Journal of Convex Analysis
Volume 22 (2015), No. 3, 733–746

Non-Archimedean Countably Injective Banach Spaces

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Received: May 18, 2013
Revised manuscript received: August 31, 2014
Accepted: September 3, 2014

The main purpose of this paper is to investigate the relationships between some classes of non-Archimedean injective Banach spaces. The results obtained reveal sharp and interesting contrasts with the classical situation (i.e. for Banach spaces over $\mathbb{R}$ or $\mathbb{C}$), recently studied in [2]. One of those contrasts has to do with a classical open problem whose roots come back to 1964. In fact, in that year J. Lindenstrauss, [6], [7], obtained that, under the continuum hypothesis, 1-separably injective Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ are 1-universally separably injective. He left open the question in the usual setting of set theory with the Axiom of Choice. A negative answer, for a Banach space of continuous functions on a compact space, was given in [2], where the authors also posed a so natural classical problem as the following one: *Without the continuum hypothesis,* 1*-separably injective classical Banach spaces must be universally separably injective?*

However, we prove in this paper that, for any non-Archimedean Banach space, all the 1-injectivity properties coincide (Theorem 3.3). Additionally, for spaces of continuous functions on a zero-dimensional compact space, we get the coincidence of all the non-Archimedean injectivity properties (Theorems 4.3, 4.4).

Keywords: Non-Archimedean Banach spaces, injective spaces, orthonormal bases, spaces of continuous functions

2010 Mathematics Subject Classification: 46S10

Introduction

Injective objects play an important role in many mathematical theories. It is maybe worthwhile to recall an abstract definition, which will suffice for our purpose here: In a given category, an object E is injective if any morphism $D \longrightarrow E$ admits an extension $F \longrightarrow E$ as soon as D embeds into F.

For the category in which objects are Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ and morphisms are linear contractions, the resulting injective spaces are called 1-*injective*; if contractions are replaced by continuous linear maps we get the so called *injective*

*Research partially supported by Ministerio de Ciencia e Innovación, MTM2010-20190-C02-02.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

Banach spaces (for the most important properties of these two kinds of injective spaces, see for example [9]). Within the subcategory of separable Banach spaces and continuous linear maps, however, the history is over, as c_0 is its only infinite-dimensional object, [13]. This last fact leads to introduce the following variants of (1)-injectivity, in which separability is involved: A real or complex Banach space is said to be (1)-*separably injective* (resp. (1)-*universally separably injective*) if it satisfies the extension property in the definition of (1)-injective space under the assumption that F (resp. D) is separable. Several structural properties and examples of these last classes of injective spaces, as well as their relations with the (1)-injective ones, were studied in [2].

This paper deals with injectivity in non-Archimedean Banach spaces, when $\mathbb{R}$ or $\mathbb{C}$ is replaced by a non-Archimedean valued field. We show that the situation differs substantially from the classical (i.e. real or complex) one.

Firstly we focus our attention on a classical open problem, whose non-Archimedean counterpart, however, has an answer, as we explain below. The roots of this classical problem come back to 1964, when J. Lindenstrauss, [6], [7], obtained what can be understood as a proof that, under the continuum hypothesis, 1-separably injective Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ are 1-universally separably injective. He left open the question in the usual setting of set theory with the Axiom of Choice. It was solved negatively very recently in [2], for a Banach space of continuous functions on a compact space. Then, it was asked in [2] whether, without the continuum hypothesis, 1-separably injective classical Banach spaces must be universally separably injective. The answer to this problem is unknown.

In contrast with the above classical picture, we prove in this paper that, for *any* non-Archimedean Banach space, the three 1-injectivity properties coincide (Theorem 3.3). Additionally, for spaces of continuous functions on a zero-dimensional compact space, we get the coincidence of *all* the non-Archimedean injectivity properties (Theorems 4.3, 4.4).

Next we pay attention to the non-Archimedean "analogous" of the basic classical result stating that the real or complex 1-injective Banach spaces are exactly those that are isometric to spaces of continuous functions on extremely disconnected compact spaces (see [4], Section D.2 and the references therein). In this paper we apply some of our previous results to prove that now the validity of the non-Archimedean version of that classical result depends mainly on the base field (Corollary 4.9, Remark 4.11.1).

The last section contains some final conclusions and an open problem.

1. Preliminaries

Throughout this paper $K := (K, |\,.\,|)$ is a non-Archimedean non-trivially valued field that is complete with respect to the metric induced by the valuation $|\,.\,| : K \longrightarrow [0, \infty)$. By $|K|$ we mean $\{|\lambda| : \lambda \in K\}$. Also, *In the sequel E, F will be non-Archimedean Banach spaces over K. The norms on E and F are both denoted by $\| \, . \, \|$.*

For fundamentals on non-Archimedean Banach spaces we refer to [10], [11] and [12]. Here we only fix some notations and recall some basic concepts that will be involved in the paper.

By $\|E\|$ we mean $\{\|x\| : x \in E\}$. For each $\varepsilon > 0$, $x \in E$, $B_\varepsilon(x) := \{y \in E : \|y - x\| \leq \varepsilon\}$ is the closed ball with radius ε about x. E is called *spherically complete* if every sequence of balls $B_{\varepsilon_1}(x_1) \supset B_{\varepsilon_2}(x_2) \supset \ldots$ in E with $\varepsilon_1 > \varepsilon_2 > \ldots$, has non-empty intersection.

E is said to be *of countable type* if it contains a countable set whose linear hull is dense in E. Banach spaces of countable type are the adequate non-Archimedean substitutes of the real or complex separable ones.

An *operator* of E to F is a continuous linear map $T : E \longrightarrow F$. The set $L(E, F)$ of all operators $T : E \longrightarrow F$ is a non-Archimedean Banach space with the norm $\|T\| := \inf\{M > 0 : \|T(x)\| \leq M \|x\| \text{ for all } x \in E\}$. If $\|E\|$ and $\|F\|$ are contained in the closure in $\mathbb{R}$ of $|K|$, then $\|T\| = \sup\{\|T(x)\| : x \in E, \, \|x\| \leq 1\}$. But this last equality does not always hold.

Let $E' := L(E, K)$ and $E'' := L(E', K)$ be the *dual* and *bidual* of E, respectively. The formula $j_E(x)(f) := f(x)$ $(x \in E, \, f \in E')$ defines the canonical operator $j_E : E \longrightarrow E''$. E is called *polar* if j_E is an homeomorphism of E onto its image. Every space of countable type is polar. Also, every non-Archimedean Banach space over a spherically complete K is polar. However, when K is not spherically complete there exist infinite-dimensional spaces E over K with trivial duals.

E is *isomorphic* to F if there exists a bijective linear homeomorphism $T : E \longrightarrow F$. If, in addition, T is an isometry we say that E is *isometric* to F and we write $E \simeq F$.

An operator $P : E \longrightarrow E$ is called a *projection* if $P^2 = P$. A closed subspace D of E is said to be *complemented* in E if there exists a surjective projection $P : E \longrightarrow D$. If, in addition, $\|P\| \leq 1$, we say that D is *orthocomplemented* in E.

For a subspace D of E, $\times_{n \in \mathbb{N}} D := \{(x_1, x_2, \ldots) \in D^{\mathbb{N}} : \sup_n \|x_n\| < \infty\}$ and $\oplus_{n \in \mathbb{N}} D := \{(x_1, x_2, \ldots) \in D^{\mathbb{N}} : \lim_n \|x_n\| = 0\}$ are non-Archimedean normed spaces over K with the norm $\|(x_1, x_2, \ldots)\| := \sup_n \|x_n\|$. They are Banach as soon as D is closed in E.

Let I be a non-empty set, let $s : I \longrightarrow (0, \infty)$. The space $\ell^\infty(I, s) := \{(\lambda_i)_{i \in I} \in K^I : \sup_i |\lambda_i| \, s(i) < \infty \}$, equipped with the norm $\|(\lambda_i)_{i \in I}\| := \sup_i |\lambda_i| \, s(i)$, is a polar Banach space, which is of countable type if and only if I is finite. The subspace $c_0(I, s) := \{(\lambda_i)_{i \in I} \in K^I : \lim_i |\lambda_i| \, s(i) = 0\}$, equipped with the norm induced by $\ell^\infty(I, s)$, is a polar Banach space, which is of countable type if and only if I is countable. If $s(i) = 1$ for all $i \in I$, we write $\ell^\infty(I) := \ell^\infty(I, s)$ and $c_0(I) := c_0(I, s)$. When $I = \mathbb{N}$ then $\ell^\infty(\mathbb{N})$ and $c_0(\mathbb{N})$ are the well-known spaces

ℓ^∞ and c_0 of all bounded sequences in K and of all sequences in K tending to 0, respectively. Observe that $\ell^\infty(\mathbb{N}) = \times_{n\in\mathbb{N}} K$ and $c_0(\mathbb{N}) = \oplus_{n\in\mathbb{N}} K$.

Let $Y := \{e_i : i \in I\}$ be a subset of non-zero vectors of E. Y is called:

- A *base* of E if every $x \in E$ can be written uniquely as $x = \sum_{i\in I} \lambda_i e_i$, $\lambda_i \in K$.
- An *orthogonal system* in E if whenever $i_1, \ldots, i_n$ are distinct elements of I, one verifies

$$\|\lambda_{i_1} e_{i_1} + \ldots + \lambda_{i_n} e_{i_n}\| = \max_{1\le k\le n} \|\lambda_{i_k} e_{i_k}\| \text{ for all } n \in \mathbb{N},\ \lambda_{i_1}, \ldots, \lambda_{i_n} \in K.$$

  If, in addition, $\|e_i\| = 1$ for all $i \in I$, we say that Y is an *orthonormal system* in E.
- An *orthogonal* (*orthonormal*) *base* of E if it is an orthogonal (orthonormal) system that is also a base of E.

For each $s : I \longrightarrow (0, \infty)$, the unit vectors of K^I form an orthogonal base of $c_0(I, s)$; this base is orthonormal for $c_0(I)$. Even more, a Banach space E has an orthogonal (resp. orthonormal) base $\{e_i : i \in I\}$ if and only if $E \simeq c_0(I, s)$ for some $s : I \longrightarrow (0, \infty)$ (resp. $E \simeq c_0(I)$). However, $\ell^\infty(I, s)$ has a base if and only if the valuation of K is discrete. When K is spherically complete, every space of countable type over K has an orthogonal base. But there exist spaces E of countable type over non-spherically complete K without non-zero mutually orthogonal elements.

Let X be a non-empty zero-dimensional compact Hausdorff space. Then the space $C(X) := \{f : X \longrightarrow K : f \text{ is continuous}\}$, equipped with the norm $\|f\| := \max_{x\in X} |f(x)|$, is a non-Archimedean Banach space over K, having an orthonormal base consisting of characteristic functions of clopen (i.e. closed and open) sets.

By "classical theory" we mean Functional Analysis over $\mathbb{R}$ or $\mathbb{C}$. For fundamentals on classical Banach spaces we refer to [8]. The classical results included in this paper are for real Banach spaces, although all of them can be extended, *mutatis mutandis*, to complex Banach spaces. Hence,

In the sequel $\mathfrak{E}, \mathfrak{F}$ *will be Banach spaces over* $\mathbb{R}$.

Real Banach spaces that will play a central role along the paper are, for any non-empty compact Hausdorff space $\mathcal{X}$, $C(\mathcal{X}) := \{f : \mathcal{X} \longrightarrow \mathbb{R} : f \text{ is continuous}\}$, equipped with the usual maximum norm.

2. Basic facts

The following classes of injective spaces $\mathfrak{E}$ were considered in [2].

Definition 2.1. (i) $\mathfrak{E}$ is said to be *injective* (resp. *universally separably injective*) if for every Banach space $\mathfrak{F}$ and each subspace (resp. each separable subspace) D of $\mathfrak{F}$, every operator $t : D \longrightarrow \mathfrak{E}$ extends to an operator $T : \mathfrak{F} \longrightarrow \mathfrak{E}$.

If some extension T exists with $\|T\| = \|t\|$, we say that $\mathfrak{E}$ is 1-*injective* (resp. 1-*universally separably injective*).

(ii) $\mathfrak{E}$ is said to be (1-)*separably injective* if it satisfies the extension property in the above definition of (1-)injective space, under the assumption that $\mathfrak{F}$ is separable.

Clearly we have

$$\begin{array}{ccccc} \text{1-injective} & \Longrightarrow & \text{1-universally separably injective} & \Longrightarrow & \text{1-separably injective} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{injective} & \Longrightarrow & \text{universally separably injective} & \Longrightarrow & \text{separably injective.} \end{array} \tag{1}$$

An analysis of diagram (1) was carried out in [2]. Now we define the natural non-Archimedean counterparts of the above classical injectivity notions. The main purpose of this paper is to investigate on the non-Archimedean couple of that diagram (see Section 5 for a summary).

Definition 2.2. (i) E is said to be *injective* (resp. *universally countably injective*) if for every Banach space F and each subspace (resp. each subspace of countable type) D of F, every operator $t : D \longrightarrow E$ extends to an operator $T : F \longrightarrow E$. If some extension T exists with $\|T\| = \|t\|$, we say that E is 1-*injective* (resp. 1-*universally countably injective*).

(ii) E is said to be 1-*countably injective* if it satisfies the extension property in the above definition of 1-injective space, under the assumption that F is of countable type.

We have not considered the non-Archimedean couple of separable injectivity, as every non-Archimedean Banach space E has this property. In fact:

Proposition 2.3. *For every Banach space of countable type F and each subspace D of F, every operator $t : D \longrightarrow E$ extends to an operator $T : F \longrightarrow E$.*

Proof. We may assume that D is closed in F. By [10], Theorem 2.3.13 there exists a surjective projection $P : F \longrightarrow D$. Then $T := t \circ P$ meets the requirements.

Remarks 2.4. 1. There exist classical spaces that are not separably injective. In fact, according to a deep result of M. Zippin, [13], c_0 is the only infinite-dimensional separable real Banach space that is separably injective. For an arbitrary infinite set I, $c_0(I)$ is also separably injective, see [3] for a rather detailed survey. But, by [2], Proposition 3.5.(b), these spaces $c_0(I)$ are not universally separably injective.

2. On the other hand, it is unknown whether some classical spaces $\mathfrak{E}$ are separably injective or not. This is, for example, the case for $\mathfrak{E} := C(\beta\mathbb{N} \times \beta\mathbb{N})$, where

$\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$, see [2], Subsection 7.2.

3. Observe that in the above definitions of injectivity, both classical and non-Archimedean, it suffices to consider closed subspaces D. Hence, all the spaces involved in those definitions belong to the category of (non-Archimedean) Banach spaces, see the introduction.

4. The non-Archimedean injective and 1-injective spaces of Definition 2.2.(i) coincide with the weakly injective and injective ones defined in [12], p. 106 and 103, respectively. Here we use the same terminology as in [2] for the classical case.

Further, non-Archimedean 1-injectivity coincides with spherical completeness. In fact:

Theorem 2.5 ([12], Theorem 4.10). *E is 1-injective if and only if E is spherically complete.*

Most of the results of [12] related to 1-injectivity that will be used along the paper are formulated in terms of spherical completeness. This is, for example, the case of the following one.

Theorem 2.6 ([12], Theorem 5.16). *For an infinite-dimensional space E the following are equivalent.*

(α) *E is spherically complete and has an orthogonal base.*

(β) *Every strictly decreasing sequence of values of the norm function of E converges to 0.*

(γ) *The valuation of K is discrete and there exist a set I and a function $s : I \longrightarrow (|\pi|, 1]$ such that $E \simeq c_0(I, s)$ while the set of values of s is well-ordered* ($\pi \in K$, $|\pi| < 1$, such that $|\pi|$ generates the multiplicative group $\{|\lambda| : \lambda \in K, \lambda \neq 0\}$).

3. Coincidence of 1-injectivity and 1-(universal) countable injectivity

In this section we prove (Theorem 3.3) the coincidence of the three non-Archimedean 1-injectivity notions considered in Definition 2.2. First a short explanation of the, strongly different, situation in the classical case.

In 1964 J. Lindenstrauss, [6], [7], obtained what can be understood as a proof that, under the continuum hypothesis, 1-separably injective spaces are 1-universally separably injective (see also [2], Proposition 6.2). He left open the question in the usual setting of set theory with the Axiom of Choice. This problem was solved negatively very recently in [2], where it was also shown that there exist 1-universally separably injective spaces that are not 1-injective. More concretely:

Theorem 3.1.

(*i*) **([2], Theorem 6.7)** *There exists a compact space $\mathcal{X}$ such that $C(\mathcal{X})$ is 1-separably injective but not 1-universally separably injective.*

(*ii*) **([2], Subsection 4.3; [5], (2.11))** *There exists a compact space $\mathcal{X}$ such that $C(\mathcal{X})$ is* 1*-universally separably injective but not injective.*

Remark 3.2. The spaces $C(\mathcal{X})$ of Theorem 3.1 are infinite-dimensional. If $C(\mathcal{X})$ is finite-dimensional then it is 1-injective. It follows from the well-known fact that, for an n-dimensional space $\mathfrak{E}$ ($n \in \mathbb{N}$), one has that $\mathfrak{E}$ is 1-injective (1-(universally) separably injective) if and only if $\mathfrak{E} \simeq \mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with its maximum norm (observe that this implies that every finite-dimensional space $\mathfrak{E}$ is injective).

However, as we have announced at the beginning of this section, for *any* non-Archimedean Banach space E, one verifies:

Theorem 3.3. *E is* 1*-injective if and only if E is* 1*-(universally) countably injective.*

This result is a consequence of the next one, which provides several characterizations of non-Archimedean 1-injective spaces, completing Theorem 4.11 of [12].

Theorem 3.4. *The following are equivalent.*

(α) *E is* 1*-injective.*

(β) *The operator $\oplus_{n\in\mathbb{N}} E \longrightarrow E$, $(x_1, x_2, \ldots) \mapsto \sum_{n=1}^{\infty} x_n$, extends to an operator $\times_{n\in\mathbb{N}} E \longrightarrow E$ of norm ≤ 1.*

(γ) *For every subspace of countable type G of E, the operator $\oplus_{n\in\mathbb{N}} G \longrightarrow E$, $(x_1, x_2, \ldots) \mapsto \sum_{n=1}^{\infty} x_n$, extends to an operator $\times_{n\in\mathbb{N}} G \longrightarrow E$ of norm ≤ 1.*

(δ) *For every $s : \mathbb{N} \longrightarrow (0, \infty)$ and each subspace D of $c_0(\mathbb{N}, s)$, every operator $t : D \longrightarrow E$ with $\|t\| \leq 1$ extends to an operator $T : c_0(\mathbb{N}, s) \longrightarrow E$ with $\|T\| \leq 1$.*

If, in addition, the valuation of K is dense then (α)–(δ) are equivalent to:

(ε) *For each subspace D of c_0, every operator $t : D \longrightarrow E$ with $\|t\| \leq 1$ extends to an operator $T : c_0 \longrightarrow E$ with $\|T\| \leq 1$.*

Proof. $(\alpha) \Longleftrightarrow (\beta)$ is proved in [12], Theorem 4.11. Clearly $(\alpha) \Longrightarrow (\gamma), (\delta)$.

$(\gamma) \Longrightarrow (\alpha)$. According to Theorem 2.5, we are done as soon as we prove that every sequence of balls in E,

$$B_{\varepsilon_1}(x_1) \supset B_{\varepsilon_2}(x_2) \supset \ldots, \quad \text{with } \varepsilon_1 > \varepsilon_2 > \ldots, \tag{2}$$

has non-empty intersection. For that, let G be the closed linear hull of $\{x_1, x_2, \ldots\}$. By assumption there exists an operator $U : \times_{n\in\mathbb{N}} G \longrightarrow E$ with $\|U\| \leq 1$ that extends the one $\oplus_{n\in\mathbb{N}} G \longrightarrow E$ considered in (γ). Since $y := (x_1, x_2 - x_1, x_3 - x_2, \ldots)$ is an element of $\times_{n\in\mathbb{N}} G$, we have that $U(y) \in E$. Then following the same proof as in Theorem 4.11 of [12] one concludes that $\|U(y) - x_n\| \leq \varepsilon_n$ for all n i.e. $U(y) \in \bigcap_n B_{\varepsilon_n}(x_n)$.

Now suppose (δ) holds and let us get (α). As above, we are done as soon as we prove that every sequence of balls as in (2) has non-empty intersection. For that, let $\varepsilon_0 > \varepsilon_1$, $s_1, s_2, \ldots \in (0, \infty)$ with $\varepsilon_n < s_n < \varepsilon_{n-1}$ for all n, and let $e_1, e_2, \ldots$ be the canonical orthogonal base of $c_0(\mathbb{N}, s)$. We have $\|x_{n+1} - x_n\| \leq \varepsilon_n < s_n = \|e_n\| < \varepsilon_{n-1}$ for all n. Next define, for each n, $y_n := e_n - e_{n+1}$. Since $\|y_n - e_n\| < \|e_n\|$ for all n, we have that $y_1, y_2, \ldots$ is an orthogonal sequence in $c_0(\mathbb{N}, s)$ ([10], Theorem 2.2.9) and that $\|y_n\| = \|e_n\|$ for all n.

Let $D \subset c_0(\mathbb{N}, s)$ be the linear hull of $\{y_1, y_2, \ldots\}$. There is a unique linear map $t : D \longrightarrow E$ with $t(y_n) = x_n - x_{n+1}$ for all n. By orthogonality of the y_n, for each $y = \sum_{n=1}^{m} \lambda_n y_n \in D$ ($m \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_m \in K$), we have

$$\|t(y)\| = \left\| \sum_{n=1}^{m} \lambda_n (x_n - x_{n+1}) \right\| \leq \max_{1 \leq n \leq m} |\lambda_n| \; \|e_n\| = \max_{1 \leq n \leq m} |\lambda_n| \; \|y_n\| = \|y\|,$$

so t is continuous and $\|t\| \leq 1$. By assumption t extends to an operator $T : c_0(\mathbb{N}, s) \longrightarrow E$ with $\|T\| \leq 1$. Set $x := x_1 - T(e_1)$. Then $x \in \bigcap_n B_{\varepsilon_n}(x_n)$. In fact, for each n we have $-e_{n+1} = -e_1 + y_1 + \ldots + y_n$ and hence $-T(e_{n+1}) = x - x_{n+1}$. Therefore,

$$\begin{aligned} \|x - x_n\| &\leq \max(\|x - x_{n+1}\|, \|x_{n+1} - x_n\|) \\ &\leq \max(\|T\| \; \|e_{n+1}\|, \varepsilon_n) \leq \max(\|e_{n+1}\|, \varepsilon_n) = \varepsilon_n. \end{aligned}$$

Finally suppose that the valuation of K is dense. Only (ε) $\Longrightarrow$ (α) needs a proof. Consider a decreasing sequence of balls as in (2), and let $\varepsilon_0 > \varepsilon_1$. The valuation of K being dense, we can choose $e_1, e_2, \ldots$ in c_0, multiples of the canonical orthogonal base of c_0, such that $\varepsilon_n < \|e_n\| < \varepsilon_{n-1}$ for all n. Then proceed as in the proof of (δ) $\Longrightarrow$ (α).

Proof of Theorem 3.3. Clearly 1-injective $\Longrightarrow$ 1-universally countably injective $\Longrightarrow$ 1-countably injective. Suppose E is 1-countably injective. Then E satisfies (δ) of Theorem 3.4, as any $c_0(\mathbb{N}, s)$ is of countable type. Hence E is 1-injective, which finishes the proof.

Remark 3.5. Let the valuation of K be discrete. Then the implication (ε) $\Longrightarrow$ (α) fails. In fact, every Banach space E over K satisfies (ε), as every closed subspace of c_0 is orthocomplemented ([10], Theorem 2.5.4). However, there exist Banach spaces over K that are not 1-injective, so (α) fails. For an example, let $E := c_0(\mathbb{N}, s)$, with $s_1, s_2, \ldots \in (0, \infty)$, $s_1 > s_2 > \ldots$ and $\lim_n s_n = 1$. By Theorems 2.5, 2.6, E is not 1-injective. Observe that by Corollary 4.14.i of [12], E is injective.

4. (Countable) injectivity and spaces of continuous functions

The results of [2] contained in Theorem 3.1 and in the next theorem show that the converses of the implications of diagram (1) fail for classical Banach spaces of the type $C(\mathcal{X})$, see Corollary 4.2.

Theorem 4.1.

(i) **([2], comments after Proposition 4.2)** *There exists a compact space $\mathcal{X}$ such that $C(\mathcal{X})$ is injective but not* 1*-separably injective.*

(ii) **([2], Proposition 4.3)** *There exists a compact space $\mathcal{X}$ such that $C(\mathcal{X})$ is separably injective but not universally separably injective.*

Putting together Theorems 3.1, 4.1 we obtain:

Corollary 4.2. *For any of the implications of diagram* (1) *there exists a compact space $\mathcal{X}$ such that its converse fails for the space $C(\mathcal{X})$.*

On the other hand, it is asked in [2], Subsection 7.2 if, without the continuum hypothesis (see the comments before Theorem 3.1), 1-separably injective spaces $\mathfrak{E}$ must be universally separably injective, even when $\mathfrak{E} = C(\mathcal{X})$. As the authors say in [2], it would be interesting to characterize (universally) separably injective $C(\mathcal{X})$ in terms of topological properties of $\mathcal{X}$. Recall that $C(\mathcal{X})$ is 1-separably injective if and only if $\mathcal{X}$ is an F-space (i.e. disjoint cozeros have disjoint closures), see [2], Proposition 4.2.

Let us analyze the non-Archimedean situation, which becomes strongly different. Indeed, we have already seen that the three non-Archimedean 1-injectivity properties always coincide (Theorem 3.3). Next we prove, for non-Archimedean Banach spaces of continuous functions, the coincidence of *all* the non-Archimedean injectivity properties. Also, we characterize (any of) them in terms of the base field K.

Theorem 4.3. *Let $E := C(X)$ be an infinite-dimensional space. Then the following are equivalent.*

(α) *E is* 1*-injective* (1*-*(*universally*) *countably injective*)*.*

(β) *E is* (*universally countably*) *injective.*

(γ) *The valuation of K is discrete.*

Proof. For $(\alpha) \Longrightarrow (\beta)$ apply Theorem 3.3.

$(\beta) \Longrightarrow (\gamma)$. Suppose E is universally countably injective. E contains an infinite-dimensional closed subspace of countable type, which is isomorphic to c_0 ([10], Corollary 2.3.9) and, since E has a base, this subspace is complemented in E ([12], Corollary 3.18). Hence c_0 is universally countably injective, so that the identity $c_0 \longrightarrow c_0$ extends to a surjective projection $\ell^\infty \longrightarrow c_0$. By [10], Theorem 8.2.1, the valuation of K must be discrete.

$(\gamma) \Longrightarrow (\alpha)$. Suppose the valuation of K is discrete. Since E has an orthonormal base, there exists an infinite set I such that $E \simeq c_0(I)$. From Theorems 2.5, 2.6, we get that E is 1-injective.

For finite-dimensional spaces (which we assume $\neq \{0\}$) we also have the above coincidence (compare Remark 3.2).

Theorem 4.4. *Let E be a finite-dimensional space. Then the following are equivalent.*

(α) *E is 1-injective (1-(universally) countably injective).*

(β) *E is (universally countably) injective.*

(γ) *K is 1-injective.*

Proof. For $(\alpha) \Longrightarrow (\beta)$ apply Theorem 3.3.

$(\beta) \Longrightarrow (\gamma)$. Suppose E is universally countably injective. Let $x \in E$, $x \neq 0$. Then Kx is complemented in E and isomorphic to K. Hence K is universally countably injective. Thus, for every Banach space F containing K, the identity $K \longrightarrow K$ extends to a surjective projection $F \longrightarrow K$ i.e. K is complemented in F. By [12], Theorems 4.12.ii, 4.15 we obtain that K is 1-injective.

For $(\gamma) \Longrightarrow (\alpha)$, recall that if K is spherically complete then so is every finite-dimensional Banach space over K ([12], Corollary 4.6). Now apply Theorem 2.5.

A key point in the proof of Theorem 4.3 is that spaces $C(X)$ have orthonormal bases. In the next result we show that in fact they are the only non-Archimedean Banach spaces with orthonormal bases.

Proposition 4.5. *E has an orthonormal base if and only if E is isometric to a space $C(X)$.*

Proof. Suppose E has an orthonormal base. Then $E \simeq c_0(I)$ for some set I. Let us see that $c_0(I)$ is isometric to a space $C(X)$, and we are done. We may assume that I is infinite.

We first construct the space X. Augment I with a new point i_∞ and take $X := I \cup \{i_\infty\}$ as a set. On X we define the topology τ as follows: A set $U \subset X$ is τ-open if either $i_\infty \notin U$ or $\{x \in X : x \notin U\}$ is finite. It is straightforward to check that (X, τ) is a zero-dimensional compact Hausdorff space.

For each $i \in I$, let $f_i : X \longrightarrow K$ be the characteristic function on the set $\{i\}$ (which is clopen in X, so f_i is continuous). Also, let $f_\infty : X \longrightarrow K$ be the constant function 1 on X. Then $\{f_i : i \in I\} \cup \{f_\infty\}$ is an orthonormal base of $C(X)$. In fact, orthonormality follows easily. Also, each $f \in C(X)$ has a unique expansion

$$f = \sum_{i \in I} \lambda_i f_i + \lambda_\infty f_\infty, \quad \lambda_\infty = f(i_\infty), \; \lambda_i = f(i) - f(i_\infty) \; (i \in I)$$

(summability of $\{\lambda_i f_i : i \in I\}$ in $C(X)$ follows from [10], Theorem 2.5.1.(ii), as $\lim_i \|\lambda_i f_i\| = \lim_i |f(i) - f(i_\infty)| = 0$). Thus, $c_0(I \cup \{i_\infty\}) \simeq C(X)$. Further, since I is infinite, $I \cup \{i_\infty\}$ has the same cardinality as I. Therefore, $c_0(I) \simeq c_0(I \cup \{i_\infty\}) \simeq C(X)$.

Since a Banach space E has a base if and only if it is isomorphic to one with an orthonormal base ([12], Corollary 3.8), we have the following isomorphic version of Proposition 4.5.

Proposition 4.6. *E has a base if and only if E is isomorphic to a space $C(X)$.*

Remark 4.7 (Compare Remark 2.4.1). By Proposition 4.5, the conclusion of Theorem 4.3 holds for any space $c_0(I)$, with I an infinite set. For finite sets I apply Theorem 4.4.

To finish this section we pay attention to the following basic classical result and to its non-Archimedean couple.

Theorem 4.8 ([4], Section D.2). *$\mathfrak{E}$ is 1-injective if and only if $\mathfrak{E}$ is isometric to a space $C(\mathcal{X})$ with $\mathcal{X}$ extremely disconnected* (i.e. the closure of any open set in $\mathcal{X}$ is open).

However, as an application of some previous results of this paper, we get the following for infinite-dimensional non-Archimedean Banach spaces. For the finite-dimensional case see Remark 4.11.1.

Corollary 4.9. *Let E be an infinite-dimensional space. Then, E is 1-injective and isometric to a space $C(X)$ if and only if $\|E\| \subset |K|$ and K is discretely valued.*

Proof. If $E \simeq C(X)$, obviously $\|E\| \subset |K|$. The rest of the "only if" follows from Theorem 4.3.

For the "if", note that the assumptions $\|E\| \subset |K|$ and K discretely valued imply that every strictly decreasing sequence of values of the norm function of E converges to 0. From Theorems 2.5, 2.6, we deduce that E has an orthonormal base (i.e. E is isometric to a space $C(X)$, Proposition 4.5) and that E is 1-injective.

The isomorphic version of Corollary 4.9 reads as follows.

Corollary 4.10. *Let E be an infinite-dimensional space. Then, E is (universally countably) injective and isomorphic to a space $C(X)$ if and only if K is discretely valued.*

Proof. For the "only if", follow the same proof as in $(\beta) \Longrightarrow (\gamma)$ of Theorem 4.3. The "if" is a consequence of the facts that if K is discretely valued then E is injective ([12], Corollary 4.14.i) and also has a base ([12], p. 181) i.e. E is isomorphic to a space $C(X)$ (Proposition 4.6).

Remarks 4.11. 1. Let E be a finite-dimensional space.

The version of Corollary 4.10 for this E is provided by $(\beta) \Longleftrightarrow (\gamma)$ of Theorem 4.4, as clearly E is isomorphic to a $C(X)$ with X finite.

For the version of Corollary 4.9 we have the following.

E is 1-injective and isometric to a space $C(X)$ if and only if $\|E\| \subset |K|$ and K is 1-injective.

The proof follows as in Corollary 4.9 by using now Theorem 4.4.(α) $\Longleftrightarrow$ (γ), Proposition 4.5 and [10], Theorem 2.3.25.

2. Every extremely disconnected regular space is zero-dimensional. However, there exist compact zero-dimensional spaces that are not extremely disconnected. For examples of such spaces, take a compact ultrametrizable space X. Clearly it is zero-dimensional; but X is not extremely disconnected unless that X is finite. See [1], Section 6.6 for these and more properties of extremely disconnected spaces.

5. Final conclusions and remarks

For any space E we clearly have the implications 1-injective $\Longrightarrow$ injective $\Longrightarrow$ universally countably injective. Also, we have proved along the paper several coincidences of the non-Archimedean injectivity notions considered in Definition 2.2. It is summarized in the next result.

Corollary 5.1.

(i) **(Theorem 3.3)** *For any non-Archimedean Banach space E,*

$$\begin{array}{ccccc} 1\text{-}\textit{injective} & \Longleftrightarrow & 1\text{-}\textit{universally countably injective} & \Longleftrightarrow & 1\text{-}\textit{countably injective} \\ \Downarrow & & & & \\ \textit{injective} & \Longrightarrow & \textit{universally countably injective.} & & \end{array} \tag{3}$$

(ii) **(Theorems 4.3, 4.4)** *If either E has an orthonormal base (i.e. E is isometric to a space $C(X)$, Proposition 4.5) or E is finite-dimensional, then all the arrows of the above diagram are equivalences.*

(iii) **(Corollary 4.10)** *If E has a base (i.e. E is isomorphic to a space $C(X)$, Proposition 4.6), then E is injective $\Longleftrightarrow$ E is universally countably injective.*

Also, (any of) the equivalent injectivity properties for the spaces E considered in (ii) and (iii) can be characterized in terms of K.

Remarks 5.2. 1. There exist infinite-dimensional injective spaces E that are not 1-injective, which shows that the converse of implication $\Downarrow$ of diagram (3) may fail. For examples of such E see [12], p. 107 when the valuation of K is dense (E is isomorphic to ℓ^∞) and Remark 3.5 when the valuation of K is discrete (E is a space of countable type with an orthogonal base).

2. Concerning the converse of implication $\Longrightarrow$ of diagram (3), clearly it holds when E satisfies the conditions of (ii) of Corollary 5.1. By using (iii) of that

corollary we have that it also holds for the larger class of spaces E with a base, for example, when E is of countable type ([10], Theorem 2.3.7) and when K is discretely valued ([12], p. 181).

Natural examples of non-Archimedean Banach spaces without bases (over densely valued K) are the spaces $\ell^\infty(I)$, with I an infinite set. For them the above converse is again true. Even more, with the same proof as in Theorem 4.4 (using now [12], 4.A for $(\gamma) \implies (\alpha)$), we get that *for* $\ell^\infty(I)$*, any of the non-Archimedean injectivity properties of Definition* 2.2 *is equivalent to "K is* 1*-injective".*

But in the general case it is unknown whether or not the converse of implication $\implies$ of diagram (3) holds, which leads to the following problem.

Problem. *Let the valuation of K be dense. Is every universally countably injective space E injective?*

Suppose we deal with polar spaces, which are the most popular non-Archimedean Banach spaces, see [10], Section 2.5. Then, from [10], Theorem 4.4.9 we derive the following equivalent formulation of the above problem.

Let the valuation of K be dense. Is every universally countably injective closed subspace of any $\ell^\infty(I)$ (I infinite) injective?

References

[1] A. V. Arkhangel′skii, V. I. Ponomarev: Fundamentals of General Topology, Mathematics and its Applications 13, D. Reidel, Dordrecht (1984).

[2] A. Avilés, F. Cabello Sánchez, J. M. F. Castillo, M. González, Y. Moreno: On separably injective Banach spaces, Adv. Math. 234 (2013) 192–216.

[3] F. Cabello Sánchez, J. M. F. Castillo, D. Yost: Sobczyk's theorems from A to B, Extr. Math. 15 (2000) 391–420.

[4] J. Diestel, J. H. Fourie, J. Swart: The metric theory of tensor products, American Mathematical Society, Providence (2008).

[5] I. E. Leonard, J. H. M. Whitfield: A classical Banach space: ℓ^∞/c_0, Rocky Mt. J. Math. 13 (1983) 531–539.

[6] J. Lindenstrauss: Extension of compact operators, Mem. Amer. Math. Soc. 48 (1964), 112 pp.

[7] J. Lindenstrauss: On the extension of operators with range in a $C(K)$ space, Proc. Amer. Math. Soc. 15 (1964) 218–225.

[8] J. Lindenstrauss, L. Tzafriri: Classical Banach Spaces I, Springer, Berlin (1977).

[9] V. Mascioni: Topics in the theory of complemented subspaces in Banach spaces, Expo. Math. 7 (1989) 3–47.

[10] C. Perez-Garcia, W. H. Schikhof: Locally Convex Spaces over Non-Archimedean Valued Fields, Cambridge University Press, Cambridge (2010).

[11] W. H. Schikhof: Banach spaces over nonarchimedean valued fields, Topol. Proc. 24 (1999) 547–581.

[12] A. C. M. van Rooij: Non-Archimedean Functional Analysis, Marcel Dekker, New York (1978).

[13] M. Zippin: The separable extension problem, Isr. J. Math. 26 (1977) 372–387.