Journal of Convex Analysis
Volume 22 (2015), No. 3, 853–858

A Sharp Quantitative Estimate for the Perimeters of Convex Sets in the Plane*

Menita Carozza
Dipartimento di Ingegneria, Universitá del Sannio,
Corso Garibaldi, 82100 Benvento, Italy
carozza@unisannio.it

Flavia Giannetti
Dipartimento di Matematica e Applicazioni,
Università di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy
giannett@unina.it

Francesco Leonetti
Dipartimento di Ingegneria e Scienze dell' Informazione e Matematica,
Università di L'Aquila, Via Vetoio, 67100 L'Aquila, Italy
leonetti@univaq.it

Antonia Passarelli di Napoli
Dipartimento di Matematica e Applicazioni,
Università di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy
antpassa@unina.it

Received: March 11, 2014
Accepted: October 1, 2014

Let $E \subset B \subset \mathbb{R}^2$ be bounded, convex sets. The monotonicity of the perimeters holds, i.e. $\mathcal{H}^1(\partial E) \leqslant \mathcal{H}^1(\partial B)$. Here we give a quantitative estimate of the difference of the perimeters: it shows how much the perimeter of B increases when B becomes larger and larger with respect to E. We give an example showing that our estimate is sharp.

Keywords: Convex sets, perimeters, Hausdorff distance

1991 Mathematics Subject Classification: 52A10

1. Introduction

Let us consider two bounded convex sets E and B, with $E \subset B \subset \mathbb{R}^2$. It is well known the monotonicity of the perimeter, i.e.

$$\mathcal{H}^1(\partial E) \leqslant \mathcal{H}^1(\partial B), \tag{1}$$

*We acknowledge the support of INdAM- GNAMPA Project - "Stime quantitative in disuguaglianze geometriche" (2014) and by the Project "Metodi matematici per la modellizzazione di problemi di localizzazione e di trasporto ottimo" (Legge5/2009 Regione Campania).

ISSN 0944-6532 / $ 2.50

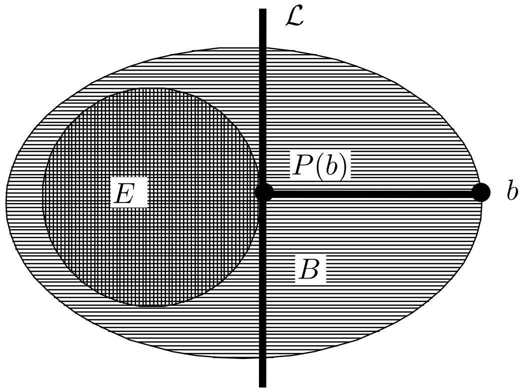

Figure 1.1.

see, for instance, page 52 of [1] and Lemma 2.4 in [2]. In (1), ∂E is the boundary of E, $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure, thus $\mathcal{H}^1(\partial E)$ is the length of the boundary of E, the perimeter of E for short. In [3], the following quantitative version of (1) has been proven

$$\mathcal{H}^1(\partial E) + \frac{[h(E,B)]^2}{\sqrt{[\mathrm{diam}(B)]^2 + [h(E,B)]^2}} \leqslant \mathcal{H}^1(\partial B), \tag{2}$$

where $h(E,B)$ is the Hausdorff distance between E and B, $\mathrm{diam}(B)$ is the diameter of B.

The deficit $\delta(E,B)$ between the perimeters in (2) involves the diameter of the set B and the Hausdorff distance between E and B. In order to obtain an improved version of estimate (2), the idea is to express $\delta(E,B)$ in terms of the measure of the section of B orthogonal to the line segment which realizes the Hausdorff distance between E and B.

More precisely, we establish the following

Theorem 1.1. *Let us consider two closed, bounded and convex sets $\varnothing \neq E \subsetneqq B \subset \mathbb{R}^2$. Then*

$$\mathcal{H}^1(\partial E) + \frac{4\,|h(E,B)|^2}{2\sqrt{\left(\frac{\mathcal{H}^1(B\cap\mathcal{L})}{2}\right)^2 + |h(E,B)|^2} + \mathcal{H}^1(B\cap\mathcal{L})} \leq \mathcal{H}^1(\partial B) \tag{3}$$

where h is the Hausdorff distance and $\mathcal{L}$ is the orthogonal line to the vector $b - P(b)$ through $P(b)$, with $b \in B$ and $P(b) \in E$ are such that $|b - P(b)| = h(E,B)$ (see Fig. 1.1).

Observe that the deficit $\delta(E,B)$ in (3) is greater than the one in (2), so we improve the inequality (2). Moreover our estimate reveals to be sharp.

Indeed, let E be a square of side length l, T the triangle of base length l and height h and $B = E \cup T$ as in Fig. 1.2.

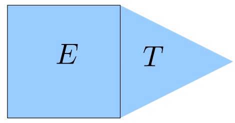

Figure 1.2.

Obviously we have

$$\mathcal{H}^1(\partial E) = 4l, \qquad \mathcal{H}^1(\partial B) = 3l + 2\sqrt{\left(\frac{l}{2}\right)^2 + h^2}$$

$$h(E, B) = h, \qquad \mathcal{H}^1(B \cap \mathcal{L}) = l.$$

In this case, in estimate (3) equality occurs.

2. Proof of Theorem 1.1

Our arguments follow the ones of [3] but we are going to be more accurate in the estimates. We recall that, since E and B are closed sets and $E \subset B$, the distance $h(E, B)$ can be written as follows

$$h(E, B) = \max_{x \in B} \min_{y \in E} |x - y|. \tag{4}$$

Let $b \in B$ and $P(b) \in E$ such that

$$h(E, B) = |b - P(b)|. \tag{5}$$

It turns out that $b \in B \setminus E$ and $P(b)$ is the projection onto the closed convex set E. We consider the line $\mathcal{L}$ through $P(b)$ orthogonal to $h = b - P(b) \neq 0$; the minimality property of the projection $P(b)$ tells us that such a line is a supporting one for the convex set E. Set

$$\mathcal{L} = \{x \in \mathbb{R}^2 \ : \ \langle h, P(b) - x\rangle = 0\}$$
$$S^+ = \{x \in \mathbb{R}^2 \ : \ \langle h, P(b) - x\rangle \geqslant 0\}$$

then $\mathcal{L} = \partial S^+$ and $E \subset S^+$. Since

$$E \subset B \cap S^+, \tag{6}$$

we can use inequality (1) as follows

$$\begin{aligned}\mathcal{H}^1(\partial E) &\leqslant \mathcal{H}^1(\partial(B \cap S^+)) \leqslant \mathcal{H}^1\left([(\partial B) \cap S^+] \ \cup \ [B \cap \partial S^+]\right) \\ &\leqslant \mathcal{H}^1((\partial B) \cap S^+) + \mathcal{H}^1(B \cap \partial S^+).\end{aligned} \tag{7}$$

On the other hand

$$\mathcal{H}^1(\partial B) = \mathcal{H}^1((\partial B) \cap S^+) + \mathcal{H}^1((\partial B) \setminus S^+).$$

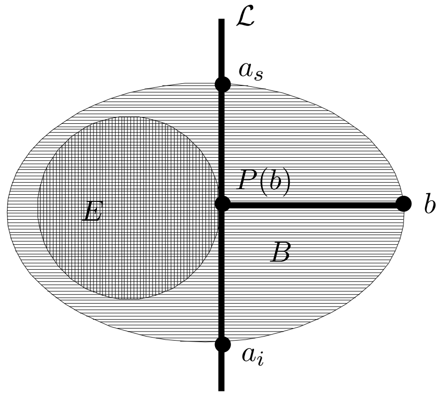

Figure 2.1.

This equality allows us to link the length of $\partial B \cap S^+$ with the length of ∂B and $\partial B \setminus S^+$ so that (7) becomes

$$\mathcal{H}^1(\partial E) \leqslant \mathcal{H}^1(\partial B) - \mathcal{H}^1((\partial B) \setminus S^+) + \mathcal{H}^1(B \cap \partial S^+). \tag{8}$$

We recall that $\mathcal{L} = \partial S^+$ and we remark that $B \cap \partial S^+$ is the line segment joining two suitable points $a_i, a_s \in \mathcal{L}$ with $a_i = ih^* + P(b)$, $a_s = sh^* + P(b)$ for some h^* verifying $|h^*| = 1$, $\langle h^*, h\rangle = 0$ and for suitable $-\infty < i \leqslant 0 \leqslant s < +\infty$, see Figure 2.1.

Let us consider the closed triangle τ with vertices $a_s,\ a_i$ and b and note that

$$\tau \subset B \cap (\overline{\mathbb{R}^2 \setminus S^+}),$$

so we can use again (1) obtaining

$$\mathcal{H}^1(\partial\tau) \leqslant \mathcal{H}^1(\partial(B \cap (\overline{\mathbb{R}^2 \setminus S^+}))). \tag{9}$$

In order to compute the perimeter of τ,

$$\mathcal{H}^1(\partial\tau) = |a_i - a_s| + |a_s - b| + |b - a_i|\,,$$

we note that

$$|a_i - a_s| = s - i$$
$$|a_s - b| = \sqrt{s^2 + |h|^2}$$
$$|b - a_i| = \sqrt{i^2 + |h|^2}.$$

In what follows (see Figure 2.2) we will denote by

$$d = s - i = |a_i - a_s|\,, \qquad b_1 = s, \qquad b_2 = -i$$

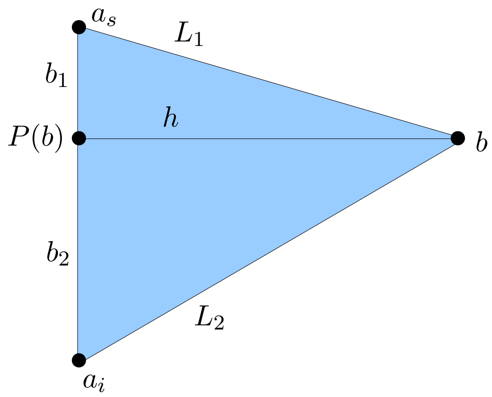

Figure 2.2.

therefore

$$\begin{aligned}\mathcal{H}^1(\partial\tau) &= \sqrt{b_1^2+|h|^2}+\sqrt{b_2^2+|h|^2}+d\\ &= \sqrt{b_1^2+|h|^2}+\sqrt{(d-b_1)^2+|h|^2}+d\\ &=: f(b_1)\geq \min_{b_1\in[0,d]} f(b_1)\\ &= f\left(\frac{d}{2}\right)=2\sqrt{\left(\frac{d}{2}\right)^2+|h|^2}+d. \end{aligned}\tag{10}$$

Recalling that $d=\mathcal{H}^1(B\cap\partial S^+)=|a_i-a_s|$, inequalities (9) and (10) merge into the following

$$\begin{aligned}\mathcal{H}^1\left((\partial B)\setminus S^+\right)+\mathcal{H}^1\left(B\cap\partial S^+\right) &\geqslant \mathcal{H}^1\big(\partial\big(B\cap(\overline{\mathbb{R}^n\setminus S^+})\big)\big)\\ &\geqslant \mathcal{H}^1(\partial\tau)\geqslant 2\sqrt{\left(\frac{d}{2}\right)^2+|h|^2}+d,\end{aligned}$$

and so

$$\mathcal{H}^1\left((\partial B)\setminus S^+\right)\geqslant 2\sqrt{\left(\frac{d}{2}\right)^2+|h|^2}.\tag{11}$$

Plugging this estimate into (8), we get

$$\mathcal{H}^1(\partial E)\leqslant \mathcal{H}^1(\partial B)-2\sqrt{\left(\frac{d}{2}\right)^2+|h|^2}+d,\tag{12}$$

i.e.,

$$\mathcal{H}^1(\partial E)+2\sqrt{\left(\frac{d}{2}\right)^2+|h|^2}-d\leqslant \mathcal{H}^1(\partial B).\tag{13}$$

Hence

$$\mathcal{H}^1(\partial E) + \frac{\left(2\sqrt{\left(\frac{d}{2}\right)^2 + |h|^2} - d\right)\left(2\sqrt{\left(\frac{d}{2}\right)^2 + |h|^2} + d\right)}{2\sqrt{\left(\frac{d}{2}\right)^2 + |h|^2} + d} \leqslant \mathcal{H}^1(\partial B), \tag{14}$$

which is equivalent to

$$\mathcal{H}^1(\partial E) + \frac{4|h|^2}{2\sqrt{\left(\frac{d}{2}\right)^2 + |h|^2} + d} \leqslant \mathcal{H}^1(\partial B). \tag{15}$$

Recalling that $|h| = h(E, B)$ and $d = \mathcal{H}^1(B \cap \mathcal{L})$, we get the conclusion. $\square$

References

[1] T. Bonnesen, W. Fenchel: Theory of Convex Bodies, BCS Associates, Moscow (USA) (1987).

[2] G. Buttazzo, V. Ferone, B. Kawohl: Minimum problems over sets of concave functions and related questions, Math. Nachr. 173 (1995) 71–89.

[3] M. La Civita, F. Leonetti: Convex components of a set and the measure of its boundary, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia 56 (2008–2009) 71–78.