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Erratum
On a Shape Derivative Formula with Respect to Convex Domains

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In [1], we extended to the case of $W^{1,1}_{\mathrm{loc}}$ functions a formula, known for C^1 functions, for the computation of the shape derivative of an integral cost functional with respect to a class of admissible convex domains. The convexity context allows one to use the support functions of the domains to express the shape derivative.

Unfortunately, few months after the publication we noticed that we made a mistake in the proof of the main result. More precisely, the argument used in the proof of part (*iii*) of Lemma 8 is false. Indeed, we applied there basically the absolute continuity property of the function f (or more precisely $\widetilde{f}$) with respect to the Lebesgue measure. Due to some lack of uniformity, that claim is not true for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and for the chosen sequence Ω^k. Take, for example, the function $f(x) = |x-1|^{-1/2}$, $n=1$, $\Omega_0 = \Omega =]-1,1[$. Then, we have $\Omega^k =]-\sqrt{k/(k+1)}, \sqrt{k/(k+1)}[$, $\Omega_\varepsilon =]-1-\varepsilon, 1+\varepsilon[$ and $\Omega^k_\varepsilon =]-1-\varepsilon\sqrt{k/(k+1)}, 1+\varepsilon\sqrt{k/(k+1)}[$. Clearly,

$$\int_{\Omega_\varepsilon} f(x)\,dx - \int_{\Omega^k_\varepsilon} f(x)\,dx = \int_{-1-\varepsilon}^{-1-\varepsilon\sqrt{k/(k+1)}} f(x)\,dx + \int_{1+\varepsilon\sqrt{k/(k+1)}}^{1+\varepsilon} f(x)\,dx$$

$$\geq \int_{1+\varepsilon\sqrt{k/(k+1)}}^{1+\varepsilon} f(x)\,dx = 2\left(\sqrt{\varepsilon} - \sqrt{\varepsilon\sqrt{k/(k+1)}}\right) = \frac{2\varepsilon\left(1-\sqrt{k/(k+1)}\right)}{\sqrt{\varepsilon} + \sqrt{\varepsilon\sqrt{k/(k+1)}}},$$

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and consequently

$$\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon} f(x)\,dx - \int_{\Omega_\varepsilon^k} f(x)\,dx\right) \geq \frac{2\left(1-\sqrt{k/(k+1)}\right)}{\sqrt{\varepsilon}+\sqrt{\varepsilon\sqrt{k/(k+1)}}}$$

$$\geq \frac{2/(k+1)}{\left(\sqrt{\varepsilon}+\sqrt{\varepsilon\sqrt{k/(k+1)}}\right)\left(1+\sqrt{k/(k+1)}\right)}$$

$$\geq \frac{2/(k+1)}{2\sqrt{\varepsilon}.2} = \frac{1}{2(k+1)\sqrt{\varepsilon}},$$

which shows that Lemma 8 (*iii*) of [1] is false.

Fortunately, we have been able to remedy to that problem and all the results of the paper remain valid. Indeed, we give here a correction where we use the full assumption $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ of Theorem 1. In fact, part (*iii*) of Lemma 8 (and also part (*iii*) of Lemma 10) of [1] has to be replaced by

Lemma 8. (*iii*) *Let f be in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. For all $\delta > 0$, there exists $k_1 \in \mathbb{N}$ such that, for all $k \geq k_1$ and all $\varepsilon \in [0,1]$,*

$$\left|\int_{\Omega_\varepsilon^k} f(x)\,dx - \int_{\Omega_\varepsilon} f(x)\,dx\right| \leq \varepsilon\,\delta.$$

Proof. First, we express our integrals by means of polar coordinates. In general, if Ω is an open, bounded and convex subset of $\mathbb{R}^n$ containing 0, we can write

$$\int_\Omega f(x)\,dx = \int_{S^{n-1}}\left(\int_0^{1/F(\omega)} f(\varrho\omega)\,\varrho^{n-1}\,d\varrho\right)d\omega,$$

where F is the distance (or gauge) function associated to Ω. Thus, if we denote by F_ε and F_ε^k the distance functions associated to Ω_ε and Ω_ε^k respectively, we have

$$\int_{\Omega_\varepsilon} f(x)\,dx - \int_{\Omega_\varepsilon^k} f(x)\,dx = \int_{S^{n-1}}\left(\int_{1/F_\varepsilon^k(\omega)}^{1/F_\varepsilon(\omega)} f(\varrho\omega)\,\varrho^{n-1}\,d\varrho\right)d\omega.$$

Of course, we are assuming that Ω_0 and Ω contain 0, but this is not a restriction of generality. Denoting the above expression by $I_{\varepsilon,k}(f)$, we can therefore estimate as follows:

$$|I_{\varepsilon,k}(f)| \leq \int_{S^{n-1}}\left|\tfrac{1}{F_\varepsilon(\omega)} - \tfrac{1}{F_\varepsilon^k(\omega)}\right| \sup_{\frac{1}{F_\varepsilon^k(\omega)}\leq\varrho\leq\frac{1}{F_\varepsilon(\omega)}} |f(\varrho\omega)\,\varrho^{n-1}|\,d\omega.$$

Now, let B and D be some balls centered at 0 and such that $B \subset \Omega_\varepsilon \subset D$ and $B \subset \Omega_\varepsilon^k \subset D$ for all ε and k. So, we have $F_D \leq F_\varepsilon \leq F_B$ and $F_D \leq F_\varepsilon^k \leq F_B$ for all ε and k, where F_B and F_D are the distance functions associated to B and D respectively. Consequently, the functions F_ε and F_ε^k are uniformly bounded from below by positive constants on S^{n-1}, $a = \inf_{S^{n-1}} (1/F_B)$ and $b = \sup_{S^{n-1}} (1/F_D)$ are positive constants and there exists a constant $C_1 > 0$ such that

$$|I_{\varepsilon,k}(f)| \le \, C_1 \int_{S^{n-1}} \left|F_\varepsilon(\omega) - F_\varepsilon^k(\omega)\right| \sup_{a \le \varrho \le b} |f(\varrho\omega)\, \varrho^{n-1}|\, d\omega.$$

Now, since Ω_ε and Ω_ε^k contain a fixed neighborhood of 0 (for example, B), the inequality

$$\left|F_\varepsilon(\omega) - F_\varepsilon^k(\omega)\right| \le C_2\, d^H(\overline{\Omega}_\varepsilon, \overline{\Omega}_\varepsilon^k)$$

holds for all $\omega \in S^{n-1}$ and all ε, k, where d^H is the Hausdorff distance and $C_2 > 0$ is a constant (see graduate textbooks on convexity). As $d^H(\overline{\Omega}_\varepsilon, \overline{\Omega}_\varepsilon^k) = \varepsilon\, d^H(\overline{\Omega}, \overline{\Omega}^k)$ (see the proof of Lemma 8 (i) of [1]), we therefore have

$$|I_{\varepsilon,k}(f)| \le \, C_1 C_2\, \varepsilon\, d^H(\overline{\Omega}, \overline{\Omega}^k) \int_{S^{n-1}} \sup_{a \le \varrho \le b} |f(\varrho\omega)\, \varrho^{n-1}|\, d\omega.$$

To finish the proof, we apply the following well known estimate for functions of one real variable:

Lemma. *For all $\varphi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ and all bounded intervals $I \subset \mathbb{R}$, we have*

$$\|\varphi\|_{L^\infty(I)} \le \, \frac{1}{|I|} \int_I |\varphi(t)|\, dt + \int_I |\varphi'(t)|\, dt,$$

where $|I|$ is the length of I.

The proof of this inequality is easy when φ is of class C^1. The general case is obtained by an argument of density and is left to the reader.

Applying the above lemma to the function $\varrho \mapsto f(\varrho\omega)\, \varrho^{n-1}$ yields

$$\begin{aligned}
|I_{\varepsilon,k}(f)| &\le C\varepsilon\, d^H(\overline{\Omega}, \overline{\Omega}^k) \int_{S^{n-1}} \int_a^b \big(|f(\varrho\omega)|(\varrho^{n-1} + \varrho^{n-2}) + |\nabla f(\varrho\omega)|\varrho^{n-1}\big)\, d\varrho\, d\omega \\
&\le C\varepsilon\, d^H(\overline{\Omega}, \overline{\Omega}^k) \int_{S^{n-1}} \int_a^b (|f(\varrho\omega)| + |\nabla f(\varrho\omega)|)\, \varrho^{n-1}\, d\varrho\, d\omega \\
&\le C\varepsilon\, d^H(\overline{\Omega}, \overline{\Omega}^k) \int_{a \le |x| \le b} (|f(x)| + |\nabla f(x)|)\, dx,
\end{aligned}$$

where C is a positive constant which may change from one inequality to the other and is independent of (ε, k). This ends the proof since by construction $d^H(\overline{\Omega}, \overline{\Omega}^k) \to 0$ as $k \to \infty$.

References

[1] A. Boulkhemair, A. Chakib: On a shape derivative formula with respect to convex domains, J. Convex Analysis 21(1) (2014) 67–87.