition 5.5. (C5) *is a sufficient condition for strong duality for* (GP) *and* (GD_c).

Proof. Taking into consideration Remark 2.7 and Lemma 5.2, we obtain

$$\mathrm{e}'\mathrm{co}\left(\operatorname{epi} \inf_{u^*, v^* \in \Theta^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot)\right) = \operatorname{epi} \Phi(\cdot, 0)^c.$$

Moreover, from Lemma 5.3, we have

$$\mathrm{e}'\mathrm{co}\left(\Pr_{W \times \mathbb{R}} (\operatorname{epi} \Phi^c)\right) = \operatorname{epi} \Phi(\cdot, 0)^c.$$

Since (C5) holds, we get

$$\operatorname{epi} \Phi(\cdot, 0)^c = \Pr_{W \times \mathbb{R}} (\operatorname{epi} \Phi^c) = \mathrm{e}'\mathrm{co}\left(\operatorname{epi} \inf_{u^*, v^* \in \Theta^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot)\right).$$

It is easy to see that $\Pr_{W \times \mathbb{R}} (\operatorname{epi} \Phi^c) \subset \operatorname{epi}(\inf_{u^*, v^* \in \Theta^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot))$, hence

$$\inf_{u^*, v^* \in \Theta^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot) = \Phi(\cdot, 0)^c. \tag{3}$$

Now, let us assume that $v(GP) = \gamma \in \mathbb{R}$. Then $-\gamma = \sup_{x \in X} -\Phi(x, 0)$.

We have, for all $\alpha > 0$,

$$-\gamma = \sup_{x \in X} \{c(x, (0, 0, \alpha)) - \Phi(x, 0)\} = \Phi(\cdot, 0)^c (0, 0, \alpha).$$

By (3), for all $\alpha > 0, -\gamma = \inf_{u^*, v^* \in \Theta^*} \Phi^c((0, u^*), (0, v^*), \alpha)$, hence, we can write

$$-\gamma = \inf_{\substack{u^*, v^* \in \Theta^* \\ \alpha > 0}} \Phi^c((0, u^*), (0, v^*), \alpha) = -v(GD_c). \tag{4}$$

Then we get $v(GP) = v(GD_c)$.

On the other hand, for all $\alpha > 0$, it holds

$$(0, 0, \alpha, -\gamma) \in \operatorname{epi} \inf_{u^*, v^* \in \Theta^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot) = \Pr_{W \times \mathbb{R}} (\operatorname{epi} \Phi^c).$$

Taking any $\overline{\alpha} > 0$, there exist $\overline{u}^*, \overline{v}^* \in \Theta^*$ such that $\Phi^c((0, \overline{u}^*), (0, \overline{v}^*), \overline{\alpha}) \leq -\gamma$, and by (4), we get that $(\overline{u}^*, \overline{v}^*, \overline{\alpha})$ is a solution of (GD_c). □

6. Regularity conditions for Fenchel duality

Let us consider the following optimization problem

$$\begin{array}{lll} (P) & \text{Inf} & f(x) \\ & \text{s.t.} & x \in A, \end{array}$$

where $A \subset X$ is a nonempty e-convex set and $f : X \to \overline{\mathbb{R}}$ is a proper e-convex function, with $\operatorname{dom} f \cap A \neq \emptyset$. The problem (P) is a particular case of (GP) with $F = f + \delta_A$, where δ_A is the indicator function of A.

We will consider the perturbation function $\Phi : X \times X \to \overline{\mathbb{R}}$ given by

$$\Phi(x, u) := \begin{cases} f(x+u) & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

In [6] the dual problem for (P) was obtained

$$\begin{array}{lll} (D) & \text{Sup} & \{-f^c(u^*, v^*, \alpha_1) - \delta_A^c(-u^*, -v^*, \alpha_2)\} \\ & \text{s.t.} & u^*, v^* \in X^*, \\ & & \alpha_1 + \alpha_2 > 0, \end{array}$$

together with the following closedness-type regularity condition:

(C6) $f + \delta_A = \sup\left\{a \mid a \in \widetilde{\mathcal{E}}_{f,\delta_A}\right\}$ and $\operatorname{epi} f^c + \operatorname{epi} \delta_A^c$ is e′-convex.

We need to reformulate (C4) and (C5) in this particular setting. First, we establish the following result.

Proposition 6.1. *The function* Φ *defined in* (5) *is proper and e-convex. Moreover,* $\operatorname{Pr}_X(\operatorname{dom}\Phi) = \operatorname{dom} f - A$.

Proof. Since $\operatorname{dom} f \cap A \neq \emptyset$, there exists $x \in A$ such that $\Phi(x, 0) < +\infty$, and clearly Φ does not take the value $-\infty$ because of the properness of f. Hence Φ is proper.

In order to prove that $\operatorname{epi}\Phi$ is e-convex, we shall use Proposition 2.1. Let $(\overline{x}, \overline{u}, \overline{\alpha}) \notin \operatorname{epi}\Phi$. We have two possibilities.

In the case $\overline{x} \notin A$, since it is e-convex, there exists $x^* \in X^* \setminus \{0\}$ verifying, for all $x \in A$,

$$\langle \overline{x} - x, x^* \rangle > 0.$$

Then, for all $(x, u, \alpha) \in \operatorname{epi}\Phi$,

$$\langle \overline{x} - x, x^* \rangle + \langle \overline{u} - u, 0 \rangle + (\overline{\alpha} - \alpha) \cdot 0 > 0.$$

If $\overline{x} \in A$ but $(\overline{x} + \overline{u}, \overline{\alpha}) \notin \operatorname{epi} f$, which means that there exist $x^* \in X^* \setminus \{0\}$ and $\delta \in \mathbb{R}$ verifying, for all $(x, \alpha) \in \operatorname{epi} f$,

$$\langle \overline{x} + \overline{u} - x, x^* \rangle + \delta \cdot (\overline{\alpha} - \alpha) > 0.$$

Since for all $(x,u,\alpha) \in \operatorname{epi}\Phi$, $(x+u,\alpha) \in \operatorname{epi} f$, we get

$$\langle \overline{x}-x, x^* \rangle + \langle \overline{u}-u, x^* \rangle + \delta \cdot (\overline{\alpha}-\alpha) > 0,$$

and hence $\operatorname{epi}\Phi$ is e-convex. Finally,

$$\begin{aligned}
\Pr_X(\operatorname{dom}\Phi) &= \{u \in X \mid \exists x \in A, \Phi(x,u) < +\infty\} \\
&= \{u \in X \mid \exists x \in A, x+u \in \operatorname{dom} f\} \\
&= \{u \in X \mid u = z-x, x \in A, z \in \operatorname{dom} f\} = \operatorname{dom} f - A. \qquad \square
\end{aligned}$$

Taking into account Proposition 6.1, conditions (C4) and (C5) in this section can be read as

($C4_F$) X is a Fréchet space and $0 \in {}^{ic}(\operatorname{dom} f - A)$.
($C5_F$) $\Pr_{W\times\mathbb{R}}(\operatorname{epi}\Phi^c)$ is e′-convex.

Our aim is to compare regularity conditions ($C4_F$), ($C5_F$) and (C6). As we shall see, the unique relationship between them is that $(C6)$ implies $(C5_F)$.

The following example shows that (C6) does not imply ($C4_F$).

Example 6.2. Let us take $X=\mathbb{R}$, $f=\delta_{[0,+\infty[}$ and $A =]-\infty,0]$.

Then $\operatorname{dom} f - A = [0,+\infty[= \operatorname{cone}[0,+\infty[$ is not a linear subspace of $\mathbb{R}$. Hence $0 \notin {}^{ic}(\operatorname{dom} f - A)$ and $(C4_F)$ does not hold.

We check now condition $(C6)$.

We have $f+\delta_A = \delta_{\{0\}}$ and let $h = \sup\left\{a \mid a \in \widetilde{\mathcal{E}}_{f,\delta_A}\right\}$. Now, an e-affine function $a_1 \in \mathcal{E}_f$ if and only if

$$a_1(x) = \begin{cases} \alpha_1 x - \beta_1 & \text{if } \gamma_1 x < \delta_1, \\ +\infty & \text{otherwise,} \end{cases}$$

with $\alpha_1 \leq 0$, $\beta_1 \geq 0$, $\gamma_1 \leq 0$ and $\delta_1 > 0$.

On the other hand, $a_2 \in \mathcal{E}_{\delta_A}$ if and only if

$$a_2(x) = \begin{cases} \alpha_2 x - \beta_2 & \text{if } \gamma_2 x < \delta_2, \\ +\infty & \text{otherwise,} \end{cases}$$

with $\alpha_2 \geq 0$, $\beta_2 \geq 0$, $\gamma_2 \geq 0$ and $\delta_2 > 0$.

Then, $a \in \widetilde{\mathcal{E}}_{f,\delta_A}$ if and only if

$$a(x) = \begin{cases} \alpha x - \beta & \text{if } \gamma x < \delta, \\ +\infty & \text{otherwise,} \end{cases}$$

with $\alpha, \gamma \in \mathbb{R}$, $\beta \geq 0$ and $\delta > 0$.

We obtain $h = \sup\left\{a \mid a \in \widetilde{\mathcal{E}}_{f,\delta_A}\right\} = \delta_{\{0\}} = f + \delta_A$. The following step is calculate $\operatorname{epi} f^c + \operatorname{epi} \delta_A^c$.

We have $(\alpha, \beta, \gamma, \delta) \in \operatorname{epi} f^c$ if and only if, for all $x \geq 0$, $\alpha x \leq \delta$ and $\beta x < \gamma$, hence $\operatorname{epi} f^c = \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_{++} \times \mathbb{R}_+$.

Similarly, $(\alpha, \beta, \gamma, \delta) \in \operatorname{epi} \delta_A^c$ if and only if, for all $x \leq 0$, $\alpha x \leq \delta$ and $\beta x < \gamma$, hence $\operatorname{epi} \delta_A^c = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$. We obtain

$$\operatorname{epi} f^c + \operatorname{epi} \delta_A^c = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+.$$

This set is e$'$-convex, according to [7, Prop. 3.6], which allows us to conclude that conditions in $(C6)$ will be fulfilled.

Proposition 6.3. *Regularity condition* (C6) *implies* (C5$_F$).

Proof. Let us assume that (C6) holds. Since $\Phi(\cdot, 0) = f + \delta_A$, we have $\operatorname{epi} \Phi(\cdot, 0)^c = \operatorname{epi}(f + \delta_A)^c$. On the other hand, by (C6), $f + \delta_A = h = \sup\left\{a \mid a \in \widetilde{\mathcal{E}}_{f,\delta_A}\right\}$, hence

$$\operatorname{epi} \Phi(\cdot, 0)^c = \operatorname{epi} h^c = \mathrm{e}'\mathrm{co}\left(\operatorname{epi} f^c + \operatorname{epi} \delta_A^c\right),$$

according to Theorem 2.9.

Again, by the fulfilment of (C6), $\operatorname{epi} f^c + \operatorname{epi} \delta_A^c$ is e$'$-convex, implying that $\operatorname{epi} \Phi(\cdot, 0)^c = \operatorname{epi} f^c + \operatorname{epi} \delta_A^c$.

By Lemma 5.3, $\operatorname{Pr}_{W \times \mathbb{R}}(\operatorname{epi} \Phi^c) \subset \operatorname{epi} \Phi(\cdot, 0)^c = \operatorname{epi} f^c + \operatorname{epi} \delta_A^c$.

We shall show that $\operatorname{epi} f^c + \operatorname{epi} \delta_A^c \subset \operatorname{Pr}_{W \times \mathbb{R}}(\operatorname{epi} \Phi^c)$, hence (C5$_F$) will fulfill.

Take any points $(x_1^*, y_1^*, \alpha_1, \beta_1) \in \operatorname{epi} f^c$ and $(x_2^*, y_2^*, \alpha_2, \beta_2) \in \operatorname{epi} \delta_A^c$. We obtain, for all $x, u \in X$,

$$c(x, (x_1^*, y_1^*, \alpha_1)) + c(u, (x_2^*, y_2^*, \alpha_2)) - f(x) - \delta_A(u) \leq \beta_1 + \beta_2.$$

Taking into account that, for all $u, x \in X$,

$$\bar{c}((u, x), (x_2^*, x_1^*), (y_2^*, y_1^*), \alpha_1 + \alpha_2) \leq c(x, (x_1^*, y_1^*, \alpha_1)) + c(u, (x_2^*, y_2^*, \alpha_2)),$$

we obtain, that, for all $u, x \in X$,

$$\bar{c}((u, x), (x_2^*, x_1^*), (y_2^*, y_1^*), \alpha_1 + \alpha_2) - f(x) - \delta_A(u) \leq \beta_1 + \beta_2.$$

Let $z := x - u \in X$. Then

$$\bar{c}((u, u + z), (x_2^*, x_1^*), (y_2^*, y_1^*), \alpha_1 + \alpha_2) - f(u + z) - \delta_A(u) \leq \beta_1 + \beta_2,$$

for all $u, z \in X$, equivalently,

$$\bar{c}((u, z), (x_1^* + x_2^*, x_1^*), (y_1^* + y_2^*, y_1^*), \alpha_1 + \alpha_2) - \Phi(u, z) \leq \beta_1 + \beta_2.$$

Hence $\Phi^c((x_1^* + x_2^*, x_1^*), (y_1^* + y_2^*, y_1^*), \alpha_1 + \alpha_2) \leq \beta_1 + \beta_2$, which means that the point $(x_1^* + x_2^*, y_1^* + y_2^*, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$ belongs to $\operatorname{Pr}_{W \times \mathbb{R}}(\operatorname{epi} \Phi^c)$. $\square$

Let us recall that for proper convex functions $f, g : X \to \overline{\mathbb{R}}$, the *infimal convolution* of f with g, denoted by $f \square g : X \to \overline{\mathbb{R}}$, is defined by

$$(f \square g)(x) := \inf_{x_1 + x_2 = x} \{f(x_1) + g(x_2)\},$$

and it is said to be *exact at* $x \in X$ if $(f \square g)(x) = f(a) + g(x - a)$ for some $a \in X$. Moreover, the infimal convolution is *exact* if it is exact at any $x \in X$.

Proposition 6.4. *It holds*

$$f^c \square \delta_A^c \geq \inf_{u^*, v^* \in X^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot).$$

Proof. Let $(x^*, y^*, \alpha) \in W$. We have

$$\begin{aligned} & (f^c \square \delta_A^c)(x^*, y^*, \alpha) && (6)\\ = & \inf_{\substack{u^*, v^* \in X^*,\\ \alpha_1 + \alpha_2 = \alpha}} \{f^c(u^*, v^*, \alpha_1) + \delta_A^c(x^* - u^*, y^* - v^*, \alpha_2)\} \\ = & \inf_{\substack{u^*, v^* \in X^*,\\ \alpha_1 + \alpha_2 = \alpha}} \sup_{z, x \in X} \{c(z, (u^*, v^*, \alpha_1)) - f(z) \\ & \quad + c(x, (x^* - u^*, y^* - v^*, \alpha_2)) - \delta_A(x)\} \\ \geq & \inf_{u^*, v^* \in X^*} \sup_{x, z \in X} \{\bar{c}((x, z), (x^* - u^*, u^*), (y^* - v^*, v^*), \alpha) - f(z) - \delta_A(x)\}. \end{aligned}$$

Let $u := z - x$. Then

$$\begin{aligned} & (f^c \square \delta_A^c)(x^*, y^*, \alpha) \\ \geq & \inf_{u^*, v^* \in X^*} \sup_{x, u \in X} \{\bar{c}((x, u), (x^*, u^*), (y^*, v^*), \alpha) - f(x + u) - \delta_A(x)\} \\ = & \inf_{u^*, v^* \in X^*} \Phi^c((x^*, u^*), (y^*, v^*), \alpha). \end{aligned}$$

$\square$

Proposition 6.5. *Assuming that* $f + \delta_A = \sup\left\{a \mid a \in \widetilde{\mathcal{E}}_{f, \delta_A}\right\}$, *regularity condition* (C5$_F$) *implies* (C6) *if and only if*

$$f^c \square \delta_A^c = \min_{u^*, v^* \in X^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot). \tag{7}$$

Proof. Let us assume that $(C5_F)$ implies $(C6)$. From [6, Th. 11] and Proposition 5.4 we get

$$f^c \square \delta_A^c = (f + \delta_A)^c = \Phi(\cdot, 0)^c = \min_{u^*, v^* \in X^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot),$$

and (7) holds.

On the other hand, if (7) is true and $(C5_F)$ fulfills, we obtain

$$f^c \square \delta_A^c = \min_{u^*, v^* \in X^*} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot) = \Phi(\cdot, 0)^c = (f + \delta_A)^c,$$

and taking into account that inequalities in the proof of Proposition 6.4 are in fact equalities, we conclude that for all $(x^*, y^*, \alpha) \in W$ and for all $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 + \alpha_2 = \alpha$, there exists $(\overline{u}^*, \overline{v}^*) \in X^* \times X^*$ such that

$$(f^c \Box \delta_A^c)(x^*, y^*, \alpha) = f^c(\overline{u}^*, \overline{v}^*, \alpha_1) + \delta_A^c(x^* - \overline{u}^*, y^* - \overline{v}^*, \alpha_2),$$

and the infimal convolution is exact. Then, according to [6, Th. 11], condition $(C6)$ is fulfilled. $\square$

The following example shows that (C5$_F$) does not imply (C6).

Example 6.6. Let us take $X = \mathbb{R}$, $A := [0, +\infty[$ and $f = \delta_{]0,+\infty[}$.

It is easy to see that $f + \delta_A = \sup\left\{a \mid a \in \widetilde{\mathcal{E}}_{f,\delta_A}\right\}$.

The most important fact to prove that

$$(f^c \Box \delta_A^c)(x^*, y^*, \alpha) > \inf_{u^*, v^* \in \mathbb{R}} \Phi^c((x^*, u^*), (y^*, v^*), \alpha)$$

at some point $(x^*, y^*, \alpha) \in \mathbb{R}^3$, implying by the above proposition that (C6) does not hold, is to find $(y^*, \alpha) \in \mathbb{R}^2$ such that, if we have, for all $x \in A$ and $u \in \operatorname{dom} f$,

$$(y^* - v^*)x + v^* u < \alpha, \tag{8}$$

for some $v^* \in \mathbb{R}$, then

$$(y^* - w^*)x < \alpha_1 \quad \text{and} \quad w^* u < \alpha_2 \tag{9}$$

whatever $w^* \in \mathbb{R}$ implies $\alpha_1 + \alpha_2 > \alpha$. This will mean that $(f^c \Box \delta_A^c)(x^*, y^*, \alpha) = +\infty$.

Take $(y^*, \alpha) = (-1, 0)$. For any $-1 \leq v^* < 0$, (8) holds. But for any $w^* \in \mathbb{R}$ verifying (9) it must be $\alpha_1 + \alpha_2 > 0$, since $\alpha_1 > 0$ and $\alpha_2 \geq 0$ necessarily.

On the other hand, if $x^* \leq 0$, using (6) for the calculus of $\inf_{u^*, v^* \in \mathbb{R}} \Phi^c((\cdot, u^*), (\cdot, v^*), \cdot)$,

$$\inf_{u^*, v^* \in \mathbb{R}} \Phi^c((x^*, u^*), (-1, v^*), 0) = \inf_{u^* \in \mathbb{R}} \left\{\sup_{x \geq 0}(x^* - u^*)x + \sup_{z > 0} u^* z\right\} = 0.$$

Now we are going to check that

$$\sup_{x \in \mathbb{R}} \{c(x, (x^*, y^*, \alpha)) - \Phi(x, 0)\} = \min_{u^*, v^* \in \mathbb{R}} \Phi^c((x^*, u^*), (y^*, v^*), \alpha), \tag{10}$$

for all $(x^*, y^*, \alpha) \in \mathbb{R}^3$, which means that $(C5_F)$ holds, in virtue of Proposition 5.4.

In the case $\sup_{x \in \mathbb{R}} \{c(x, (x^*, y^*, \alpha)) - \Phi(x, 0)\} = +\infty$, (10) holds trivially, according to Lemma 5.1.

Hence let us assume that $\sup_{x\in\mathbb{R}}\{c(x,(x^*,y^*,\alpha))-\Phi(x,0)\}<+\infty$. It is equivalent to say that $(x^*,y^*,\alpha)\in\mathbb{R}_-\times(\mathbb{R}_-\times\mathbb{R}_+\setminus\{(0,0)\})$ and then

$$\sup_{x\in\mathbb{R}}\{c(x,(x^*,y^*,\alpha))-\Phi(x,0)\}=0.$$

We compute $\Phi^c((x^*,u^*),(y^*,v^*),\alpha)$, for $(x^*,y^*,\alpha)\in\mathbb{R}_-\times(\mathbb{R}_-\times\mathbb{R}_+\setminus\{(0,0)\})$ and any $u^*,v^*\in\mathbb{R}$. By (6),

$$\begin{aligned}
&\Phi^c((x^*,u^*),(y^*,v^*),\alpha)\\
=&\sup_{x,z\in\mathbb{R}}\{\bar{c}((x,z),(x^*-u^*,u^*),(y^*-v^*,v^*),\alpha)-f(z)-\delta_A(x)\}\\
=&\sup_{x\geq 0,z>0}\{\bar{c}((x,z),(x^*-u^*,u^*),(y^*-v^*,v^*),\alpha)\}.
\end{aligned}$$

Since we are interested in those suprema which are finite, if $y^*<0$ and $\alpha\geq 0$, take any $v^*\in[y^*,0[$ and if $y^*=0$ and $\alpha>0$, take $v^*=0$. Then

$$\begin{aligned}
&\inf_{u^*,v^*\in\mathbb{R},}\sup_{x\geq 0,z>0}\{\bar{c}((x,z),(x^*-u^*,u^*),(y^*-v^*,v^*),\alpha)\}\\
=&\inf_{u^*\leq x^*}\sup_{x\geq 0,z>0}\{(x^*-u^*)x+u^*z\}=0,
\end{aligned}$$

and this infimum is a minimum.

We finish the comparison with an example showing that $(C4_F)$ does not imply $(C5_F)$.

Example 6.7. Let us take $X=\mathbb{R}$, $A:=[0,+\infty[$ and $f=\delta_{]1,+\infty[}$. At the point $(x^*,-1,-1)\in\mathbb{R}^3$, $x^*\leq 0$, we obtain

$$\sup_{x\in\mathbb{R}}\{c(x,(-1,-1,-1))-\Phi(x,0)\}=\sup_{x>1}\{c(x,(x^*,-1,-1))\}=x^*.$$

On the other hand,

$$\begin{aligned}
&\inf_{u^*,v^*\in\mathbb{R}}\Phi^c((x^*,u^*),(-1,v^*),-1)\\
=&\inf_{u^*,v^*\in\mathbb{R}}\sup_{z>1,x\geq 0}\{\bar{c}((x,z),(x^*-u^*,u^*),(-1-v^*,v^*),-1)\},
\end{aligned}$$

in virtue of formula (6). Let us observe that a necessary condition (depending on $v^*\in\mathbb{R}$) for these suprema to be finite is that, for all $x\geq 0$ and $z>1$,

$$(-1-v^*)x+v^*z<-1.$$

In particular for $x=0$ and $z>1$, it must be $v^*<-1$ but this implies that $(-1-v^*)>0$ and hence $(-1-v^*)x$ cannot be bounded from above, since $x\geq 0$. We conclude that

$$\bar{c}((x,z),(x^*-u^*,u^*),(-1-v^*,v^*),-1)=+\infty,$$

for all $u^*, v^* \in \mathbb{R}$ and

$$\min_{u^*,v^* \in \mathbb{R}} \Phi^c\left((x^*, u^*), (-1, v^*), -1\right) = +\infty.$$

According to Proposition 5.4, ($C5_F$) does not hold despite ($C4_F$) fulfills, since $\operatorname{dom} f - A = \mathbb{R}$ hence $0 \in {}^{ic}(\operatorname{dom} f - A)$.

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