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A Generalised Korovkin Theorem

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The main result of this paper is a general Korovkin-type approximation theorem for an injective isometry between $C(X)$ and $C(Y)$, where X, Y are compact Hausdorff spaces.

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1. Introduction

Throughout the paper, X will denote a compact Hausdorff space and $C(X)$ (respectively, $C_{\mathbb{R}}(X)$) the space of continuous complex-valued (resp., real-valued) functions on X equipped with the supremum norm.

Korovkin proved that a sequence of positive linear operators T_n on $C_{\mathbb{R}}[0, 1]$ that converges to the identity operator, I, on the quadratic polynomials, also converges to I for all $f \in C_{\mathbb{R}}[0, 1]$.

The Korovkin Theorem has beautiful applications and many generalisations. In the early literature on Korovkin-type approximations, many results were proved for subspaces of $C_{\mathbb{R}}(X)$ (for a small sampling, see [3, 4, 12]: for a discussion of such problems in the setting of normed linear spaces, see [11]). With the exception of [1], we are unaware of analogous results for complex subspaces of $C(X)$. The main purpose of the present paper is to formulate and prove a general Korovkin-type approximation theorem for an injective isometry between $C(X)$ and $C(Y)$, where Y is also compact Hausdorff, and this yields as a special case results for complex subspaces of $C(X)$ which do not necessarily contain the constants. Several results in the literature follow as easy consequences of our main Theorem 3.3. Many examples that illustrate our points of departure

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have been given. Several results in [9], which rely heavily on Choquet theory, provided the impetus for the solutions of the problems discussed here.

For background information on the subject, the well-written account [2] may be consulted.

2. Notations and preliminary remarks

For a normed linear space V, let $V_1 = \{x \in V : \|x\| \leq 1\}$ denote the closed unit ball of V. For normed linear spaces E and F, let $B(E, F)$ denote the space of all bounded linear operators $T : E \to F$. Let $\mathbb{T} \subseteq \mathbb{C}$ denote the unit circle. Let X, Y be compact Hausdorff spaces. To avoid confusion, we will denote the sup norm on $C(X)$, $C(Y)$ etc. as $\|\cdot\|_X$, $\|\cdot\|_Y$ instead of $\|\cdot\|_\infty$.

We will be following, with some exceptions explained below, the notations of [9]. We summarise them here for the reader's convenience.

Let $M(X)$ be the Banach space of all complex regular Borel measures on X endowed with the total variation norm. By the Riesz Representation Theorem, $M(X)$ is isometrically isomorphic to the dual space of $C(X)$.

Let A be a complex subspace of $C(X)$. For $x \in X$, let ε_x be Dirac measure at x, and $\phi(x) \in A^*$ denote the *evaluation functional* at x, *i.e.*, $\phi(x) = \varepsilon_x|_A$. The evaluation map $\phi : X \to A^*$ is a homeomorphism of X into the w*-compact set A_1^*.

Definition 2.1. The Choquet boundary of A is defined as

$$\partial A = \{x \in X : \phi(x) \in \mathrm{ext}(A_1^*)\},$$

where $\mathrm{ext}(D)$ denotes the set of extreme points of a convex set D.

This definition coincides with the classical definition of the Choquet boundary when A separates points of X and contains constants [8, Proposition 6.2]. We use the same definition even otherwise. One sees easily that ∂A is non-empty [5, p. 441] and

$$\mathrm{ext}(A_1^*) = \mathbb{T}\phi(\partial A).$$

Assumption 2.2. For the rest of the paper, unless otherwise specified, we assume that A is a subspace of $C(X)$ with the following two properties:

(I) A is *point-separating*, i.e., given two points $x, y \in X$, $x \neq y$, there exists $g \in A$ such that $g(x) \neq g(y)$;

(II) $\|\phi(x)\| = 1$ for all $x \in X$.

The following will be a very important space for this paper [9, p. 306]:

Definition 2.3. Let A be a subspace of $C(X)$ satisfying our assumptions. $\widetilde{A} = \{f \in C(X) : f(x) = \alpha f(y)$ whenever $g(x) = \alpha g(y)$ for all $g \in A$, for some $\alpha \in \mathbb{T}$, $x, y \in X\}$.

$\widetilde{A}$ is a closed linear subspace of $C(X)$. Clearly, $A \subseteq \widetilde{A} \subseteq C(X)$.

Definition 2.4. Let A be a subspace of $C(X)$ satisfying our assumptions. If $\lambda, \mu \in M(X)$ with $\|\lambda\| = \|\mu\| = 1$, we write $\lambda \approx \mu$ if $\mu(h) = \lambda(h)$ for all $h \in \widetilde{A}$.

Definition 2.5. Let A be a subspace of $C(X)$ satisfying our assumptions. For $x \in X$, let $M_{x,A} = \{\mu : \mu \in M(X),\ \|\mu\| = 1,\ \mu|_A \equiv \phi(x)\}$ and $M^+_{x,A}$ the probability measures in $M_{x,A}$. We call the members of $M_{x,A}$ the complex *representing measures* of x. If A is fixed, as is most often the case in this paper, we will abbreviate $M_{x,A}$ and $M^+_{x,A}$ by M_x and M^+_x respectively.

All other unexplained terms below, if any, will be found in [9].

We recall the following characterization of ∂A.

Theorem 2.6 ([9, Proposition 3.7]). *Let A be a subspace of $C(X)$ satisfying our assumptions. Let $x \in X$. Then $x \in \partial A$ if and only if $\mu \in M_x$ implies $\mu \approx \varepsilon_x$.*

We shall find it convenient in our subsequent work to use the following particular version of a representation theorem from [5, p. 490]:

Theorem 2.7. *If $T : C(X) \to C(Y)$ is a bounded linear operator, then there is a unique continuous map $\tau : Y \to (M(X), w^*)$ such that*

$$(Tf)(y) = \tau(y)(f), \quad (y \in Y, f \in C(X)) \tag{1}$$

$$\|T\| = \sup\{\|\tau(y)\| : y \in Y\} \tag{2}$$

where $\|\tau(y)\|$ is the total variation norm *of the measure $\tau(y)$.*

Conversely, any continuous map $\tau : Y \to (M(X), w^*)$ uniquely defines a bounded linear operator $T : C(X) \to C(Y)$ with both (1) *and* (2) *satisfied.*

The following result will be used crucially and repeatedly in our proofs.

Definition 2.8. A set $K \subseteq X$ is said to be a boundary for a subspace $A \subseteq C(X)$, if for every $g \in A$, there exists $t \in K$ such that $|g(t)| = \|g\|_X$.

Theorem 2.9 (Holsztyński's Theorem, [6]). *Let $T : C(X) \to C(Y)$ be a linear isometry. Then there is a closed boundary $Y_0 \subseteq Y$ for $T(C(X)) \subseteq C(Y)$, $u : Y_0 \to \mathbb{T}$ continuous and $\lambda : Y_0 \to X$ continuous onto such that*

$$(Tf)(z) = u(z) f(\lambda(z)), \quad \textit{for all } f \in C(X),\ z \in Y_0. \tag{3}$$

3. The Main Results

We begin with an example that motivates the results that follow and illustrates their points of departure.

Example 3.1. Let $A = \{f \in C[0,1] : f(0) = if(1)\}$. Then A satisfies our assumptions and $1 \notin A$.

Clearly, $A = \widetilde{A}$, hence by Theorem 2.6, $\partial A = [0, 1]$.

It is instructive to compute M_x for $x \in [0, 1]$.

Let $\mu_x \in M_x$. Then $\mu_x - \varepsilon_x = \alpha(\varepsilon_0 - i\varepsilon_1)$ for some $\alpha \in \mathbb{C}$. Hence, for $x \in (0, 1)$, $\mu_x = \varepsilon_x$, that is, $M_x = \{\varepsilon_x\}$.

On the other hand, for $x = 0$, this implies $\mu_0 = (1 + \alpha)\varepsilon_0 - i\alpha\varepsilon_1$ and therefore, $1 = |1 + \alpha| + |\alpha|$. It follows that $\alpha = -\gamma$ with $0 \leq \gamma \leq 1$. Thus,

$$M_0 = \{(1 - \gamma)\varepsilon_0 + i\gamma\varepsilon_1 : 0 \leq \gamma \leq 1\}.$$

Similarly,

$$M_1 = \{(1 - \gamma)\varepsilon_1 - i\gamma\varepsilon_0 : 0 \leq \gamma \leq 1\}.$$

If $f \in C[0, 1]\backslash A$, we can write $f = g + \lambda 1$ where $g \in A$ and $\lambda \in \mathbb{C}$. If $\{T_n\}$ is a sequence of operators on $C[0, 1]$ with $\|T_n\| \leq 1$ for each n and $\lim_{n\to\infty} T_n(g) = g$ uniformly for each $g \in A$, then it is clear that $\lim_n T_n(f) = f$ uniformly for each $f \in C[0, 1]$ if and only if $\lim_{n\to\infty} T_n(1) = 1$ uniformly, but this fails as we now see by defining $T_n \in B(C[0, 1], C[0, 1])$:

$$T_n(f) = 1/2(\varepsilon_0 + i\varepsilon_1)(f)h_n + (1 - h_n)f, \quad f \in C[0, 1],$$

where

$$h_n(t) = \begin{cases} 1 - nt, & 0 \leq t \leq \frac{1}{n} \\ 0, & \frac{1}{n} \leq t \leq 1 \end{cases} \quad n = 1, 2, 3, \ldots.$$

Recall that $1/2(\varepsilon_0 + i\varepsilon_1) \in M_0$. One checks easily that $\|T_n\| \leq 1$, that

$$T_n(g) - g = (g(0) - g)h_n \to 0 \text{ uniformly for each } g \in A,$$

but

$$T_n(1)(0) - 1 = \frac{1}{2}(i - 1) \not\to 0.$$

This suggests that the Korovkin closure of A (Definition 3.2) need not necessarily be the whole of $C(X)$ in general and this will be vindicated in Theorem 3.3 below.

We will now ready to state and prove our main theorem.

Definition 3.2. Let X, Y be compact Hausdorff spaces. Let A be a subspace of $C(X)$ satisfying our assumptions. Let $T : C(X) \to C(Y)$ be a linear isometry and $Z \subseteq Y$ a closed set. The (T, Z)-Korovkin closure of A is defined as:

$$\begin{aligned} \widehat{A}_Z^T = \{f \in C(X) : \text{ if } \{T_\alpha\} \subseteq B(C(X), C(Z))_1 \text{ is such that for all } g \in A, \\ \lim_\alpha \|T_\alpha(g) - T(g)\|_Z = 0, \text{ then } \lim_\alpha \|T_\alpha(f) - T(f)\|_Z = 0\}. \end{aligned}$$

In what follows, we will most often use $Z = Y_0$ as in Theorem 2.9.

If $X = Y$ and $T = I$, the identity operator, then $Z = Y_0 = Y$ and we simply write $\widehat{A}$ instead of $\widehat{A}_Y^I$ and call it the Korovkin closure of A.

We are now ready for our main theorem:

Theorem 3.3. *Let X, Y be compact Hausdorff spaces. Let A be a subspace of $C(X)$ satisfying our assumptions. Let $T : C(X) \to C(Y)$ be a linear isometry. Let $\widetilde{A}$ and $\widehat{A}^T_{Y_0}$ be as in Definitions* 2.3 *and* 3.2 *respectively. The following statements are equivalent:*

(a) $\partial A = X$.

(b) $\widehat{A}^T_{Y_0} = \widetilde{A}$.

Proof. (a) $\Rightarrow$ (b): We adapt to our context the proof of Saskin's theorem [8, Section 9].

Let $Y_0 \subseteq Y$, $u : Y_0 \to \mathbb{T}$ and $\lambda : Y_0 \to X$ be as given by Theorem 2.9. Let $\{T_\alpha\} \subseteq B(C(X), C(Y_0))_1$ be such that $\lim_\alpha \|T_\alpha(g) - T(g)\|_{Y_0} = 0$ for all $g \in A$. By Theorem 2.7, let τ_α, $\tau : Y_0 \to (M(X)_1, w^*)$ correspond to T_α and T respectively. Then for each $g \in A$,

$$\lim_\alpha \sup\{|\tau_\alpha(z)(g) - \tau(z)(g)| : z \in Y_0\} = 0. \tag{4}$$

Let $f \in \widetilde{A}$. We have to show that

$$\lim_\alpha \sup\{|\tau_\alpha(z)(f) - \tau(z)(f)| : z \in Y_0\} = 0.$$

It suffices to show that every subnet τ_β of τ_α has a further subnet τ_γ such that $\lim_\gamma \sup_{Y_0} |(\tau_\gamma(z)(f) - \tau(z)(f))| = 0$.

Now, since $z \to \tau_\beta(z)(f) - \tau(z)(f)$ is continuous on Y_0,

$$\sup_{Y_0} |\tau_\beta(z)(f) - \tau(z)(f)| = |\tau_\beta(z_\beta)(f) - \tau(z_\beta)(f)|$$

for some $z_\beta \in Y_0$. By compactness, there is a subnet $\{z_\gamma\}$ such that $z_\gamma \to z_0 \in Y_0$ and $\tau_\gamma(z_\gamma) \xrightarrow{w^*} \tau_0 \in M(X)_1$. If $g \in A$,

$$\begin{aligned} |\tau_\gamma(z_\gamma)(g) - \tau(z_0)(g)| &\leq |\tau_\gamma(z_\gamma)(g) - \tau(z_\gamma)(g)| + |\tau(z_\gamma)(g) - \tau(z_0)(g)| \\ &\leq \sup_{Y_0} |\tau_\gamma(z)(g) - \tau(z)(g)| + |\tau(z_\gamma)(g) - \tau(z_0)(g)| \end{aligned}$$

and (4) implies, RHS $\xrightarrow{\gamma} 0$. So $\tau_\gamma(z_\gamma)(g) \xrightarrow{\gamma} \tau(z_0)(g)$. But, since $\tau_\gamma(z_\gamma) \xrightarrow{w^*} \tau_0$,

$$\tau_0(g) = \tau(z_0)(g) = (Tg)(z_0) = u(z_0)g(\lambda(z_0)), \quad \text{for all } g \in A,$$

by (3). It follows that $\overline{u(z_0)}\tau_0 \in M_{\lambda(z_0)}$. By ($a$), $\lambda(z_0) \in \partial A$. Therefore, by Theorem 2.6, $\overline{u(z_0)}\tau_0(f) = f(\lambda(z_0))$ and so

$$\tau_0(f) = u(z_0)f(\lambda(z_0)) = (Tf)(z_0) = \tau(z_0)(f).$$

Now,

$$\sup_{z\in Y_0} |\tau_\gamma(z)(f) - \tau(z)(f)| = |\tau_\gamma(z_\gamma)(f) - \tau(z_\gamma)(f)|$$
$$\leq |\tau_\gamma(z_\gamma)(f) - \tau_0(f)| + |\tau_0(f) - \tau(z_\gamma)(f)| \to 0$$

as $\tau_\gamma(z_\gamma) \xrightarrow{w^*} \tau_0$ and $|\tau_0(f) - \tau(z_\gamma)(f)| \to |\tau_0(f) - \tau(z_0)(f)| = 0$.

This proves that $\widetilde{A} \subseteq \widehat{A}^T_{Y_0}$. To prove the equality, we will need the following construction that will be used later again.

Let $x_0 \in X$. Let $\{U_\alpha\}$ be a net of neighbourhood base at x_0 ordered by reverse inclusion. By Urysohn's lemma, for each α, we can find $h_\alpha \in C(X)$ such that $0 \leq h_\alpha \leq 1$, $h_\alpha|_{U^c_\alpha} \equiv 0$ and $h_\alpha(x_0) = 1$. For $\mu \in M(X)_1$, define $T_\alpha \in B(C(X), C(Y_0))$ by

$$(T_\alpha f)(z) = \mu(f)u(z)h_\alpha(\lambda(z)) + [1 - h_\alpha(\lambda(z))](Tf)(z) \quad \text{for } z \in Y_0. \tag{5}$$

It is easily checked that $\|T_\alpha\| \leq 1$. For $z \in Y_0$ and $f \in C(X)$,

$$|(T_\alpha f)(z) - (Tf)(z)| = |\mu(f)u(z)h_\alpha(\lambda(z)) - u(z)f(\lambda(z))h_\alpha(\lambda(z))|$$
$$= h_\alpha(\lambda(z))|\mu(f) - f(\lambda(z))|, \text{ as } |u(z)| = 1. \tag{6}$$

Now, let $f_0 \in C(X)\backslash\widetilde{A}$. By definition of $\widetilde{A}$, there are points $x_0 \neq x_1 \in X$ and $t \in \mathbb{T}$ such that $\phi(x_0) = t\phi(x_1)$ and $f_0(x_0) \neq tf_0(x_1)$. Thus, the measure $1/2(\varepsilon_{x_0} + t\varepsilon_{x_1}) \in M_{x_0}$.

Let $\{U_\alpha\}$ be as above. Define $T_\alpha \in B(C(X), C(Y_0))$ by (5) with $\mu = 1/2(\varepsilon_{x_0} + t\varepsilon_{x_1})$. Then, by (6), for $z \in Y_0$ and $g \in A$,

$$|(T_\alpha g)(z) - (Tg)(z)| = h_\alpha(\lambda(z))|\mu(g) - g(\lambda(z))| = h_\alpha(\lambda(z))|g(x_0) - g(\lambda(z))|$$

Let $\varepsilon > 0$. There is a neighbourhood U of x_0 such that $|g(x) - g(x_0)| < \varepsilon$ for $x \in U$. There is an α_0 so that $U_{\alpha_0} \subseteq U$. If $z \notin \lambda^{-1}(U_{\alpha_0})$, then $h_{\alpha_0}(\lambda(z)) = 0$ and hence, $(T_{\alpha_0}g)(z) - (Tg)(z) = 0$. It follows that for $\alpha \geq \alpha_0$, $|(T_\alpha g)(z) - (Tg)(z)| < \varepsilon$ for all $z \in Y_0$. That is, $\lim_\alpha \|T_\alpha(g) - T(g)\|_{Y_0} = 0$ for each $g \in A$. On the other hand, if $\lambda(z_0) = x_0$,

$$|(T_\alpha f_0)(z_0) - (Tf_0)(z_0)| = h_\alpha(x_0)|\mu(f_0) - f_0(x_0)| = 1/2|tf_0(x_1)) - f_0(x_0)| \neq 0.$$

Hence, $f_0 \notin \widehat{A}^T_{Y_0}$.

$(b) \Rightarrow (a)$. Fix $x_0 \in X$ and $\mu \in M_{x_0}$. Let $\{U_\alpha\}$ and $\{h_\alpha\}$ be as above. Let $\lambda(z_0) = x_0$. Define $T_\alpha \in B(C(X), C(Y_0))$ by (5). It follows as above that $\|T_\alpha g - Tg\|_{Y_0} \xrightarrow{\alpha} 0$.

By (b), for any $f \in \widetilde{A}$ also, $\|T_\alpha f - Tf\|_{Y_0} \xrightarrow{\alpha} 0$. In particular,

$$|(T_\alpha f)(z_0) - (Tf)(z_0)| = |\mu(f) - f(x_0)| \to 0.$$

So $\mu(f) = f(x_0)$ for each $f \in \widetilde{A}$ and one concludes by Theorem 2.6 that $x_0 \in \partial A$. $\square$

Remark 3.4. Notice that in $(b) \Rightarrow (a)$, we only use $\widetilde{A} \subseteq \widehat{A}^T_{Y_0}$.

Corollary 3.5. *If $A = \widetilde{A}$, then $\partial A = X$.*

Remark 3.6. This clearly also follows from Theorem 2.6. Example 3.1 is a case in point.

Since the statement (a) in Theorem 3.3 does not involve the operator T, the statement can be strengthened to read:

Corollary 3.7. *The following statements are equivalent:*

(a) $\partial A = X$.

(b) *For some compact Hausdorff space Y and some linear isometry $T : C(X) \to C(Y)$, $\widehat{A}^T_{Y_0} = \widetilde{A}$.*

(c) *For all compact Hausdorff spaces Y and all linear isometries $T : C(X) \to C(Y)$, $\widehat{A}^T_{Y_0} = \widetilde{A}$.*

(d) $\widehat{A} = \widetilde{A}$.

Let us now study the situation where $\widetilde{A} = C(X)$. [9, Proposition 3.4] characterizes this phenomenon. It follows that

Corollary 3.8 ([1]). *If $1 \in A$, then $\widehat{A} = C(X)$ if and only if $\partial A = X$.*

Corollary 3.9. *If A is a function algebra, then $\widehat{A} = C(X)$ if and only if $\partial A = X$.*

Proof. If A is a function algebra, then $1 \in \widetilde{A}$ and hence, $\widetilde{A} = C(X)$ [9, Proposition 3.4]. □

Another example where $\widetilde{A} = C(X)$ is

Example 3.10. Let $A = \{f \in C[0,1] : \int_0^1 f(x)\ dx = 0\}$. A satisfies our assumptions and $1 \notin A$. It is clear that $\widetilde{A} = C(X)$. It is also easy to see that $\partial A = [0,1]$. And therefore, $\widehat{A} = C(X)$.

Now we give an example such that $A \subsetneq \widetilde{A} \subsetneq C(X)$.

Example 3.11. Let $A = \{f \in C[0,1] : \int_0^1 f(x)\ dx = 0 \text{ and } f(0) + f(1) = 0\}$. Arguing as in Example 3.1, and noting that A is of codimension two, one sees that $\partial A = [0,1]$. And hence A satisfies our assumptions and $1 \notin A$. It is clear that $\widetilde{A} = \{f \in C[0,1] : f(0) + f(1) = 0\}$ and therefore, $A \subsetneq \widetilde{A} = \widehat{A} \subsetneq C(X)$.

Remark 3.12. It is an interesting question whether one can replace $\widehat{A}^T_{Y_0}$ in Theorem 3.3 by $\widehat{A}^T_Y$. Our next example shows that the answer, in general, is "no"!

Example 3.13. Let X and A be as in Example 3.11 above and $Y = [0,1] \cup \{2\}$. We may identify $C(Y) = C(X) \oplus_\infty \mathbb{C}$. Define $T : C(X) \to C(Y)$ by

$$Tf = \left(f, \int_0^1 f(x)\, dx \right)$$

Then $Y_0 = X$ and hence, as noted above, $\widehat{A}^T_{Y_0} = \widetilde{A}$.

Notice that if $f \in \widehat{A}^T_Y$, then $Tf \in \widehat{T(A)}$. Therefore, if $\widetilde{A} = \widehat{A}^T_Y$, then $T(\widetilde{A}) \subseteq \widehat{T(A)}$.

Now, $T(A) = \{(g,0) : g \in A\}$. Define $U_n : C(Y) \to C(Y)$ by

$$U_n(h,\gamma) = \left(\left(1 - \frac{1}{n} \right) h, 0 \right), \quad h \in C(X), \gamma \in \mathbb{C}.$$

Clearly, $\|U_n\| \leq 1$ and for $(g,0) \in T(A)$,

$$\|U_n(g,0) - (g,0)\|_Y = \frac{1}{n}\|g\|_X \to 0.$$

Now suppose $(f,\alpha) \in \widehat{T(A)}$. Then

$$\|U_n(f,\alpha) - (f,\alpha)\|_Y = \max\left\{ \frac{1}{n}\|f\|_X, |\alpha| \right\} \to 0.$$

It follows that $\alpha = 0$.

Since $T(\widetilde{A}) = \{(h, \int_0^1 h(x)\, dx) : h \in \widetilde{A}\}$, we cannot have $T(\widetilde{A}) \subseteq \widehat{T(A)}$. That is, in general, $\widehat{A}^T_{Y_0} \neq \widehat{A}^T_Y$, even when $\partial A = X$.

The natural next question is: *when* can one replace $\widehat{A}^T_{Y_0}$ by $\widehat{A}^T_Y$ in Theorem 3.3?

This can surely be done if T is onto. In this case, $Y_0 = Y$ and $\widehat{A}^T_{Y_0} = \widehat{A}$.

On the other hand, if we define T to satisfy (3), that is, suppose $u : Y \to \mathbb{T}$ is continuous and $\lambda : Y \to X$ is continuous onto and we define $T : C(X) \to C(Y)$ by

$$(Tf)(y) = u(y) f(\lambda(y)), \quad \text{for all } f \in C(X),\ y \in Y. \tag{7}$$

then we can surely have $Y_0 = Y$. If λ is not 1-1, such a T cannot be onto.

Example 3.14. As a concrete example, define $T : C(\mathbb{T}) \to C(\mathbb{T})$ by

$$(Tf)(z) = f(z^2), \quad \text{for all } f \in C(\mathbb{T}),\ z \in \mathbb{T}. \tag{8}$$

Let us now obtain a necessary and sufficient condition for $Y_0 = Y$.

Let $T : C(X) \to C(Y)$ be a linear isometry. Let $T(C(X)) = R$. Then R is a closed subspace of $C(Y)$. Instead of $T : C(X) \to C(Y)$, we prefer to look at the isometry $T : C(X) \to R$. Note that $T^* : R^* \to C(X)^* = M(X)$ is also an onto isometry and that $T^*(\text{ext}(R_1^*)) = \text{ext}(M(X)_1)$.

We will need to recall some definitions from [6]. For $x \in X$ and $y \in Y$, define

$$S_x = \{f \in C(X) : \|f\|_X = 1 \text{ and } |f(x)| = 1\} \quad \text{and}$$
$$R_y = \{g \in C(Y) : \|g\|_Y = 1 \text{ and } |g(y)| = 1\}.$$

For $x \in X$, define

$$Q_x = \{y \in Y : T(S_x) \subseteq R_y\}.$$

And finally,

$$Y_0 = \bigcup_{x \in X} Q_x.$$

Proposition 3.15. $Y_0 = \partial(R)$.

Proof. Let $y \in \partial(R)$. Then $\phi(y) \in \text{ext}(R_1^*)$ and hence, $T^*(\phi(y)) = \alpha\varepsilon_x$, where $x \in X$ and $\alpha \in \mathbb{T}$. It follows that for any $f \in C(X)$, $(Tf)(y) = \alpha f(x)$. Thus, if $f \in S_x$, then $|(Tf)(y)| = |f(x)| = 1$. Therefore, $y \in Q_x \subseteq Y_0$.

Conversely, let $y \in Y_0$. Then, by (3), for all $f \in C(X)$, $(Tf)(y) = u(y)f(\lambda(y))$. It follows that $T^*(\phi(y)) = u(y)\varepsilon_{\lambda(y)} \in \text{ext}(M(X)_1)$. It follows that $y \in \partial(R)$. $\square$

Corollary 3.16. *$Y = Y_0$ if and only if $Y = \partial(R)$.*

Remark 3.17. It is not clear if R, in general, would satisfy our assumptions, but if it does, it follows that $Y = Y_0$ if and only if $\widehat{R} = \widetilde{R}$.

We can now ask for similar results when T and $\{T_\alpha\}$ are *positive operators*.

Definition 3.18. Let T be a positive isometry. Let $A \subseteq C(X)$ and $Y_0 \subseteq Y$ be as before. Define

$$\begin{aligned} \widehat{A}_+^T = \{f \in C(X) :\ & \text{if } \{T_\alpha\} \subseteq B(C(X), C(Y_0))_1 \text{ is such that } T_\alpha \geq 0 \\ & \text{and } \lim_\alpha \|T_\alpha(g) - T(g)\|_{Y_0} = 0 \text{ for all } g \in A, \text{ then} \\ & \lim_\alpha \|T_\alpha(f) - T(f)\|_{Y_0} = 0\}. \end{aligned}$$

Clearly, $\widehat{A}_{Y_0}^T \subseteq \widehat{A}_+^T$.

Theorem 3.19. *Let T be a positive isometry. Then the following statements are equivalent:*

(*a*) *For all $x \in X$, $M_x^+ = \{\varepsilon_x\}$.*
(*b*) $\widehat{A}_+^T = C(X)$.

Proof. Notice that if T is a positive isometry, then in Theorem 2.9, $u \equiv 1$. The proof is almost the same as that of Theorem 3.3 once we observe:

(i) If T is positive and $\tau : Y_0 \to (M(X)_1, w^*)$ corresponds to T, then $\tau(z)$ is a non-negative measure for each $z \in Y_0$. Similarly, $\tau_\alpha(z)$ is a non-negative measure for each α and each $z \in Y_0$. As a consequence, the measure $\tau_0 = \lim_\alpha \tau_\alpha(z_\alpha) \in M(X)_1$ that was obtained in the proof of $(a) \Rightarrow (b)$ in Theorem 3.3 is necessarily non-negative.

(ii) If T is positive and μ is a non-negative measure, then the operators T_α constructed in (5) are positive. □

Remark 3.20. Notice that the condition $\partial A = X$ implies the condition (a) above. To see this, note that if $\phi(x) \in \mathrm{ext}(A_1^*)$, then M_x^+ is a w*-closed face of $M(X)_1$ and hence, any extreme point of M_x^+ is necessarily of the form ε_y. Since A is point-separating, $M_x^+ = \{\varepsilon_x\}$. Moreover, the two are equivalent if $1 \in A$. But not in general, as the following example shows.

Example 3.21. Let $A = \{f \in C[0,1] : f(0) + \int_0^1 f(x)\,dx = 0\}$. It is not difficult to see that A satisfies our assumptions and $1 \notin A$. It is clear that $\widetilde{A} = C(X)$. It also follows from Theorem 2.6 that $\partial A = (0,1]$. But, clearly, $M_0^+ = \{\varepsilon_0\}$. And therefore, $\widehat{A} \subsetneq \widetilde{A} = C(X) = \widehat{A}_+$. Notice that since A is of codimension 1, we have $A = \widehat{A}$.

Essentially similar proof also yields

Theorem 3.22. *Let T be a positive isometry. Then the following statements are equivalent:*

(a) *If $x \in X$ and $\mu \in M_x^+$, then $\mu \approx \varepsilon_x$.*
(b) $\widetilde{A} \subseteq \widehat{A}_+^T$.

Remark 3.23.

(a) Theorem 3.22 was suggested by a reading of [7].
(b) Unlike Theorem 3.3, in this case we do not have $\widetilde{A} = \widehat{A}_+$.
(c) In fact, Example 3.1 satisfies the condition (a) above and shows that $A = \widetilde{A} = \widehat{A} \subsetneq C(X) = \widehat{A}_+$.

We can interpret the preceding theorem in terms of extreme points. First, a bit of notation. If $\mu \in M(X)$, then $\mu \circ \phi^{-1}$ is the image measure defined on A_1^* by $\mu \circ \phi^{-1}(E) = \mu(\phi^{-1}(E))$, E a Borel subset of A_1^* with its weak* topology. Let $D = \overline{\mathrm{conv}}^{w^*}\{\phi(x) : x \in X\}$. We will need the following

Lemma 3.24. $\phi(x) \in \mathrm{ext}(D)$ *if and only if* $\mu \in M_x^+$ *implies* $\mu \approx \varepsilon_x$.

Proof. $(\Rightarrow)$ First note by Milman's theorem, $\mathrm{ext}(D) \subseteq \{\phi(x) : x \in X\}$ since the latter set is w*-closed. Let $\phi(x) \in \mathrm{ext}(D)$ and suppose $\mu \in M_x^+$. To show $\mu \approx \varepsilon_x$, we have to prove that $\mathrm{hom}(\mu \circ \phi^{-1}) = \mathrm{hom}(\varepsilon_x \circ \phi^{-1}) = \mathrm{hom}\,\varepsilon_{\phi(x)}$ (by [9,

p. 308]). Consider the measure $\lambda = R(\mu \circ \phi^{-1})$ defined by

$$\lambda(g) = \int_U g(\phi(x)) d(\mu \circ \phi^{-1}), \quad g \in C(U)$$

where $U = (A_1^*, w^*)$ (Note that the definition works equally well with $g \in C(D)$ as $\mu \circ \phi^{-1}$ is supported on $\phi(X)$). λ is a probability measure on $\phi(X)$ with resultant $\phi(x) \in D$, consequently $\lambda = \varepsilon_{\phi(x)}$ as $\phi(x) \in \mathrm{ext}(D)$. Now $\hom R(\mu \circ \phi^{-1}) = \hom(\mu \circ \phi^{-1})$, by [9, p. 309], i.e., $\hom \lambda = \hom(\mu \circ \phi^{-1}) = \hom(\varepsilon_{\phi(x)})$ as was to be shown.

($\Leftarrow$) As this proof is an easier version of the proof of ($\Leftarrow$) in [9, Proposition 3.7], we will take the liberty of omitting it. □

Remark 3.25. Thus, we can replace condition (a) in Theorem 3.22 by for each $x \in X$, $\phi(x) \in \mathrm{ext}(D)$, where $D = \overline{\mathrm{conv}}^{w^*}\{\phi(x) : x \in X\}$.

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