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Monge-Ampère Type Function Splittings

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Given convex $u \in C(\bar{\Omega})$ with Monge-Ampère measure Mu, and finite Borel measures μ and ν satisfying $\mu + \nu = Mu$, consider the problem of determining a 'splitting' $u = v + w$ for u where $v, w \in C(\bar{\Omega})$ are convex functions satisfying $Mv = \mu$, $Mw = \nu$, so that $Mu = M(v + w) = Mv + Mw$. It is shown that although this problem is not in general solvable, a best L^p approximation $v^* + w^*$ for u may always be found. In particular, letting $U = \sup_{(v,w)\in\mathcal{F}} (v + w)$, there exist optimal sums $v^* + w^*$ achieving $\inf_{(v,w)\in\mathcal{F}} \|u - (v + w)\|_p$ and $\inf_{(v,w)\in\mathcal{F}} \|U - (v+w)\|_p$, $p \geq 1$, for appropriately constrained classes $\mathcal{F}$ of feasible pairs (v, w) of convex functions satisfying $Mv = \mu$, $Mw = \nu$ and $v + w = u$ on $\partial\Omega$. Moreover, U may be written as $U = \bar{v} + \bar{w}$ within $\bar{\Omega}$, $(\bar{v}, \bar{w}) \in \mathcal{F}$. The analysis depends upon basic properties of convex functions and the measures they determine. We also consider the related problem of characterizing functions $u \in W^{2,n}(\Omega)$ which may be realized as differences $u = v - w$ of convex functions $v, w \in W^{2,n}(\Omega)$ with $Mu = Mv - Mw$. Here Mu is the signed measure defined by $dMu = \det D^2u\,dx$. Letting $U^- = \sup_{(v,w)\in\mathcal{F}}(v - w)$ and $U_- = \inf_{(v,w)\in\mathcal{F}}(v - w)$, we show that optimal differences $v^* - w^*$ exist for the problems $\inf_{(v,w)\in\mathcal{F}} \|u - (v - w)\|_p$, $\inf_{(v,w)\in\mathcal{F}} \|U^- - (v - w)\|_p$ and $\inf_{(v,w)\in\mathcal{F}} \|U_- - (v - w)\|_p$. Also, $U^- = v^- - w^-$ and $U_- = v_- - w_-$ for appropriate pairs $(v^-, w^-), (v_-, w_-) \in \mathcal{F}$. Finally, the relaxed problem of finding $v + w = u$ for general Mv and Mw with $Mv + Mw = Mu$ (no fixed μ and ν), is considered. Topological properties of the collection of these relaxed splitting pairs (v, w), and those for the unrelaxed problem, for a given u, are developed.

Keywords: Monge-Ampère equations, additive solution, optimization characterizations, convex functions

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1. Introduction

The theory of the solution of elliptic Monge-Ampère equations has a rich, interesting and varied history dating back at least to Alexandrov [2, 3]. The weak formulation of the Monge-Ampère partial differential equation identifies convex functions on a bounded convex domain with Borel measures. In this form, the existence, uniqueness and smoothness of solutions have been studied extensively in the literature [2]–[10], [13], [14].

In this paper we consider two novel problems based upon the theory of Monge-

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Ampère equations. The first problem concerns splitting a convex function into the sum of two convex functions whose associated Borel measures sum to the given function's Monge-Ampère measure. The second problem concerns appropriately constructing a difference of convex functions (see [1], [11], [12], [16].) such that the difference of their measures coincides with a given signed measure. Both of these problems are motivated by the problem that naturally arises when the Borel measure of a Monge-Ampère partial differential equation is replaced with a more general signed measure. In this case, standard theory breaks down since the solution of such an equation cannot be simply a convex function, and it is unclear what the association between a more general function and a related signed measure should be. This obstacle can be surmounted in the setting of $W^{2,n}$ functions, and it is natural to explore the hypothesis that a difference of convex functions solves the equation.

We do not provide a complete solution for either general problem posed above. We pursue the following more modest goals. We first precisely formulate our problem of splitting convex functions u as sums $u = v+w$ or differences $u = v-w$, with v and w convex. We illustrate by examples that many, but certainly not all appropriately chosen functions may be realized as such splittings. We proceed to show however, that the splitting problem may be solved approximately in both a pointwise and an L^p sense. This allows us to determine maximal sums and differences for pairs (v, w) chosen from a feasible class. We conclude by describing further properties of splittings and properties of certain classes of generalized splittings.

2. The Splitting Problems

Throughout this paper, let $\Omega \subset \mathbb{R}^n$ be open, bounded and strictly convex. Let $u \in C(\Omega)$ be convex with associated Monge-Ampère measure Mu defined by $Mu(E) = m(\partial u(E))$ for Borel $E \subset \Omega$, where $m(\cdot)$ denotes Lebesgue measure. The *normal mapping* ∂u is defined by $\partial u(E) = \{p \in \mathbb{R}^n \mid u(x) \geq < p, x - x_0 > + u(x_0),\ x_0 \in E,\ x \in \Omega\}$, the collection of subgradients of u at points in E. Mu is a Borel measure, finite on compact subsets of Ω. The open set Ω is *strictly convex* when for any $x, y \in \bar{\Omega} = \mathrm{cl}\ \Omega$, the open line segment joining x and y lies entirely within Ω. This is a technical requirement necessary for the existence of appropriate solutions to Monge-Ampère equations (1)–(3) below.

Consider first the problem: Given convex $u \in C(\bar{\Omega})$, and a 'splitting' $\mu+\nu = Mu$ for the measure Mu, μ and ν Borel measures, finite on compact sets, find a 'splitting' $u = v+w$ for u where $v, w \in C(\bar{\Omega})$ are convex functions satisfying the Monge-Ampère partial differential equations $Mv = \mu$ and $Mw = \nu$, respectively.

This splitting problem may be precisely formulated as:

Monge-Ampère splitting problem (**MAS**): Given convex $u \in C(\bar{\Omega})$, $Mu(\Omega) <$

$+\infty$, satisfying (in the weak sense)

$$\begin{aligned} Mu(x) &= \mu + \nu \quad \text{in } \Omega \\ u(x) &= g(x) \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

find convex v and w in $C(\bar{\Omega})$ satisfying

$$\begin{aligned} Mv(x) &= \mu \quad \text{in } \Omega \\ v(x) &= f(x) \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

and

$$\begin{aligned} Mw(x) &= \nu \quad \text{in } \Omega \\ w(x) &= g(x) - f(x) \quad \text{on } \partial\Omega \end{aligned} \tag{3}$$

for which $v(x) + w(x) = u(x)$ for all $x \in \bar{\Omega}$, where μ and ν are given Borel measures in Ω, finite on compact sets, $g \in C(\partial\Omega)$ is a given function, and $f \in C(\partial\Omega)$ is chosen from an initial class $\mathcal{C} \subset C(\partial\Omega)$.

Under the given conditions, since $Mu(\Omega) < +\infty$ implies that $\mu(\Omega) < +\infty$ and $\nu(\Omega) < +\infty$, the Dirichlet problems (1)–(3) have unique, weak convex solutions (see Gutiérrez [10]). Note explicitly that $f \in \mathcal{C}$ determines both v and w through (2) and (3), and on $\partial\Omega$, $v + w = f + (g - f) = g$ as required. The 'data' triple (μ, ν, g) and the class $\mathcal{C}$, uniquely and succinctly characterize **MAS**. That is, given (μ, ν, g), u solving (1) is uniquely defined, and we seek $C(\bar{\Omega})$ convex v and w satisfying (2) and (3) for which $v + w = u$.

To fix the ideas, simple examples of solutions of the splitting problem are given below. In Examples 2.1 and 2.2, let Ω be the open unit disk in the xy-plane.

Example 2.1. Let $g(x,y) = |y|$, and let $v(x,y) = |y|$ for $y \geq 0$ and $v(x,y) = 0$ for $y < 0$. This determines $f = |y|$, $y \geq 0$, $f = 0$, $y < 0$ on $\partial\Omega$. Similarly set $w(x,y) = |y|$ for $y \leq 0$ and $y > 0$. Trivially, $Mv = Mw = 0$, so that $\mu = \nu = 0$. Setting $u = v + w$, one has $Mu = Mv + Mw = 0$ with $u = v + w = g$ on $\partial\Omega$. Replacing $|y|$ by y^2 above provides an example with smooth functions. □

Example 2.2. Let $g(x,y) = 2x^2 + 8xy + 8y^2$ and let $v(x,y) = \frac{3}{2}x^2 + 6xy + 6y^2$. This determines $f = \frac{3}{2}x^2 + 6xy + 6y^2$ on $\partial\Omega$. Set $w(x,y) = \frac{1}{2}x^2 + 2xy + 2y^2$. Again, $Mv = Mw = M(v+w) = 0$ so that setting $u = v + w$, $Mu = Mv + Mw$ and $v + w = g$ on $\partial\Omega$. □

In Examples 2.1 and 2.2, $Mv = Mw = M(v+w) = 0$. The *homogenous* splitting problem $(0, 0, g)$ ($\mu = \nu = 0$), always has the trivial solution $v + w$ where $Mv = Mw = 0$, $v|_{\partial\Omega} = g$ and consequently $w = 0$ throughout $\bar{\Omega}$ ($v \equiv u$). Similarly, $v' = 0$ and w' satisfying $Mw' = 0$, $w'|_{\partial\Omega} = g$ is another splitting, as are the convex combinations (v^λ, w^λ) where $v^\lambda = (1-\lambda)v + \lambda v'$ and $w^\lambda = (1-\lambda)w + \lambda w'$, $0 \leq \lambda \leq 1$ (see proof of Theorem 7.4, Section 7). Interestingly, in two dimensions

$(n = 2)$, for splittings $v, w \in C^2(\bar{\Omega})$ to exist, it is necessary that $Mv = 0$ and $Mw = 0$. Thus, in $\mathbb{R}^2$, only homogeneous $C^2(\bar{\Omega})$ splittings are possible. In particular, choose $\mu \neq 0$ and $\nu \neq 0$ so that v, w and u satisifying (1)–(3) are $C^2(\bar{\Omega})$ solutions (e.g. μ and ν constant in Ω.). Then, if u satisfying (1) has **MAS** $u = v + w$, by Theorem A.2 in the Appendix, necessarily $Mu = Mv = Mw = 0$, contradicting $\mu \neq 0$ and $\nu \neq 0$. Thus no splitting exists for the chosen μ and ν for any boundary data g.

The following example presents a splitting for a nonhomogeneous problem.

Example 2.3. Let $\Omega = \{(x, y) \mid x^2 + 4y^2 < 4\}$. $\partial\Omega$ is tangent to the unit disk at the points $(0, \pm 1)$. Write $\Omega = \cup_{i=1}^{3} \Omega_i$ where Ω_1 is the open unit disk, $\Omega_2 = \{(x, y) \in \Omega \mid \Omega \setminus \Omega_1,\ x < 0\}$ and $\Omega_3 = \{(x, y) \in \Omega \mid \Omega \setminus \Omega_1,\ x > 0\}$. Let $v = x^2 + y^2 - 1$ on $\bar{\Omega}_2$ and $v = 0$ on $\bar{\Omega} \setminus \bar{\Omega}_2$, and let $w = x^2 + y^2 - 1$ on $\bar{\Omega}_3$ and $w = 0$ on $\bar{\Omega} \setminus \bar{\Omega}_3$. It is easily checked that $\mu = Mv$ and $\nu = Mw$ are mutually singular, both zero on the open unit disk. It follows directly that for $u = v + w$, $Mu = \mu + \nu$ and so $v + w$ is a splitting for u. $\square$

If μ and ν are mutually singular ($\mu \perp \nu$) then the splitting $u = v + w$ may be viewed as a natural 'orthogonal' splitting of u. In general, as remarked above, the nonhomogeneous **MA** (Monge-Ampère) splitting problem will not always have a solution, even when $\mu \perp \nu$. This should not be surprising since Monge-Ampère equations are fully nonlinear.

The collection of convex functions on Ω, continuous up to the boundary, is very rich. The principle difficulty in the **MAS** problem is that for convex v and w, it may happen that $M(v + w)(E) > Mv(E) + Mw(E)$ for Borel set $E \subset \Omega$. (For example, $v = x^2$, $w = y^2$, $m(E) > 0$.) If this occurs for all v and w satisfying (2) and (3), for some E depending on v and w, then there can be no **MAS** based upon μ and ν. Consider the following less trivial example, in which $\mu \perp \nu$.

Example 2.4. Let $v = |(x, y)|$ and $w = |(x, y) - (2, 0)|$. For Borel set E with $(0, 0) \notin E$, $\partial v(E) \subset S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ and hence $Mv(E) = 0$. Similarly, $\partial w(E) \subset S^1$ so that $Mw(E) = 0$. Let $\mu = \delta_{(0,0)}$ and $\nu = \delta_{(2,0)}$ be Dirac measures concentrated at $(0, 0)$ and $(2, 0)$ in the plane. It is natural to conjecture that $v + w$ is a **MA** splitting for the function u defined by $u = v + w$ (I.e., $Mu = Mv + Mw$.).

Let $E = \{(x, y) \mid .5 \leq x \leq 1.5, 1 \leq y \leq 2\}$, a rectangle and Borel set in $\mathbb{R}^n$. We have $Mv(E) = Mw(E) = 0$. It is easy to check that $M(v + w)(E) = m(\partial(v + w)(E)) > 0$ since the set $\partial(v + w)(E)$ has nonempty interior. Thus $M(v + w)(E) > Mv(E) + Mw(E)$. $\square$

Note that this example does not show that u as defined cannot be split as desired. We provide no test here for failure to be splittable. The example does indicate that 'obvious' splittings may fail.

Remark 2.5. The general question of the existence of examples of splittings with both $Mv \neq 0$ and $Mw \neq 0$ is difficult. Demonstrating the existence of $C^2(\bar{\Omega})$ splittings $v + w = u$, $v, w \in C^2(\bar{\Omega})$, is equivalent to showing that symmetric, positive semi-definite matrices A may be split as $A = B + C$ using symmetric, positive semi-definite matrices B and C satisfying $\det(A) = \det(B) + \det(C)$. This is of course because convex $z \in C^2(\Omega)$ has $dMz = \det D^2 z \, dx$. As shown in the Appendix, in $\mathbb{R}^2$, $\det(B + C) = \det(B) + \det(C)$ only when both $\det(B) = 0$ and $\det(C) = 0$. Hence, since all convex functions $v, w \in C(\Omega)$ may be approximated by sequences of $C^2(\Omega)$ convex functions converging uniformly on compact subsets of Ω, it may be that splittings exist in general only under restrictive conditions. □

Even though **MAS** is not always solvable, a main goal here is to prove that optimal approximations $v + w$ exist for u satisfying (1), v and w satisfying (2) and (3). To state these results, let $C(\partial\Omega)$ be equipped with the supremum norm and let $\mathcal{C} \subset C(\partial\Omega)$ be a family of boundary functions f. Given (μ, ν, g) and family $\mathcal{C}$, the pair (v, w) will be said to be *feasible for* (μ, ν, g) if v and w satisfy (2) and (3) for some $f \in \mathcal{C}$. Let the corresponding *feasible class* $\mathcal{F}_{(\mu,\nu,g;\mathcal{C})}$ be defined by

$$\mathcal{F}_{(\mu,\nu,g;\mathcal{C})} = \{(v, w) \mid (v, w) \text{ is feasible for } (\mu, \nu, g),\ v|_{\partial\Omega} \in \mathcal{C}\}.$$

Since each function $v|_{\partial\Omega} = f \in \mathcal{C}$ uniquely determines v solving (2), and hence w solving (3), write $f \leftrightarrow (v, w)$ (or $(v, w) \leftrightarrow f$) if $(v, w) \in \mathcal{F}_{(\mu,\nu,g;\mathcal{C})}$ and $v|_{\partial\Omega} = f$. Let the convex function U be defined by

$$U = \sup_{(v,w)\in\mathcal{F}_{(\mu,\nu,g;\mathcal{C})}} (v + w) = \sup_{(v,w)\leftrightarrow f\in\mathcal{C}} (v + w). \tag{4}$$

Importantly, throughout this paper, U depends on the data (μ, ν, g) and the boundary value family $\mathcal{C}$.

Let data (μ, ν, g) be given, and let u solve (1). Since the set $\mathcal{C}$ of boundary values will be a focus of our development below, to simplify notation, set $\mathcal{F}_{\mathcal{C}} = \mathcal{F}_{(\mu,\nu,g;\mathcal{C})}$. Topologize the continuous functions $C(\Omega)$ using the topology of uniform convergence on compact subsets of Ω. Let $\bar{\mathcal{F}}_{\mathcal{C}}$ be the closure of $\mathcal{F}_{\mathcal{C}}$ as a subset of $C(\Omega) \times C(\Omega)$ equipped with the product topology. Under the regularity assumption (CBE) on $\mathcal{C}$, as formulated in Section 4, we prove first the following results:

1. For each p, $1 \leq p \leq \infty$, there exists a L^p-feasible pair $(v^*, w^*) \in \mathcal{F}_{\mathcal{C}}$ satisfying
$$\inf_{(v,w)\in\mathcal{F}_{\mathcal{C}}} \|u - (v + w)\|_p = \|u - (v^* + w^*)\|_p; \tag{5}$$
2. There exists an feasible pair $(\bar{v}, \bar{w}) \in \mathcal{F}_{\mathcal{C}}$ such that $U = \bar{v} + \bar{w}$ in $\bar{\Omega}$. In particular,
$$\inf_{(v,w)\in\mathcal{F}_{\mathcal{C}}} \|U - (v + w)\|_p = \|U - (\bar{v} + \bar{w})\|_p = 0, \quad p \geq 1. \tag{6}$$

   Furthermore, in the case that $\mathcal{C} = C(\partial\Omega)$ and certain technical conditions are satisfied (see Section 5), we have:

3. There exists a pair $(\bar{v}, \bar{w}) \in \bar{\mathcal{F}}_{\mathcal{C}}$ such that $U = \bar{v} + \bar{w}$ in Ω and equation (6) holds for all $p \geq 1$.

Note that since $M(v+w) \geq \mu+\nu$ for $(v,w) \in \mathcal{F}_{\mathcal{C}}$, as recalled in Section 2 below, we have only $U \leq u$ above. Moreover, letting $\bar{u} = \bar{v} + \bar{w}$ from (6), $M\bar{u} \geq \mu + \nu$ so that $\bar{u} \leq u$. Thus, even if $\mathcal{C} = C(\partial\Omega)$, neither U nor $\bar{u}$ may solve **MAS**.

The second problem studied here considers convex differences $v - w$. It is well known that $\det D^2 u\, dx$ defines a signed measure Mu in Ω for $u \in W^{2,n}(\Omega)$ via $Mu(E) = \int_E \det D^2\, u(x)\, dx$. Recall that the Sobolev Embedding theorem guarantees that $u \in W^{2,n}(\Omega)$ satisfies $u \in C^{0,\alpha}(\Omega)$, all $0 < \alpha < 1$. Define a modified feasible set $\mathcal{F}^-_{\mathcal{C}}$, with closure $\bar{\mathcal{F}}^-_{\mathcal{C}}$, by

$$\mathcal{F}^-_{\mathcal{C}} = \{(v,w) \mid v \text{ solves (2)},\ f \in \mathcal{C},\ Mw = \nu,\ w|_{\partial\Omega} = f - g\}.$$

For $(v,w) \in \mathcal{F}^-_{\mathcal{C}}$, $v - w = g$ on $\partial\Omega$. The difference splitting problem is defined as follows.

Monge-Ampère difference splitting problem (**MADS**): Given data (μ,ν,g) and class $\mathcal{C}$, $u \in C(\bar{\Omega}) \cap W^{2,n}(\Omega)$, find a 'difference splitting' $u = v - w$ for u, $(v,w) \in \mathcal{F}^-_{\mathcal{C}}$, for which $Mu = M(v-w) = \mu - \nu$.

In **MADS** we replace the sum $\mu + \nu$ in (1) by the difference $\mu - \nu$ and, for given u, seek $u = v - w$ instead of $u = v + w$ (see Section 6). This problem is again characterized by the data (μ,ν,g) and the initial class $\mathcal{C}$. It is easy to see that Examples 2.1–2.3 provide examples of difference splittings of functions u, if sums $v + w$ in the examples are replaced by differences $v - w$. Say that $(v,w) \in \mathcal{F}^-_{\mathcal{C}}$ is *feasible* for **MADS** as determined by the data (μ,ν,g).

In analogy with the definition of U above, set

$$\begin{aligned} U^- &= \sup\nolimits_{(v,w)\in\mathcal{F}^-_{\mathcal{C}}}(v-w) = \sup\nolimits_{(v,w)\leftrightarrow f\in\mathcal{C}}(v-w) \\ U_- &= \inf\nolimits_{(v,w)\in\mathcal{F}^-_{\mathcal{C}}}(v-w) = \inf\nolimits_{(v,w)\leftrightarrow f\in\mathcal{C}}(v-w). \end{aligned} \tag{7}$$

Let data (μ,ν,g) be given, and let $u \in C(\bar{\Omega}) \cap W^{2,n}$, $u = g$ on $\partial\Omega$, with $\mathcal{C}$ satisfying (CBE). In analogy with the **MAS** splitting theorems, we will prove in Section 6 the following.

1′. For each p, $1 \leq p \leq \infty$, there exists an L^p- feasible pair $(v^*, w^*) \in \mathcal{F}^-_{\mathcal{C}}$ satisfying

$$\inf\nolimits_{(v,w)\in\mathcal{F}^-_{\mathcal{C}}} \|u - (v-w)\|_p = \|u - (v^* - w^*)\|_p;$$

2′. If U^- and U_- are bounded, there exist feasible pairs (v^-, w^-) and (v_-, w_-) such that $U^- = v^- - w^-$ and $U_- = v_- - w_-$ in $\bar{\Omega}$. In particular, under appropriate conditions,

$$\inf\nolimits_{(v,w)\in\mathcal{F}^-_{\mathcal{C}}} \|U^- - (v+w)\|_p = \|U^- - (v^- - w^-)\|_p = 0, \quad p \geq 1, \tag{8}$$

and

$$\inf\nolimits_{(v,w)\in\mathcal{F}^-_{\mathcal{C}}} \|U_- - (v+w)\|_p = \|U_- - (v_- - w_-)\|_p = 0, \quad p \geq 1. \tag{9}$$

Furthermore, in the case that $\mathcal{C} = C(\partial\Omega)$ and certain technical conditions are satisfied, we have:

3'. If U^- and U_- are bounded, there exists pairs (v^-, w^-) and (v_-, w_-) in $\bar{\mathcal{F}}_{\mathcal{C}}^-$ such that $U^- = v^- - w^-$ and $U_- = v_- - w_-$ in Ω and equations (8) and (9) above hold for $p \geq 1$.

In order to establish the above results, the remainder of the paper is organized as follows. We first show how to approximate u solving (1) using sums $v + w$ satisfying (2) and (3). We do this by considering the optimization problems $\inf_{(v,w)\in\mathcal{F}_{\mathcal{C}}} \|u - (v + w)\|_p$, $p \geq 1$. We use these results to establish the **MAS** splitting results above. By analogy, we establish the **MADS** splitting results by focusing on the problems $\inf_{(v,w)\in\mathcal{F}_{\mathcal{C}}^-} \|u-(v-w)\|_p$ and $\sup_{(v,w)\in\mathcal{F}_{\mathcal{C}}^-} \|u-(v-w)\|_p$, $p \geq 1$. We close by considering the more general problem of finding, for given u, splittings $u = v + w$, $Mu = Mv + Mw$, with no fixed measures μ and ν specified. We derive topological properties of the collection of all such splittings for a given u.

3. Preliminaries

We begin by recalling basic properties of the measure $M(\cdot)$ as applied to convex functions. Given data (μ, ν, g), let u solve (1). For any convex functions v and w, it is well-known that $M(v+w) \geq Mv+Mw$ since $\det(A+B) \geq \det(A)+\det(B)$ for positive semidefinite matrices A and B, and general convex functions in $C(\Omega)$ may be approximated uniformly on compact subsets by convex functions in $C^2(\Omega)$. Thus, for all feasible (v, w), since $M(v + w) \geq \mu + \nu = Mu$ and $v+w = g$ on $\partial\Omega$, $(v+w)(x) \leq u(x)$ for all $x \in \bar{\Omega}$, so that $U \leq u$. This is a result of the following well-known comparison principle, Theorem 1.4.6, Gutiérrez [10].

Fact 3.1. *Let $u_1, u_2 \in C(\bar{\Omega})$ be convex functions satisfying $Mu_2 \geq Mu_1$. Then*

$$\min_{x\in\bar{\Omega}} (u_1(x) - u_2(x)) = \min_{x\in\partial\Omega} (u_1(x) - u_2(x)). \qquad \square$$

We will need the following results for our analysis. Fact 3.2 is Theorem 3.7, Rausch and Taylor [14] (cf. the slightly weaker Lemma 1.6.1 [10]). Fact 3.3 is Lemma 1.2.3, Gutiérrez [10]. The notation $\rightharpoonup$ denotes weak convergence of measures.

Fact 3.2. *Let Ω' be a bounded, open convex set. Let $u_k, u \in C(\bar{\Omega}')$ be convex functions satisfying:*

(i) $u_k \to u$ *uniformly on* $\partial\Omega'$ *as* $k \to \infty$*;*
(ii) $Mu_k \rightharpoonup Mu$ *as* $k \to \infty$*;*
(iii) $Mu_k(\Omega') \leq A < +\infty$ *for* $k \geq 1$*.*

Then, $u_k \to u$ *in* $C(\bar{\Omega}')$. $\square$

Fact 3.3. *Let $u_k \in C(\Omega')$ be convex functions converging to u uniformly on compact subsets of a domain Ω'. Then $Mu_k \rightharpoonup Mu$.* $\square$

It will be convenient in what follows to recast the **MAS** problem in the following slightly modified equivalent form.

Monge-Ampère splitting problem (**MAS**): Given a function $g \in C(\partial\Omega)$, an initial class $\mathcal{C} \subset C(\partial\Omega)$, and finite Borel measures μ and ν defined in Ω, find feasible $(v, w) \in \mathcal{F}_{\mathcal{C}}$ for which $M(v + w) = \mu + \nu$.

This formulation is equivalent to the earlier one since if v and w solve this problem, then setting $u = v + w$, one has $Mu = \mu + \nu$ and $u = g$ on $\partial\Omega$. Hence $v + w$ is the unique convex function solving (1). Viewed from this new perspective, however, we do not begin with a given function u to be 'split'. Rather, we begin simply with data (μ, ν, g). The 'split' function (if it exists) is automatically $u = v + w$. As indicated above, we also refer to this formulation as problem **MAS**. Given convex u, a feasible pair (v, w) for (μ, ν, g) for which $u = v + w$ and $Mu = M(v + w) = \mu + \nu$ will be called a *Monge-Ampère* **(MA)** *splitting* for u. The splitting problem seeks feasible pairs (v, w) on whose sums $v + w$ the nonlinear map $M(\cdot)$ acts additively.

4. $L^p(\Omega)$ Optimal Splittings; $\mathcal{C}$ Satisfies (CBE); U is Splittable in $\bar{\Omega}$

Our goal in this section is to approximate u solving (1), and U from (4), in $L^p(\Omega)$. For $(v, w) \in \mathcal{F}_{\mathcal{C}}$, $v, w \in C(\bar{\Omega})$, so that $v, w \in L^p(\Omega)$ for $1 \le p \le \infty$, consider the $L^p(\Omega)$ approximation problem:

Let data (μ, ν, g) and initial values $\mathcal{C}$ be given and let u solve (1). Find $f^* \in \mathcal{C}$, $f^* \leftrightarrow (v^*, w^*)$, satisfying

$$\|u - (v^* + w^*)\|_p = \inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|u - (v + w)\|_p. \tag{10}$$

Here of course, for $h : \Omega \to \mathbb{R}$, $\|h\|_p = (\int_\Omega |h(x)|^p \, dx)^{\frac{1}{p}}$ for $1 \le p < \infty$, and for continuous h, $\|h\|_\infty = \sup_{x\in\Omega} |h(x)|$. Note that since $\mathcal{C}$ can be chosen arbitrarily, there is no reason to expect the infimum in (10) will be 0. For each $(v, w) \leftrightarrow f \in \mathcal{C}$ we have only

$$\|u - (v + w)\|_p \ge \|u - \sup_{(v,w)\leftrightarrow f\in\mathcal{C}} (v + w)\|_p = \|u - U\|_p.$$

Moreover, if (10) is zero, it need not follow that (v^*, w^*) solves **MAS** since only $M(v^* + w^*) \ge Mv^* + Mw^* = \mu + \nu$ implying $v^* + w^* \le u$. The infimum in (10) is zero if and only if $U = u = v^* + w^*$.

As will be seen below, the issue of splittings centers on approximating sequences $\{(v_n, w_n)\} \subset \mathcal{F}_{\mathcal{C}}$ with $\{v_n\}$ and $\{w_n\}$ bounded. This requirement is met if $\mathcal{C}$ is a bounded collection since then all v satisfying (2) are uniformly bounded on $\bar{\Omega}$ by Alexandroff's maximum principle (Theorem 1.4.2 [10]). Since w has $w = g - v$ on $\partial\Omega$, it follows that all w solving (3) are also uniformly bounded on $\bar{\Omega}$. In this section we will impose the following condition.

(CBE) $\mathcal{C}$ is a closed and bounded, equicontinuous family.

Given data (μ, ν, g) and class $\mathcal{C}$, if a convex function $u \in C(\bar{\Omega})$, $u = g$ on $\partial\Omega$, may be written as $u = v + w$ in $\bar{\Omega}$, for $(v, w) \in \mathcal{F}_{\mathcal{C}}$, we will say that u is *splittable in* $\bar{\Omega}$. As above, since $Mu \geq Mv + Mw = \mu + \nu$, $u = v + w$ need not solve **MAS**. If in addition, $Mu = M(v + w) = \mu + \nu$, so that u solves (1), $u = v + w$ is a **MA** splitting for u. We observe that if u solving (1) is **MA** splittable, then necessarily $U = u$ is **MA** splittable and hence $MU = Mu$. We explore in detail below criteria guaranteeing that U is splittable. The following proposition encapsulates what appears to be a basic requirement for splittings to exist.

Proposition 4.1. *Let data (μ, ν, g) and $\mathcal{C}$ satisfying (CBE) be given. Let $u \in C(\bar{\Omega})$ be convex and let $\{(v_n, w_n)\} \subset \mathcal{F}_{\mathcal{C}}$ be a feasible sequence with $v_n + w_n$ converging pointwise to u in Ω. Then, u is splittable in $\bar{\Omega}$.*

Proof. Since condition (CBE) implies that the sequence $\{v_n\}$ is bounded, by the Ascoli theorem, we may extract a subsequence $\{v_{n_k}\}$ converging to v uniformly on compact subsets of Ω. For each $k \geq 1$, let w_{n_k} solve (3) with $w_{n_k}|_{\partial\Omega} = g - v_{n_k}|_{\partial\Omega}$. By extracting further subsequences, since $\{w_{n_k}\}$ is bounded, we may assume $w_{n_k} \to w$ uniformly on compact subsets of Ω as well. We conclude that $v_{n_k} + w_{n_k} \to v + w = u$ pointwise in Ω. If v and w extend continuously to $\bar{\Omega}$, then $(v, w) \in \mathcal{F}$ is feasible as $\mathcal{C}$ is closed in $C(\partial\Omega)$. It would then follow that $u = v + w$ in $\bar{\Omega}$. We show that these extensions do exist.

Since $\mathcal{C}$ is a closed, bounded equicontinuous family, again by the Ascoli theorem, the sequence $\{v_{n_k}|_{\partial\Omega}\}$ has a subsequence, also denoted $\{v_{n_k}|_{\partial\Omega}\}$, converging uniformly on $\partial\Omega$ to a limit $f' \in \mathcal{C}$ as $k \to \infty$. Let v' be the unique solution to (2) with $v'|_{\partial\Omega} = f'$. Since all $Mv_{n_k} = \mu$, by Fact 3.2, v_{n_k} converges to v' in $C(\bar{\Omega})$. Since we showed above that $v_{n_k} \to v$ in Ω, it follows that $v' = v$ in Ω. Similarly, let w' solve (3) with $w'|_{\partial\Omega} = g - f'$. Since $w_{n_k}|_{\partial\Omega} = g - v_{n_k}$ converges uniformly on $\partial\Omega$ to $g - f' \in C(\partial\Omega)$, repeating the argument given above shows that w_{n_k} converges to w' in $C(\bar{\Omega})$ with $Mw' = \nu$ and $w' = w$ in Ω. Thus $u = v' + w' \in C(\bar{\Omega})$ extends $v + w \in C(\Omega)$ and the proposition is proved. □

Theorem 4.2. *Let data (μ, ν, g) be given, and let $\mathcal{C}$ satisfy (CBE). Let $F : C(\bar{\Omega}) \to I\!R$ be lower semi-continuous and bounded below. Then, there exists $f^* \in \mathcal{C}$, $f^* \leftrightarrow (v^*, w^*) \in \mathcal{F}_{\mathcal{C}}$, with*

$$\inf_{(v,w)\leftrightarrow f\in\mathcal{C}} F(v + w) = F(v^* + w^*) \tag{11}$$

Proof. Clearly $\inf_{(v,w)\leftrightarrow f\in\mathcal{C}} F(v + w)$ is finite for all collections $\mathcal{C}$. Choose a minimizing sequence $\{v_n + w_n\}$, $F(v_n + w_n) \searrow \inf_{f\in\mathcal{C}} F(v + w)$. Extracting subsequences as in the proof of Proposition 4.1, we may assume, after reindexing, that subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$ converge to convex v^* and w^*, respectively in $C(\bar{\Omega})$, with $(v^*, w^*) \in \mathcal{F}_{\mathcal{C}}$. Thus since $\{v_{n_k} + w_{n_k}\}$ remains a minimizing sequence,

$$F(v^* + w^*) \leq \liminf_{k\to\infty} F(v_{n_k} + w_{n_k}) = \inf_{f\in\mathcal{C}} F(v + w). \qquad \square$$

Corollary 4.3 (Optimal L^p Approximations: $\mathcal{C}$ Satisfies (CBE)). *Let data (μ, ν, g) be given, and let $\mathcal{C}$ satisfy (CBE). Let u solve* (1). *Then, for each $p \geq 1$, including $p = \infty$, there exists $f^* \in \mathcal{C}$, $f^* \leftrightarrow (v^*, w^*) \in \mathcal{F}_{\mathcal{C}}$, with*

$$\|u - (v^* + w^*)\|_p = \min_{(v,w)\leftrightarrow f\in\mathcal{C}} \|u - (v + w)\|_p. \tag{12}$$

Moreover, the theorem remains valid if U replaces u.

Proof. The Corollary follows directly from Theorem 4.2 after setting $F(\cdot) = \|u-(\cdot)\|_p$, and considering the problem $\inf_{(v,w)\leftrightarrow f\in\mathcal{C}} F(v+w) = \inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|u-(v + w)\|_p$. Repeating this argument with $F(\cdot) = \|U - (\cdot)\|_p$ competes the proof. $\square$

The optimal pair (v^*, w^*) guaranteed by Corollary 4.3 appears to depend on p. We show below that when $\mathcal{C}$ satisfies (CBE), a single pair may be chosen that is optimal for all p.

Theorem 4.4 (U is Splittable in $\bar{\Omega}$: $\mathcal{C}$ Satisfies (CBE)). *Let $\mathcal{C}$ satisfy (CBE), and let data (μ, ν, g) be given. Then U is splittable in $\bar{\Omega}$.*

Proof. The proof is a straightforward application of Zorn's Lemma. Construct a partial ordering of the feasible pairs $(v, w) \in \mathcal{F}_{\mathcal{C}}$ by setting $(v, w) \geq (v', w')$ if $M(v + w) \leq M(v' + w')$, and consequently $v + w \geq v' + w'$ in $\bar{\Omega}$. Choose an ascending chain (totally ordered subset) $\mathcal{T} = \{(v_\alpha, w_\alpha)\}_{\alpha\in\mathcal{A}} \subset \mathcal{F}_{\mathcal{C}}$, under the above partial ordering, $\mathcal{A}$ an index set. Clearly each $v_\alpha + w_\alpha \leq U' = \sup_{\alpha\in\mathcal{A}}(v_\alpha + w_\alpha)$. We show that $U' = v' + w'$ for $(v', w') \in \mathcal{F}_{\mathcal{C}}$ and hence that (v', w') serves as an upper bound for $\mathcal{T}$ in $\mathcal{F}_{\mathcal{C}}$.

First choose a countable dense subset $D \subset \Omega$. Enumerate the points in D (redundantly) as $\{x_j\}$, such that each $x \in D$ occurs infinitely often (an infinite number of times) in the enumeration. (For example, if $D = \{y_j\}_{j=1}^{\infty} \subset \mathbb{R}$ initially, set $x_1 = y_1, x_2 = y_2, x_3 = y_1, x_4 = y_2, x_5 = y_3, x_6 = y_1, x_7 = y_2, x_8 = y_3, x_9 = y_4, x_{10} = y_1, \ldots$) Let $\{\varepsilon_n\}$ satisfy $0 < \varepsilon_{n+1} < \varepsilon_n$ and $\varepsilon_n \downarrow 0$. Now choose a subset $\mathcal{T}' = \{v_n+w_n\} \subset \mathcal{T}$ with v_n+w_n increasing to $U' = \sup_{(v_n,w_n)\in\mathcal{T}'} (v_n+w_n)$ as follows. (i) Select (v_1, w_1) such that $U'(x_1) - (v_1 + w_1)(x_1) < \varepsilon_1$. (ii) If $U'(x_2) - (v_1 + w_1)(x_2) < \varepsilon_2$, set $v_2 = v_1$, $w_2 = w_1$. Otherwise select (v_2, w_2) such that $U'(x_2) - (v_2 + w_2)(x_2) < \varepsilon_2 < \varepsilon_1$. Note that $(v_2 + w_2)(x_2) > (v_1 + w_1)(x_2)$ implies that $v_2 + w_2 \geq v_1 + w_1$ throughout $\bar{\Omega}$ as $\mathcal{T}$ is totally ordered by $M(\cdot)$. Continue in like manner for $n = 3, 4, \ldots$. We have $v_n + w_n \uparrow U'$. By Proposition 4.1 there exists $(v', w') \in \mathcal{F}_{\mathcal{C}}$ such that $U' = v' + w'$. Thus, by the way the enumeration of D was chosen, $\mathcal{T}$ has the upper bound (v', w') in $\mathcal{F}_{\mathcal{C}}$. Conclude by Zorn's Lemma that $\mathcal{F}_{\mathcal{C}}$ contains a maximal element $(\bar{v}, \bar{w})$with $\bar{v}+\bar{w} \geq v+w$ for all $(v, w) \in \mathcal{F}_{\mathcal{C}}$. Thus $U = \bar{v} + \bar{w}$ is splittable in $\bar{\Omega}$. $\square$

We may strengthen Corollary 4.3 as follows.

Corollary 4.5. *The optimal pair (v^*, w^*) in Corollary* 4.3 *may chosen independently of $p \geq 1$, with $U = v^* + w^*$.*

Proof. The proof is immediate. By Theorem 4.4, U is splittable in $\bar{\Omega}$. Thus there exists a feasible pair (v^*, w^*) such that $U = v^* + w^*$ and hence

$$\min_{(v,w)\leftrightarrow f\in\mathcal{C}} \|U - (v+w)\|_p = \|U - (v^* + w^*)\|_p = 0$$

for all $p \geq 1$. Furthermore $u - (v+w) \geq u - U = u - (v^* + w^*) \geq 0$ for all $(v, w) \in \mathcal{F}_{\mathcal{C}}$ so that

$$\min_{(v,w)\leftrightarrow f\in\mathcal{C}} \|u - (v+w)\|_p = \|u - (v^* + w^*)\|_p \qquad \square$$

Remark 4.6. Throughout our development, the problem of splitting in $\bar{\Omega}$ reduces to whether or not limiting functions v and w as above can be extended continuously from Ω to $\bar{\Omega}$. The question of continuous extensions is rather a delicate one. In Theorem 10.3 [15], Rockafellar shows that if $\bar{\Omega}$ is a locally simplicial convex set and convex v is bounded on compact subsets of Ω, then it extends uniquely to $\bar{\Omega}$. Unfortunately, $\bar{\Omega}$ here does not satisfy this condition. Thus, we must impose extra conditions on $\mathcal{C}$, or on sequences converging to optimal pairs (v^*, w^*). $\square$

Remark 4.7. The standard existence theory for weak solutions of Monge-Ampère equations (see [10], [14]) relies on 'improving' pairs of convex functions chosen from a feasible class using their maxima, which are again in the feasible class. Such a strategy fails here, thus preventing the comparable construction of the existence of general splittings for u. None the less, approximations by feasible pairs may be made as shown in Corollary 4.3 and the discussion which follows below. $\square$

5. $L^p(\Omega)$ Optimal Splittings; $\mathcal{C} = C(\partial\Omega)$; U is Splittable In Ω.

We now consider the case $\mathcal{C} = C(\partial\Omega)$. Given data (μ, ν, g), if a convex function $u \in C(\Omega)$ may be written as $u = v + w$ in Ω for $(v, w) \in \bar{\mathcal{F}}_{\mathcal{C}}$ (see Section 2.), we will say that u is *splittable in* Ω. Note that $(v, w) \in \bar{\mathcal{F}}_{\mathcal{C}}$ may not satisfy $v+w = g$ on $\partial\Omega$. Indeed, v and w need not be well–defined on $\partial\Omega$. However, for each $(v, w) \in \bar{\mathcal{F}}_{\mathcal{C}}$, there exist sequences from $\mathcal{F}_{\mathcal{C}}$ converging on compact subsets to v and w, respectfully, so that $Mv = \mu$ and $Mw = \nu$ by Fact 3.3.

In this section, with $\mathcal{C} = C(\partial\Omega)$, we show that U is splittable in Ω, but not necessarily in $\bar{\Omega}$. As with Proposition 4.1, we have the following basic convergence result.

Proposition 5.1. *Let data (μ, ν, g) be given, and let $\mathcal{C} = C(\partial\Omega)$. Let $u \in C(\Omega)$ be convex and suppose that $\{(v_n, w_n)\} \subset \mathcal{F}_{\mathcal{C}}$ is a sequence with (i) $v_n + w_n$ converging pointwise to u in Ω with (ii) $\{v_n\}$ bounded on compact subsets of Ω. Then, u is splittable in Ω.*

Proof. Since the sequences $\{v_n\}$ and hence $\{w_n\}$ are bounded on compact subsets, as in the proof of Proposition 4.1, we immediately have subsequences $\{v_{n_k}\}$ and $\{w_{n_k}\}$ converging on compact subsets, hence pointwise, to limit convex functions v and w, respectively, $v, w \in C(\Omega)$ (Theorem 10.7, Rockafellar [15].). Necessarily then $u = v+w$ in Ω with $Mv = \mu$, $Mw = \nu$, so that $(v, w) \in \bar{\mathcal{F}}_{\mathcal{C}}$. $\square$

Let $\mathcal{C} = C(\partial\Omega)$. For positive integer N define $\mathcal{C}_N = \{f \in \mathcal{C} \mid g(\sigma) - N \leq f(\sigma) \leq g(\sigma) + N,\ |f(\sigma_1) - f(\sigma_2)| \leq N|\sigma_1 - \sigma_2|,\ \text{all}\ \ \sigma, \sigma_1, \sigma_2 \in \partial\Omega\}$. Set $U_N = \sup_{(v,w)\leftrightarrow f\in\mathcal{C}_N} (v + w)$. For $1 \leq p \leq \infty$, since $\mathcal{C}_N$ satisfies (CBE), let (v_N, w_N) achieve $\min_{(v,w)\leftrightarrow f\in\mathcal{C}_N} \|U_N - (v + w)\|_p$, as guaranteed by Corollary 4.5, where $U_N = v_N + w_N$. Since $\mathcal{C}_N \subset \mathcal{C}_{N+1}$, $\{v_N + w_N\}$ is an increasing sequence, the closure of $\mathcal{C}^\circ = \cup_{N=1}^\infty \mathcal{C}_N$ in $C(\partial\Omega)$ is $\mathcal{C}$, and we have, in view of Fact 3.2 ($f \in \mathcal{C}$ may be uniformly approximated by functions in $\mathcal{C}^\circ$), for $u \in \mathcal{F}'_{\mathcal{C}}$

$$\inf_{(v,w)\leftrightarrow f\in\mathcal{C}}\|u - (v + w)\|_p = \inf_{(v_N,w_N)\leftrightarrow f_N}\ \|u - (v_N + w_N)\|_p, \quad 1 \leq p \leq \infty.$$

Since $\|u-(v_N+w_N)\|_p$ decreases, v_N+w_N is a minimizing sequence for the first infimum above. The analogous result holds when U replaces u.

Using the pairs (v_N, w_N), we now show that U may be arbitrarily closely approximated in L^p by feasible pairs (v, w).

Theorem 5.2 (Approximation in L^p : $\mathcal{C} = C(\partial\Omega)$). *Let data (μ,ν,g) be given, let u solve* (1)*, and let $\mathcal{C} = C(\partial\Omega)$. Then, for each $p \geq 1$, including $p = \infty$,*

$$\begin{aligned} \inf_{(v,w)\leftrightarrow f\in\mathcal{C}}\ \|U - (v + w)\|_p &= 0 \\ \inf_{(v,w)\leftrightarrow f\in\mathcal{C}}\ \|u - (v + w)\|_p &= \|u - U\|_p. \end{aligned} \tag{13}$$

Proof. As defined above, set $U_N = \sup_{(v,w)\leftrightarrow f\in\mathcal{C}_N} (v + w) = v_N + w_N$. Since $v_N+w_N \leq v_{N+1}+w_{N+1}$, we have $U_N = v_N+w_N \uparrow U$. Suppose $\inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|U - (v + w)\|_p = \inf_{(v_N,w_N)\leftrightarrow f_N\in\mathcal{C}^\circ}\ \|U - (v_N + w_N)\|_p \neq 0$ for some $p \geq 1$. Then

$$\inf_{(v_N,w_N)\leftrightarrow f_N\in\mathcal{C}^\circ}\ \|U - (v_N + w_N)\|_\infty \neq 0. \tag{14}$$

Let open sets $\Omega_n \subset\subset \Omega$ satisfy $\cup_{n=1}^\infty \Omega_n = \Omega$. It suffices to show that for each fixed n, $\lim_{N\to\infty} \sup_{x\in\Omega_n}\ (U(x) - (v_N(x) + w_N(x)) = 0$, contradicting (14). Suppose this does not hold for some n. Then there exists an $\varepsilon > 0$ and a subsequence $\{N_k\}_{K=1}^\infty$ of the natural numbers $N = 1, 2, \ldots$ with corresponding sequence $\{x_{N_k}\}_{k=1}^\infty \subset \Omega_n$ such that

$$U(x_{N_k}) - (v_N(x_{N_k}) + w_N(x_{N_k})) > \varepsilon, \quad k \geq 1. \tag{15}$$

Assume that $x_N \to x \in \bar{\Omega}_n$. Otherwise extract a convergent subsequence and renumber. There exists a closed neighborhood of x contained in Ω on which the convex functions $v_N + w_N$ are uniformly Lipschitz continuous (see Rockafellar [15]). Hence as $k \to \infty$

$$\begin{aligned} &U(x_{N_k}) - (v_{N_k}(x_{N_k}) + w_{N_k}(x_{N_k})) \\ &\leq |U(x_{N_k}) - U(x)| + |U(x) - (v_{N_k}(x) + w_{N_k}(x))| \\ &\qquad + |(v_{N_k}(x) + w_{N_k}(x)) - (v_{N_k}(x_{N_k}) + w_{N_k}(x_{N_k}))| \to 0 \end{aligned}$$

contradicting (15) and hence (14).

To finish the proof, since $0 \leq u - U \leq u - (v + w)$ for all feasible (v, w), it follows that

$$\begin{aligned}\|u - U\|_p &\leq \inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|u - (v + w)\|_p \\ &\leq \|u - U\|_p + \inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|U - (v + w)\|_p = \|u - U\|_p. \qquad \square\end{aligned}$$

We note here that there is no guarantee that U may be split when $\mathcal{C} = C(\partial\Omega)$. The characterization of U may well require an unbounded collection of feasible pairs (v, w). Under the boundedness assumption below, however, U may be split in Ω.

Corollary 5.3 (U is splittable in $\Omega : \mathcal{C} = C(\partial\Omega)$). *Let data (μ, ν, g) be given, and let $\mathcal{C} = C(\partial\Omega)$. Let the sequence $\{(v_n, w_n)\} \subset \mathcal{F}_\mathcal{C}$ be a minimizing sequence for $\inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|U - (v + w)\|_\infty$ with $\{v_n\}$ bounded on compact subsets of Ω. Then U is splittable in Ω.*

Proof. By the first equality in Theorem 5.2, $v_n + w_n \to U$. The corollary follows from Proposition 5.1. $\square$

Note that the sequence $\{(v_N, w_N)\}$ as defined above is a natural candidate for the minimizing sequence in Corollary 5.3.

Importantly, observe that the limiting functions v and w in the proof above, though elements of $C(\Omega)$, may fail to be elements of $C(\bar{\Omega})$. Hence they are not candidates for a **MA** splitting for u. In analogy with Theorem 4.2 we have the following.

Theorem 5.4. *Let data (μ, ν, g) be given, and let $\mathcal{C} = C(\partial\Omega)$.*

(1.) Let $F : C(\Omega) \to I\!R$ be lower semi-continuous and bounded below. Let $\{v_n, w_n\} \subset \mathcal{F}_\mathcal{C}$ be a minimizing sequence for $\inf_{(v,w)\in\mathcal{F}_\mathcal{C}} F(v + w)$ and suppose that $\{v_n\}$ is bounded on compact subsets of Ω. Then, there exists $(v^*, w^*) \in \bar{\mathcal{F}}_\mathcal{C}$, with

$$\inf_{(v,w)\in\mathcal{F}_\mathcal{C}} F(v + w) = F(v^* + w^*) \tag{16}$$

(2.) Let $F : C(\Omega) \to I\!R$ be weakly lower semi-continuous and bounded below, and suppose that for $1 < p \leq \infty$, $\{v_n\}$ and $\{w_n\}$ from (1) are L^p bounded with $v_{n_k} \rightharpoonup v^*$ and $w_{n_k} \rightharpoonup w^*$ by weak compactness, $v, w \in L^p$. Suppose further that $\|v_n\|_p \to \|v^*\|_p$ and $\|w_n\|_p \to \|w^*\|_p$. Then (16) above remains valid.

Proof. Let $\{v_n + w_n\}$ be a minimizing sequence for $\inf_{(v,w)\in\mathcal{F}_\mathcal{C}} F(v + w)$ with $\{v_n\}$ bounded on compact subsets. Then $\{v_n + w_n\}$ is bounded on compact subsets so that as argued above there exists a minimizing subsequence $\{v_{n_k} + w_{n_k}\}$, converging uniformly on compact subsets, hence pointwise, to a limit u^* in Ω.

By Proposition 5.1, we conclude that $u^* = v^* + w^*$ is splittable in Ω with $(v^*, w^*) \in \bar{\mathcal{F}}_{\mathcal{C}}$. By lower semi-continuity then

$$F(v^* + w^*) \leq \liminf_{k\to\infty} F(v_{n_k} + w_{n_k}) = \inf_{(v,w)\in\mathcal{F}_{\mathcal{C}}} F(v+w). \tag{17}$$

and (*1.*) is established.

As for (*2.*), by a theorem of Radon-Riesz we conclude that $v_n \to v^*$ and $w_n \to w^*$ in L^p. By extracting subsequences, we may assume as above that $v_n \to v^*$ and $w_n \to w^*$ pointwise so again $(v^*, w^*) \in \bar{\mathcal{F}}_{\mathcal{C}}$. Inequality (17) then follows by weak lower semi-continuity and (*2.*) is established. □

Theorem 5.5 (Optimal Approximation in L^p: $\mathcal{C} = C(\partial\Omega)$). *Let $\mathcal{C} = C(\partial\Omega)$ and let data (μ, ν, g) be given. Let u solve* (1)*.*

(*1.*) *Suppose that for fixed $p \geq 1$, including $p = \infty$, there exists a minimizing sequence $\{v_n + w_n\}$ for $\inf_{(v,w)\in\mathcal{F}_{\mathcal{C}}} \|u - (v+w)\|_p$ with $\{v_n\}$ bounded on compact subsets of Ω. Then there exists $(v^*, w^*) \in \bar{\mathcal{F}}_{\mathcal{C}}$ with*

$$\|u - (v^* + w^*)\|_p = \min_{(v,w)\in\mathcal{F}_{\mathcal{C}}} \|u - (v+w)\|_p.$$

(*2.*) *If $\{v_n\}$ and $\{w_n\}$ from (1.) are bounded in L^p, $p > 1$, with $v_{n_k} \rightharpoonup v^*$ and $w_{n_k} \rightharpoonup w^*$, $v, w \in L^p$, and with $\|v_n\|_p \to \|v^*\|_p$ and $\|w_n\|_p \to \|w^*\|_p$, then result (1.) holds.*

Moreover, in either case, the theorem remains valid if U replaces u.

Proof. For fixed p, $1 \leq p \leq \infty$, let $\{v_n + w_n\}$ be a minimizing sequence for $\inf_{(v,w)\in\bar{\mathcal{F}}_{\mathcal{C}}} \|u-(v+w)\|_p$ with $\{v_n\}$ bounded on compact subsets of Ω. Then the result (*1.*) follows from Theorem 5.2(*1.*) with $F(\cdot) = \|u - \cdot\|$ and $F(\cdot) = \|U - \cdot\|$. Result (*2.*) follows from Theorem 5.2(*2.*). □

Remark 5.6. For analysis purposes, in Section 4, we focused our arguments on the class $\mathcal{C}$ of initial conditions f for functions v. Proposition 4.1 relies upon extensions to $\bar{\Omega}$ of appropriate limit functions v and w. Condition (CBE) appears to be most germaine to our development. It guarantees boundedness, hence local Lipschitz continuity and extendability of appropriate limit functions through Fact 3.2. Alternatively, instead of requiring (CBE), we could have focused on functions v which are Lipschitz in Ω. These functions automatically extend continuously to $\bar{\Omega}$ and hence satisfy (2). The corresponding functions w satisfying (3) are automatically determined. The results of Section 4 then follow provided $\mathcal{C}$ is a bounded collection since then v and w are bounded. The results of Section 5 also remain valid if we replace the initial condition classes $\mathcal{C}_N$ with the convex function classes $\mathcal{E}_N = \{(v,w) \ : \ v, w \in C(\bar{\Omega}),\ g(\sigma) - N \leq v|_{\partial\Omega} \leq g(\sigma) + N,\ |v(x) - v(y)| \leq N,$ all $\sigma \in \partial\Omega$ and $x, y \in \Omega\}$, rather than $\mathcal{C}_N$. Minimizers of $F(\cdot)$ over these classes, as in Theorem 4.2 above, automatically extend to $\bar{\Omega}$. One shows that $\sup\{(v+w) \mid (v,w) \in \cup\mathcal{E}_N\} = \sup\{(v+w) \mid (v,w) \in \cup\mathcal{C}_N\}$ and the proof of Theorem 5.2 follows. Note that limits of sequences $\{v_n\}$ and

$\{w_n\}$, $\{(v_n, w_n)\} \subset \cup_{N=1}^{\infty} \mathcal{E}_N$ are not guaranteed to be Lipschitz in Ω and hence may not extend continuously to $\bar{\Omega}$. Details are omitted here. $\square$

Remark 5.7. We make the following additional observations concerning **MAS**.

(1.) Let u_{aff} be an affine function. Then (v, w) is feasible for (μ, ν, g) if and only if $(v + u_{aff}, w)$ is feasible for $(\mu, \nu, g + u_{aff})$. In particular, the **MAS** splitting problems (μ, ν, g) and $(\mu, \nu, g + u_{aff})$ are equivalent.

(2.) The **MAS** problems (μ, ν, g) and $(\mu, \nu, g + C)$ are equivalent for any constant C.

(3.) In the **MAS** problem, we may assume that $v|_{\partial\Omega} = f \geq 0$ (or $f \leq 0$) for all $f \in \mathcal{C}$.

(4.) Instead of (3.), in the **MAS** problem we may assume that $v|_{\partial\Omega} = f \leq g$ (or $f \geq g$) for all $f \in \mathcal{C}$.

(5.) **MA** splittings are not unique.

Note that by (4.) above, we could have restricted our attention in our analysis above to initial conditions $f \leq g$. Accordingly, our boundedness assumptions can be weakened to bounded below assumptions only.

To understand observations (1.)–(5.), let (v, w) be feasible for (μ, ν, g). Then for affine function u_{aff}, $M(v + u_{aff}) = \mu$ and $(v + u_{aff}) + w = g + u_{aff}$ on $\partial\Omega$. For the problem $(\mu, \nu, g + u_{aff})$, the functions f, g, U in the problem (μ, ν, g) are replaced, respectively, by $f + u_{aff}, g + u_{aff}$, and $U + u_{aff}$. The pairing $(v, w) \leftrightarrow (v + u_{aff}, w)$ is a 1–1 identification of feasible pairs whose sums differ by u_{aff}. Thus (1.) follows. Choosing $u_{aff} = C$ in (1.), (2.) follows. As for (3.), if $v|_{\partial\Omega} < 0$ for some $\sigma \in \partial\Omega$, choose constant C so that $v|_{\partial\Omega} + C \geq 0$ (or ≤ 0) on $\partial\Omega$. Set $v' = v + C$ and $w' = w - C$. Then $(v', w') = (v + C, w - C)$ is feasible for (μ, ν, g), and $v' + w' = (v + C) + (w - C) = v + w$. Since we are interested only in values of sums $v + w$, we may assume $f \geq 0$ and (3.) holds. As for (4.), choose C with $v|_{\partial\Omega} + C \leq g$ (or $\geq g$). Then again for $v' = v + C$ and $w' = w - C$, $v' + w' = v + w$ and (4.) follows. Finally, since $(v + u_{aff}) + (w - u_{aff}) = v + w$, with $M(v + u_{aff}) = Mv = \mu$ and $M(w + u_{aff}) = Mw = \nu$, **MA** splittings are not unique. $\square$

6. The Difference Splitting Problem MADS

A natural extension of the **MAS** splitting problem poses the question of solving a Monge-Ampère type equation

$$\begin{aligned} Mu(x) &= \mu - \nu \quad \text{in } \Omega \\ u(x) &= g(x) \quad \text{on } \partial\Omega, \end{aligned} \tag{18}$$

for given data (μ, ν, g) and signed measure $\mu - \nu$. As observed above, $dMu = \det D^2 u \, dx$ defines a signed measure Mu for $u \in W^{2,n}(\Omega)$. Let us say that $u \in C(\bar{\Omega}) \cap W^{2,n}(\Omega)$ *solves the* **MA** *signed measure equation* if it satisfies (18) with $dMu = \det D^2 u \, dx$. One might further explore the possibility that a convex difference $v - w$ might satisfy (18) with $M(v-w) = Mv - Mw$, where $Mv = \mu$, $Mw = \nu$ and $v - w = g$ on $\partial\Omega$. There is a sizable literature on *difference convex functions* $v - w$ dating back at least to Alexandrov [1] in 1950 (see [11], [12], [16] also). It is straightforward to see that each $u \in C^2(\bar{\Omega})$ may be written in this form, since $h(x) = u(x) + \frac{1}{2}\alpha|x|^2$ is convex for sufficiently large $\alpha > 0$. Rausch and Taylor [14] showed how to compare $W^{2,n}$ functions u to convex functions w when Mw is appropriately comparable to Mu (see Fact 6.2 below). The extended map $M(\cdot)$ is no longer elliptic throughout Ω, so it is unlikely that a useful maximum principle is provable for the comparison of general functions in $W^{2,n}(\Omega)$.

In the classical setting, to solve **MADS** as defined in Section 2, one could seek $v, w \in C^2(\bar{\Omega})$ for which $\det D^2(v-w) = \det D^2 v - \det D^2 w$. This clearly cannot be satisfied in general since generically $\det(A - B) \neq \det A - \det B$. (See the Appendix for conditions under which $\det(A-B) = \det A - \det B$ in $\mathbb{R}^{2\times 2}$.) It does however define a difference splitting problem for $u \in C^2(\bar{\Omega})$ whenever μ and ν guarantee solutions $v, w \in C(\bar{\Omega})$. One then attempts to generalize this to the problem of characterizing those functions $u \in W^{2,n}(\Omega)$ which may be expressed as a difference $u = v - w$ of convex functions.

As mentioned earlier, it is easily seen that the differences $v - w$ constructed from the v and w in Examples 2.1–2.3, Section 1, provide examples of $C^2(\bar{\Omega})$ functions $u = v - w$ which can be split as convex differences. In Example 2.4, let $D = \{(0,0), (2,0)\}$, a two point set. Given a Borel set $F \subset \mathbb{R}^2$, consider defining a signed measure $M(v-w)$ by setting $M(v-w)(F) = \int_{F\setminus D} \det D^2(v-w)\,dx + \delta_{(0,0)}(F) - \delta_{(2,0)}(F)$. This extended definition seems necessary here to ensure that $M(v-w)$ is a signed measure. (Note that v and w as defined in Example 2.4 fail to be in $W^{2,n}(\Omega)$.) For the set E given in Example 2.4, $M(v-w)(E) = \int_E \det D^2(v-w)(x)\,dx < 0$, as it is easily seen that $D^2(v-w)(x) < 0$ for $x \in E$. But $\mu(E) = \delta_{(0,0)}(E) = 0$ and $\nu(E) = \delta_{(2,0)}(E) = 0$, so that $M(v-w)(E) \neq \mu(E) - \nu(E) = 0$. This shows that the 'natural' difference splitting of the function $u = v - w \in C(\Omega)$ is actually not a difference splitting.

Before proceeding to a discussion of **MADS**, let us briefly consider the comparison of differences of convex functions. The comparison $v_1 - w_1 \leq v_2 - w_2$ is equivalent to the comparison $v_1 + w_2 \leq v_2 + w_1$. If $v_1 - w_1 = v_2 - w_2 = g$ on $\partial\Omega$, then $v_1 + w_2 = v_2 + w_1 = v_1 + v_2 - g$ on $\partial\Omega$. Thus by Fact 3.1, if $M(v_1 + w_2) \geq M(v_2 + w_1)$, this implies that $v_1 + w_2 \leq v_2 + w_1$ so that $v_1 - w_1 \leq v_2 - w_2$. This relates the difference convex splitting problem **MADS** to the convex sum problem **MAS**. This relationship served as the original motivation for the Monge-Ampère type splittings discussed here. We have shown the following.

Fact 6.1 (Difference Comparison Principle). *Given data (μ, ν, g), initial class $\mathcal{C}$, and feasible pairs (v_1, w_1) and (v_2, w_2) in $\mathcal{F}_{\mathcal{C}}^-$, if $M(v_1 + w_2) \geq M(v_2 + w_1)$, then $v_1 - w_1 \leq v_2 - w_2$ in $\bar{\Omega}$.* □

The following useful comparison principle providing a lower bound for solutions of (18) was proved by Rausch and Taylor ([14], Theorem 5.1).

Fact 6.2. *Let Ω be strictly convex and bounded and let $u \in W^{2,n}(\Omega)$. If $v \in C(\bar{\Omega})$ is convex in Ω and*

$$Mv \geq [Mu]^+ \equiv \max\,(Mu, 0)$$

then

$$\min\,\{u(x) - v(x) \mid x \in \Omega\} = \min\,\{u(x) - v(x) \mid x \in \partial\Omega\}.$$

In particular, if $v \leq u$ on $\partial\Omega$ then $v \leq u$ in Ω. □

A consequence of this is the following interesting result relating differences $v-w$, $(v,w) \in \mathcal{F}_{\mathcal{C}}^-$, the feasible class defined in Section 1, to corresponding sums $v+w$.

Theorem 6.3. *Given initial class $\mathcal{C}$, data (μ, ν, g) and u solving* (18)*, we have*

$$u \geq \sup_{(v,w)\in\mathcal{F}_{\mathcal{C}}^-}(v+w).$$

Proof. By Remark 6.7(4.), given $f \leftrightarrow (v,w) \in \mathcal{F}_{\mathcal{C}}^-$, we may assume that $f \leq g$. By Fact 6.2 then $v \leq u$. Since $w|_{\partial\Omega} = f - g \leq 0$, $w \leq 0$ throughout $\bar{\Omega}$. Thus

$$u - w \geq v - w \geq v$$

from which follows

$$u \geq v + w \quad \text{hence } u \geq \sup_{(v,w)\in\mathcal{F}_{\mathcal{C}}^-}(v+w).$$ □

We must be careful here not to confuse $\mathcal{F}_{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{C}}^-$. For $(v,w) \in \mathcal{F}_{\mathcal{C}} \cap \mathcal{F}_{\mathcal{C}}^-$, we must have $v + w = v - w = g$ on $\partial\Omega$, so that $w|_{\partial\Omega} = 0$ and hence $v|_{\partial\Omega} = g$. This suggests that the pair (v,w) with $v = g$ may play an important role in the problems considered here. It would be very interesting to show that <u>some</u> feasible pair $(v,w) \in \mathcal{F}_{\mathcal{C}}^-$ had $v-w \leq u$ or $u \leq v-w$. Then, we could consider the 'lower solution' $u_- = \sup_{v-w\leq u}\, v - w$, and the 'upper solution' $u^- = \inf_{v-w\geq u}\, v - w$, and establish results analogous to those in Sections 4 and 5 with u_- and u^- replacing U. Unfortunately, we are unable to do this, except in the special case $u = v - w$. Thus, we focus on U^- and U_- below.

We now present results for **MADS** analogous to those derived above for **MAS**. First recast **MADS** in the following equivalent form.

Monge-Ampère difference splitting problem (**MADS**): Given data (μ, ν, g) and initial class $\mathcal{C}$, find convex v and w satisfying $Mv = \mu$ and $Mw = \nu$, for which $u = v - w \in W^{2,n} \cap C(\bar{\Omega})$, $(v-w)|_{\partial\Omega} = g$, and $M(v-w) = \mu - \nu$ in Ω.

Recall the definitions (7) $U^- = \sup_{(v,w)\leftrightarrow f\in\mathcal{C}}\,(v-w)$ and $U_- = \inf_{(v,w)\leftrightarrow f\in\mathcal{C}}\,(v-w)$, with now $f \leftrightarrow (v,w) \in \mathcal{F}_{\mathcal{C}}^-$. In order to show that these extremal functions are splittable, we will need to know that they are bounded above and below, respectively. We are unable to prove boundedness in general, even though it is suspected. We can verify it in special cases only. As an example, consider the following.

Fact 6.4. *If $\mu \geq \nu$, then $v - w \leq 0$ for all $(v, w) \in \mathcal{F}_{\mathcal{C}}^-$. Hence, U^- is bounded. If $\nu \geq \mu$, then all $v - w \geq 0$. Hence, U_- is bounded.*

Proof. By Remark 6.7(2.) below we may assume that $g \leq 0$. Then $f \leq f - g$ so that $v \leq w$ on $\partial\Omega$ and hence, since $\mu \geq \nu$, $v \leq w$ on $\bar{\Omega}$. But this says $v - w \leq 0$ so that $U^- \leq 0$. An analogous argument with $\nu \geq \mu$ and $g \geq 0$ completes the proof. $\square$

If a general function $u \in C(\bar{\Omega}) \cap W^{2,n}$ may be written as $u = v - w$ in $\bar{\Omega}$ for $(v, w) \in \mathcal{F}_{\mathcal{C}}^-$, say that u is *difference splittable in* $\bar{\Omega}$. If $u = v - w$ in Ω for convex v and w satisfying only $Mv = \mu$ and $Mw = \nu$, but $v - w \neq g$ on $\partial\Omega$, say that $u = v - w$ is *difference splittable in* Ω. Clearly, such functions u need not solve **MADS**. If u is *difference splittable in* $\bar{\Omega}$ and in addition, $Mu = M(v-w) = \mu-\nu$, we will say u is *Monge-Ampère difference (***MAD***) splittable* (u solves **MADS**).

Most of the results from Sections 4 and 5 remain valid for **MADS**, as the interested reader may verify. The proofs of these results essentially follow directly from earlier proofs after simply replacing sums $v + w$ by differences $v - w$, replacing u solving (1) by u solving (18), replacing U by U^- or U_-, and then verifying that the logic remains intact. Additionally, in Section 4, the following modifications are needed: (i) in Proposition 4.1, remove 'u is convex' as an assumption; (ii) the partial ordering in Theorem 4.4 should be defined using Fact 6.1, namely, for feasible pairs $(v, w), (v', w') \in \mathcal{F}_{\mathcal{C}}^-$, set $(v, w) \geq (v', w')$ if $M(v + w') \leq M(v' + w)$, so that $v - w \geq v' - w'$. Also, when considering U_-, observe that

$$-U_- = -\inf_{(v,w)\leftrightarrow f\in\mathcal{C}}(v - w) = \sup_{(v,w)\leftrightarrow f\in\mathcal{C}}(w - v)$$

and apply the result for U^-; (iii) in Corollary 4.5 the second statement in the proof remains true for U^- (or U_-) only if we assume $u \geq U^-$ (or $u \leq U_-$). In Section 5, in the proof of Theorem 5.2 observe that differences $v_N - w_N$ are uniformly Lipschitz as required since v and w are separately, and again the second statement remains true only if we assume $u \geq U^-$ (or $u \leq U_-$). We summarize these results in the following theorems.

First we consider the case $\mathcal{C}$ satisfies (CBE).

Theorem 6.5. *Let $\mathcal{C} \subset C(\Omega)$ satisfy (CBE), let data (μ, ν, g) be given, and let u solve* (18).

(*1.*) **[Optimal L^p Approximations: $\mathcal{C}$ Satisfies (CBE).]** *(cf. Corollary* 4.3*)* *For each $p \geq 1$, including $p = \infty$, there exists $f^* \in \mathcal{C}$, $f^* \leftrightarrow (v^*, w^*) \in \mathcal{F}_{\mathcal{C}}^-$, with*

$$\|u - (v^* - w^*)\|_p = \min_{(v,w)\in\mathcal{F}_{\mathcal{C}}^-} \|u - (v - w)\|_p. \tag{19}$$

Moreover, if either U^- is bounded above, or U_- is bounded below, then the result (19) *holds with either U^- or U_-, respectively, replacing u.*

(*2.*) [U^- **and** U_- **Difference Splittable in** $\bar{\Omega}$**:** $\mathcal{C}$ **Satisfies (CBE).**] *(cf. Theorem* 4.4*) If U^- is bounded above, it is difference splittable in $\bar{\Omega}$. If U_- is bounded below, it is difference splittable in $\bar{\Omega}$.* □

Now we consider $\mathcal{C} = C(\partial\Omega)$.

Theorem 6.6. *Let data (μ, ν, g) be given, let $\mathcal{C} = C(\partial\Omega)$ and let u solve* (18).

(*1.*) [**Approximation in** L^p **:** $\mathcal{C} = C(\partial\Omega)$**.**] *(cf. Theorem* 5.2*) For each $p \geq 1$, including $p = \infty$, if U^- is bounded above, then*

$$\inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|U^- - (v - w)\|_p = 0.$$

If $u \geq U^-$ then

$$\inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|u - (v - w)\|_p = \|u - U^-\|_p. \tag{20}$$

This also holds when U_- bounded below replaces U^-, the second equality requiring $u \leq U_-$.

(*2.*) (*i*) [**Optimal Approximation in** L^p **:** $\mathcal{C} = C(\partial\Omega)$**.**] *(cf. Theorem* 5.5*) Suppose that for fixed $p \geq 1$, including $p = \infty$, there exists a minimizing sequence for $\inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|u - (v - w)\|_p$ with $\{v_n\}$ bounded on compact subsets of Ω. Then there exists $(v^*, w^*) \in \bar{\mathcal{F}}_{\mathcal{C}}^-$ with*

$$\|u - (v^* + w^*)\|_p = \min_{(v,w)\leftrightarrow f\in\mathcal{C}} \|u - (v + w)\|_p.$$

If U^- and U_- are bounded, then the result holds with these functions replacing u.

(*ii*) *If $\{v_n\}$ and $\{w_n\}$ from* (*1.*) *are bounded in L^p, $p > 1$, with $v_{n_k} \rightharpoonup v^*$ and $w_{n_k} \rightharpoonup w^*$, $v, w \in L^p$, and with $\|v_n\|_p \to \|v^*\|_p$ and $\|w_n\|_p \to \|w^*\|_p$, then result* (*i*) *holds.*

(*3.*) [U^- **and** U_- **Difference Splittable in** Ω **:** $\mathcal{C} = C(\partial\Omega)$**.**] *(cf. Corollary* 5.3*) Let U^- be bounded, and let the sequence $\{v_n - w_n\}$, $(v_n, w_n) \in \mathcal{F}_{\mathcal{C}}^-$ be a minimizing sequence for $\inf_{(v,w)\leftrightarrow f\in\mathcal{C}} \|U^- - (v - w)\|_\infty$ with $\{v_n\}$ bounded on compact subsets of Ω. Then U^- is difference splittable in Ω. If U_- is bounded, this result holds with U_- replacing U^-.* □

Remark 6.7. The observations (1.)–(5.) for **MAS** in Remark 5.7 remain valid for **MADS**. To see this, first observe that for feasible pair (v, w) for (μ, ν, g), $M(v + u_{aff}) = \mu$, and $(v + u_{aff}) - w = g + u_{aff}$ on $\partial\Omega$. Thus $(v + u_{aff}, w)$ is feasible for $(\mu, \nu, g + u_{aff})$ and the problems (μ, ν, g) and $(\mu, \nu, g + u_{aff})$ are equivalent. Since we are only interested in differences $v - w$ in **MADS**, and $(v + C) - (w + C) = v + w$ for constant C, $(v + C, w + C)$ is feasible for (μ, ν, g) and we may assume $v|_{\partial\Omega} = f \geq 0$ or $v|_{\partial\Omega} = f \leq g$. □

7. General Discussion and Splitting Families

A more general splitting problem seeks, for given fixed convex $u \in C(\bar{\Omega})$, *any* sum $v + w = u$ with v and w satisfying $Mv + Mw = Mu$ (no fixed μ and ν). Thus effectively, out of all possible splittings for the measure Mu, (at least) one particular choice $Mu = \mu' + \nu'$ is sought with corresponding convex v and w satisfying $Mv = \mu'$, $Mw = \nu'$, $v + w = u$ in $\bar{\Omega}$ and $M(v + w) = \mu' + \nu' = Mu$. The pair (v, w) is a **MA** splitting for u given the data $(\mu', \nu', u|_{\partial\Omega})$.

Let $\mathbf{C}(\bar{\Omega})$ denote the linear space $C(\bar{\Omega}) \times C(\bar{\Omega})$ equipped with the product metric topology derived from the supremum norm for $C(\bar{\Omega})$. Given convex $u \in C(\Omega)$, $u = g$ on $\partial\Omega$, and initial class $\mathcal{C}$, enlarge the feasible class to

$$\begin{aligned}\mathcal{F}'_{\mathcal{C}} = \{(v, w) \in \mathbf{C}(\bar{\Omega}) \mid\ & v \text{ and } w \text{ convex},\\ & v|_{\partial\Omega} \in \mathcal{C},\ (v + w)|_{\partial\Omega} = g,\ Mv + Mw \geq Mu\}.\end{aligned}$$

Note that since $M(v + w) \geq Mv + Mw \geq Mu$, we are assured that $v + w \leq u$. We formulate the *generalized* splitting problem.

Generalized Monge-Ampère splitting problem for convex functions (**GMAS**): Given convex $u \in C(\bar{\Omega})$, find $(v, w) \in \mathcal{F}'_{\mathcal{C}}$ such that $u = v + w \in \bar{\Omega}$ and $Mu = Mv + Mw$.

Note that **GMAS** is uniquely determined by the class $\mathcal{C}$ and the data (Mu, g), where $u = g$ on $\partial\Omega$. In this more general setting let

$$\mathcal{F}'_{u,\mathcal{C},N} = \{(v, w) \in \mathcal{F}'_{\mathcal{C}} \mid Mv(\Omega) \leq N, Mw(\Omega) \leq N\}$$

for chosen integer $N > 0$. Set $U_N = \sup_{(v,w)\in\mathcal{F}'_{u,\mathcal{C},N}}(v+w)$, and replace $\inf_{(v,w)\leftrightarrow f\in\mathcal{C}}$ with $\inf_{(v,w)\in\mathcal{F}'_{u,\mathcal{C},N}}$ in the results of Sections 4 and 5. Then, the results of those sections are valid with U_N replacing U since Fact 3.2 guarantees that $\mathcal{F}'_{u,\mathcal{C},N}$ is closed under taking limits.

We present the following results which are valid for either the problem of fixed data (μ, ν, g), or for the more general problem **GMAS** described above.

Theorem 7.1. *Let data (μ, ν, g) and initial class $\mathcal{C}$ be given. For fixed $(v, w) \in \mathcal{F}_{\mathcal{C}}$, set $u = v + w$. The following statements are equivalent for* **MAS**.

(1.) (v, w) *is a* **MA** *splitting for u.*
(2.) $M(v + w) = Mv + Mw$.
(3.) $\int_{\partial(v+w)(E)} dp = \int_{\partial v(E)} dp + \int_{\partial w(E)} dp$ *for all Borel sets $E \subset \Omega$.*
(4.) $M(v + w)(\Omega) = Mv(\Omega) + Mw(\Omega)$.

If $v, w \in C^2(\Omega)$, then (1.)–(4.) are equivalent to

(5.) $\det\, D^2(v + w)(x) = \det\, D^2(v)(x) + \det\, D^2(w)(x)$ *for $x \in \Omega$.*

Proof. *(1.)*–*(3.)* are equivalent since *(2.)* is the requirement for $v + w$ to be a **MA** splitting for u and *(3.)* simply rewrites *(1.)* in terms of the definitions of

Monge-Ampère measures. Consider (*4.*). (*2.*) clearly implies (*4.*). Assume (*4.*). For Borel set $E \subset \Omega$, $M(v+w)(E) \geq Mv(E) + Mw(E)$. Suppose there exists F such that $M(v+w)(F) > Mv(F) + Mw(F)$. Then, since $M(v+w)(F^c) \geq Mv(F^c) + Mw(F^c)$, one has $M(v+w)(\Omega) = M(v+w)(F) + M(v+w)(F^c) > Mv(\Omega) + Mw(\Omega)$, a contradiction. Thus $M(v+w)(F) = M(v)(F) + M(w)(F)$ for all Borel sets F and (*2.*) and (*4.*) are equivalent.

(*1.*) and (*5.*) are equivalent by continuity since convex $z \in C^2(\Omega)$ has $dMz(x) = \det D^2 z(x)\, dx$. □

Corollary 7.2. *Let convex* $u \in C(\bar{\Omega})$ *and* $\mathcal{C}$ *be given. Then, the results of* Theorem 7.1 *remain valid if* $\mathcal{F}_{\mathcal{C}}$ *is replaced by* $\mathcal{F}'_{\mathcal{C}}$. *That is,* Theorem 7.1 *is valid for* **GMAS**.

Proof. Suppose $(v,w) \in \mathcal{F}'_{\mathcal{C}}$ with $u = v + w$ in $\bar{\Omega}$ and $M(v+w) = Mv + Mw = Mu$. Set $\mu = Mv$ and $\nu = Mw$. Then Theorem 7.1 holds since its proof is valid for the specific measures μ and ν. □

Given convex $u \in C(\bar{\Omega})$, let

$$\mathcal{F}^*_{u,\mathcal{C}} = \{(v,w) \in \mathcal{F}'_{\mathcal{C}} \mid M(v+w) = Mv + Mw = Mu\}$$

the collection of all *generalized* splittings for u. As we have observed above, the class $\mathcal{F}^*_{u,\mathcal{C}}$ may be empty. If not, note that for $(v,w) \in \mathcal{F}^*_{u,\mathcal{C}}$, $u = v + w$ in $\bar{\Omega}$ and $Mu = Mv + Mw$. Given $g \in C(\partial\Omega)$ and class $\mathcal{C}$, let

$$\begin{aligned} \mathcal{F}^*_{\mathcal{C}} = \{(v,w) \in \mathbf{C}(\bar{\Omega}) \mid\ & v \text{ and } w \text{ convex}, \\ & v|_{\partial\Omega} \in \mathcal{C},\ (v+w)|_{\partial\Omega} = g,\ M(v+w) = Mv + Mw\} \end{aligned}$$

the collection of all splittings for the data g and $\mathcal{C}$ (independent of u). The classes defined above and **GMAS** may be further studied elsewhere. We establish only the following topological facts regarding them here.

Fact 7.3. *Let* $\mathcal{C}$ *be a closed subset of* $C(\partial\Omega)$. *Then the sets* $\mathcal{F}^*_{u,\mathcal{C}}, \mathcal{F}^*_{\mathcal{C}}$ *and* $\mathcal{F}'_{\mathcal{C}}$ *are closed subsets of* $\mathbf{C}(\bar{\Omega})$.

Proof. If $\{(v_n, w_n)\} \subset \mathcal{F}^*_{\mathcal{C}}$ with $(v_n, w_n) \to (v,w) \in \mathbf{C}(\bar{\Omega})$, then trivially $v_n \to v$ and $w_n \to w$ uniformly in Ω. Since $\mathcal{C}$ is closed, $v|_{\partial\Omega} \in \mathcal{C}$. Since $M(v_n + w_n) = Mv_n + Mw_n$, passing to weak limits and using Fact 3.3, we have $M(v+w) = \lim_{n\to\infty} M(v_n + w_n) = \lim_{n\to\infty}(M(v_n) + M(w_n)) = Mv + Mw$. Thus $(v,w) \in \mathcal{F}^*_{\mathcal{C}}$ and $\mathcal{F}^*_{\mathcal{C}}$ is closed. The same argument shows $\mathcal{F}^*_{u,\mathcal{C}}$ is closed.

If $\{(v_n, w_n)\} \subset \mathcal{F}'_{\mathcal{C}}$, $M(v+w) = \lim_{n\to\infty} M(v_n + w_n) \geq \lim_{n\to\infty}(M(v_n) + M(w_n)) \geq Mu$ and $\mathcal{F}'_{\mathcal{C}}$ is closed. □

An additional interesting observation is the following. Given data (μ, ν, g), initial class $\mathcal{C}$, and u solving (1), let

$$\mathcal{F}_u = \{(v,w) \in \mathcal{F}^*_{u,\mathcal{C}} \mid Mv = \mu, Mw = \nu\}$$

the collection of all **MA** splittings for u with respect to the given data and $\mathcal{C}$. Note that $\mathcal{F}_u \subset \mathcal{F}^*_{u,\mathcal{C}} \subset \mathcal{F}'_{\mathcal{C}}$.

Theorem 7.4. *Let data (μ, ν, g) be given, let u solve (1), and let $\mathcal{C}$ be closed and convex as a subset of $C(\partial\Omega)$. Then $\mathcal{F}_u$ is a closed and convex subset of $\mathbf{C}(\bar{\Omega})$.*

Proof of Theorem 7.4. Arguing as in the proof of Fact 7.3 shows that $\mathcal{F}_u$ is closed. To establish convexity, choose feasible $(v_1, w_1), (v_2, w_2) \in \mathcal{F}_u$. For $0 \leq \lambda \leq 1$ set $v^\lambda = (1-\lambda)v_1 + \lambda v_2$, and let $w^\lambda = (1-\lambda)w_1 + \lambda w_2$. The pair (v^λ, v^λ) is a well-defined element of $\mathbf{C}(\bar{\Omega})$. Since $v_1 + w_1 = v_2 + w_2 = u$, we have $u = v^\lambda + w^\lambda$. Moreover, since $\mathcal{C}$ is convex, $v^\lambda|_{\partial\Omega} \in \mathcal{C}$, and $w^\lambda|_{\partial\Omega} = g - v^\lambda|_{\partial\Omega}$.

Claim. $M(v^\lambda) = \mu$.

Proof of Claim. Suppose the claim is false. Let $v' \in C(\bar{\Omega})$ solve $Mv' = \mu$, $v'|_{\partial\Omega} = v^\lambda$. Since $Mv^\lambda \geq \mu$ by Lemma A.3, automatically $v^\lambda \leq v'$ in Ω by Fact 3.1. There must then exist $x_0 \in \Omega$ for which $v^\lambda(x_0) < v'(x_0)$. Otherwise, $v^\lambda = v'$ in Ω and $Mv^\lambda = \mu$, contrary to assumption. Let $w' \in C(\bar{\Omega})$ solve $Mw' = \nu$, $w'|_{\partial\Omega} = w^\lambda$. Since $Mw^\lambda \geq \nu$, $w^\lambda(x) \leq w'(x)$ throughout Ω. Putting all of this together gives

$$\begin{aligned} u(x_0) &= (1-\lambda)(v_1 + w_1)(x_0) + \lambda(v_2 + w_2)(x_0) \\ &= (v^\lambda + w^\lambda)(x_0) < (v' + w')(x_0). \end{aligned} \tag{21}$$

But, since $(v', w') \in \mathcal{F}'_{\mathcal{C}}$, we must have $v' + w' \leq u$ in Ω, contradicting (21). This contradiction establishes the claim. □

A symmetric argument shows that $Mw^\lambda = \nu$. Since $v_1 + w_1 = v_2 + w_2 = u$ on $\bar{\Omega}$, $v^\lambda + w^\lambda = (1-\lambda)(v_1 + w_1) + \lambda(v_2 + w_2) = u$ on $\bar{\Omega}$. Thus $(v^\lambda, w^\lambda) \in \mathcal{F}_u$ and $\mathcal{F}_u$ is convex. □

We establish two further results. Given data (μ, ν, g) and initial class $\mathcal{C}$, for feasible $(v, w) \leftrightarrow f \in \mathcal{C}$, define

$$F : \mathcal{C} \to C(\bar{\Omega}), \quad F(f)(x) = (v + w)(x) \ \text{ for } x \in \bar{\Omega}, \quad (v, w) \leftrightarrow f \in \mathcal{C}.$$

Theorem 7.5. *Let data data (μ, ν, g) and convex class $\mathcal{C}$ be given . Then, the mapping F is concave on $\mathcal{C}$.*

Proof. Let $f^\lambda = (1-\lambda)f_1 + \lambda f_2$ for given $f_i \in \mathcal{C}$, $f_i \leftrightarrow (v_i, w_i) \in \mathcal{F}_{\mathcal{C}}$, $i = 1, 2$. Since $\mathcal{C}$ is convex, $f^\lambda \in \mathcal{C}$. As above, let $v^\lambda = (1-\lambda)v_1 + \lambda v_2$ and $w^\lambda = (1-\lambda)w_1 + \lambda w_2$, $0 \leq \lambda \leq 1$. Again by Lemma A.3, $Mv^\lambda \geq \mu$ and $Mw^\lambda \geq \nu$. On $\partial\Omega$, $v^\lambda = f^\lambda$ and $w^\lambda = g - f^\lambda$. Let $Mv' = \mu$ and $Mw' = \nu$, with $v'|_{\partial\Omega} = f^\lambda$ and $w'|_{\partial\Omega} = w^\lambda$, so that $f^\lambda \leftrightarrow (v', w') \in \mathcal{F}_{\mathcal{C}}$. We have

$$\begin{aligned} F(f^\lambda) &= v' + w' \geq v^\lambda + w^\lambda \\ &= (1-\lambda)(v_1 + w_1) + \lambda(v_2 + w_2) = (1-\lambda)F(f_1) + \lambda F(f_2) \end{aligned}$$

and F is concave. $\square$

Note that the concavity of F implies that the mapping

$$F' : \mathcal{C} \to \mathbb{R}, \quad F'(f) = \int_\Omega (v+w)(x)\,dx \;\text{ for } (v,w) \leftrightarrow f \in \mathcal{C}$$

is also concave on $\mathcal{C}$. If $(v,w) \leftrightarrow f \in \mathcal{C}$ is a **MA** splitting for u solving (1), then (v,w) maximizes both F and F' on $\mathcal{C}$.

Theorem 7.6. *Let data (μ,ν,g) and initial class $\mathcal{C}$ be given. For fixed $(v,w) \in \mathcal{F}_\mathcal{C}$, $(v,w) \leftrightarrow f^* \in \mathcal{C}$, set $u = v+w$. The following statements are equivalent for* **MAS**.

(*1.*) (v,w) *is a* **MA** *splitting for* u.
(*2.*) $M(v+w) = Mv + Mw$.
(*3.*) $F(f^*) = \sup_{f\in\mathcal{C}} F(f)$.
(*4.*) $F'(f^*) = \sup_{f\in\mathcal{C}} F'(f)$.

Proof. Follows directly from Theorem 5.2. $\square$

Theorems 7.5 and 7.6 permit the power of convex analysis to be brought to bear upon the **MA** splitting problem. Unfortunately, this analysis has not as yet yielded useful results.

A. Appendix

When seeking splittings $u = v + w$ for $u \in C^2(\Omega)$, one seeks $v, w \in C^2(\Omega)$ for which $M(v+w) = Mv + Mw$. This implies that $|D^2(v+w)(x)| = |D^2v(x)| + |D^2w(x)|$ for all $x \in \Omega$, where for square matrix A, $|A| = \det(A)$.

Lemma A.1. *Let $A, B \in \mathbb{R}^{2\times 2}$ be symmetric positive semidefinite matrices. Then:*

(*1.*) *$A+B$ is symmetric positive semidefinite, and $|A+B| = |A| + |B|$ only if $|A| = |B| = 0$;*
(*2.*) *$A-B$ is symmetric, and if $|B|=0$, then $|A-B| = |A| - |B|$ only if $|A| = 0$;*
(*3.*) *Let symmetric $C = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ with $|C| = 0$ be given. Set*

$$A_r = \begin{pmatrix} a-r & \sqrt{(a-r)(c-\frac{b^2}{a^2}r)} \\ \sqrt{(a-r)(c-\frac{b^2}{a^2}r)} & c - \frac{b^2}{a^2}r \end{pmatrix}$$
$$\text{and} \quad B_r = r\begin{pmatrix} 1 & \frac{b}{a} \\ \frac{b}{a} & \frac{b^2}{a^2} \end{pmatrix}, \quad -\infty < r < \infty. \tag{22}$$

Then:

(*i*) *For each r, $|A_r| = |B_r| = 0$;*

(*ii*) *For each* r, $C = A_r + B_r$ *and hence* $|C| = |A_r| + |B_r| = 0$;

(*iii*) A_r *is symmetric positive semidefinite for* $r \leq a$. B_r *is symmetric positive semidefinite for* $r \geq 0$;

(*iv*) *For each* $r \leq \min\{0, a\}$, $C = A_r - B_{-r}$ *and hence* $|C| = |A_r| - |B_{-r}| = 0$, *with* A_r *and* B_{-r} *both positive semidefinite.*

Proof. Let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

By definition $(A+B)^T = A+B$ and for $x \in \mathbb{R}^2$, $x^T(A+B)x = x^TAx + x^TBx$. Thus $A+B$ is symmetric positive semidefinite. One has $|A+B| = |A| + |B|$ if and only if

$$(a+\alpha)(c+\gamma) - (b+\beta)^2 = (ac - b^2) + (\alpha\gamma - \beta^2), \tag{23}$$

that is

$$a\gamma + \alpha c = 2b\beta. \tag{24}$$

Since A and B are positive semidefinite, $a \geq 0$, $d \geq 0$, $\alpha \geq 0$ and $\gamma \geq 0$. Also, since A and B have nonnegative eigenvalues,

$$|A| = ac - b^2 \geq 0 \quad \text{and} \quad |B| = \alpha\gamma - \beta^2 \geq 0. \tag{25}$$

Thus by (25),

$$a\gamma + \alpha c \geq \frac{b^2}{c}\gamma + \frac{\beta^2}{\gamma}c \geq 2b\beta \tag{26}$$

since

$$\left(b\sqrt{\frac{\gamma}{c}} - \beta\sqrt{\frac{c}{\gamma}}\right)^2 = \frac{b^2}{c}\gamma - 2b\beta + \beta^2\frac{c}{\gamma} \geq 0. \tag{27}$$

In view of (25), $a\gamma \geq \frac{b^2}{c}\gamma$ and $\alpha c \geq \frac{\beta^2}{\gamma}c$. Thus equalities can hold in (26), so that (24), hence (23) holds, only if $ac = b^2$ and $\alpha\gamma = \beta^2$, that is, $|A| = |B| = 0$. This establishes (*1.*).

To show (*2.*), expanding the equality $|A - B| = |A| - |B|$ gives

$$a\gamma + \alpha c = 2b\beta + 2(\alpha\gamma - \beta^2). \tag{28}$$

If $\alpha\gamma - \beta^2 = |B| = 0$, then (28) reduces to (24), and the argument establishing (*1.*) again applies to show that necessarily $|A| = 0$.

Now let C be as in (*3.*), with $|C| = ac - b^2 = 0$ and let A_r and B_r be as in (22). A_r is well-defined for $r \leq a$ as $a - r \geq 0$, and $c - \frac{b^2}{a^2}r \geq c - \frac{b^2}{a^2} = 0$, since $|C| = 0$. For $r \geq a$, A_r remains well-defined since $a - r \geq 0$, and $c - \frac{b^2}{a^2}r \leq c - \frac{b^2}{a^2} = 0$. By definition, $|A_r| = |B_r| = 0$ for all r and (*i*) holds. Simple algebraic simplification using $b^2 = ac$ shows $C = A_r + B_r$ and (*ii*) follows. (*iii*) then follows directly since $a - r \geq 0$ and $|A_r| = 0$, which together show that A_r is positive semi-definite. B_r is obviously positive semi-definite when $r \geq 0$. Since $A_r - B_{-r} = A_r + B_r$, (*iv*) follows from (*i*)–(*iii*). □

Note that when $r = 0$ in (22), $A_0 = C$ and $B_0 = 0$, a trivial splitting of C. Similarly, when $r = a$, $A_a = 0$ and $B_a = C$. Also note that if $A = B = I$, then $|A - B| = |A| - |B|$ so that conclusion (*2.*) above is necessarily weaker than (*1.*).

Theorem A.2. *In $\mathbb{R}^2$, if $u = v + w$ is a* **MA** *splitting for u with $v, w \in C^2(\Omega)$, then necessarily, $Mu = Mv = Mw = 0$. That is, there exist no nonhomogeneous $C^2(\Omega)$ splittings in $\mathbb{R}^2$.*

Proof. Let $u = v+w$ satisfy $M(v+w) = Mv+Mw$ with $v, w \in C^2(\Omega)$. We have $dMv = \det\, D^2v\; dx$, $dMw = \det\, D^2w\; dx$, and $dM(v+w) = \det\, D^2(v+w)\; dx = \det\,(D^2v+D^2w)\; dx = \det\, D^2v\; dx + \det\, D^2w\; dx$. Thus $\det\,(D^2v(x)+D^2w(x)) = \det\, D^2v(x) + \det\, D^2w(x)$ for all $x \in \Omega$ by continuity. By Lemma A.1 then, $\det\, D^2u(x) = \det\,(D^2v(x) + D^2w(x)) = \det\, D^2v(x) = \det\, D^2w(x) = 0$ for all $x \in \Omega$. Hence $Mu = Mv = Mw = 0$ and the theorem is proved. $\square$

Lemma A.1 provides a convenient tool for constructing examples of quadratic functions $u = \frac{1}{2}x^TQx + Rx$ which possess splittings. For example, in Example 2.2, Section 1, after adding linear terms, consider $u(x,y) = 2x^2 + 8xy + 8y^2 + x + 2y + 3$ with splitting $u = v + w$ where $v(x,y) = \frac{3}{2}x^2 + 6xy + 6y^2 + 2y$ and $w(x,y) = \frac{1}{2}x^2 + 2xy + 2y^2 + x + 3$. The Hessian $C = D^2u$ shown below has $|C| = 0$. Using Lemma A.1 with $r = 1$, $C = A_1 + B_1$ where

$$C = \begin{pmatrix} 4 & 8 \\ 8 & 16 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

For Borel set $E \subset \mathbb{R}^2$, we have

$$\begin{aligned} Mv(E) &= \int_E |D^2v|\, dx = \int_E \left| \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \right| dx = 0 \\ Mw(E) &= \int_E |D^2w|\, dx = \int_E \left| \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \right| dx = 0 \\ M(v+w)(E) &= \int_E |D^2(v+w)|\, dx = \int_E \left| \begin{pmatrix} 4 & 8 \\ 8 & 16 \end{pmatrix} \right| dx = 0. \end{aligned}$$

Similarly, choosing $r = -1$ in (iv) of Lemma A.1,

$$C = \begin{pmatrix} 4 & 8 \\ 8 & 16 \end{pmatrix}, \qquad A_{-1} = \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

we have $C = A_{-1} - B_1$ and $|C| = |A| = |B| = 0$. It follows that $u(x,y) = 2x^2 + 8xy + 8y^2 + x + 2y + 3$ has the difference splitting $u = v - w$ where $v(x,y) = \frac{5}{2}x^2 + 10xy + 10y^2 + 2y$ and $w(x,y) = \frac{1}{2}x^2 + 2xy + 2y^2 - x - 3$.

Lemma A.3. *Let Ω be an open, bounded strictly convex set and let σ be a Borel measure in Ω. Let $u_1, u_2, \ldots, u_N \in C(\bar{\Omega})$, $N \geq 2$, be convex functions satisfying*

$Mu_i \geq \sigma$, $Mu_i(\Omega) < +\infty$, $i = 1, 2, \ldots, N$. *Then*

$$M\left(\sum_{i=1}^{N} \lambda_i u_i\right) \geq \sigma \quad \text{for all } \lambda_i \geq 0 \text{ with } \sum_{i=1}^{N} \lambda_i \geq 1. \tag{29}$$

□

Proof of Lemma A.3. Consider the case $N = 2$. If $\lambda_1 + \lambda_2 > 1$, then there exist $\lambda_1' \geq 0$, $\lambda_2' \geq 0$, with $\lambda_1' \leq \lambda_1$, $\lambda_2' \leq \lambda_2$ and $\lambda_1' + \lambda_2' = 1$, so that

$$M(\lambda_1 u_1 + \lambda_2 u_2) \geq M(\lambda_1' u_1 + \lambda_2' u_2) + M((\lambda_1 - \lambda_1')u_1 + (\lambda_2 - \lambda_2')u_2).$$

Since the second term on the right is nonnegative, by induction it suffices to show (29) when $\lambda_1 = \lambda$, and $\lambda_2 = 1 - \lambda$, $0 \leq \lambda \leq 1$.

Suppose first that $u_1, u_2 \in C^2(\Omega)$. Then $dMu_i(x) = \det\, D^2 u_i(x)\; dx$, $x \in \Omega$, $i = 1, 2$. Define absolutely continuous measures infinitesimally by $\mu_k^{(i)}(x)\; dx$, $i = 1, 2$, using continuous density functions

$$\mu_k^{(i)}(x) = \int_\Omega \eta_{\frac{1}{k}}(x-y)\; dMu_i(y) = \int_\Omega \eta_{\frac{1}{k}}(x-y) \det\, D^2 u_i(y)\; dy, \quad i = 1, 2, \tag{30}$$

where η_ε is the standard mollifier. Define sequences of convex functions $\{u_k^{(1)}\}$, $\{u_k^{(2)}\} \subset C^2(\bar{\Omega})$ as the solutions to the Monge-Ampère equations

$$\begin{aligned} Mu_k^{(i)}(x) &= \mu_k^{(i)}(x)\; dx \quad \text{in } \Omega \\ u_k^{(i)}(x) &= u_i(x) \quad \text{on } \partial\Omega,\; i = 1, 2. \end{aligned} \tag{31}$$

As a function of x,

$$\lim_{k\to\infty} \int_\Omega \eta_{\frac{1}{k}}(x-y) \det\, D^2 u_1(y)\; dy = \det\, D^2 u_1(x)$$

uniformly on compact subsets of Ω. Thus for compactly supported $\varphi \in C(\Omega)$,

$$\begin{aligned} &\lim_{k\to\infty} \int_\Omega \varphi(x) \mu_k^{(1)}(x)\, dx \\ &= \lim_{k\to\infty} \int_\Omega \int_\Omega \varphi(x) \eta_{\frac{1}{k}}(x-y) \det\, D^2 u_1(y)\, dy dx \\ &= \int_\Omega \varphi(x) \det\, D^2 u_1(x)\, dx = \int_\Omega \varphi(x)\, dMu_1(x) \end{aligned} \tag{32}$$

which implies that $\mu_k^{(1)}(x)\; dx \rightharpoonup dMu_1(x)$. Analogously, $\mu_k^{(2)}(x)\; dx \rightharpoonup dMu_2(x)$. Since $\int_\Omega \eta_{\frac{1}{k}}(x-y)\; dy \leq 1$ for all k and $x \in \Omega$ and $Mu_i(\Omega) < +\infty$, using (30) we find that the sequences $\{Mu_k^{(1)}(\Omega)\}$ and $\{Mu_k^{(2)}(\Omega)\}$ are bounded by

$\max\{Mu_1(\Omega), Mu_2(\Omega)\}$. By Fact 3.2 then, $u_k^{(1)} \to u_1$ and $u_k^{(2)} \to u_2$ uniformly in Ω.

Set $\rho_k(x) = \int_\Omega \eta_{\frac{1}{k}}(x-y)\ dM\sigma(y)$. Immediately we have $\rho_k(x)\ dx \rightharpoonup \sigma$. By definition, $\mu_k^{(1)}(x) \geq \rho_k(x)$ and $\mu_k^{(2)}(x) \geq \rho_k(x)$ since $Mu_1 \geq \sigma$ and $Mu_2 \geq \sigma$. Note that since $\mu_k^{(i)}(x)\,dx = \det\ D^2u_k^{(i)}(x)\,dx$ by (30), $\mu_k^{(i)}(x) = \det\ D^2u_k^{(i)}(x)$ for $x \in \Omega$, $i = 1, 2$. Since the function $\det(\cdot)^{\frac{1}{n}}$ is concave on the collection of positive semi-definite matrices,

$$\begin{aligned}
&(\det\ D^2((1-\lambda)u_k^{(1)}(x) + \lambda u_k^{(2)}(x)))^{\frac{1}{n}}\\
&\geq (1-\lambda)(\det\ D^2u_k^{(1)}(x))^{\frac{1}{n}} + \lambda(\det\ D^2u_k^{(2)}(x))^{\frac{1}{n}}\\
&\geq (1-\lambda)\rho_k(x)^{\frac{1}{n}} + \lambda\rho_k(x)^{\frac{1}{n}}\\
&= \rho_k(x)^{\frac{1}{n}}
\end{aligned}\tag{33}$$

so that

$$\det\ D^2((1-\lambda)u_k^{(1)}(x) + \lambda u_k^{(2)}(x))\ dx \geq \rho(x)\ dx.$$

Passing to weak limits using Fact 3.3 gives

$$M((1-\lambda)u_1 + \lambda u_2) \geq \sigma. \tag{34}$$

Thus (29) holds for convex $u_1, u_2 \in C^2(\Omega)$. Since any convex $u_1, u_2 \in C(\Omega)$ may be approximated uniformly on compact subsets of Ω by $C^2(\Omega)$ convex functions, (29) holds in general by Fact 3.3. A straightforward induction on k completes the proof of the lemma. □

References

[1] A. D. Alexandrov: On functions which may be represented by differences of convex surfaces, Dokl. Akad. Nauk SSSR, n. Ser. 72 (1950) 613–616.

[2] A. D. Alexandrov: Certain estimates for the dirichlet problem, Sov. Math., Dokl. 1 (1961) 1151–1154.

[3] A. D. Alexandrov: Majorization of solutions of second-order linear equations, Amer. Math. Soc., Transl., II. Ser. 68 (1968) 120–143.

[4] I. J. Bakelman: Convex Analysis and Nonlinear Geometric Elliptic Equations, Springer, Berlin (1994).

[5] L. A. Caffarelli: Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation, Ann. Math. 131 (1990) 135–150.

[6] L. A. Caffarelli: Some regularity properties of solutions of Monge-Ampère equation, Commun. Pure Appl. Math. 44 (1991) 965–969.

[7] L. A. Caffarelli, C. E. Gutièrrez: Real analysis related to the Monge-Ampère equation, Trans. Amer. Math. Soc. 348(3) (1996) 1075–1092.

[8] L. A. Caffarelli, C. E. Gutièrrez: Properties of the solutions of the linearized Monge-Ampère equation, Amer. J. Math. 119(2) (1997) 423–465.

[9] C. E. Gutièrrez, Q. Huang: Geometric properties of the sections of solutions to the Monge-Ampère equation, Trans. Amer. Math. Soc. 352(2) (2000) 4381–4396.

[10] C. E. Gutièrrez: The Monge-Ampère Equation, Progress in Nonlinear Differential Equations and their Applications 44, Birkhäuser, Boston (2001).

[11] P. Hartman: On functions representable as a difference of convex functions, Pac. J. Math. 9 (1959) 707–713.

[12] E. M. Landis: On functions representable as the difference of two convex functions, Dokl. Akad. Nauk SSSR, n. Ser. 72 (1950) 613–616.

[13] A. V. Pogorelov: Monge-Ampère Equations of Elliptic Type, P. Noordhoff, Groningen (1964).

[14] J. Rauch, B. A. Taylor: The dirichlet problem for the multidimensional Monge-Ampère equation, Rocky Mt. J. Math. 7(2) (1977) 345–364.

[15] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[16] H. Tuy: Convex Analysis and Global Optimization, Nonconvex Optimization and Global Optimization 22, Kluwer, Dordrecht (1998) 707–713.