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On Constant Width Sets in Hilbert Spaces and Around

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In this paper we present some results concerning the family $\mathcal{CW}$ of sets with constant width and the family $\mathcal{DM}$ of diametrically maximal (or diametrically complete) sets. In particular we prove (Theorem 5.1) that $\mathcal{CW} = \mathcal{DM}$ in Hilbert spaces.

Keywords: Constant width, diametrically maximal sets, arcs

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1. Introduction

Convex bodies of constant width in Euclidean spaces have been studied for many years.

In Minkowskian spaces, the class of diametrically complete, or diametrically maximal sets, has been introduced and studied around one century ago.

It is well known that in Euclidean spaces of any dimension this class coincides with the class of convex sets of constant width. (See the definitions below).

For two-dimensional spaces these two classes again coincide, but already in three-dimensional (also "good", but not Euclidean) normed spaces in general the two classes are different.

These two notions can be extended to all infinite-dimensional Banach spaces; in this setting they have been considered only rather recently.

It is more or less known, or at least guessed, that in (infinite-dimensional) Hilbert spaces, for convex sets diametrical completeness is the same as constant width;

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the classical proof of this equivalence given in Euclidean spaces uses compactness arguments, so does not extend directly to the infinite-dimensional situation. This fact was pointed out by R. Phelps (see [9, p. 832]). In fact, a proof of this equivalence in Hilbert spaces is contained in a recent paper (see [15], also for a short history of the problem); but it is based on results utilizing a rather awkward notion.

Here we start from the classical proof concerning the equivalence in the Euclidean case, which has a clear geometrical flavour, and we extend it to Hilbert spaces. Also, we discuss the intermediate results, pointing out which among them hold in general and which have an essentially Hilbertian character. In particular, we discuss in details circular arcs, and the spaces where they satisfy certain properties.

After indicating in Section 2 the notations and definitions needed, in Section 3 we prove some preliminary results and properties of circular arcs. Section 4 deals with width and related notions. The main result is proved in Section 5. In Section 6 we discuss diametral points and we give some other results concerning sets belonging to our classes. Finally, in the last Section 7 we discuss some other general facts.

2. Definitions and notations

Let X be a Banach space of dimension at least two, and X^* its dual.

A *plane* is a two-dimensional linear variety (not necessarily containing the origin 0).

For $x \in X$ and $r \in R^+$, we set:

$$B(x,r) = \{y \in X : ||x-y|| \leq r\}; \qquad S(x,r) = \{y \in X : ||x-y|| = r\}.$$

We also set: $B_X = B(0,1)$; $S_X = S(0,1)$.

Given a nonempty bounded set A, let bdry(A) denote its boundary and $\delta(A)$ its diameter:

$\delta(A) = \sup\{||x-y|| : x, y \in A\}$.

We shall assume that the sets A, or C (this will usually denote a convex set), considered in the following, are always nonempty and bounded.

If $f \in S_{X^*}$, set

$$\alpha_f(A) = \sup_{x \in A} f(x); \qquad \beta_f(A) = \inf_{x \in A} f(x); \qquad w_f(A) = \alpha_f(A) - \beta_f(A);$$

$w_f(A)$ is called the *width of A in the direction of* f; we also define the *minimal width*, or *thickness* of A:

$$w(A) = \inf_{f \in S_{X^*}} w_f(A).$$

Similarly, we can define the *(maximal) width* of A:

$$W(A) = \sup_{f \in S_{X^*}} w_f(A).$$

We say that A has *constant width* if $w_f(A) = w(A)$ for all $f \in S_{X^*}$.

In general, in the literature, only convex bodies of constant width are considered: in fact, the above property for a set does not imply its convexity (think of a ball without the origin).

Recall that in infinite-dimensional spaces convex sets of constant width can have empty interior (see [8, Section 3]); so we shall consider the following class.

We denote by $\mathcal{CW}$ the class of closed convex sets of constant width.

A bounded set A is said to be *diametrically maximal*, or also *diametrically complete*, if

$$\delta(A \cup \{x\}) > \delta(A) \quad \text{for every } x \notin A.$$

An equivalent formulation for this property is the following (see for example [2, Proposition 3.1]):

$$A = \bigcap_{a \in A} B(a, \delta(A)).$$

We denote by $\mathcal{DM}$ the class of diametrically maximal sets (these sets are always closed and convex).

As a general reference concerning these notions, we still suggest the classical survey [3], together with Section 2 in [10].

If $u \in S_X$, set

$$l_u(A) = \sup\{||a_1 - a_2|| : \ a_1, a_2 \in A; \ a_1 - a_2 = \lambda u \text{ for some } \lambda \in R\};$$

this is called the *length of A in the direction of u*.

Also, set

$$l(A) = \inf_{u \in S_X} l_u(A).$$

3. Preliminary results; circular arcs

It is well known that $\mathcal{CW} \subset \mathcal{DM}$ always; the converse inclusion is true in Euclidean spaces: see for example [18, Section 7.6], where one can find (in English) more or less the original proof of this fact.

Proposition 3.1. *Let A be a closed, bounded set; then*

- $w(A) = \inf\{w_f(A) : f \in S_{X^*},\ f$ *attains its norm at* $S_X\}$;
- $W(A) = \sup\{w_f(A) : f \in S_{X^*},\ f$ *attains its norm at* $S_X\}$.

Proof. It is enough to recall that every Banach space is subreflexive (Bishop-Phelps Theorem): the set of norm attaining functionals is dense in X^*. $\square$

Given a bounded set A, we say that $x \in A$ is *diametral* if $\sup_{a\in A} ||x-a|| = \delta(A)$.

Lemma 3.2. *Let $A \in \mathcal{DM}$; then every $x \in \mathrm{bdry}(A)$ is diametral.*

Proof. Trivial: in fact, $\sup_{a\in A} ||x-a|| = \delta(A) - \varepsilon$ ($\varepsilon > 0$) for some $x \in \mathrm{bdry}(A)$ would imply that $B(x, \varepsilon)$ could be added to A without increasing its diameter. □

Definition. Let $c \in X$, $r > 0$, and $a, b \in S(c, r)$; a, b, c not collinear. Set $z_\lambda = \lambda a + (1-\lambda)b$. We define the *(minor) circular arc* (of $S(c, r)$) as

$$\overset{\frown}{ab}_{(c,r)} = \left\{ y = c + r\frac{z_\lambda - c}{||z_\lambda - c||} \;:\; 0 \le \lambda \le 1 \right\}.$$

Definition. We say that X has the *arc inclusion property* (AIP) if given $a, b \in S_X$, any circular arc $\overset{\frown}{ab}_{(c,r)}$ with $r \ge 1$ is contained in B_X.

Remark 3.3. Let X have the AIP; then, any circular arc $\overset{\frown}{ab}_{(c,r)}$ with $a, b \in B_X$ and $r \ge 1$ is contained in B_X: in fact, a point in such an arc not contained in B_X would also belong to an arc with end points on S_X. Moreover, if A is an intersection of balls of radii $\le r$ and $\overset{\frown}{ab}_{(c,r)}$ is a circular arc with $a, b \in A$, then $\overset{\frown}{ab}_{(c,r)} \subset A$.

The next fact is known: see for example [18, Theorem 7.6.4.].

Lemma 3.4. *Hilbert spaces have the AIP.*

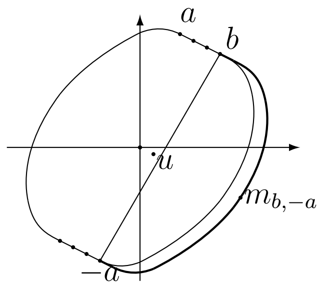
The last part of Section 2 in [16] contains some results on AIP. We shall now characterize Minkowskian planes with the AIP, precising in particular [16, Lemma 2.12].

We first prove the following general result.

Proposition 3.5. *If a space X satisfies the AIP, then it is strictly convex.*

Proof. Assume that X is not strictly convex. This means that there is a segment joining two different points a, b, which is contained in S_X. Set $u = (b-a)/3$: then we can see that $-a, b \in S_X$, but $\overset{\frown}{-ab}_{(u,1)} \notin B_X$. In fact: note first that $-a, u, b$ are not collinear; moreover, let $m_{b,-a} = u + \frac{1/2(-a)+(1/2)b-u}{||(1/2)(-a)+(1/2)b-u||}$. Clearly $m_{b,-a} \in \overset{\frown}{-ab}_{(u,1)}$ but $||m_{b,-a}|| = 1 + ||u|| > 1$.

Thus X lacks the AIP. □



Proposition 3.6. *Let* $\dim(X) = 2$*; then* X *has the AIP if and only if* X *is strictly convex.*

Proof. The "only if" part is contained in the previous proposition.

"If" part. Assume that X is strictly convex, $\dim(X) = 2$. Consider $a, b \in S_X$ and let $\overset{\frown}{ab}_{(c,r)}$ be a circular arc with $r \geq 1$.

We have: $r = ||c - a|| \leq ||c|| + 1$; $||c|| \leq ||c - a|| + ||a|| = r + 1$.

If $c = \mathit{0}$, then $r = 1$ and there is nothing to prove.

If $c \neq \mathit{0}$, then according to [11, Proposition 14], the two spheres $S(\mathit{0}, 1)$ and $S(c, r)$ meet in at most two points, in this case a and b. These points split $S(c, r)$ into two components: one of them (closed) is inside $B(\mathit{0}, 1)$, the other one is into its complement.

To be convinced that one of the two components is inside $B(\mathit{0}, 1)$, we indicate a point of $S(c, r)$ which is in $B(\mathit{0}, 1)$.

Take, on the half-line from c and going to $\mathit{0}$, a point c' such that $||c - c'|| = r$.

If $||c - \mathit{0}|| \leq r$, then $r = ||c - \mathit{0}|| + ||\mathit{0} - c'||$ and so $||\mathit{0} - c'|| = r - ||c - \mathit{0}|| \leq r - (r - 1) = 1$.

If $||c - \mathit{0}|| > r$, then $||c' - \mathit{0}|| = ||c - \mathit{0}|| - ||c - c'|| = ||c|| - r \leq 1$.

In any case, $c' \in B_X$. □

Remark 3.7. Note that if X is strictly convex, then for $r > 1$ ($c \neq \mathit{0}$), the points in $\overset{\frown}{ab}_{(c,r)}$ different from a and b lie in the interior of B_X.

Corollary. *Let* $\dim(X) = 2$*. Then given* $a, b \in S_X$*, any circular arc* $\overset{\frown}{ab}_{(c,r)}$ *with* $r > 1$ *is contained in* B_X*.*

Proof. It is enough to observe that given X, there is a sequence of strictly convex unit balls converging to B_X. □

The next example shows that general Minkowskian spaces need not have the AIP.

Example 3.8. Let $X = R^3$ with the max norm. Let $1 < r = 3/2$; $c = (-\frac{1}{2}, 0, -\frac{1}{2})$; $a = (1,1,0)$; $b = (0,1,1)$. Consider the circular arc with center c and radius $3/2$, which contains a and b; both a and b also belong to $S(0,1)$. Take $y = c + \frac{3}{4}(a-c) + \frac{3}{4}(b-c) = (-\frac{1}{2}, 0, -\frac{1}{2}) + \frac{3}{4}\big[(\frac{3}{2}, 1, \frac{1}{2}) + (\frac{1}{2}, 1, \frac{3}{2})\big] = (-\frac{1}{2}, 0, -\frac{1}{2}) + (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}) = (1, \frac{3}{2}, 1)$; thus $y \notin B(0,1)$.

Remark 3.9. Since it is possible to renorm R^3 with a strictly convex (and smooth) norm, arbitrarily near to the max norm, the previous example shows that when $\dim(X) > 2$, the "if" portion of the Proposition 3.6 fails (and this, even if the inequality for the radius of the arc is strict).

We conjecture that the following is true.

Conjecture. If $\dim(X) \geq 3$, then X has the AIP (if and) only if X is Hilbertian.

4. Length, width, thickness

Given a point $u \in X$, $u \neq 0$, let $J(u)$ be the set of norm-one functionals for which $f(u) = ||u||$; according to Hahn-Banach theorem, this set is always nonempty.

Take a set A; let $||u|| = 1$ and $f \in J(u)$ $(f(u) = 1)$. Then $a - b = \lambda u,\ a, b \in A$, implies that $||a - b|| = |\lambda| = |f(a-b)| \leq \alpha_f(A) - \beta_f(A) = w_f(A)$, so:

$$l_u(A) \leq w_f(A).$$

Remark 4.1. Note that even for a convex set C, we can have (for $f \in J(u)$) : $l_u(C) < w_f(C)$; moreover, if also $f \in J(u')$ with $u \neq u'$, in general we can have $l_u(C) \neq l_{u'}(C)$.

Lemma 4.2. *$l(A) \leq w(A)$ always.*

Proof. Let $u \in S_X$; if we take any $f \in J(u)$, then $l_u(A) \leq w_f(A)$. By passing to the inf and using Proposition 3.1 we obtain the asserted inequality. □

Remark 4.3. It is clear that in general, in any space, if A is not convex we can have $l(A) < w(A)$.

Theorem 4.4. *Let C be a bounded convex (closed) set. Moreover, assume that one of the following assumptions holds: X is reflexive, or C has nonempty interior.*

Then $l(C) = w(C)$.

Proof. In view of the previous result, we must prove that we always have $l(C) \geq w(C)$.

Take $\varepsilon > 0$. Let $u \in S_X$ give a direction such that $l_u(C) < l(C) + \varepsilon$.

Now set $z_\varepsilon = (l_u(C) + \varepsilon)u$ and $T : z \to z + z_\varepsilon$.

We *claim* that $C \cap T(C) = \emptyset$ (for a related fact see [6, (1.6)]).

In fact $C \cap T(C) \neq \emptyset$ would imply the existence of $x \in C$ such that $x = Ty = y + z_\varepsilon$ for some $y \in C$. Therefore $x - y = \lambda u$ for $\lambda = l_u(C) + \varepsilon$; $||x - y|| = ||z_\varepsilon|| = l_u(C) + \varepsilon$, a contradiction proving the claim.

Thus there exists a hyperplane, say $\Pi = g^{-1}(c)$ ($g \in S_{X^*}$), separating C and $T(C)$: $g(x) \leq c \leq g(z)$ for all $x \in C$, $z \in T(C)$.

So $y \in C$ implies $y + z_\varepsilon \in T(C) \Longrightarrow g(y + z_\varepsilon) \geq c \Longrightarrow g(y) \geq c - g(z_\varepsilon)$.

Thus $c - g(z_\varepsilon) \leq g(y) \leq c \; \forall y \in C$, and then $\alpha_g(C) - \beta_g(C) \leq c - (c - g(z_\varepsilon)) \leq ||z_\varepsilon|| = l_u(C) + \varepsilon$.

Therefore $w(C) \leq w_g(C) \leq l_u(C) + \varepsilon < l(C) + 2\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, this proves that $w(C) \leq l(C)$, as claimed; so the conclusion. $\square$

The previous result has been proved, for convex bodies in Minkowskian spaces, for example in [1, Th. 3] and in [6, (1.5)]; the last paper, also contains, for the same setting, Proposition 4.5 below, and the fact that a constant width set C satisfies the equality in Remark 4.6 below (see [6, (1.5) and (1.4)]).

For the sake of completeness, we prove the following simple result, that will be used later (for a proof of the first equality, see [4, Lemma 3.5]).

Proposition 4.5. *We always have:*

$$W(A) = \delta(A) = \sup_{u \in S_X} l_u(A).$$

Proof. The equality $\delta(A) = \sup_{u \in S_X} l_u(A)$ is trivial.

Let $\varepsilon > 0$. Take $x, y \in A$ with $||x - y|| > \delta(A) - \varepsilon$. If $f \in J(x - y)$, then $w_f(A) = \alpha_f(A) - \beta_f(A) \geq |f(x - y)| = ||x - y|| > \delta(A) - \varepsilon$; this implies $W(A) \geq \delta(A)$.

Conversely, given $\varepsilon > 0$, choose $f \in S_{X*}$ such that $w_f(A) \geq W(A) - \varepsilon$; then there exist x, $y \in A$ such that $f(x) - f(y) > w_f(A) - \varepsilon \geq W(A) - 2\varepsilon$. Thus $\delta(A) \geq ||x - y|| \geq f(x) - f(y) > W(A) - 2\varepsilon$, hence $\delta(A) \geq W(A)$. This concludes the proof. $\square$

Remark 4.6. According to the previous result, a (closed, bounded convex) set C is of constant width if and only if $w_f(C) = \delta(C)$ for every $f \in S_X$; Theorem 4.4 shows that, if X is reflexive, this is also equivalent to

$$l_u(C) = \delta(C) \text{ for all } u \in S_X.$$

But this is true in every space. In fact, the last condition implies (by Lemma 4.2) $\delta(C) = l_u(C) \leq w(C) \leq W(C) \leq \delta(C)$. Conversely, if $C \in \mathcal{CW}$, then according to [14, Proposition 1.1] the closure of $C - C := C + (-C)$ coincides with $B(0, \delta(C))$, so $l_u(C) = \delta(C)$ for every $u \in S_X$.

5. Main result

We are proving now the main result, extending to Hilbert spaces the equality $\mathcal{CW} = \mathcal{DM}$.

Theorem 5.1. *In any Hilbert space, $\mathcal{CW} = \mathcal{DM}$.*

Proof. Since $\mathcal{CW} \subset \mathcal{DM}$ always, we must prove that in Hilbert spaces $\mathcal{DM} \subset \mathcal{CW}$.

Let $C \in \mathcal{DM}$. We can assume, eventually after a rescaling, that $\delta(C) = 1$.

Assume, by contradiction, that $C \notin \mathcal{CW}$. According to Theorem 4.4, $w(C) = l(C)$ while $C \notin \mathcal{CW}$ implies that $w(C) < 1$.

Set $w(C) = 1 - d$ $(0 < d \leq 1)$. Let $\varepsilon \in (0, d/3)$; take $f \in S_{X^*}$ such that $w_f(C) < 1 - d + \varepsilon$. Let $u \in S_X$ be a point where f assumes its norm; we have $l_u(C) \leq w_f(C)$. Now take $a, b \in C$ such that $a - b = \lambda u$ for some $\lambda > 0$ and $l_u(C) - \varepsilon < ||a - b||$; we can also assume that $a \in \mathrm{bdry}(C)$. We have

$$w_f(C) - \varepsilon < 1 - d = w(C) = l(C) \leq l_u(C) < ||a - b|| + \varepsilon \tag{1}$$

and therefore:

$$1 - d - \varepsilon < ||a - b|| \leq l_u(C) \leq w_f(C); \qquad w_f(C) - ||a - b|| < 2\varepsilon. \tag{1'}$$

Now we can assume that $a = \mathit{0}$ (so $f(b) < 0$); if

$$\Pi_2 = \{x \in X : f(x) = \beta_f(C)\ (\leq 0)\}; \qquad \Pi_1 = \{x \in X : f(x) = \alpha_f(C)\ (\geq 0)\},$$

then we have (by (1')):

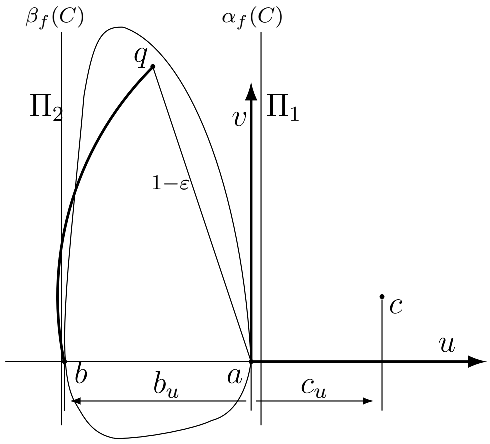
$$\mathrm{dist}(\mathit{0}, \Pi_1) + ||a - b|| \leq \alpha_f(C) - \beta_f(C) = w_f(C) \Longrightarrow \alpha_f(C) = \mathrm{dist}(\mathit{0}, \Pi_1) < 2\varepsilon;$$

$$\mathrm{dist}(b, \Pi_2) + ||a - b|| \leq \alpha_f(C) - \beta_f(C) = w_f(C) \Longrightarrow \mathrm{dist}(b, \Pi_2) < 2\varepsilon.$$

But $\sup_{c \in C} ||a - c|| = \delta(C) = 1$ (Lemma 3.2); so there exists $q \in C$ such that $||q|| = ||q - a|| = 1 - \varepsilon$.

Note that q, b, a cannot be collinear since $q = \lambda u$ for some $\lambda \in R$ would imply $||q - a|| \leq l_u(C) \leq w_f(C) < 1 - d + \varepsilon < 1 - \varepsilon$.

So we can reason on the 2-dimensional space Y generated by b and q. In Y we consider an orthonormal basis consisting of u, v, where $f(v) = 0$: then $b = b_u u$ with $b_u < 0$. Set $q = q_u u + q_v v$, and we can assume $q_v > 0$.



Take $c = c_u u + c_v v \in Y$ such that $||c - q|| = ||c - b|| = 1$.

We want to prove that $\operatorname{dist}(c, \Pi_2) < 1$. If $c_u \leq 0$, this is trivial; so let $c_u > 0$.

We have: $1 = ||c - q||^2 = ||c||^2 + ||q||^2 - 2(c_u q_u + c_v q_v) \geq$

$$\geq \begin{cases} ||c||^2 + (1-\varepsilon)^2 - 2c_v q_v & \text{if } q_u < 0 \\ ||c||^2 + (1-\varepsilon)^2 - 2c_u q_u - 2c_v q_v & \text{also if } q_u \geq 0. \end{cases}$$

In any case, since $q_u \leq \alpha_f(C) < 2\varepsilon$ ($q \in C$), we have $-2c_u q_u \geq -4\varepsilon c_u$, and then:

$$2c_v q_v \geq ||c||^2 + \varepsilon^2 - 2\varepsilon - 4\varepsilon ||c||.$$

Now observe that $t^2 - 4\varepsilon t$ strictly increases with t if $t > 2\varepsilon$: since $||c|| \geq ||b - c|| - ||b|| \geq 1 - w_f(C) > 1 - (1 - d + \varepsilon) = d - \varepsilon$ ($> 2\varepsilon$), we have:

$2c_v q_v > (d - \varepsilon)^2 + \varepsilon^2 - 2\varepsilon - 4\varepsilon(d - \varepsilon) > d^2 - 6d\varepsilon - 2\varepsilon$. But $q_v \leq ||q|| = 1 - \varepsilon < 1$; so, if $\varepsilon > \frac{d^4}{30}$, we obtain: $2c_v > 2c_v q_v > d^2 - \frac{d^5}{5} - \frac{d^4}{15} \geq \frac{11}{15} d^2 > 4\sqrt{\varepsilon}$.

Since $q_v > 0$, then $c_v > 0$; thus $c_v > c_v q_v > 2\sqrt{\varepsilon}$.

Therefore, since $1 = ||c - b||^2 = (c_u - b_u)^2 + (c_v)^2$, we have $1 > (c_u - b_u)^2 + 4\varepsilon$.

So $\operatorname{dist}(c, \Pi_2) = c_u - \beta_f(C) \leq c_u - b_u + 2\varepsilon < \sqrt{1 - 4\varepsilon} + 2\varepsilon < 1$.

This shows that the circular arc of center c and radius $1 = \delta(C)$, passing through b and q, is not contained in the strip between Π_1 and Π_2; so neither in C. But C contains b and q, so this gives a contradiction with the AIP (see Lemma 3.4). This proves the theorem. □

6. Diametral points

Given a set A and $x \in \operatorname{bdry}(A)$, we say that x is *strongly diametral* if there exists $a \in A$ such that $||x - a|| = \delta(A)$.

In any infinite-dimensional Banach space, given $\varepsilon \in (0, 1)$ there exists a symmetric, bounded closed and convex body C without diametral pairs, such that $(1 - \varepsilon)B_X \subset C \subset B_X$ (see [17]).

The next example shows that an infinite-dimensional space X can contain a set $C \in \mathcal{CW}$ with a point $x \in \mathrm{bdry}(C)$ such that x is not strongly diametral (compare with Lemma 3.2).

Example 6.1. Consider c_0; let C be the set of all sequences with all components in $[0,1]$: we have $C \in \mathcal{CW}$. If $x = (1/2, 1/3, 1/4, ..., 1/n, ...)$, then $\sup_{a \in C} ||x - a|| = 1 > ||x - y||$ for every $y \in C$.

We can observe that, in the previous example, if we consider the functional $f = (1/2, 1/4, ..., 1/2^n, ...)$, then $w_f(C)$ is not attained.

Other quantities are attained in reflexive spaces, as indicated by the next proposition.

Proposition 6.2. *Let C be a closed, bounded and convex set of a reflexive Banach spaces X. For every $f \in X^*$, $w_f(C)$ is attained; also, for every $u \in S_X$, $l_u(C)$ is attained.*

Proof. Let X be reflexive; thus C is weakly compact, and then so is $C - C$. Therefore any continuous linear functional f attains its supremum on $C - C$. We also have $w_f(C) = \sup f(C - C)$. Moreover $l_u(C) = \sup\{||x|| : x \in (C - C) \cap Ru\}$, which is compact. □

We conclude by proving the implication $C \in \mathcal{DM} \Longrightarrow C \in \mathcal{CW}$ under different assumptions; the next theorem is partially known: see [13, Proposition 2.1] (see also [10, p. 98]).

We say that a convex set C is *smooth* if for every $x \in \mathrm{bdry}(C)$ there is at most one hyperplane supporting C at x.

Theorem 6.3. *If C is smooth and every point $x \in \mathrm{bdry}(C)$ is strongly diametral, then $C \in \mathcal{CW}$. In particular, in finite-dimensional spaces, smooth diametrically complete sets are of constant width.*

Proof. Suppose that some set C, satisfying the assumptions, is not in $\mathcal{CW}$: so, by Proposition 4.5, $w(C) < \delta(C)$. Since support functionals of C are dense, there also exists $f \in S_{X^*}$, supporting C at some point x, such that $\alpha_f(C) - \beta_f(C) < \delta(C)$.

Take $y \in C$ such that $||y - x|| = \delta(C)$, then let $g \in J(y - x)$. We have: $\delta(C) \geq w_g(C) \geq g(y) - g(x) = ||y - x|| = \delta(C)$, so $\alpha_g(C) - \beta_g(C) = w_g(C) = \delta(C)$. But then g supports C at x: in fact $g(c) \geq g(x)$ for all $c \in C$ since $g(c) < g(x)$ for some $c \in C$ would imply $\delta(C) \geq ||y - c|| \geq g(y) - g(c) > g(y) - g(x) = \delta(C)$.

Then the smoothness of C implies $g = f$: so we have a contradiction, proving that $C \in \mathcal{CW}$. □

7. Final considerations

In this paper we have studied the condition $\mathcal{CW} = \mathcal{DM}$.

Denote by $\mathcal{B}$ the class of closed balls. Clearly, $\mathcal{B} \subset \mathcal{CW}$.

There are spaces where $\mathcal{DM} = \mathcal{B}$, so where the three classes $\mathcal{DM}$, $\mathcal{CW}$ and $\mathcal{B}$ coincide: these have been characterized as the $\mathcal{P}_1$ spaces (see [5]).

The equality $\mathcal{CW} = \mathcal{B}$ holds in $C(K)$ spaces (see [8, Theorem 4.3]), as well as in some finite-dimensional spaces (see [7, Proposition 3.8]).

It seems to be rather difficult to characterize the spaces where $\mathcal{CW} = \mathcal{B}$, or $\mathcal{CW} = \mathcal{DM}$. For some recent results on this, see [12]; see also [9].

We finally note that there can be a big difference between $w(C)$ and $W(C)$ for a a set $C \in \mathcal{DM}$, as the following example shows.

Example 7.1. Consider, in $C[-1, 1]$, the set A of all functions $f(x)$ such that $-1 \le f(x) \le 0$, if $-1 \le x \le 0$; $0 \le f(x) \le 1$, if $0 \le x \le 1$. It is simple to see that $A \in \mathcal{DM}$; $w(A) = 0$; $W(A) = 1$.

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References

[1] G. Averkov: On cross-section measures in Minkowski spaces, Extr. Math. 18 (2003) 201–208.

[2] M. Baronti, P. L. Papini: Diameters, centers and diametrically maximal sets, Suppl. Rend. Circ. Mat. Palermo, II. Ser. 38 (1995) 11–24.

[3] G. D. Chakerian, H. Groemer: Convex bodies of constant width, in: Convexity and its Applications, P. M. Gruber, J. M Wills (eds.), Birkhäuser, Basel (1983) 49–96.

[4] L. Dalla, N. K. Tamvakis: Sets of constant width and diametrically complete sets in normed spaces, Bull. Greek Math. Soc. 26 (1985) 27–39.

[5] C. Franchetti: Relationship between the Jung constant and a certain projection constant in Banach spaces, Ann. Univ. Ferrara, N. Ser., Sez. VII 33 (1977) 39–44.

[6] P. Gritzmannn, V. Klee: Inner and outer j-radii of convex bodies in finite-dimensional normed spaces, Discrete Comput. Geom. 7 (1992) 255–280.

[7] A. S. Granero, J. P. Moreno, R. R. Phelps: Mazur sets in normed spaces, Discrete Comput. Geom. 31 (2004) 411–420.

[8] J. P. Moreno, P. L. Papini, R. R. Phelps: Diametrically maximal and constant width sets in Banach spaces, Can. J. Math. 58 (2006) 820–842.

[9] J. P. Moreno, P. L. Papini, R. R. Phelps: New families of convex sets related to diametral maximality, J. Convex Analysis 13 (2006) 823–837.

[10] H. Martini, K. J. Swanepoel: The geometry of Minkowski spaces - a survey. Part II, Expo. Math. 22 (2004) 93–144.

[11] H. Martini, K. J. Swanepoel, G. Weiss: The geometry of Minkowski spaces - a survey. Part I, Expo. Math. 19 (2001) 97–142.

[12] J. P. Moreno, R. Schneider: Diametrically complete sets in Minkowski spaces, Isr. J. Math. 191 (2012) 701–720.

[13] M. Naszódi, B. Visy: Sets with a unique extension to a set of constant width, in: Discrete Geometry, K. Bezdek (ed.), Marcel Dekker, New York (2003) 373–380.

[14] R. Payá, Á. Rodríguez-Palacios: Banach spaces which are semi L-summands in their biduals, Math. Ann. 289 (1991) 529–542.

[15] E. S. Polovinkin: Convex bodies of constant width, Dokl. Mat. 70 (2004) 560–562.

[16] M. Spirova: On a theorem of G. D. Chakerian, Contrib. Discrete Math. 5 (2010) 107–118.

[17] L. Veselý: Convex sets without diametral pairs, Extr. Math. 24 (2009) 271–280.

[18] R. Webster: Convexity, Oxford Science Publications, Oxford Univ. Press, New York (1994).