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The Degree of Convex Nondensifiability in Banach Spaces

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It is shown in this paper that the notion of α-dense curve in a bounded set of a metric space produces a degree of nondensifiability ϕ_d which extended to convex hulls of bounded sets in a Banach space generates a measure of noncompactness (briefly MNC) Φ that we will call degree of convex nondensifiability. We also prove that Φ dominates any other MNC and, in particular, dominates the MNC of Hausdorff of the best possible form in infinite dimensional Banach spaces.

Keywords: α-dense curves, degree of nondensifiability, degree of convex nondensifiability, measures of noncompactness

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1. Introduction

In the whole paper I will be the closed interval $I := [0, 1]$ in $\mathbb{R}$. By $B(y, \epsilon)$ we will denote the *closed ball* of a metric space centered at y with radius $\epsilon > 0$. If X is a Banach space B_X will be its *closed unit ball*, that is, its closed ball centered at the origin with radius 1.

In 1997 was introduced [16] the concept of α-dense curve γ_α in a bounded set B of a metric space (E, d) as a continuous map $\gamma_\alpha : I \to E$ with image $\gamma_\alpha(I) \subset B$ and satisfying that for every $x \in B$ there is some $t \in I$ such that the distance

$$d\left(\gamma_\alpha(t), x\right) \leq \alpha, \tag{1}$$

for a given nonnegative real number α.

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Given a continuous real map f defined on a bounded set B of (E, d), an α-dense curve γ_α allows us to obtain an approximation method (for details see [17], [18]) to turn the global optimization problem

$$\inf\{f(x) : x \in B\}$$

into the unidimensional

$$\inf\{f(\gamma_\alpha(t)) : t \in I\},$$

provided that it is possible to define such a curve for arbitrarily small α.

The infimum value of α for which the inequality (1) holds, coincides with the Hausdorff distance [13, *IV*, Problem D] from the image of the curve $\gamma_\alpha(I)$ to the set B. Hence, in any metric space (E, d), the parameter α measures how different is a bounded set B from a special subset of B of type $\gamma_\alpha(I)$, compact, connected and locally connected, i.e., a Peano continuum, as it is known in the classical book of Hocking-Young [10, p. 124].

It is immediate to check that given a bounded set B of (E, d) it is always possible to define an α-dense curve γ_α in B for any value $\alpha \geq \operatorname{diam} B$, the diameter of B. This curve γ_α may be obtained, for instance, by defining a constant map from I to a fixed point of B. However, for small values of α, it may not be possible to define an α-dense curve in a bounded set, even compact and connected, as we will point out by means of an example (see the set defined in (6)).

The infimum of the values of α for which there exists an α-dense curve in a bounded set B of a metric space expresses the degree of nondensifiability (briefly DND) ϕ_d of B (see Definition 2.2). Whenever B is a set such that for arbitrarily small α it contains an α-dense curve, B is said to be *densifiable*. That means that a bounded set B is densifiable if and only if B is the limit, in the Hausdorff metric, of Peano continua of the form $\gamma_\alpha(I)$ when α tends to zero. The densifiable sets will then have a DND equal to 0 and they form a special class of precompact and connected sets [20, Proposition 1] which contains in particular the family of curves of Peano type or space-filling curves, as also known from [21].

In 1930 Kuratowski [14] defined the notion of measure of noncompactness, briefly MNC, related with some problems of General Topology and Darbo [7] used that measure to extend the Shauder theorem to noncompact operators. Another relevant MNC is that of Hausdorff which was introduced by Gohberg, Goldenshtein and Markus [9].

In 1977 appeared the measure of De Blasi [8], β, a regular weak MNC (briefly σMNC), which vanishes on the class of relatively weakly compacts of a Banach space X. The theorem of Banas-Rivero [5, Theorem 4] states that β acts as an upper bound for any σMNC, say ω, in the sense that there exists $A > 0$ such that $\omega(B) \leq A\beta(B)$ for all $B \in \mathcal{B}$, the class of bounded sets of X (we can also say that β dominates ω). Nevertheless, noticing every bounded set is relatively weakly compact in a reflexive Banach space [6, Proposition 12.20], any σMNC

is identically 0 in a such space and then the above inequality becomes a trivial equality. Consequently it must be raised in nonreflexive Banach spaces.

The natural question, inspirated on Banas-Rivero's result, is the problem of the existence of an MNC (in the strong sense) such that to act as an upper bound for any other MNC. In this paper we have given a positive answer to the above question, firstly by settling an inequality of Banas-Rivero type satisfied by our degree of nondensifiability ϕ_d (see inequality (28) and compare it with [5, Theorem 4]). Secondly, by acting ϕ_d on convex hulls of bounded sets we have obtained a new MNC, that we have called degree of convex nondensifiability, denoted by Φ, which dominates any MNC and, in particular, dominates the MNC of Hausdorff of the best possible form in infinite dimensional Banach spaces (Theorem 3.7).

Finally, noticing the MNC's are very often used in the theory of functional equations and in particular the fixed point theorems derived from them, as we can see in the literature devoted to this subject [1, 3, 4, 12, 22], we also present in this paper some relations between the most usual MNC's and the degree of nondensifiability ϕ_d. Especifically we have compared ϕ_d with the Hausdorff and Kuratowski MNC's by means of inequalities (see Theorem 2.3, 2.5 and 2.7) which are the best possible in infinite dimensional Banach spaces.

2. Degree of nondensifiability

To facilitate the comprehension of this manuscript we first introduce the notion of MNC.

Definition 2.1. Let (E, d) be a complete metric space and $\mathcal{B}$ the family of bounded subsets of E. A map $\phi : \mathcal{B} \rightarrow [0, +\infty)$ is said to be a measure of noncompactness if it satisfies:

i) Regularity, $\phi(B) = 0$ if and only if B is a precompact set.
ii) Invariant under closure, $\phi(B) = \phi(\overline{B})$, for all $B \in \mathcal{B}$.
iii) Semi-additivity, $\phi(B_1 \cup B_2) = \max\{\phi(B_1), \phi(B_2)\}$, for all $B_1, B_2 \in \mathcal{B}$.

To have a measure of noncompactness in a Banach space it is needed to add the two properties:

iv) Semi-homogeneity, $\phi(\lambda B) = |\lambda|\,\phi(B)$ for any scalar λ and $B \in \mathcal{B}$.
v) Invariant under translations, $\phi(x + B) = \phi(B)$ for any vector x and $B \in \mathcal{B}$.

Given a nonvoid bounded set B in a metric space, since it is always possible to find an α-dense curve in B for certain $\alpha \geq 0$, the family $\Gamma_{\alpha,B}$ of α-dense curves in B is nonvoid for sufficiently large α. That allows us to introduce the notion of degree of nondensifiability, briely DND, as follows.

Definition 2.2. Let (E, d) be a metric space and $\mathcal{B}$ the family of nonvoid bounded subsets of E. Given a set $B \in \mathcal{B}$, let $\Gamma_{\alpha,B}$ be the class of all α-dense curves γ_α in B for a given $\alpha \geq 0$. Then we define the degree of nondensifiability as a map $\phi_d : \mathcal{B} \rightarrow [0, +\infty)$ such that

$$\phi_d(B) := \inf \{\alpha \geq 0 : \Gamma_{\alpha,B} \neq \emptyset\}, \quad \text{for every } B \in \mathcal{B}. \tag{2}$$

For example, in a finite dimensional space E with the metric induced by the Euclidean norm, the DND of its closed unit ball B_E is 0. Indeed, since B_E is compact, connected and locally connected, by Hahn-Mazurkiewitz's theorem [21, Theorem 6.8], B_E is the continuous image of the unit interval $I = [0, 1]$. Then B_E is itself a curve, so it has density $\alpha = 0$ and consequently $\phi_d(B_E) = 0$.

By recalling that in a complete metric space (E, d), the Hausdorff measure of noncompactness [3, p. 18] of a bounded subset B of E is defined as

$$\chi(B) := \inf \{\epsilon > 0 : B \text{ is covered by finitely many balls of radius} \leq \epsilon\}, \tag{3}$$

the next result proves that χ is upper bounded by the DND defined in (2).

Theorem 2.3. *On the family $\mathcal{B}$ of bounded sets of a complete metric space (E, d) one has*

$$\chi(B) \leq \phi_d(B), \quad \text{for all } B \in \mathcal{B}. \tag{4}$$

Furthermore, the above inequality is the best possible in infinite dimensional Banach spaces.

Proof. Let B be a bounded set of E and $\delta := \phi_d(B)$. By (2), for a given $\epsilon > 0$ there exists a curve γ_α in B with density $\alpha = \delta + \frac{\epsilon}{2}$. Since $\gamma_\alpha(I)$ is compact, given $\frac{\epsilon}{2}$ there is a finite set $F = \{y_i \in \gamma_\alpha(I) : i = 1, ..., n\}$ such that

$$\gamma_\alpha(I) \subset \cup_{i=1}^n B\left(y_i, \frac{\epsilon}{2}\right).$$

Since γ_α is α-dense in B, for any $x \in B$ there exists $y \in \gamma_\alpha(I)$ such that $d(x, y) \leq \alpha$. The triangular inequality implies that $B \subset \cup_{i=1}^n B(y_i, \delta + \epsilon)$. Then, from (3), and taking into account that ϵ is arbitrary, one has $\chi(B) \leq \delta$, so the inequality

$$\chi(B) \leq \phi_d(B)$$

holds.

The sharpness of (4) is obtained by considering the closed unit ball B_X of any infinite dimensional Banach space X. Indeed, by [3, p. 23], the MNC of Hausdorff verifies

$$\chi(B_X) = 1.$$

On one hand, because of the curve $\gamma(t) := 0$ for all $t \in I$ is $1-$dense in B_X, from (2), one has $\phi_d(B_X) \leq 1$. On the other hand, by [19, Theorem 12], we get $\phi_d(B_X) \geq 1$ and then it follows that $\phi_d(B_X) = 1$. Now we conclude that

$$\chi(B_X) = \phi_d(B_X) = 1 \tag{5}$$

in any infinite dimensional Banach space X. The proof of the theorem is now completed. □

Observe that the MNC of Hausdorff χ and the DND ϕ_d, defined in (2), are distinct even in finite dimensional spaces. Indeed, the set

$$B := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x \in [-1,0) \cup (0,1] \right\} \cup \{(0,y) : y \in [-1,1]\} \qquad (6)$$

is connected and compact, so precompact, but is not locally connected. A simple observation shows that it is impossible to define an α-dense curve in B with $\alpha < 1$, so $\phi_d(B) \geq 1$. However, from the regularity of χ, it follows that $\chi(B) = 0$.

Remark 2.4. The set defined in (6) proves that ϕ_d is not a MNC because the regularity fails. Likewise the example (6) shows that in a complete metric space (E, d), the property

$$\chi(B) \leq \phi_d(B) \leq 2\chi(B), \text{ for all } B \in \mathcal{B}, \qquad (7)$$

is false.

However, on the subclass of bounded and arc-connected sets, the inequalities (7) are true as we prove in the next result.

Theorem 2.5. *Let χ be the Hausdorff measure of noncompactness and ϕ_d the degree of nondensifiability in a complete metric space (E, d). Then on $\mathcal{B}_a$, the family of bounded and arc-connected sets in E, one has*

$$\chi(B) \leq \phi_d(B) \leq 2\chi(B) \text{ for all } B \in \mathcal{B}_a. \qquad (8)$$

Furthermore, the inequalities (8) *are the best possible in infinite dimensional Banach spaces.*

Proof. The first inequality of (8) is exactly Theorem 2.3, then it is only needed to prove the second inequality of (8) and its sharpness. Consider a set B of $\mathcal{B}_a$ and let $\lambda := \chi(B)$. By (3), for arbitrary ϵ, B can be covered by finitely many balls $B(c_i, \lambda + \frac{\epsilon}{2})$. In every $B(c_i, \lambda + \frac{\epsilon}{2})$ we select a point of B, say b_i, and since B is arc-connected there exists a curve $\gamma : I \to B$ passing through the b_i's. Hence, there is at least a value $t_i \in I$ such that $\gamma(t_i) = b_i$ for each i. Now, since given an arbitrary point $x \in B$ there is some i such that $x \in B(c_i, \lambda + \frac{\epsilon}{2})$, one has

$$d(x, \gamma(t_i)) \leq d(x, c_i) + d(c_i, b_i) \leq 2\lambda + \epsilon.$$

This means that the curve γ is α-dense in B with density $2\lambda + \epsilon$. Because ϵ is arbitrary, γ densifies B with density $2\lambda = 2\chi(B)$, so, by (2), we get

$$\phi_d(B) \leq 2\chi(B)$$

and then the second inequality of (8) follows.

Now, it remains to prove the existence of a set in an infinite dimensional Banach space for which the above inequality becomes an equality. An example is the bounded convex set

$$\mathcal{D} := \left\{ f \in L_1[0,1] : f \geq 0 \text{ and } \int_I f(x)dx = 1 \right\}$$

in the space $L_1[0,1]$ of all real Lebesgue-measurable functions defined on $[0,1]$ of integrable modulus, endowed with the usual norm $\|f\| := \int_{[0,1]} |f(x)|\, dx$. About the set $\mathcal{D}$ we claim that its degree of nondensifiablity

$$\phi_d(\mathcal{D}) = 2. \tag{9}$$

Indeed, since $\mathcal{D} \subset B_{L_1[0,1]}$, its diameter $\operatorname{diam} \mathcal{D} \leq 2$. Then any curve α-dense in $\mathcal{D}$ has density $\alpha \leq 2$ and consequently, by (2), $\phi_d(\mathcal{D}) \leq 2$. By assuming that $\phi_d(\mathcal{D}) < 2$, let η be a real number such that

$$\phi_d(\mathcal{D}) < \eta < 2. \tag{10}$$

Consider in $\mathcal{D}$ the sequence $\{f_n := n\varphi_{[0,\frac{1}{n}]},\ n \geq 1\}$, where $\varphi_{[0,\frac{1}{n}]}$ is the characteristic function of the interval $\left[0, \frac{1}{n}\right]$. It is immediate that this sequence does not have any convergent subsequence. Hence $\mathcal{D}$ is not compact and then necessarily the density of any curve $\gamma_\alpha : I \to L_1[0,1]$ contained in $\mathcal{D}$, because $\gamma_\alpha(I)$ is compact, must be positive. In particular $\gamma_\alpha(I)$ is weakly compact, so uniformly integrable [6, Proposition 13.1]. That is, given $\epsilon > 0$ there exists $\delta > 0$ such that for all $g \in \gamma_\alpha(I)$ and for all Borel set $A \subset I$ with Lebesgue measure $\Lambda(A) < \delta$ one has

$$\int_A g(x)dx < \epsilon.$$

Thus given a curve γ_α in $\mathcal{D}$, since its density $\alpha > 0$, noticing (10), we can choose an ϵ such that $0 < \epsilon < \frac{\alpha}{2}(\frac{2}{\eta} - 1)$. From the uniform integrability of $\gamma_\alpha(I)$ determine the corresponding $\delta > 0$ that allows us to define an integer $m > \max\left\{1, \frac{1}{\delta}\right\}$ and the Borel set $A := \left[0, \frac{1}{m}\right]$. Then, given the function $f_m := m\varphi_{[0,\frac{1}{m}]} \in \mathcal{D}$, by α-density and uniform integrability, there exists $g_m \in \gamma_\alpha(I)$ such that

$$\begin{aligned}
\alpha \geq\ & \|f_m - g_m\| = \int_{[0,1]} |f_m(x) - g_m(x)|\, dx \\
=\ & \int_{[0,\frac{1}{m}]} |f_m(x) - g_m(x)|\, dx + \int_{[\frac{1}{m},1]} |f_m(x) - g_m(x)|\, dx \\
\geq\ & \int_{[0,\frac{1}{m}]} (f_m(x) - g_m(x))dx + \int_{[\frac{1}{m},1]} g_m(x)dx \\
=\ & \int_{[0,\frac{1}{m}]} f_m(x)dx - 2\int_{[0,\frac{1}{m}]} g_m(x)dx + \int_{[0,1]} g_m(x)dx \\
>\ & 2 - 2\epsilon > 2 - \alpha\left(\frac{2}{\eta} - 1\right),
\end{aligned}$$

which means that $\alpha > \eta$. Then, since γ_α is arbitrary, by (2), the degree of nondensifiability $\phi_d(\mathcal{D}) \geq \eta$, which contradicts (10). Therefore (9) is true, as claimed. On the other hand, since $\mathcal{D} \subset B_{L_1[0,1]}$, from (3), it is immediate that $\chi(\mathcal{D}) \leq 1$. Then, by (8) and (9), necessarily $\chi(\mathcal{D}) = 1$, which proves that

$$\phi_d(\mathcal{D}) = 2\chi(\mathcal{D}) = 2.$$

Hence, the second inequality of (8) is the best possible in infinite dimensional Banach spaces. Now the proof is completed. □

In Remark 2.4, we noted that ϕ_d is not a measure of noncompactness. However, in the next result we prove that ϕ_d has almost all properties of an MNC.

Proposition 2.6. *Let (E, d) be a complete metric space and $\mathcal{B}$ the family of bounded sets. Then, the degree of nondensifiability ϕ_d defined in (2) satisfies:*

1) *It is regular on the subfamily $\mathcal{B}_a$ of bounded and arc-connected sets.*

2) *It is invariant under closure.*

3) *It is semi-additive on sets B_1, $B_2 \neq \emptyset$ of $\mathcal{B}_a$, provided that $B_1 \cap B_2 \neq \emptyset$.*

Furthermore, if E is a Banach space, then ϕ_d also satisfies:

4) *It is semi-homogene, that is $\phi_d(\lambda B) = |\lambda|\, \phi_d(B)$ for any scalar λ and $B \in \mathcal{B}$.*

5) *It is invariant under translations, that is, $\phi_d(x + B) = \phi_d(B)$, for any $x \in E$, $B \in \mathcal{B}$.*

Proof. *1*) Let B be a set of $\mathcal{B}_a$ such that $\phi_d(B) = 0$. Then, given $\epsilon > 0$, there exists an ϵ-dense curve γ_ϵ in B, so $\gamma_\epsilon(I)$ is a compact contained in B. Hence, there is a finite set

$$F := \{x_i \in \gamma_\epsilon(I) : i = 1, ..., n\}$$

such that for any $x \in \gamma_\epsilon(I)$ the distance $d(x, F) \leq \epsilon$. Now, from the definition of ϵ-dense curve and the triangular inequality, it follows that for any $y \in B$ the distance $d(y, F) \leq 2\epsilon$, which means, because ϵ is arbitrarily small, that B is precompact.

Conversely, assume that B is a precompact set of $\mathcal{B}_a$ and let ϵ be a positive number. Then there is a finite set $G := \{y_j \in B : j = 1, ..., m\}$ such that for any $y \in B$ the distance

$$d(y, G) \leq \epsilon. \tag{11}$$

On the other hand, as B is arc-connected, there exists a continuous function $\gamma_\epsilon : I \to B$ which passes through the points of G. By (11), γ_ϵ is ϵ-dense in B and, since ϵ is arbitrary, we then have $\phi_d(B) = 0$. Consequently the regularity of ϕ_d on the subclass $\mathcal{B}_a$ follows.

2) The invariance under closure of ϕ_d is a direct consequence of the fact that the distance $d(B, \overline{B}) = 0$ for any subset B of (E, d).

3) To prove the semi-additivity when $B_1 \cap B_2 \neq \emptyset$, we put $\alpha_j := \phi_d(B_j)$, $j = 1$, 2, and assume that $\alpha_1 \leq \alpha_2$. Then, by (2), given $\epsilon > 0$, there exist curves γ_j which are $(\alpha_j + \epsilon)$-dense in B_j for $j = 1, 2$. Therefore, by connecting γ_1 with γ_2 passing through a point of $B_1 \cap B_2$, we obtain a curve γ that densifies the set $B_1 \cup B_2$ with density $\alpha \leq \alpha_2 + \epsilon$. Noticing ϵ is arbitrary, we get

$$\phi_d(B_1 \cup B_2) = \alpha_2 = \max\{\phi_d(B_1), \phi_d(B_2)\},$$

which proves the semi-additivity of ϕ_d.

4) Now assume that $(E, \|\|)$ is a Banach space. Then, if $\lambda = 0$, the set $\lambda B = \{0\} \in \mathcal{B}_a$ and by regularity $\phi_d(\lambda B) = 0$, so the equality $\phi_d(\lambda B) = |\lambda| \phi_d(B)$ trivially follows in this case. Suppose that $\lambda \neq 0$ and let $\delta := \phi_d(B)$; then, from (2), given $\epsilon > 0$ there exists an $(\delta + \frac{\epsilon}{|\lambda|})$-dense curve γ_ϵ in B. Now define a curve γ_λ as $\gamma_\lambda(t) := \lambda\gamma_\epsilon(t)$, $t \in I$. Then it is immediate that $\gamma_\lambda(I) \subset \lambda B$. On the other hand, given a point $y \in \lambda B$ one has $\frac{1}{\lambda} y \in B$ and then, since γ_ϵ is $(\delta + \frac{\epsilon}{|\lambda|})$-dense in B, there is a value $t_\lambda \in I$ such that

$$\left\| \frac{1}{\lambda} y - \gamma_\epsilon(t_\lambda) \right\| \leq \delta + \frac{\epsilon}{|\lambda|}.$$

Now multiplying by $|\lambda|$ we have

$$\|y - \lambda\gamma_\epsilon(t_\lambda)\| \leq |\lambda| \delta + \epsilon$$

and noticing $\gamma_\lambda(t) = \lambda\gamma_\epsilon(t)$ for all $t \in I$, we get

$$\|y - \gamma_\lambda(t_\lambda)\| \leq |\lambda| \delta + \epsilon,$$

which means that the curve γ_λ is $(|\lambda| \delta + \epsilon)$-dense in λB. Since ϵ is arbitrary, γ_λ is $|\lambda| \delta$-dense in λB and then, by (2), $\phi_d(\lambda B) \leq |\lambda| \delta$. Finally, by supposing that the above inequality is strict, let μ be a number so that $\phi_d(\lambda B) < \mu < |\lambda| \delta$; then, by (2), determine a μ-dense curve in λB, γ_μ, so for every $x \in B$ there exists $t \in I$ such that

$$\|\lambda x - \gamma_\mu(t)\| \leq \mu.$$

By dividing the above inequality by $|\lambda|$ we have $\left\|x - \frac{1}{\lambda}\gamma_\mu(t)\right\| \leq \frac{\mu}{|\lambda|}$. That means that the curve $\frac{1}{\lambda}\gamma_\mu(t)$ is $\frac{\mu}{|\lambda|}$-dense in B and then, by (2), $\phi_d(B) \leq \frac{\mu}{|\lambda|}$. But by the choice of μ, one deduces that $\phi_d(B) \leq \frac{\mu}{|\lambda|} < \delta$, which is a contradiction since $\delta := \phi_d(B)$. Hence $\phi_d(\lambda B) = |\lambda| \delta$ and consequently the semi-homogeneity of ϕ_d follows.

5) The invariance for translation is immediate if we take into account that if a curve γ is α-dense in B, given a vector x, the curve $\gamma_x(t) := x + \gamma(t)$ is also α-dense in the set $x + B$. Conversely, if γ_x is an α-dense curve in $x + B$, the curve $\gamma(t) := -x + \gamma_x(t)$ is also α-dense in B. $\square$

By recalling that the Kuratowski MNC [3, p. 20] is defined by

$$\kappa(B) := \inf\{\epsilon > 0 : B \text{ is covered by finitely many sets with diameter} \leq \epsilon\},$$

in the next result we find a relation between κ and ϕ_d.

Theorem 2.7. *Let (E, d) be a complete metric space, κ the Kuratowski MNC and ϕ_d the degree of nondensifiability on the family $\mathcal{B}$ of bounded sets of (E, d). Then*

$$\kappa(B) \leq 2\phi_d(B), \text{ for all } B \in \mathcal{B}. \tag{12}$$

Furthermore, inequality (12) is the best possible in infinite dimensional Banach spaces.

Proof. Let B be a bounded set and $\delta := \phi_d(B)$. Then, by (2), given $\epsilon > 0$ there exists an α-dense curve γ_α in B with density $\alpha = \delta + \frac{\epsilon}{4}$. Since $\gamma_\alpha(I)$ is compact, given $\frac{\epsilon}{4}$ there is a finite set $F = \{y_i \in \gamma_\alpha(I) : i = 1, ..., n\}$ such that

$$\gamma_\alpha(I) \subset \cup_{i=1}^n B\left(y_i, \frac{\epsilon}{4}\right).$$

By α-density, for any $x \in B$ there exists $y \in \gamma_\alpha(I)$ such that $d(x, y) \leq \alpha$. Then from the triangular inequality we get

$$B \subset \cup_{i=1}^n B\left(y_i, \delta + \frac{\epsilon}{2}\right),$$

which means that B is covered by finitely many set with diameter $\leq 2\delta + \epsilon$. Consequently (12) follows.

About the second part of the theorem, consider the unit closed ball B_{l^∞} of the space l^∞ of all real bounded sequences with the supremum norm. Thus we have $\kappa(B_{l^\infty}) = 2$ and $\chi(B_{l^\infty}) = 1$, by virtue of [3, Theorem 2.5, p. 23]. On the other hand, from (5), we get $\chi(B_{l^\infty}) = \phi_d(B_{l^\infty}) = 1$ and then

$$\kappa(B_{l^\infty}) = 2\phi_d(B_{l^\infty}) = 2.$$

This completes the proof. $\square$

Remark 2.8. The set B defined in (6) is so that $\phi_d(B) \geq 1$ but, from the compactness of B, one has $\kappa(B) = 0$. Therefore, in general, a relation of type

$$\phi_d(B) \leq \kappa(B) \leq 2\phi_d(B), \text{ for all } B \in \mathcal{B}$$

is false.

3. Degree of convex nondensifiability in Banach spaces

From now on $(X, \|\cdot\|)$ will be an infinite dimensional Banach space and, as usual, $\mathrm{conv}(B)$ will denote the convex hull of a set $B \subset X$, while $\bar{B}$ will mean the closure of B. $\mathcal{B}$ will be the family of bounded subsets of X. We recall that the closed unit ball of X is denoted by B_X.

By means of the notion of degree of nondensifiability ϕ_d given in Definition 2.2, we now define the concept of degree of convex nondensifiability of a bounded set of a Banach space $(X, \|\cdot\|)$.

Definition 3.1. The degree of convex nondensifiability is a map $\Phi : \mathcal{B} \longrightarrow [0, +\infty)$ defined as

$$\Phi(B) := \sup\{\phi_d(\mathrm{conv}(A)) : A \subset \overline{B}\}, \tag{13}$$

for every $B \in \mathcal{B}$.

Firstly, we observe that Φ is well defined because for every bounded set B of a Banach space X, the closure set $\overline{B}$ is also bounded. Then every subset $A \subset \overline{B}$ is bounded and, by [11, Lemma B, p. 60], $\mathrm{conv}(A)$ is also bounded, so, in particular, $\mathrm{conv}(\overline{B})$ is bounded. Choose a real $M > 0$ such that

$$B \subset MB_X \ \text{ hence } \ \mathrm{conv}(\overline{B}) \subset MB_X.$$

Therefore, for any $A \subset \overline{B}$, given an arbitrary curve γ in A, for every $x \in A$ we get

$$\|x - \gamma(t)\| \leq \|x\| + \|\gamma(t)\| \leq 2M, \ \text{ for any } t \in [0,1]\,.$$

That implies that any curve γ in A is $2M$-dense, so by (2),

$$\phi_d(\mathrm{conv}(A)) \leq 2M \ \text{ for all } A \subset \overline{B},$$

consequently Φ is well defined.

Secondly, we emphasize that, in general, Φ and ϕ_d are different as we point out by means of the following example.

Example 3.2. Let B_{L_1} be the closed unit ball of L_1. Then $\phi_d(B_{L_1}) = 1$, but $\Phi(B_{L_1}) = 2$.

Indeed, by (5), B_{L_1} satisfies $\phi_d(B_{L_1}) = 1$, so the first part of the example follows. To prove the second part, we firstly observe that for any $A \subset B_X$, the closed unit ball of any Banach space X, an arbitrary curve $\gamma : I = [0,1] \longrightarrow X$ with $\gamma(I) \subset \mathrm{conv}(A) \subset B_X$ satisfies

$$\|x - \gamma(t)\| \leq \|x\| + \|\gamma(t)\| \leq 1 + 1 = 2, \ \text{ for any } x \in \mathrm{conv}(A) \text{ and all } t \in [0,1]\,,$$

which means that γ is 2-dense in $\mathrm{conv}(A)$. Then, by (2), $\phi_d(\mathrm{conv}(A)) \leq 2$ for all $A \subset B_X$. Hence, by (13), $\Phi(B_X) \leq 2$ and in particular we get

$$\Phi(B_{L_1}) \leq 2.$$

On the other hand, the set

$$\mathcal{D} := \left\{ f \in L_1[0,1] : f \geq 0 \text{ and } \int_I f(x)dx = 1 \right\}$$

satisfies, by (9), $\phi_d(\mathcal{D}) = 2$. Then as $\mathcal{D}$ is a convex set contained in B_{L_1}, from (13), $\Phi(B_{L_1}) \geq 2$. Hence we deduce that

$$\Phi(B_{L_1}) = 2$$

and consequently the example follows.

A relevant property of Φ is showed in the following result.

Theorem 3.3. *The degree of convex nondensifiability Φ defined in* (13) *is an MNC on the bounded sets of any Banach space X.*

Proof. Let $\mathcal{B}$ be the family of bounded subsets of X. Firstly we claim that Φ is monotone on $\mathcal{B}$ with respect to the inclusion. That is, given B, $C \in \mathcal{B}$ with $B \subset C$, then $\Phi(B) \leq \Phi(C)$. Indeed, by (13),

$$\phi_d(\text{conv}(A)) \leq \Phi(C) \text{ for all } A \subset \overline{C}.$$

Then, since $\overline{B} \subset \overline{C}$, in particular we have

$$\phi_d(\text{conv}(A)) \leq \Phi(C), \text{ for all } A \subset \overline{B},$$

so, by (13), we get

$$\Phi(B) := \sup\{\phi_d(\text{conv}(A)) : A \subset \overline{B}\} \leq \Phi(C).$$

Assume that $B \in \mathcal{B}$ is precompact. Then, since X is complete, $\overline{B}$ is compact. Then $\text{conv}(\overline{B})$ is also compact by virtue of [11, Theorem C, p. 61]. Therefore, given any subset $A \subset \overline{B}$, since $\text{conv}(A) \subset \text{conv}(\overline{B})$, one deduces that $\text{conv}(A)$ is precompact and convex, so $\text{conv}(A) \in \mathcal{B}_a$, the family of bounded and arc-connected sets of X. Noticing ϕ_d is regular on $\mathcal{B}_a$, by Proposition 2.6, we have

$$\phi_d(\text{conv}(A)) = 0 \text{ for all } A \subset \overline{B}$$

and, from (13), that means that

$$\Phi(B) = 0.$$

Conversely, let $B \in \mathcal{B}$ be so that $\Phi(B) = 0$. Then, by (13), $\phi_d(\text{conv}(A)) = 0$ for any $A \subset \overline{B}$, so, in particular, $\phi_d(\text{conv}(B) = 0$, which means that $\text{conv}(B)$ is densifiable. Now, from [20, Proposition 1], $\text{conv}(B)$ is precompact which implies that B is also precompact. Then the regularity of Φ is proved.

That Φ is invariant under closure is an immediate consequence of the invariance under closure of ϕ_d.

About the semi-additivity, let B_1, B_2 be bounded sets of X. Firstly, we claim that for any subsets $A_1 \subset \overline{B_1}$, $A_2 \subset \overline{B_2}$ one has

$$\phi_d(\mathrm{conv}(A_1 \cup A_2)) \leq \max\{\phi_d(\mathrm{conv}(A_1)), \phi_d(\mathrm{conv}(A_2))\}. \tag{14}$$

Indeed, by denoting $\delta_1 := \phi_d(\mathrm{conv}(A_1))$ and $\delta_2 := \phi_d(\mathrm{conv}(A_2))$, from (2), given $\epsilon > 0$ there exist two curves γ_1 and γ_2 in $\mathrm{conv}(A_1)$, $\mathrm{conv}(A_2)$, respectively, such that

$$\mathrm{conv}(A_1) \subset \gamma_1(I) + (\delta_1 + \epsilon)B_X \tag{15}$$

and

$$\mathrm{conv}(A_2) \subset \gamma_2(I) + (\delta_2 + \epsilon)B_X. \tag{16}$$

Noticing $\mathrm{conv}(A_1) \cup \mathrm{conv}(A_2) \subset \mathrm{conv}(A_1 \cup A_2)$, it follows

$$\gamma_1(I) \cup \gamma_2(I) \subset \mathrm{conv}(\gamma_1) \cup \mathrm{conv}(\gamma_2) \subset \mathrm{conv}(A_1) \cup \mathrm{conv}(A_2) \subset \mathrm{conv}(A_1 \cup A_2)$$

and then

$$\mathrm{conv}(\gamma_1(I) \cup \gamma_2(I)) \subset \mathrm{conv}(A_1 \cup A_2).$$

Since $\gamma_1(I) \cup \gamma_2(I)$ is compact, by [11, Theorem C, p. 61], $\mathrm{conv}(\gamma_1(I) \cup \gamma_2(I))$ is a compact and convex set of X, so is, in particular, compact, connected and locally connected. Then, by Hahn-Mazurkiewicz's theorem [21, Theorem 6.8], there exists a continuous map $\gamma : I \longrightarrow X$ such that

$$\gamma(I) = \mathrm{conv}(\gamma_1(I) \cup \gamma_2(I)). \tag{17}$$

Now, we see that γ is a δ-dense curve in $\mathrm{conv}(A_1 \cup A_2)$, where $\delta := \max\{\delta_1, \delta_2\}$. Indeed, let x be a point of $\mathrm{conv}(A_1 \cup A_2)$, then

$$x := \sum_{i=1}^{m} \lambda_i x_i, \;\; \lambda_i \geq 0, \; \sum_{i=1}^{m} \lambda_i = 1 \tag{18}$$

with $x_1, \ldots, x_r \in A_1$ and $x_{r+1}, \ldots, x_m \in A_2$ for some $1 < r < m$. Hence, from (15) and (16), there exist $t_1, \ldots, t_m \in I$ such that

$$\|x_i - \gamma_1(t_i)\| \leq \delta_1 + \epsilon \;\text{ for all } i = 1, \ldots, r$$

and

$$\|x_i - \gamma_2(t_i)\| \leq \delta_2 + \epsilon \;\text{ for all } i = r+1, \ldots, m.$$

Now by defining

$$y := \sum_{i=1}^{r} \lambda_i \gamma_1(t_i) + \sum_{i=r+1}^{m} \lambda_i \gamma_2(t_i)$$

it follows that $y \in \mathrm{conv}(\gamma_1(I) \cup \gamma_2(I))$. Taking into account (17), there exists $t \in I$ such that $\gamma(t) = y$. Then, from (18), we have

$$\begin{aligned}\|x - y\| \leq & \sum_{i=1}^{r} \lambda_i \|x_i - \gamma_1(t_i)\| + \sum_{i=r+1}^{m} \lambda_i \|x_i - \gamma_2(t_i)\| \\ & \leq (\delta_1 + \epsilon) \sum_{i=1}^{r} \lambda_i + (\delta_2 + \epsilon) \sum_{i=r+1}^{m} \lambda_i \leq \delta + \epsilon,\end{aligned}$$

which proves that γ is a $(\delta+\epsilon)$-dense curve in $\mathrm{conv}(A_1 \cup A_2)$. Then, since ϵ is arbitrary, γ is a δ-dense curve in $\mathrm{conv}(A_1 \cup A_2)$ as we said. Therefore, by (2), $\phi_d(\mathrm{conv}(A_1 \cup A_2)) \leq \delta$ and, in consequence, inequality (14) holds as we claimed.

On the other hand, from (13), given any $\epsilon > 0$, there exists $A_\epsilon \subset \overline{B_1 \cup B_2} = \bar{B}_1 \cup \bar{B}_2$ such that

$$\Phi(B_1 \cup B_2) \leq \phi_d(\mathrm{conv}(A_\epsilon)) + \epsilon. \tag{19}$$

By defining $A_1 := \bar{B}_1 \cap A_\epsilon$ and $A_2 := \bar{B}_2 \cap A_\epsilon$ it follows $A_\epsilon = A_1 \cup A_2$. Then, since $A_1 \subset \bar{B}_1$ and $A_2 \subset \bar{B}_2$, from (14) and (13), we have

$$\phi_d(\mathrm{conv}(A_\epsilon)) \leq \max\{\phi_d(\mathrm{conv}(A_1)),\, \phi_d(\mathrm{conv}(A_2))\} \leq \max\{\Phi(B_1),\, \Phi(B_2)\},$$

which means, noticing (19), that

$$\Phi(B_1 \cup B_2) \leq \max\{\Phi(B_1),\, \Phi(B_2)\} + \epsilon.$$

Thus, since ϵ is arbitrary, we get

$$\Phi(B_1 \cup B_2) \leq \max\{\Phi(B_1),\, \Phi(B_2)\}. \tag{20}$$

Now, by assuming that the inequality (20) is strict and supposing, for instance, that $\max\{\Phi(B_1),\, \Phi(B_2)\} = \Phi(B_1)$, from the monotonicity of Φ, we have

$$\Phi(B_1) \leq \Phi(B_1 \cup B_2) < \max\{\Phi(B_1),\, \Phi(B_2)\} = \Phi(B_1),$$

a contradiction. Thus (20) is an equality, so the semi-additivity of Φ follows.

About the semi-homogeneity of Φ, from (13) and noticing that from Proposition 6 ϕ_d is semi-homogene, by assuming $\lambda \neq 0$, we put

$$|\lambda|\, \Phi(B) := |\lambda| \sup\{\phi_d(\mathrm{conv}(A)) : A \subset \overline{B}\} = \sup\{\phi_d(\lambda\, \mathrm{conv}(A)) : A \subset \overline{B}\}.$$

Then, taking into acount that $\mathrm{conv}(\lambda A) = \lambda\, \mathrm{conv}(A)$ and $\overline{\lambda A} = \lambda \overline{A}$ for any subset A of X, we obtain

$$\begin{aligned} &\sup\{\phi_d(\lambda\, \mathrm{conv}(A)) : A \subset \overline{B}\} = \sup\{\phi_d(\mathrm{conv}(\lambda A)) : A \subset \overline{B}\} \\ = \;&\sup\{\phi_d(\mathrm{conv}(\lambda A)) : \lambda A \subset \lambda\overline{B}\} = \sup\{\phi_d(\mathrm{conv}(\lambda A)) : \lambda A \subset \overline{\lambda B}\} = \Phi(\lambda B). \end{aligned}$$

This proves that $|\lambda|\, \Phi(B) = \Phi(\lambda B)$ for $\lambda \neq 0$. Since the equality trivially follows when $\lambda = 0$, Φ is semi-homogene.

Finally, it remains to prove the invariance under translations. We need to note the two immediate properties, $\mathrm{conv}(x + A) = x + \mathrm{conv}(A)$ and $\overline{x + A} = x + \overline{A}$, for any vector x and any subset A. Then, from (13) and taking into account that from Proposition 2.6 ϕ_d is invariant under translations, we have

$$\begin{aligned} \Phi(x + B) = \;&\sup\{\phi_d(\mathrm{conv}(A)) : A \subset \overline{x + B}\} \\ = \;&\sup\{\phi_d(\mathrm{conv}(A)) : A \subset x + \overline{B}\} \\ = \;&\sup\{\phi_d(\mathrm{conv}(A)) : -x + A \subset \overline{B}\} \\ = \;&\sup\{\phi_d(x + \mathrm{conv}(-x + A)) : -x + A \subset \overline{B}\} \\ = \;&\sup\{\phi_d(\mathrm{conv}(-x + A)) : -x + A \subset \overline{B}\} = \Phi(B). \end{aligned}$$

This completes the proof. $\square$

Now, with the aim to prove that the degree of convex nondensifiability dominates any MNC and to compare this result with the corresponding theorem for a regular weak MNC, briefly σMNC, we reproduce here the definition of De Blasi's measure [8] and Banas-Rivero's theorem [5, Theorem 4].

Definition 3.4. On the bounded subsets of a Banach space X, the measure of De Blasi is a map $\beta : \mathcal{B} \longrightarrow [0, +\infty)$ defined as

$$\beta(B) := \inf\{\epsilon > 0 : B \subset K + \epsilon B_X, \text{ for some relatively weakly compact } K \text{ of } X\}. \tag{21}$$

The measure of De Blasi is an upper bound to any σMNC defined on the class $\mathcal{B}$ of bounded sets of a Banach space X in the following sense:

Theorem 3.5 (Banas-Rivero). *Let $\phi : \mathcal{B} \longrightarrow [0, +\infty)$ be any σMNC. Thus,*

$$\phi(B) \leq \phi(B_X)\beta(B) \tag{22}$$

for any $B \in \mathcal{B}$.

The validity of the preceding theorem cannot be extended to a general MNC as proves the following example.

Example 3.6. In the space $L_1[0, 1]$ the inequality (22) fails by taking $\phi = \chi$, the measure of noncompactness of Hausdorff, and $B := \{r_n : n \in \mathbb{N}\}$, the set of Rademacher functions [3, p. 32].

Indeed, for every n the norm $\|r_n\| = 1$ [3, p. 185], so B is a bounded set of $L_1[0, 1]$. Moreover, by [2, Lemma 6.3.2], the sequence $(r_n)_{n\geq 1}$ weakly converges to 0. Thus $\{r_n : n \in \mathbb{N}\}$ is a weakly compact set of $L_1[0, 1]$. Therefore, from (21), $\beta(B) = 0$. Now, noticing [3, p. 185], one has $\chi(B) = 1$, which proves that (22) fails and then the example follows.

Now we are ready to prove that the degree of convex nondensifiability satisfies an inequality similar to (22) for any MNC in any Banach space.

Theorem 3.7. *Let $(X, \|\cdot\|)$ be a Banach space, $\mathcal{B}$ the family of bounded subsets of X and Φ the degree of convex nondensifiability defined in (13). Then Φ dominates any MNC ϕ, i.e.*

$$\phi(B) \leq \phi(B_X)\Phi(B), \quad \text{for all } B \in \mathcal{B}. \tag{23}$$

Furthermore for $\phi = \chi$, the Hausdorff MNC, the above inequality is the best possible in Banach spaces of infinite dimension.

Proof. Given a bounded set B, let $\delta := \phi_d(B)$. Then taking into account (2), given $\epsilon > 0$ there exists a curve $\gamma_\epsilon : I \longrightarrow B$ such that $B \subset \gamma_\epsilon(I) + (\delta + \epsilon)B_X$. Then noticing any MNC, even in metric spaces, is monotone [3, p. 19], we have

$$\phi(B) \leq \phi(\gamma_\epsilon(I) + (\delta + \epsilon)B_X). \tag{24}$$

Now we claim that

$$\phi(\gamma_\epsilon(I) + (\delta+\epsilon)B_X) \leq (\delta+\epsilon)\phi(B_X). \tag{25}$$

Indeed, $\gamma_\epsilon(I)$ is compact, so precompact. Then, for arbitrary small η, $\gamma_\epsilon(I)$ can be covered by finitely many closed balls $B(x_i, \eta)$, that is

$$\gamma_\epsilon(I) \subset \cup_{i=1}^n B(x_i, \eta) = \cup_{i=1}^n (x_i + \eta B_X).$$

Hence, again by monotonicity, it follows

$$\begin{aligned}\phi(\gamma_\epsilon(I) + (\delta+\epsilon)B_X) &\leq \phi(\cup_{i=1}^n (x_i + \eta B_X) + (\delta+\epsilon)B_X) \\ &= \phi(\cup_{i=1}^n (x_i + \eta B_X + (\delta+\epsilon)B_X)).\end{aligned} \tag{26}$$

By the semi-additivity and invariance under translations of ϕ, we have

$$\begin{aligned}\phi(\cup_{i=1}^n (x_i + \eta B_X + (\delta+\epsilon)B_X)) &= \max_{i=1,..n} \{\phi(x_i + \eta B_X + (\delta+\epsilon)B_X)\} \\ &= \phi(\eta B_X + (\delta+\epsilon)B_X).\end{aligned} \tag{27}$$

Now, since $\eta B_X + (\delta+\epsilon)B_X \subset (\eta+\delta+\epsilon)B_X$, by the monotonicity and semi-homogeneity of ϕ, we get

$$\phi(\eta B_X + (\delta+\epsilon)B_X) \leq \phi((\eta+\delta+\epsilon)B_X) = (\eta+\delta+\epsilon)\phi(B_X).$$

Then, from (26), (27) and noticing η is arbitrary, the inequality (25) follows as claimed. Therefore, according to (24), we obtain

$$\phi(B) \leq (\delta+\epsilon)\phi(B_X), \text{ for any } \epsilon > 0,$$

which means that $\phi(B) \leq \delta\phi(B_X)$. Now, since $\delta := \phi_d(B)$, we conclude that

$$\phi(B) \leq \phi(B_X)\phi_d(B), \text{ for all } B \in \mathcal{B}. \tag{28}$$

Let A be any subset of $\overline{B}$, by the monotonicity of ϕ, from (28) and noticing (13), we have

$$\phi(A) \leq \phi(\mathrm{conv}(A)) \leq \phi(B_X)\phi_d(\mathrm{conv}(A)) \leq \phi(B_X)\Phi(B).$$

In particular, by taking $A = B$, the inequality (23) follows.

About the second part of the theorem, let $B := \{e_n : n = 1, 2, ...\}$ be the standard basis of the Banach space c_0 of null sequences at infinity with the supremum norm and let us take $\phi = \chi$, the Hausdorff MNC. Then, as $\|e_i - e_j\| = 1$ for all $i \neq j$, and the distance from any infinite subset of B to any vector of c_0 is not smaller than 1, from (3), we have $\chi(B) = 1$. On the other hand, from (5), $\chi(B_X) = \phi_d(B_X) = 1$. Then, because (23), necessarily

$$\Phi(B) \geq 1. \tag{29}$$

Now we claim that $\Phi(B) = 1$. Indeed, we firstly note that B is closed because of the property $\|e_i - e_j\| = 1$ for all $i \neq j$. Let A be a subset of B and γ an arbitrary curve in $\text{conv}(A)$. Given any $x \in \text{conv}(A)$ and any $\gamma(t)$, $t \in I$, we write $x = \sum_{i=1}^{n} \alpha_i e_i$ and $\gamma(t) = \sum_{i=1}^{m} \beta_i e_i$ with α_i, $\beta_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{m} \beta_i = 1$. Then, since $|\alpha_i - \beta_j| \leq 1$ for any i, j, it follows that $\|x - \gamma(t)\| \leq 1$. That means that any curve γ in $\text{conv}(A)$ is 1-dense, so, by (2), $\phi_d(\text{conv}(A)) \leq 1$. Since this is valid for any $A \subset B = \overline{B}$, by (13), we obtain

$$\Phi(B) \leq 1,$$

which jointly with (29) allows us to conclude that $\Phi(B) = 1$, as claimed. Then, we have

$$\chi(B) = \chi(B_X) = \Phi(B) = 1,$$

which proves the sharpness of (23). The proof is now complete. □

Remark 3.8. The inequality (23) can be strict.

Indeed, by virtue of Example 3.2, in the Banach space $X = L_1$ the closed unit ball B_{L_1} satisfies

$$\phi_d(B_{L_1}) = 1; \qquad \Phi(B_{L_1}) = 2.$$

On the other hand, from (5), in any Banach space the Hausdorff MNC χ, satisfies

$$\chi(B_X) = \phi_d(B_X) = 1.$$

Hence, by taking $B = B_{L_1}$ and $\phi = \chi$, the left-side of (23) is equal to 1 whereas the right-side is equal to 2.

Corollary 3.9. *In infinite dimensional Banach spaces the degree of convex nondensifiability Φ and the Kuratowski and Hausdorff MNC's are comparable.*

Proof. If in (23) we take $\phi = \kappa$, the Kuratowski MNC, we have

$$\kappa(B) \leq \kappa(B_X)\Phi(B), \text{ for all } B \in \mathcal{B}.$$

On the other hand, in [15, Proposition 2] one proves that κ dominates any MNC, so in particular κ dominates Φ. Then

$$\Phi(B) \leq \Phi(B_X)\kappa(B), \text{ for all } B \in \mathcal{B}.$$

Hence, since $\kappa(B_X) = 2$ because of [3, Theorem 2.5, p. 23], by dividing the above inequalities by $\kappa(B_X)$, we get

$$\frac{1}{2}\kappa(B) \leq \Phi(B) \leq \Phi(B_X)\kappa(B), \text{ for all } B \in \mathcal{B},$$

which means that Φ and κ are comparable. Now, noticing [3, Theorem 2.7, p. 23], κ and χ, the Hausdorff MNC, are comparable. Therefore Φ is comparable with both MNC's, κ and χ. □

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