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Uniform Rotundity and k-Uniform Rotundity in Musielak-Orlicz-Bochner Function Spaces and Applications *

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In this paper, criteria for uniform rotundity of Musielak-Orlicz-Bochner function spaces equipped with Orlicz norm are given. We also prove that, in Musielak-Orlicz-Bochner function spaces, uniform rotundity and k-uniform rotundity are equivalent. Moreover, we give a sufficient condition for a Musielak-Orlicz-Bochner function space to have the fixed point property.

Keywords: Musielak-Orlicz-Bochner function spaces, uniform rotundity, k-uniform rotundity, Orlicz norm, fixed point

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1. Introduction

A lot of rotundity concepts in Banach spaces are known. Among them rotundity, uniform rotundity and k-uniformly rotundity play essential role in geometry of Banach spaces and its numerous applications. In particular each of these properties guarantees normal structure, which in turn implies the fixed point property for nonexpansive mappings. Therefore, it was natural and interesting to look for criteria of uniform rotundity and k-uniformly rotundity in various well-known classes of Banach spaces. The criteria for uniform rotundity in the classical Orlicz function spaces (but under some restrictions on Orlicz function generating

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the space) have been given in [3] already. A general result on rotundity of Orliczs paces can be found in [4]. In [11] H. Hudzik gave some sufficient conditions in order that the Cartesian product of Musielak-Orlicz-Bochner function spaces endowed with Luxemburg norm be uniformly convex. However, because of the complicated structure of Musielak-Orlicz-Bochner function spaces endowed with Orlicz norm, up to now the criteria for uniform rotundity have not been discussed yet. In the paper, we will discuss criteria for uniform rotundity in Musielak-Orlicz-Bochner function spaces endowed with Orlicz norm. We also prove that, in Musielak-Orlicz-Bochner function spaces, uniform rotundity and k-uniform rotundity are equivalent. The topic of this paper is related to the topic of [6], [16], [24] and [7], [9], [10], [12], [14], [15], [18], [19], [20], [22], [23].

Let $(X, \|\cdot\|)$ be a real Banach space. $S(X)$ and $B(X)$ denote the unit sphere and unit ball, respectively. By X^* denote the dual space of X. Let N, R and R^+ denote the set natural number, reals and nonnegative reals, respectively. Let us recall some geometrical notions concerning rotundity. A Banach space X is said to be rotund if for any $x, y \in S(X)$ and $\|x+y\| = 2$ we have $x = y$. A Banach space X is said to be uniformly rotund if for any $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in $S(X)$ with $\|x_n + y_n\| \to 2$ we have $\|x_n - y_n\| \to 0$ as $n \to \infty$. A Banach space X is said to be weak uniformly rotund if for any $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in $S(X)$ with $\|x_n + y_n\| \to 2$ we have $x_n - y_n \stackrel{w}{\to} 0$ as $n \to \infty$. It is easy to see that

uniformly rotund space $\Rightarrow$ weak uniformly rotund space $\Rightarrow$ rotund space

A Banach space X is said to be k-uniformly rotund if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x_1, x_2, ..., x_{k+1} \in S(X)$ and $\|x_1 + x_2 + ... + x_{k+1}\| \geq k + 1 - \delta$ implies

$$\Delta(x_1, x_2, ..., x_{k+1}) = \sup_{\substack{x_i^* \in B(X^*) \\ i \leq k}} \begin{vmatrix} 1 & 1 & ... & 1 \\ x_1^*(x_1) & x_1^*(x_2) & ... & x_1^*(x_{k+1}) \\ ... & ... & ... & ... \\ x_k^*(x_1) & x_k^*(x_2) & ... & x_k^*(x_{k+1}) \end{vmatrix} < \varepsilon.$$

It is well known that 1-uniformly rotund space and uniformly rotund space are equivalent. A Banach space X is said to be k-rotund if for any $k+1$ elements $x_1, x_2, ..., x_{k+1}$ in $S(X)$ and $\|x_1 + x_2 + ... + x_{k+1}\| = k+1$ implies $x_1, x_2, ..., x_{k+1}$ are linearly dependent. It is easy to see that 1-rotund space and rotund space are equivalent.

Let (T, Σ, μ) be complete nonatomic measurable space. Suppose that a function $M : T \times [0, \infty) \to [0, \infty]$ statisfies the following conditions.

(1) for μ-a.e., $t \in T$, $M(t, 0) = 0$, $\lim_{u\to\infty} M(t, u) = \infty$ and $M(t, u') < \infty$ for some $u' > 0$.
(2) for μ-a.e., $t \in T$, $M(t, u)$ is convex on $[0, \infty)$ with respect to u.
(3) for each $u \in [0, \infty)$, $M(t, u)$ is a μ-measurable function of t on T.

Every such a function M is called a Musielak-Orlicz-function. Let $p(t, u)$ denote the right derivative of $M(t, \cdot)$ at $u \in R^+$ (where we assume that $p(t, u) = \infty$ if

$M(t,u)=\infty$) and $q(t,\cdot)$ be the generalized inverse function of $p(t,\cdot)$ defined on R^+ by

$$q(t,v)=\sup_{u\geq 0}\{u\geq 0: p(t,u)\leq v\}.$$

Then the function $N(t,v)$ defined by $N(t,v)=\int_0^{|v|}q(t,s)ds$ for any $v\in R$ and μ-a.e. $t\in T$ is called the complementary function to M in the sense of Young. It is well known that the Young inequality $uv\leq M(t,u)+N(t,v)$ holds for all $u,v,\in R$ and μ-a.e. $t\in T$. Moreover, for any $u\in R$ the equality $uv=M(t,u)+N(t,u)$ holds if and only if $v\in[p_-(t,u),p(t,u)]$. Let

$$e(t)=\sup\{u>0: M(t,u)=0\},\qquad E(t)=\sup\{u>0: M(t,u)<\infty\}.$$

For a fixed $t\in T$ and $v\geq 0$, if there exists $\varepsilon\in(0,1)$ such that

$$M(t,v)=\frac{1}{2}M(t,v+\varepsilon)+\frac{1}{2}M(t,v-\varepsilon)<\infty,$$

then v is called a point of affinity of $M(t,\cdot)$. The set of all points of affinity of $M(t,\cdot)$ for a fixed $t\in T$ is denoted by K_t.

Moreover, given any Banach space $(X,\|\cdot\|)$, we denote by X_T the set of all strongly Σ-measurable functions from T to X, and for each $u\in X_T$, we define the modular of u by

$$\rho_M(u)=\int_T M(t,\|u(t)\|)dt.$$

Let us define the Musielak-Orlicz-Bochner function space $L_M(X)$ by

$$L_M(X)=\{u\in X_T:\rho_M(\lambda u)<\infty \text{ for some } \lambda>0\}$$

and its subspace

$$E_M(X)=\{u\in X_T:\rho_M(\lambda u)<\infty \text{ for all } \lambda>0\}.$$

It is well known that the spaces $L_M(X)$ and their subspaces $E_M(X)$ are Banach spaces when they are equipped with the Luxemburg norm

$$\|u\|=\inf\left\{\lambda>0:\rho_M\left(\frac{u}{\lambda}\right)\leq 1\right\},$$

or with the Orlicz norm

$$\|u\|^0=\inf_{k>0}\frac{1}{k}[1+\rho_M(ku)].$$

Note that the Luxemburg norm and the Orlicz norm are equivalent.

The spaces $(L_M(X),\|\cdot\|),(L_M(X),\|\cdot\|^0),(E_M(X),\|\cdot\|_M)$ and $(E_M(X),\|\cdot\|^0)$ are denoted shortly by $L_M(X),L_M^0(X),E_M(X)$ and $E_M^0(X)$, respectively. $L_M(R)$

and $L^0_M(R)$ are called Musielak-Orlicz function spaces, and $E_M(R)$ and $E^0_M(R)$ are the subspaces of order continuous elements from $L_M(R)$ and $L^0_M(R)$, respectively. Set

$$K(u) = \left\{ k > 0 : \frac{1}{k}(1 + \rho_M(ku)) = \|u\|^0 \right\}.$$

In general the set $K(u)$ is nonempty. To show that, we give a proposition.

Proposition 1.1 (see [11]). *If* $\lim_{u\to\infty} \frac{M(t,u)}{u} = \infty$ μ*−a.e.* $t \in T$*, then* $K(u) \neq \emptyset$ *for any* $u \in L^0_M(X)$*.*

In some strange cases the set $K(u)$ is empty. For example, the Lebesgue space L^1.

The regularity condition Δ_2 for a Musielak-Orlicz function M plays an important role in the theory of Musielak-Orlicz function spaces (see [3], [24]). We say that M satisfies condition Δ_2 ($M \in \Delta_2$ for short) if there are a set $B \in \Sigma$ of measure zero, a number $K \geq 2$ and a nonnegative real function $h \in L^0(R) := R_T$ with $\int_T h(t)dt < \infty$ such that the inequality

$$M(t, 2u) \leq KM(t, u) + h(t)$$

holds for any $t \in T \setminus B$ and any $u \in R$. It is easy to see that if $M \in \Delta_2$ then $L_M(X) = E_M(X)$ and $L^0_M(X) = E^0_M(X)$.

Definition 1.2. A Musielak-Orlicz function M is said to be almost uniformly convex if for any ε, there exists a nonnegative measurable function $f_\varepsilon(t)$ with $\int_T M(t, f_\varepsilon(t))dt < \varepsilon$ and $\delta_\varepsilon \in (0, 1)$ such that

$$M\left(t, \frac{u+v}{2}\right) \leq (1 - \delta_\varepsilon)\frac{M(t, u) + M(t, v)}{2}$$

for almost $t \in T$ and for all u, v satisfying $|u - v| \geq \varepsilon \max\{|u|, |v|\} \geq \varepsilon f_\varepsilon(t)$.

Definition 1.3 (see [3]). A Musielak-Orlicz function $M(t, u)$ is said to be strictly convex with respect u for $t \in T$, a.e, if for almost every $t \in T$, a.e, and any $u, v \in R$, $u \neq v$, we have

$$M\left(t, \frac{u+v}{2}\right) < \frac{M(t, u) + M(t, v)}{2}.$$

First let us recall some results that will be used in the further part of the paper.

Lemma 1.4 (see [6]). *Suppose* $M \in \triangle_2$ *and* $e(t) = 0$ $\mu - a.e.$ *on* T*. Then*

$$\rho_M(u_n) \to 0 \Leftrightarrow \|u_n\| \to 0, \qquad \rho_M(u_n) \to 1 \Leftrightarrow \|u_n\| \to 1 \quad as\ n \to \infty.$$

Since the Luxemburg norm and Orlicz norm are equivalent, the first equivalence holds true for Orlicz norm as well.

Lemma 1.5 (see [24]).

(a) $L_N^0(X^*)$ *is isometrically isomorphic to a subspace of* $(E_M(X))^*$.
(b) $L_N(X^*)$ *is isometrically isomorphic to a subspace of* $(E_M^0(X))^*$.

Lemma 1.6 (see [6]). *If* $N \in \Delta_2$ *and* $M(t,u)$ *is strictly convex with respect* u *for almost all* $t \in T$, *then*

$$\sup_{\|u\|^0=1}\left\{k>0 : \|u\|^0 = \frac{1}{k}(1+\rho_M(ku))\right\} < \infty.$$

Lemma 1.7. *A Musielak-Orlicz function* M *is almost uniformly convex if and only if for any* $\varepsilon > 0$ *and* $[a,b] \subset (0,1)$, *there exists a nonnegative measurable function* $f_\varepsilon(t)$ *with* $\int_T M(t, f_\varepsilon(t))dt < \varepsilon$ *and* $\delta_\varepsilon \in (0,1)$ *such that*

$$M\left(t, \lambda u + (1-\lambda)v\right) \le (1-\delta_\varepsilon)\lambda M(t,u) + (1-\lambda)M(t,v)$$

for almost $t \in T$ *and for all* u, v *satisfying* $|u-v| \ge \varepsilon \max\{|u|, |v|\} \ge \varepsilon f_\varepsilon(t)$, *where* $\lambda \in [a,b]$.

Proof. Since M is almost uniformly convex if for any $\varepsilon > 0$, there exists a nonnegative measurable function $f_\varepsilon(t)$ with $\int_T M(t, f_\varepsilon(t))dt < \varepsilon$ and $r_\varepsilon \in (0,1)$ such that

$$M\left(t, \frac{u+v}{2}\right) \le (1-r_\varepsilon)\frac{M(t,u)+M(t,v)}{2} \tag{1}$$

for almost $t \in T$ and for all u, v satisfying $|u-v| \ge \varepsilon \max\{|u|, |v|\} \ge \varepsilon f_\varepsilon(t)$. For clarity, we will divide the proof into two parts.

I. $\lambda \in [a, \frac{1}{2}]$. By the convexity of M and inequality (1), we have

$$\begin{aligned}
M(t, \lambda u + (1-\lambda)v) &= M(t, \lambda u + \lambda v + (1-2\lambda)v) \\
&= M\left(t, 2\lambda \cdot \frac{\lambda u + \lambda v}{2\lambda} + (1-2\lambda)v\right) \\
&\le 2\lambda M\left(t, \frac{\lambda u + \lambda v}{2\lambda}\right) + (1-2\lambda)M(t,v) \\
&= 2\lambda M\left(t, \frac{u+v}{2}\right) + (1-2\lambda)M(t,v) \\
&\le \lambda(1-r_\varepsilon)[M(t,u)+M(t,v)] + (1-2\lambda)M(t,v) \\
&= \lambda(1-r_\varepsilon)M(t,u) + \lambda(1-r_\varepsilon)M(t,v) + (1-2\lambda)M(t,v) \\
&< \lambda(1-r_\varepsilon)M(t,u) + (1-\lambda)M(t,v).
\end{aligned}$$

II. $\lambda \in (\frac{1}{2}, b]$. By the convexity of M, we have

$$\begin{aligned}
& M(t, \lambda u + (1-\lambda)v) \\
&= M(t, (1-\lambda)(u+v) + (2\lambda - 1)u) \\
&= M\left(t, (2-2\lambda)\frac{(1-\lambda)(u+v)}{2-2\lambda} + (2\lambda-1)u\right) \\
&\le (2-2\lambda)M\left(t, \frac{(1-\lambda)(u+v)}{2(1-\lambda)}\right) + (2\lambda-1)M(t,u) \\
&= (2-2\lambda)M\left(t, \frac{u+v}{2}\right) + (2\lambda-1)M(t,u) \\
&\le (1-\lambda)(1-r_\varepsilon)[M(t,u) + M(t,v)] + (2\lambda-1)M(t,u) \\
&= (1-\lambda)(1-r_\varepsilon)M(t,u) + (1-\lambda)(1-r_\varepsilon)M(t,v) + (2\lambda-1)M(t,u) \\
&= [(1-\lambda) + (2\lambda-1) - r_\varepsilon(1-\lambda)]M(t,u) + (1-\lambda)(1-r_\varepsilon)M(t,v) \\
&= [\lambda - r_\varepsilon(1-\lambda)]M(t,u) + (1-\lambda)(1-r_\varepsilon)M(t,v) \\
&\le \lambda\left[1 - \left(\frac{r_\varepsilon}{\lambda} - r_\varepsilon\right)\right]M(t,u) + (1-\lambda)M(t,v) \\
&\le \lambda\left[1 - \left(\frac{r_\varepsilon}{b} - r_\varepsilon\right)\right]M(t,u) + (1-\lambda)M(t,v).
\end{aligned}$$

Finally, combining the considerations from Parts I and II and denoting

$$\delta_\varepsilon = \min\left\{r_\varepsilon, \frac{r_\varepsilon}{b} - r_\varepsilon\right\}.$$

We get inequality

$$M(\lambda u + (1-\lambda)v) \le \lambda(1-\delta_\varepsilon)M(u) + (1-\lambda)M(v),$$

which finishes the proof. □

Lemma 1.8 (see [6]). *Let $L_M^0(R)$ be weak uniformly rotund space. Then $M \in \Delta_2$, $N \in \Delta_2$, $M(t,u)$ is strictly convex with respect u for almost all $t \in T$ and M is almost uniformly convex.*

2. Main results

Theorem 2.1. *Let $L_M^0(X)$ be Musielak-Orlicz-Bochner function space. Then the following are equivalent:*

(a) *$L_M^0(X)$ is uniformly rotund space;*
(b) *$L_M^0(X)$ is k-uniformly rotund space;*
(c) (1c) *For any $u \in S(L_M^0(X))$, the set $K(u) \neq \emptyset$.*
  (2c) *$M \in \Delta_2$, $N \in \Delta_2$;*
  (3c) *$M(t,u)$ is strictly convex with respect u for almost all $t \in T$ and M is almost uniformly convex;*
  (4c) *X is uniformly rotund space.*

In order to prove the theorem, we give some Lemmas and define a new function $\delta(h,\alpha,l)$ from $R^+\times(0,1)\times R^+$ into R by

$$\delta(h,\alpha,l)=\inf_{x,y\in X}\{l-\|\alpha x+(1-\alpha)y\|:\|x-y\|\geq h,\ \ \|x\|=\|y\|=l\}.$$

Lemma 2.2. *Let X be a uniformly rotund space. Then for any $\varepsilon\in(0,1)$, $\alpha\in(0,1)$ $r\in[0,\frac{1}{8}\varepsilon]$, if $\|y-x\|\geq\varepsilon$, $r\geq|\|y\|-\|x\||$ then*

$$\begin{aligned}&\alpha\|x\|+(1-\alpha)\|y\|-\|\alpha x+(1-\alpha)y\|\\ \geq\ &\min\{\|y\|,\|x\|\}\delta\left(\frac{\varepsilon-r}{\min\{\|y\|,\|x\|\}},\alpha,1\right).\end{aligned}$$

Proof. Since X is uniformly rotund space, it is easy to see that if $h\leq 2l$ then $\delta(h,\alpha,l)>0$. Let $\|y-x\|\geq\varepsilon,\ r\geq|\|y\|-\|x\||$. We may assume, without loss of generality, that $\|y\|\geq\|x\|$. Therefore

$$\begin{aligned}\left\|\frac{\|x\|}{\|y\|}y-x\right\|&=\left\|y-x-\left(1-\frac{\|x\|}{\|y\|}\right)y\right\|\\ &\geq\|y-x\|-\left\|\left(1-\frac{\|x\|}{\|y\|}\right)y\right\|\\ &\geq\|y-x\|-(\|y\|-\|x\|)\\ &\geq\varepsilon-r.\end{aligned}$$

This implies that

$$\left\|\alpha x+(1-\alpha)\frac{\|x\|}{\|y\|}y\right\|\leq\alpha\|x\|+(1-\alpha)\left\|\frac{\|x\|}{\|y\|}y\right\|-\delta(\varepsilon-r,\alpha,\|x\|).$$

By $\|y\|\geq\|x\|$, we have $\|x\|=\min\{\|x\|,\|y\|\}$. Hence we have

$$\left\|\alpha x+(1-\alpha)\frac{\|x\|}{\|y\|}y\right\|\leq\alpha\|x\|+(1-\alpha)\left\|\frac{\|x\|}{\|y\|}y\right\|-\delta(\varepsilon-r,\alpha,\min\{\|x\|,\|y\|\}).\quad(2)$$

Therefore, by inequality (2), we obtain

$$\begin{aligned}\|\alpha x+(1-\alpha)y\|&\leq\left\|(1-\alpha)\left(1-\frac{\|x\|}{\|y\|}\right)y\right\|+\left\|(1-\alpha)\frac{\|x\|}{\|y\|}y+\alpha x\right\|\\ &\leq(1-\alpha)\left\|\left(1-\frac{\|x\|}{\|y\|}\right)y\right\|+(1-\alpha)\left\|\frac{\|x\|}{\|y\|}y\right\|+\alpha\|x\|\\ &\quad-\delta(\varepsilon-r,\alpha,\min\{\|x\|,\|y\|\})\\ &=(1-\alpha)\|y\|+\alpha\|x\|-\delta(\varepsilon-r,\alpha,\min\{\|x\|,\|y\|\}).\end{aligned}$$

We complete the proof by showing that

$$\min\{\|y\|,\|x\|\}\delta\left(\frac{\varepsilon-r}{\min\{\|y\|,\|x\|\}},\alpha,1\right)=\delta(\varepsilon-r,\alpha,\min\{\|y\|,\|x\|\}).$$

In fact,

$$\begin{aligned}
&\delta(\varepsilon - r, \alpha, \min\{\|y\|, \|x\|\}) \\
= &\inf_{x',y'\in X}\{\alpha\|x'\| + (1-\alpha)\|y'\| - \|\alpha x' + (1-\alpha)y'\| : \\
&\qquad \|x'-y'\| \geq \varepsilon - r, \|x'\| = \|y'\| = \min\{\|y\|, \|x\|\}\} \\
= &\min\{\|y\|, \|x\|\} \inf_{x'',y''\in X}\Big\{\|\alpha x''\| + \|(1-\alpha)y''\| - \|\alpha x'' + (1-\alpha)y''\| : \\
&\qquad \|x''-y''\| \geq \frac{\varepsilon - r}{\min\{\|y\|, \|x\|\}}, \|x''\| = \|y''\| = 1\Big\} \\
= &\min\{\|y\|, \|x\|\}\delta\left(\frac{\varepsilon - r}{\min\{\|y\|, \|x\|\}}, \alpha, 1\right).
\end{aligned}$$

This completes the proof. □

Lemma 2.3. *For any $\varepsilon > 0$, if $\alpha_n \to \alpha$, then $\delta(\varepsilon, \alpha_n, 1) \to \delta(\varepsilon, \alpha, 1)$ as $n \to \infty$, where $\alpha \in (0, 1)$.*

Proof. By definitions of $\delta(\varepsilon, \alpha_n, 1)$ and $\delta(\varepsilon, \alpha, 1)$, it is easy to see that

$$\delta(\varepsilon, \alpha_n, 1) = \inf\{1 - \|\alpha_n x + (1-\alpha_n)y\| : \|x - y\| \geq \varepsilon, \|x\| = \|y\| = 1\},$$

$$\delta(\varepsilon, \alpha, 1) = \inf\{1 - \|\alpha x + (1-\alpha)y\| : \|x - y\| \geq \varepsilon, \|x\| = \|y\| = 1\}.$$

Suppose that $\limsup_{n\to\infty} \delta(\varepsilon, \alpha_n, 1) > \delta(\varepsilon - r, \alpha, 1)$. Then there exist $l > 0$ and passing to a subsequence, if necessity, we can assume that $\delta(\varepsilon, \alpha_n, 1) > \delta(\varepsilon, \alpha, 1) + l$. By definition of $\delta(\varepsilon, \alpha, 1)$, there exist $x_0, y_0 \in S(X)$ such that

$$\|x_0 - y_0\| \geq \varepsilon, \qquad \|x_0\| = \|y_0\| = 1,$$

$$1 - \|\alpha x_0 + (1-\alpha)y_0\| \leq \delta(\varepsilon, \alpha, 1) + \frac{1}{8}l.$$

By $\alpha_n \to \alpha$, there exists $n_0 \in N$ such that $|\alpha_{n_0} - \alpha| < \frac{1}{8}l$. Hence we have

$$\begin{aligned}
&|(1 - \|\alpha x_0 + (1-\alpha)y_0\|) - (1 - \|\alpha_{n_0}x_0 + (1-\alpha_{n_0})y_0\|)| \\
= &|\|\alpha x_0 + (1-\alpha)y_0\| - \|\alpha_{n_0}x_0 + (1-\alpha_{n_0})y_0\||| \\
\leq &\|(\alpha x_0 + (1-\alpha)y_0) - (\alpha_{n_0}x_0 + (1-\alpha_{n_0})y_0)\| \\
\leq &|\alpha_{n_0} - \alpha| \cdot \|x_0 - y_0\| < l/4.
\end{aligned}$$

This implies that

$$\begin{aligned}
\delta(\varepsilon, \alpha_{n_0}, 1) &\leq 1 - \|\alpha_{n_0}x_0 + (1-\alpha_{n_0})y_0\| \\
&< 1 - \|\alpha x_0 + (1-\alpha)y_0\| + \frac{1}{4}l \\
&\leq \delta(\varepsilon, \alpha, 1) + \frac{1}{8}l + \frac{1}{4}l \\
&= \delta(\varepsilon, \alpha, 1) + \frac{3}{8}l.
\end{aligned}$$

Which contradicts $\delta(\varepsilon, \alpha_{n_0}, 1) > \delta(\varepsilon, \alpha, 1) + l$. Hence we have $\limsup_{n\to\infty} \delta(\varepsilon, \alpha_n, 1) \leq \delta(\varepsilon, \alpha, 1)$.

Suppose that $\liminf_{n\to\infty} \delta(\varepsilon, \alpha_n, 1) < \delta(\varepsilon, \alpha, 1)$. Then there exist $l' > 0$ and a subsequence of $\{n\}$ denoted again by $\{n\}$, such that $\delta(\varepsilon, \alpha_n, 1) + l' < \delta(\varepsilon, \alpha, 1)$. By $\alpha_n \to \alpha$, there exists $n_0 \in N$ such that $|\alpha_{n_0} - \alpha| \leq \frac{1}{8}l'$. By definition of $\delta(\varepsilon, \alpha_{n_0}, 1)$, there exist $x_0, y_0 \in S(X)$ such that $\|x_0 - y_0\| \geq \varepsilon$, $\|x_0\| = \|y_0\| = 1$ and

$$1 - \|\alpha_{n_0} x_0 + (1 - \alpha_{n_0}) y_0\| \leq \delta(\varepsilon, \alpha_{n_0}, 1) + \frac{1}{8} l.$$

Similarly, we have $\delta(\varepsilon, \alpha_{n_0}, 1) \leq \delta(\varepsilon, \alpha_{n_0}, 1) + \frac{3}{8} l$, a contradiction! Hence we have $\liminf_{n\to\infty} \delta(\varepsilon, \alpha_n, 1) \geq \delta(\varepsilon, \alpha, 1)$. This implies that $\lim_{n\to\infty} \delta(\varepsilon, \alpha_n, 1) = \delta(\varepsilon, \alpha, 1)$. This completes the proof. $\square$

Lemma 2.4 (see [21]). *$\|u\|^0 \leq \int_T A(t) \cdot \|u(t)\|\, dt$ and if $K(u) = \emptyset$, then $\|u\|^0 = \int_T A(t) \cdot \|u(t)\|\, dt$, where $A(t) = \lim_{u\to\infty} \frac{M(t,u)}{u}$.*

Proof of Theorem 2.1. $(b) \Rightarrow (c)$. Suppose that X is not k-uniformly rotund space. Then there exist $\varepsilon_0 > 0$, $\{x_{1,n}\}_{n=1}^\infty, \{x_{2,n}\}_{n=1}^\infty, ..., \{x_{k+1,n}\}_{n=1}^\infty \subset S(X)$ and $\{x^*_{1,n}\}_{n=1}^\infty, \{x^*_{2,n}\}_{n=1}^\infty, ..., \{x^*_{k,n}\}_{n=1}^\infty \subset S(X^*)$ such that

$$\|x_{1,n} + x_{2,n} + \cdots + x_{k+1,n}\| \to k+1 \quad \text{as } n \to \infty,$$

$$\begin{vmatrix} 1 & 1 & ... & 1 \\ x^*_{1,n}(x_{1,n}) & x^*_{1,n}(x_{2,n}) & ... & x^*_{1,n}(x_{k+1,n}) \\ ... & ... & ... & ... \\ x^*_{k,n}(x_{1,n}) & x^*_{k,n}(x_{2,n}) & ... & x^*_{k,n}(x_{k+1,n}) \end{vmatrix} \geq \varepsilon_0.$$

Pick $h \in S(L^0_M(X))$, then there exists $d > 0$ such that $\mu T_1 > 0$, where $T_1 \subset \{t \in T : \|h(t)\| \geq d\}$. Put $h_1(t) = d \cdot \chi_{T_1}(t)$. It is easy to see that $h_1 \in L^0_M(R) \backslash \{0\}$. Hence there exists $l > 0$ such that $l \cdot h_1 \in S(L^0_M(R))$. We may assume, without loss of generality, that $l = 1$. Put

$$u_{1,n} = d \cdot x_{1,n} \chi_{T_1}\ , ...,\ u_{k+1,n} = d \cdot x_{1,n} \chi_{T_1} \quad n = 1, 2, ...$$

Then $\{u_{1,n}\}_{n=1}^\infty, ..., \{u_{k+1,n}\}_{n=1}^\infty \subset S(L^0_M(X))$. It is easy to see that $\|u_{1,n} + u_{2,n} + \cdots + u_{k+1,n}\| \leq k+1$. Pick $l_n \in K(u_n)$. Since $N \in \Delta_2$, by Lemma 1.6, we obtain that $\{l_n\}_{n=1}^\infty$ is bounded. We may assume, without loss of generality, that $l_n \to l$

as $n \to \infty$. By dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty} \left\| \frac{u_{1,n} + ... + u_{k+1,n}}{k+1} \right\|^0 \\
&= \lim_{n\to\infty} \frac{1}{l_n} \left[1 + \int\limits_{T_1} M\left(t, l_n d \left\| \frac{x_{1,n} + ... + x_{k+1,n}}{k+1} \right\| \right) dt \right] \\
&= \frac{1}{l} \left[1 + \int\limits_{T_1} \lim_{n\to\infty} M\left(t, l_n d \left\| \frac{x_{1,n} + ... + x_{k+1,n}}{k+1} \right\| \right) dt \right] \\
&= \frac{1}{l} \left[1 + \int\limits_{T_1} M\left(t, \lim_{n\to\infty} l_n d \left\| \frac{x_{1,n} + ... + x_{k+1,n}}{k+1} \right\| \right) dt \right] \\
&= \frac{1}{l} \left[1 + \int\limits_{T_1} M(t, ld) dt \right] \\
&\geq \|d\chi_{T_1}\|^0 = 1.
\end{aligned}$$

Hence we have $\|u_{1,n} + u_{2,n} + \cdots + u_{k,n} + u_{k+1,n}\|^0 = k+1$ as $n \to \infty$. By $(L_M(R))^* = (E_M(R))^* = L_N(R)$ and Hahn-Banach theorem, there exists $w \in S(L_N(R))$ such that $\int_T w(t)h_1(t)dt = 1$. Put

$$u^*_{1,n} = x^*_{1,n} w\chi_{T_1},\ u^*_{2,n} = x^*_{2,n} w\chi_{T_1}\ , ...,\ u^*_{k,n} = x^*_{k,n} w\chi_{T_1} \quad n = 1, 2, ...$$

It is easy to see that $u^*_{1,n}, ..., u^*_{k+1,n} \in S(L_N(X^*))$. By Lemma 1.5, we obtain $u^*_{1,n}, ..., u^*_{k+1,n} \in S((L^0_M(X))^*)$. Moreover, it is easy to see that

$$\begin{array}{ccc}
u^*_{1,n}(u_{1,n}) = x^*_{1,n}(x_{1,n}) & ... & u^*_{1,n}(u_{k+1,n}) = x^*_{1,n}(x_{k+1,n}), \\
... & ... & ... \\
u^*_{k,n}(u_{1,n}) = x^*_{k,n}(x_{1,n}) & ... & u^*_{k,n}(u_{k+1,n}) = x^*_{k,n}(x_{k+1,n}).
\end{array}$$

Therefore,

$$\begin{aligned}
&\begin{vmatrix}
1 & 1 & ... & 1 \\
u^*_{1,n}(u_{1,n}) & u^*_{1,n}(u_{2,n}) & ... & u^*_{1,n}(u_{k+1,n}) \\
... & ... & ... & ... \\
u^*_{k,n}(u_{1,n}) & x^*_{k,n}(u_{2,n}) & ... & x^*_{k,n}(u_{k+1,n})
\end{vmatrix} \\
= &\begin{vmatrix}
1 & 1 & ... & 1 \\
x^*_{1,n}(x_{1,n}) & x^*_{1,n}(x_{2,n}) & ... & x^*_{1,n}(x_{k+1,n}) \\
... & ... & ... & ... \\
x^*_{k,n}(x_{1,n}) & x^*_{k,n}(x_{2,n}) & ... & x^*_{k,n}(x_{k+1,n})
\end{vmatrix} \\
\geq &\ \varepsilon_0,
\end{aligned}$$

a contradiction! Hence we obtain that X is $k-$uniformly rotund space.

Assume that X is $k-$uniformly rotund space and suppose that X is not uniformly rotund space. Then there exist $x, y, z \in S(X)$ with $2x = y + z$ and $y \neq z$. Since 1-rotund space and rotund space are equivalent, it is easy to see that $k \geq 2$. By Hahn-Banach theorem, there exists $x^* \in S(X^*)$ such that $x^*(x) = 1$. Hence we have $x^*(y) = x^*(z) = x^*(x) = 1$. Pick $h \in S(L_M^0(X))$, then there exists $d > 0$ such that $\mu H > 0$, where

$$H = \{t \in T : \|h(t)\| \geq d\}.$$

Put $h_1 = d \cdot x \cdot \chi_H$. It is easy to see that $h_1 \in L_M^0(X)\backslash\{0\}$. Hence there exists $l > 0$ such that $l \cdot h_1 \in S(L_M^0(X))$. By $(E_M^0(R))^* = L_N(R)$ and Hahn-Banach theorem, there exists $h_2 \in S(L_N(R))$ such that $\int_T ld \cdot \chi_H(t) \cdot h_2(t)dt = 1$. We may assume, without loss of generality, that $l = d = 1$. Decompose H into H_1, H_2,..., H_{k+1} such that

$$\int\limits_{H_1} h_2(t)dt = \int\limits_{H_2} h_2(t)dt = \cdots = \int\limits_{H_k} h_2(t)dt = \int\limits_{H_{k+1}} h_2(t)dt.$$

Put

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \ldots \\ u_k \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} y & x & x & \ldots & x & z \\ z & y & x & \ldots & x & x \\ x & z & y & \ldots & x & x \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x & x & x & \ldots & y & x \\ x & x & x & \ldots & z & y \end{pmatrix} \begin{pmatrix} \chi_{H_1} \\ \chi_{H_2} \\ \chi_{H_3} \\ \ldots \\ \chi_{H_k} \\ \chi_{H_{k+1}} \end{pmatrix}$$

Notice that $h_1 = x \cdot \chi_H \in S(L_M^0(X))$, we have

$$\begin{aligned} \|u_i\|^0 &= \inf_{c>0} \frac{1}{c}[1 + \rho_M(cu_i)] = \inf_{c>0} \frac{1}{c}\left[1 + \sum_{i=1}^{k+1} \int\limits_{H_i} M(t, c)dt\right] \\ &= \inf_{c>0} \frac{1}{c}\left[1 + \int\limits_{H} M(t, c)dt\right] = \|h_1\|^0 = 1. \end{aligned}$$

for any $i \in \{1, 2, ..., k + 1\}$. Moreover, we have

$$\frac{u_1 + u_2 + ... + u_{k+1}}{k + 1} = x \cdot \chi_H = h_1 \in S(L_M^0(X)).$$

By Hahn-Banach theorem, there exists $x_1^* \in S(X^*)$ such that $x_1^*(y) = c$ and $x_1^*(z) = -c$. We may assume, without loss of generality, that $c = 1$. Then $x_1^*(y) = 1$, $x_1^*(z) = -1$ and $x_1^*(x) = 0$. Define

$$v_1 = x_1^* \cdot h_2\chi_{H_2},\ v_2 = x_1^* \cdot h_2\chi_{H_3}\ , ...,\ v_k = x_1^* \cdot h_2\chi_{H_{k+1}}.$$

Hence, for any $i \in \{1, ..., k+1\}$ and $j \in \{1, ..., k\}$, we have

$$
\begin{aligned}
&\Delta(u_1, u_2, ..., u_{k+1}) \\
\geq &\begin{vmatrix} 1 & 1 & 1 & ... & 1 \\ (v_1, u_1) & (v_1, u_2) & (v_1, u_3) & ... & (v_1, u_{k+1}) \\ (v_2, u_1) & (v_2, u_2) & (v_2, u_3) & ... & (v_2, u_{k+1}) \\ ... & ... & ... & ... & ... \\ (v_k, u_1) & (v_k, u_2) & (v_k, u_3) & ... & (v_k, u_{k+1}) \end{vmatrix}_{k+1} \\
= &\begin{vmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & (k+1)^{-1} & -(k+1)^{-1} & ... & 0 \\ 0 & 0 & (k+1)^{-1} & ... & 0 \\ ... & ... & ... & ... & ... \\ -(k+1)^{-1} & 0 & 0 & ... & (k+1)^{-1} \end{vmatrix}_{k+1} \\
= &(k+1)^{-k} \left(\begin{vmatrix} 1 & -1 & 0 & ... & 0 \\ 0 & 1 & -1 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{vmatrix}_k + (-1)^{k+2} \begin{vmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & -1 & 0 & ... & 0 \\ 0 & 1 & -1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & -1 \end{vmatrix}_k \right) \\
= &(k+1)^{-k}(1 + k(-1)^{2k+1}) \\
\neq &\, 0,
\end{aligned}
$$

a contradiction! This implies that X is uniformly rotund space. $(4c)$ is true.

Suppose that $(1c)$ is not true. Then there exists $u \in S(L_M^0(X))$ such that $K(u) = \emptyset$. By Lemma 2.4, we have $\|u\|^0 = \int_T A(t) \cdot \|u(t)\| dt = 1$. Decompose T into T_1, T_2 ,..., T_{n+1} such that

$$\int\limits_{T_1} A(t) \cdot \|u(t)\|\, dt = \int\limits_{T_2} A(t) \cdot \|u(t)\|\, dt = \cdots = \int\limits_{T_{k+1}} A(t) \cdot \|u(t)\|\, dt. \tag{3}$$

Pick $\varepsilon \in (0, 1)$ and put

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ ... \\ u_k \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1+\varepsilon & 1 & 1 & ... & 1 & 1-\varepsilon \\ 1-\varepsilon & 1+\varepsilon & 1 & ... & 1 & 1 \\ 1 & 1-\varepsilon & 1+\varepsilon & ... & 1 & 1 \\ ... & ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 1+\varepsilon & 1 \\ 1 & 1 & 1 & ... & 1-\varepsilon & 1+\varepsilon \end{pmatrix} \begin{pmatrix} u\chi_{T_1} \\ u\chi_{T_2} \\ u\chi_{T_3} \\ ... \\ u\chi_{T_k} \\ u\chi_{T_{k+1}} \end{pmatrix}$$

By Lemma 2.4 and (3), we obtain that $\|u_1\|^0 \leq 1$, $\|u_2\|^0 \leq 1$,..., $\|u_{k+1}\|^0 \leq 1$. Moreover, it is easy to see that

$$u = \frac{u_1 + u_2 + ... + u_{k+1}}{k+1} \in S(L_M^0(X))$$

and

$$\begin{vmatrix} 1+\varepsilon & 1 & 1 & ... & 1 & 1-\varepsilon \\ 1-\varepsilon & 1+\varepsilon & 1 & ... & 1 & 1 \\ 1 & 1-\varepsilon & 1+\varepsilon & ... & 1 & 1 \\ ... & ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 1+\varepsilon & 1 \\ 1 & 1 & 1 & ... & 1-\varepsilon & 1+\varepsilon \end{vmatrix}_{k+1} = (k+1)^2\varepsilon^k > 0.$$

This implies that $u_1, u_2, ..., u_{k+1}$ are linearly independent. Hence $L^0_M(X)$ is not k-rotund space. This means that $L^0_M(X)$ is not k-uniformly rotund space, a contradiction!

Since $L^0_M(R)$ is k-uniformly rotund space, we have that $L^0_M(R)$ is reflexive space. This means that $M \in \Delta_2$ and $N \in \Delta_2$. This implies that $(2c)$ is true.

We next will prove that $(3c)$ is true. Since $L^0_M(R)$ is subspace of $L^0_M(X)$, we obtain that $L^0_M(R)$ is k-uniformly rotund space. Moreover we know that if Banach space X is k-uniformly rotund space, then X is k-rotund space. Hence $L^0_M(R)$ is k-rotund space. We will show that $L^0_M(R)$ is rotund space. In fact, suppose that $L^0_M(R)$ is not rotund. Then there exists $u \in S(L^0_M(R))$ such that u is not an extreme point of $B(L^0_M(R))$. Hence we obtain $\mu\{t \in T : |ku(t)| \in K_t\} > 0$, where $k \in K(u)$ (see [21]). It is easy to see that there exist $G \subset \{t \in T : |ku(t)| \in K_t\}$ and $0 < \varepsilon < \frac{1}{16}$ such that

$$M(t, |ku(t)|) = \frac{1}{2}M(t, |ku(t)| + k\varepsilon) + \frac{1}{2}M(t, |ku(t)| - k\varepsilon) < \infty, \quad t \in G$$

and $\mu G > 0$. Moreover, it is easy to see that there exist $p(t)$ and $q(t)$ such that

$$M(t, |ku(t)|) = p(t)\,|ku(t)| + q(t),$$
$$M(t, |ku(t)| \pm k\varepsilon) = p(t)(|ku(t)| \pm k\varepsilon) + q(t)$$

whenever $t \in G$. Decompose G into G_1, G_2 ,..., G_{n+1} such that $\int_{G_1} p(t)dt = \int_{G_2} p(t)dt = \cdots = \int_{G_{k+1}} p(t)dt$. Let

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ ... \\ u_k \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1+\varepsilon & 1 & 1 & ... & 1 & 1-\varepsilon \\ 1-\varepsilon & 1+\varepsilon & 1 & ... & 1 & 1 \\ 1 & 1-\varepsilon & 1+\varepsilon & ... & 1 & 1 \\ ... & ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 1+\varepsilon & 1 \\ 1 & 1 & 1 & ... & 1-\varepsilon & 1+\varepsilon \end{pmatrix} \begin{pmatrix} u\chi_{G_1} \\ u\chi_{G_2} \\ u\chi_{G_3} \\ ... \\ u\chi_{G_k} \\ u\chi_{G_{k+1}} \end{pmatrix}$$

Then we have

$$\begin{aligned}
\|u_1\|^0 &\leq \frac{1}{k}\left[1+\int_G M(t,|ku(t)|)dt\right] \\
&= \frac{1}{k}\left[1+\int_{G_1} M(t,|ku(t)|+k\varepsilon)dt+\int_{G_2} M(t,|ku(t)|)dt+\cdots\right. \\
&\qquad \left.+\int_{G_{k+1}} M(t,|ku(t)|-k\varepsilon)dt\right] \\
&= \frac{1}{k}\left[1+\int_{G_1} (p(t)(|ku(t)|+k\varepsilon)+q(t))dt+\int_{G_2} M(t,|ku(t)|)dt+\cdots\right. \\
&\qquad \left.+\int_{G_{k+1}} (p(t)(|ku(t)|-k\varepsilon)-q(t))dt\right] \\
&= \frac{1}{k}\left[1+\int_{G_1} M(t,|ku(t)|)dt+\int_{G_2} M(t,|ku(t)|)dt+\int_{G_{k+1}} M(t,|ku(t)|)dt\right] \\
&= \frac{1}{k}\left[1+\int_G M(t,|ku(t)|)dt\right] \\
&= \|u\|^0 = 1.
\end{aligned}$$

Similarly, we have $\|u_2\|^0 \leq \|u\|^0, ..., \|u_{k+1}\|^0 \leq \|u\|^0$. Moreover, it is easy to see that

$$u = \frac{u_1+u_2+...+u_{k+1}}{k+1} \in S(L_M^0(R))$$

and

$$\begin{vmatrix}
1+\varepsilon & 1 & 1 & ... & 1 & 1-\varepsilon \\
1-\varepsilon & 1+\varepsilon & 1 & ... & 1 & 1 \\
1 & 1-\varepsilon & 1+\varepsilon & ... & 1 & 1 \\
... & ... & ... & ... & ... & ... \\
1 & 1 & 1 & ... & 1+\varepsilon & 1 \\
1 & 1 & 1 & ... & 1-\varepsilon & 1+\varepsilon
\end{vmatrix}_{k+1} = (k+1)^2\varepsilon^k > 0.$$

This implies that $u_1, u_2, ..., u_{k+1}$ are linearly independent. Hence we obtain that $L_M^0(R)$ is not k-rotund space, a contradiction. So $L_M^0(R)$ is rotund space. Since $L_M^0(R)$ is k-uniformly rotund space and rotund space, we obtain that $L_M^0(R)$

is uniformly rotund space. This implies that $L_M^0(R)$ is weak uniformly rotund space. By Lemma 1.8, we obtain that $(3c)$ is true.

$(c) \Rightarrow (a)$. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty} \subset S(L_M^0(X))$ satisfy $\|u_n+v_n\|^0 \to 2$ and $k_n \in K(u_n)$, $h_n \in K(v_n)$. Since $N \in \Delta_2$, by Lemma 1.6, there exists $d \in (1,\infty)$ such that $d = \sup_n\{k_n, h_n\}$. Moreover, we know that $\inf_n\{k_n, h_n\} \leq 1$. Set

$$a = \inf_n \left\{ \frac{k_n}{k_n+h_n}, \frac{h_n}{k_n+h_n} \right\}, \qquad b = \sup_n \left\{ \frac{k_n}{k_n+h_n}, \frac{h_n}{k_n+h_n} \right\}$$

Then $[a,b] \subset (0,1)$. Since M is almost uniformly convex, by Lemma 1.7, for any $0 < \varepsilon < \frac{1}{16}$, there exist $\delta_\varepsilon > 0$ and a nonnegative measurable function $f_\varepsilon(t)$ such that

$$\lambda \in [a,b] \quad \text{and} \quad |u-v| \geq \frac{\varepsilon}{8}\max\{|u|,|v|\} \geq \frac{\varepsilon}{8}\cdot f_\varepsilon(t)$$

implies

$$M(t, \lambda u + (1-\lambda)v) \leq (1-\delta_\varepsilon)\lambda M(t,u) + (1-\lambda)M(t,v)$$

and

$$\rho_M(f_\varepsilon) = \int_T M(t, f_\varepsilon(t))dt < \frac{1}{8}\varepsilon.$$

For each $n \in N$, let

$$G_n = \left\{ t \in T : \max(\|k_n u_n(t)\|, \|h_n v_n(t)\|) \leq \frac{1}{2} f_\varepsilon(t) \right\},$$

$$E_n = \left\{ t \in T : \|k_n u_n(t) - h_n v_n(t)\| < \varepsilon \max(\|k_n u_n(t)\|, \|h_n v_n(t)\|) > \frac{\varepsilon}{2} f_\varepsilon(t) \right\},$$

$$F_n = \Big\{ t \in T : \|k_n u_n(t) - h_n v_n(t)\| \geq \varepsilon \max(\|k_n u_n(t)\|, \|h_n v_n(t)\|) > \varepsilon \cdot \frac{1}{2} f_\varepsilon(t), \\ |\|k_n u_n(t)\| - \|h_n v_n(t)\|| \leq \frac{\varepsilon}{8}\cdot\frac{1}{2} f_\varepsilon(t) \Big\},$$

$$H_n = \Big\{ t \in T : \|k_n u_n(t) - h_n v_n(t)\| \geq \varepsilon \max(\|k_n u_n(t)\|, \|h_n v_n(t)\|) > \varepsilon \cdot \frac{1}{2} f_\varepsilon(t), \\ |\|k_n u_n(t)\| - \|h_n v_n(t)\|| > \frac{\varepsilon}{8}\cdot\frac{1}{2} f_\varepsilon(t) \Big\}$$

It is easy to see that $H_n = T \backslash (G_n \cup E_n \cup F_n)$. From $k_n h_n/(k_n+h_n) \leq d^2/2$ and

$\rho_M(k_n u_n) + \rho_M(h_n v_n) = k_n + h_n - 2 \le 2d - 2$, we deduce

$$\begin{aligned}\int_{G_n} M(t, \|k_n u_n(t) - h_n v_n(t)\|)dt &\le \int_{G_n} M(t, \|k_n u_n(t)\| + \|h_n v_n(t)\|)dt \\ &\le \int_{G_n} M\left(t, \frac{1}{2}f_\varepsilon(t) + \frac{1}{2}f_\varepsilon(t)\right)dt \\ &\le \int_{G_n} M\left(t, f_\varepsilon(t)\right)dt \le \frac{1}{8}\varepsilon\end{aligned}$$

for all $n \in N$ and

$$\begin{aligned}&\|(k_n u_n - h_n v_n)\chi_{E_n}\|^0 \\ &\le \sqrt{\varepsilon}\left[1 + \int_{E_n} M\left(t, \frac{1}{\sqrt{\varepsilon}}\|k_n u_n(t) - h_n v_n(t)\|\right)dt\right] \\ &\le \sqrt{\varepsilon}\left[1 + \int_{E_n} M\left(t, \frac{1}{\sqrt{\varepsilon}}\varepsilon \max\left(\|k_n u_n(t)\|, \|h_n v_n(t)\|\right)\right)dt\right] \\ &\le \sqrt{\varepsilon}\left[1 + \int_{E_n} M\left(t, \sqrt{\varepsilon}\left(\|k_n u_n(t)\| + \|h_n v_n(t)\|\right)\right)dt\right] \\ &= \sqrt{\varepsilon}\left[1 + \int_{E_n} M\left(t, 2\sqrt{\varepsilon}\left(\frac{\|k_n u_n(t)\| + \|h_n v_n(t)\|}{2}\right)\right)dt\right] \\ &\le \sqrt{\varepsilon}\left[1 + 2\sqrt{\varepsilon}\int_{E_n} M\left(t, \frac{\|k_n u_n(t)\| + \|h_n v_n(t)\|}{2}\right)dt\right] \\ &\le \sqrt{\varepsilon}\left[1 + \sqrt{\varepsilon}\left(\int_{E_n} M\left(t, \|k_n u_n(t)\|\right)dt + \int_{E_n} M\left(t, \|h_n v_n(t)\|\right)dt\right)\right] \\ &\le \sqrt{\varepsilon}\left[1 + \sqrt{\varepsilon}(\rho_M(k_n u_n) + \rho_M(h_n v_n))\right] \\ &\le \sqrt{\varepsilon}\left[1 + \sqrt{\varepsilon}(2d - 2)\right] < \sqrt{\varepsilon}(1 + 2d).\end{aligned}$$

for all $n \in N$. We may assume, without loss of generality, that $\sqrt{\varepsilon}(1 + 2d) < 1$. Then

$$\begin{aligned}\rho_M((k_n u_n - h_n v_n)\chi_{E_n}) \le \|(k_n u_n - h_n v_n)\chi_{E_n}\| &\le \|(k_n u_n - h_n v_n)\chi_{E_n}\|^0 \\ &< \sqrt{\varepsilon}(1 + 2d).\end{aligned}$$

for all $n \in N$. Next, we will prove that $\|k_n u_n \chi_{F_n}\|^0 \to 0$ and $\|h_n v_n \chi_{F_n}\|^0 \to 0$ as $n \to \infty$. Decompose F_n into F_n^1 and F_n^2, where

$$F_n^1 = \{t \in F_n : \|k_n u_n(t)\| - \|h_n v_n(t)\| \geq 0\},$$

$$F_n^2 = \{t \in F_n : \|k_n u_n(t)\| - \|h_n v_n(t)\| < 0\}.$$

It is well known that function $f(\lambda) = (\lambda - \frac{1}{16}\varepsilon f_\varepsilon(t))/\lambda$ is nondecreasing on $\lambda \in (\frac{1}{16}\varepsilon f_\varepsilon(t), \infty)$. Hence, for any $t \in F_n^1$, we have

$$1 \geq \frac{\|h_n v_n(t)\|}{\|k_n u_n(t)\|} \geq \frac{\|k_n u_n(t)\| - \frac{1}{16}\varepsilon f_\varepsilon(t)}{\|k_n u_n(t)\|} \geq \frac{\varepsilon \frac{1}{2} f_\varepsilon(t) - \frac{1}{16}\varepsilon f_\varepsilon(t)}{\varepsilon \frac{1}{2} f_\varepsilon(t)} = \frac{7}{8}$$

whenever $f_\varepsilon(t) > 0$. Moreover, it is easy to see that $\|h_n v_n(t)\| = \|k_n u_n(t)\|$ whenever $f_\varepsilon(t) = 0$ and $t \in F_n$. This implies that

$$\|h_n v_n(t)\| \geq \frac{7}{8} \|k_n u_n(t)\| \quad \text{whenever } t \in F_n^1.$$

Similarly, for any $t \in F_n^2$, we have

$$\|h_n v_n(t)\| \geq \frac{7}{8} \|k_n u_n(t)\| \quad \text{whenever } t \in F_n^2.$$

Hence, by Lemma 2.2, for any $t \in F_n^1$, we have

$$\begin{aligned}
&\frac{h_n}{k_n + h_n} \|k_n u_n(t)\| + \frac{k_n}{k_n + h_n} \|h_n v_n(t)\| \\
&\qquad - \left\| \frac{h_n}{k_n + h_n} k_n u_n(t) + \frac{k_n}{k_n + h_n} h_n v_n(t) \right\| \\
&\geq \|h_n v_n(t)\| \cdot \delta \left(\frac{\varepsilon \|k_n u_n(t)\| - \frac{\varepsilon f_\varepsilon(t)}{16}}{\|h_n v_n(t)\|}, \frac{h_n}{k_n + h_n}, 1 \right) \\
&\geq \|h_n v_n(t)\| \cdot \delta \left(\frac{\varepsilon \|k_n u_n(t)\| - \frac{1}{8}\varepsilon \|k_n u_n(t)\|}{\|k_n u_n(t)\|}, \frac{h_n}{k_n + h_n}, 1 \right) \\
&= \|h_n v_n(t)\| \cdot \delta \left(\frac{7}{8}\varepsilon, \frac{h_n}{k_n + h_n}, 1 \right).
\end{aligned}$$

Similarly, for any $t \in F_n^2$, we have

$$\begin{aligned}
&\frac{h_n}{k_n + h_n} \|k_n u_n(t)\| + \frac{k_n}{k_n + h_n} \|h_n v_n(t)\| \\
&\qquad - \left\| \frac{h_n}{k_n + h_n} k_n u_n(t) + \frac{k_n}{k_n + h_n} h_n v_n(t) \right\| \\
&\geq \min(\|k_n u_n(t)\|, \|h_n v_n(t)\|) \cdot \delta \left(\frac{7}{8}\varepsilon, \frac{h_n}{k_n + h_n}, 1 \right) \\
&= \|k_n u_n(t)\| \cdot \delta \left(\frac{7}{8}\varepsilon, \frac{h_n}{k_n + h_n}, 1 \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|u_n\|^0 + \|v_n\|^0 - \|u_n + v_n\|^0 \\
&\geq \frac{1}{k_n}[1 + \rho_M(k_n u_n)] + \frac{1}{h_n}[1 + \rho_M(h_n v_n)] \\
&\qquad - \frac{k_n + h_n}{k_n h_n}\left[1 + \rho_M\left(\frac{k_n h_n}{k_n + h_n}(u_n + v_n)\right)\right] \\
&= \frac{k_n + h_n}{k_n h_n}\left[\frac{h_n}{k_n + h_n}\int\limits_T M(t, \|k_n u_n(t)\|)dt + \frac{k_n}{k_n + h_n}\int\limits_T M(t, \|h_n v_n(t)\|)\right. \\
&\qquad \left. - \int\limits_T M\left(t, \frac{k_n h_n}{k_n + h_n}\|u_n(t) + v_n(t)\|\right)dt\right. \\
&= \frac{k_n + h_n}{k_n h_n}\int\limits_T \left(\frac{h_n}{k_n + h_n}M(t, \|k_n u_n(t)\|) + \frac{k_n}{k_n + h_n}M(t, \|h_n v_n(t)\|)\right. \\
&\qquad \left. - M\left(t, \frac{k_n h_n}{k_n + h_n}\|u_n(t) + v_n(t)\|\right)\right)dt \\
&\geq \frac{k_n + h_n}{k_n h_n}\int\limits_{F_n}\left(M\left(t, \frac{h_n}{k_n + h_n}\|k_n u_n(t)\| + \frac{k_n}{k_n + h_n}\|h_n v_n(t)\|\right)\right. \\
&\qquad \left. - M\left(t, \frac{k_n h_n}{k_n + h_n}\|u_n(t) + v_n(t)\|\right)\right)dt \\
&\geq \frac{k_n + h_n}{k_n h_n}\int\limits_{F_n}\left(M\left(t, \frac{h_n}{k_n + h_n}\|k_n u_n(t)\| + \frac{k_n}{k_n + h_n}\|h_n v_n(t)\|\right.\right. \\
&\qquad \left.\left. - \left\|\frac{h_n}{k_n + h_n}(k_n u_n(t)) + \frac{k_n}{k_n + h_n}(h_n v_n(t))\right\|\right)\right)dt \\
&\geq \frac{k_n + h_n}{k_n h_n}\int\limits_{F_n^1} M\left(t, \|h_n v_n(t)\| \cdot \delta\left(\frac{7}{8}\varepsilon, \frac{h_n}{k_n + h_n}, 1\right)\right)dt \\
&\qquad + \frac{k_n + h_n}{k_n h_n}\int\limits_{F_n^2} M\left(t, \|k_n u_n(t)\| \cdot \delta\left(\frac{7}{8}\varepsilon, \frac{h_n}{k_n + h_n}, 1\right)\right)dt \\
&\geq \frac{k_n + h_n}{k_n h_n}\int\limits_{F_n} M\left(t, \frac{7}{8}\|k_n u_n(t)\| \cdot \delta\left(\frac{7}{8}\varepsilon, \frac{h_n}{k_n + h_n}, 1\right)\right)dt.
\end{aligned}$$

By $\|u_n\|^0 + \|v_n\|^0 - \|u_n + v_n\|^0 \to 0$, we have

$$\frac{k_n + h_n}{k_n h_n}\int\limits_{F_n} M\left(t, \frac{7}{8}\|k_n u_n(t)\| \cdot \delta\left(\frac{7}{8}\varepsilon, \frac{h_n}{k_n + h_n}, 1\right)\right)dt \to 0 \quad \text{as } n \to \infty.$$

We may assume, without loss of generality, that $\frac{h_n}{k_n+h_n} \to \alpha$ as $n \to \infty$, where $\alpha \in (0,1)$. By Lemma 1.4, we have

$$\left\| k_n u_n \chi_{F_n} \cdot \delta\left(\frac{7}{8}\varepsilon, \frac{h_n}{k_n+h_n}, 1\right) \right\| \to 0 \quad \text{as } n \to \infty.$$

By Lemma 2.3, we have $\delta(\frac{7}{8}\varepsilon, \frac{h_n}{k_n+h_n}, 1) \to \delta(\frac{7}{8}\varepsilon, \alpha, 1) > 0$. This implies that $\|k_n u_n \chi_{F_n}\| \to 0$ as $n \to \infty$. Similarly, we have $\|h_n v_n \chi_{F_n}\| \to 0$ as $n \to \infty$. Since $\|k_n u_n \chi_{F_n}\| \to 0$ and $\|h_n v_n \chi_{F_n}\| \to 0$, we have $\|(k_n u_n - h_n v_n)\chi_{F_n}\| \to 0$ as $n \to \infty$. By Lemma 1.4, we have $\rho_M((k_n u_n - h_n v_n)\chi_{F_n}) \to 0$ as $n \to \infty$.

Now, we will consider the sets H_n. For any $t \in H_n$, we have

$$\begin{aligned} &M\left(t, \frac{h_n}{k_n+h_n}\|k_n u_n(t)\| + \frac{k_n}{k_n+h_n}\|h_n v_n(t)\|\right) \\ &\le (1-\delta_\varepsilon)\frac{h_n}{k_n+h_n}M(t, \|k_n u_n(t)\|) + \frac{k_n}{k_n+h_n}M(t, \|h_n v_n(t)\|) \end{aligned} \tag{4}$$

and

$$\begin{aligned} &M\left(t, \frac{h_n}{k_n+h_n}\|k_n u_n(t)\| + \frac{k_n}{k_n+h_n}\|h_n v_n(t)\|\right) \\ &\le (1-\delta_\varepsilon)\frac{k_n}{k_n+h_n}M(t, \|h_n v_n(t)\|) + \frac{h_n}{k_n+h_n}M(t, \|k_n u_n(t)\|). \end{aligned} \tag{5}$$

Therefore, by inequality (4), we have

$$\begin{aligned} &\|u_n\|^0 + \|v_n\|^0 - \|u_n + v_n\|^0 \\ \ge\ &\frac{k_n+h_n}{k_n h_n}\Bigg[\int_{H_n} \frac{h_n}{k_n+h_n}M(t, \|k_n u_n(t)\|) + \frac{k_n}{k_n+h_n}M(t, \|h_n v_n(t)\|) \\ &\quad - M\left(t, \frac{k_n h_n}{k_n+h_n}\|u_n(t)+v_n(t)\|\right) dt \\ \ge\ &\frac{k_n+h_n}{k_n h_n}\Bigg[\int_{H_n} \frac{h_n}{k_n+h_n}M\left(t, \|k_n u_n(t)\|\right) + \frac{k_n}{k_n+h_n}M\left(t, \|h_n v_n(t)\|\right) \\ &\quad - M\left(t, \frac{h_n}{k_n+h_n}\|k_n u_n(t)\| + \frac{k_n}{k_n+h_n}\|h_n v_n(t)\|\right) dt \\ \ge\ &\frac{k_n+h_n}{k_n h_n}\delta_\varepsilon \int_{H_n} \frac{h_n}{k_n+h_n}M(t, \|k_n u_n(t)\|)dt \\ =\ &\frac{1}{k_n}\delta_\varepsilon \int_{H_n} M(t, \|k_n u_n(t)\|)dt \\ \ge\ &\frac{1}{d}\delta_\varepsilon \int_{H_n} M(t, \|k_n u_n(t)\|)dt \end{aligned}$$

Similarly, by inequality (5), we have

$$\|u_n\|^0 + \|v_n\|^0 - \|u_n + v_n\|^0 \geq \frac{1}{d}\delta_\varepsilon \int\limits_{H_n} M(t, \|h_n v_n(t)\|)dt.$$

By $\|u_n\|^0 + \|v_n\|^0 - \|u_n + v_n\|^0 \to 0$ as $n \to \infty$, we have

$$\int\limits_{H_n} M(t, \|k_n u_n(t)\|)dt \to 0, \qquad \int\limits_{H_n} M(t, \|h_n v_n(t)\|)dt \to 0 \ \text{ as } n \to \infty.$$

By $M \in \Delta_2$, we have $\|k_n u_n \chi_{H_n}\|^0 \to 0$ and $\|h_n v_n \chi_{H_n}\|^0 \to 0$ as $n \to \infty$. By $M \in \Delta_2$, we have $\rho_M((k_n u_n - h_n v_n)\chi_{H_n}) \to 0$ as $n \to \infty$. Moreover, we have proved $\rho_M((k_n u_n - h_n v_n)\chi_{F_n}) \to 0$ as $n \to \infty$. Hence there exists natural number N_0 such that $\rho_M((k_n u_n - h_n v_n)\chi_{H_n}) < \varepsilon$ and $\rho_M((k_n u_n - h_n v_n)\chi_{F_n}) < \varepsilon$ whenever $n > N_0$. We have proved that

$$\rho_M((k_n u_n - h_n v_n)\chi_{G_n}) < \frac{1}{8}\varepsilon \quad \text{and} \quad \rho_M((k_n u_n - h_n v_n)\chi_{E_n}) < \sqrt{\varepsilon}(1 + d).$$

for all $n \in N$, whence

$$\begin{aligned} \rho_M(k_n u_n - h_n v_n) &= \rho_M((k_n u_n - h_n v_n)\chi_{G_n}) + \rho_M((k_n u_n - h_n v_n)\chi_{E_n}) \\ &\qquad + \rho_M(k_n u_n - h_n v_n \chi_{H_n}) + \rho_M(k_n u_n - h_n v_n \chi_{F_n}) \\ &\leq \frac{1}{8}\varepsilon + \sqrt{\varepsilon}(1 + d) + \varepsilon + \varepsilon \\ &\leq \frac{1}{8}\sqrt{\varepsilon} + \sqrt{\varepsilon}(1 + d) + \sqrt{\varepsilon} + \sqrt{\varepsilon} \end{aligned}$$

whenever $n > N_0$. This implies that $\rho_M(k_n u_n - h_n v_n) \to 0$ as $n \to \infty$. By Lemma 1.4, $\|k_n u_n - h_n v_n\|^0 \to 0$ as $n \to \infty$. Moreover, we have

$$|k_n - h_n| = \left| \|k_n u_n\|^0 - \|h_n v_n\|^0 \right| \leq \|k_n u_n - h_n v_n\|^0 \to 0 \ \text{ as } n \to \infty.$$

This implies that $\|u_n - v_n\|^0 \to 0$ as $n \to \infty$. Hence we obtain that $L^0_M(X)$ is uniformly rotund space.

$(a) \Rightarrow (b)$ is obvious. This completes the proof. □

3. Applications to fixed point property of Musielak-Orlicz-Bochner function spaces

One of the classical problems of metric fixed point theory concerns existence of fixed points of nonexpansive mappings. Let C be a nonempty bounded closed and convex subset of a Banach space X. A mapping $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A Banach space X is said to have the fixed point property (FPP) if every such mapping has a fixed point. Adding the assumption that C is weakly compact in this condition, we obtain the definition of the weak fixed point property (WFPP). In 1965, F. Browder [1] proved that Hilbert spaces have FPP. In the same year, Browder [2] and D. Gohde [8] showed independently that uniformly convex spaces have FPP, and W. A. Kirk [17] proved a more general result stating that all Banach spaces with weak normal structure have WFPP. In 2004, Dowling, Lennard and Turett [5] proved that a nonempty closed bounded convex subset of c_0 has FPP if and only if it is weakly compact. Using Theorem 2.1, we give a sufficient condition for a Musielak-Orlicz-Bochner function space $L_M^0(X)$ to have the fixed point property.

Theorem 3.1. *Let $L_M^0(X)$ be a Musielak-Orlicz-Bochner function space. If*

(a) *For any $u \in S(L_M^0(X))$, the set $K(u) \neq \emptyset$;*

(b) *$M \in \Delta_2$, $N \in \Delta_2$;*

(c) *$M(t, u)$ is strictly convex with respect u for almost all $t \in T$ and M is almost uniformly convex;*

(d) *X is uniformly rotund space;*

then Musielak-Orlicz-Bochner function space $L_M^0(X)$ have the fixed point property.

Proof. By [2], we know that uniformly rotund Banach spaces have the fixed point property for nonexpansive mappings. Using Theorem 2.1, it is easy to see that $L_M^0(X)$ have the fixed point property. $\square$

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