

Characterizing P-spaces X in Terms of $C_p(X)^*$

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Received: March 3, 2014

Revised manuscript received: July 7, 2014

Accepted: September 3, 2014

Dual weak barrelledness led us to prove X is a P -space if and only if every pointwise eventually zero sequence in $C_p(X)$ is summable, and other better known characterizations. Novel ones recall utility functions from economics and Arkhangel'skii's (strict) τ -continuity. Mackey $\aleph_0$ -barrelled duality leads us to prove X is discrete if and only if every bounded σ -compact set in $C_p(X)$ is relatively compact. We relax the σ -compact hypothesis of Velichko and the σ -countably compact hypothesis of Tkachuk/Shakhmatov: X is a P -space if and only if $C_p(X)$ is σ -relatively sequentially complete.

Keywords: P -spaces, $C_p(X)$, relatively compact, weak barrelledness

2010 Mathematics Subject Classification: 54C35, 46A08

1. Introduction

A completely regular Hausdorff (Tichonov) space X is a P -space if every countable intersection of open sets is open. $C_p(X)$ is the space $C(X)$ of continuous real-valued functions on X endowed with the topology of pointwise convergence. A sequence $\{f_n\}_n$ in $C_p(X)$ is *pointwise eventually zero* if for each point x in X , the set $\{n \in \mathbb{N} : f_n(x) \neq 0\}$ is finite. If we note that a locally convex space E is primitive if and only if each pointwise eventually zero sequence in $(E', \sigma(E', E))$

*Ferrando and Kąkol supported by the Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, Grant PROMETEO/2013/058; Kąkol, also by the National Center of Science, Poland, grant no. N N201 605340.

is summable, our [5, Theorem 6] translates into a natural scale of characterizations (1)–(6), to which (7) is added:

Theorem 1.1. *The following seven statements are equivalent.*

- (1) X is a P -space.
- (2) Every bounded sequence in $C_p(X)$ is relatively compact.
- (3) Every bounded set in $C_p(X)$ is relatively countably compact.
- (4) $C_p(X)$ is sequentially complete. (Buchwalter/Schmets)
- (5) $C_p(X)$ is locally complete.
- (6) Every pointwise eventually zero sequence in $C_p(X)$ is summable.
- (7) Every uniformly bounded eventually zero sequence in $C_p(X)$ is summable.

Proof. (1) $\Rightarrow$ (2). Let $\{f_n\}_n$ be a bounded sequence in $C_p(X)$ with closure K in the product space $\mathbb{R}^X$. By the Tichonov theorem, K is compact; we must show that $K \subset C(X)$. For each $x \in X$ we set

$$N(x) = \bigcap_{m,n \in \mathbb{N}} \{z \in X : |f_n(z) - f_n(x)| < 1/m\},$$

a neighborhood of x , by (1), on which each f_n has the constant value $f_n(x)$. If $h \in \mathbb{R}^X$ and $h(y) \neq h(x)$ for some y in $N(x)$, then, clearly, x, y and $\varepsilon := |h(y) - h(x)|/2$ determine an open neighborhood of h in $\mathbb{R}^X$ that misses $\{f_n\}_n$. Therefore every h in K is constant on each $N(x)$, and thus continuous at x ; i.e., $K \subset C(X)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). Trivially.

(5) $\Rightarrow$ (6). If $\{f_n\}_n$ is a pointwise eventually zero sequence in $C_p(X)$, then $\{2^n f_n\}_n$ is also, and thus is bounded. By (5), the series $\sum_n 2^{-n} (2^n f_n) = \sum_n f_n$ converges in $C_p(X)$ [14, 2.1].

(6) $\Rightarrow$ (7). Trivially.

(7) $\Rightarrow$ (1). We must show that the intersection U of any decreasing sequence $\{U_n\}_n$ of neighborhoods of a point y in X is, itself, a neighborhood of y . We can inductively choose a decreasing sequence $\{V_n\}_n$ of neighborhoods of y and a sequence $\{h_n\}_n \subset C_p(X)$ such that, for all $n \in \mathbb{N}$, we have $|h_n| \leq 1$, $V_n \subset U_n$, $h_n(y) = 1$, h_n vanishes off V_n , and $|h_k(x) - 1| < 1/n$ for all $x \in V_{n+1}$ when $k \leq n$.

Clearly, the pointwise limit h of $\{h_n\}_n$ is identically 1 on $V := \bigcap_k V_k$ and vanishes elsewhere. Moreover, each $h_n - h_{n+1}$ vanishes on V , and if x is not in some V_k , then $(h_n - h_{n+1})(x) = 0 - 0 = 0$ for all $n > k$. Thus $\{h_n - h_{n+1}\}_n$ is eventually zero at all points of X , and $|h_n - h_{n+1}| \leq 2$ for each n . The pointwise sum $h_1 - h$ of the telescoping series $\sum_n (h_n - h_{n+1})$ is in $C(X)$ by (7), and so, then, is h . Therefore $h^{-1}((1/2, 3/2)) = V$ is a neighborhood of y contained in U . $\square$

Besides $[(1) \Leftrightarrow (4)]$, Buchwalter/Schmets proved $C_p(X)$ is quasicomplete if and only if X is discrete. Since non-discrete P -spaces exist, between (1) and (2)

we cannot insert (1^+) Every bounded set in $C_p(X)$ is relatively compact. Two acceptable conditions:

- (1) Every sequence in $C(X)$ is equicontinuous (Buchwalter/Schmets);
- (2) The closure in $\mathbb{R}^X$ of each countable set in $C(X)$ is contained in $C(X)$.

We labor in weak barrelledness conditions mirrored by (2)–(6), from ℓ^∞ -barrelled to *primitive* (see [5, Theorem 6], [12, Theorem 2]). With an arbitrary locally convex space E replacing $C_p(X)$ and with (6) suitably interpreted, condition (n) is strictly stronger than condition $(n+1)$ for $n = 2, \dots, 5$, as [15] shows, making (2) very much stronger than (6). Therefore to us, the equivalence of (2) and (6) is a rare and wonderful thing, dual to the (infrequent) equivalence of ℓ^∞ -barrelled and *primitive*, a serendipitous union of topology and analysis.

However, it turns out that the equivalence of (1)–(7) is not really new. Indeed, reading [2, 2.2] more carefully, one sees that Buchwalter and Schmets, with older terminology, explicitly state and prove the equivalence of (1), (1), (4), and (5). Moreover, their *Preuve* shows $[(1) \Rightarrow (1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6 \wedge 7) \Rightarrow (1')]$, where $(1')$ is the zero-set condition [7] famously equivalent to (1). Thanks to Profs. Gruenhage and Tkachuk, we found $[(1) \Leftrightarrow (2)]$ and a paraphrase of $[(1) \Leftrightarrow (3)]$ in [16, S. 355, S. 397] and [11, p. 102]. But $[(1) \Rightarrow (2)]$ is elemental, corollary to [10, 8.12]; Tichonov's theorem makes $[(2) \Rightarrow (2)]$ immediate; and $[(2) \Rightarrow (3) \Rightarrow (4)]$ is completely trivial.

Thus we must cede Theorem 1.1 to Buchwalter and Schmets. Nevertheless, our proof of $[(7) \Rightarrow (1)]$ is decidedly simpler and more direct than their proof of $[(5) \Rightarrow (1')]$, about which they wrote *C'est là le point délicat* [2, 2.2].

2. More weak barrelledness motivations

Ferrando and Sánchez Ruiz [6] articulated what is now known as the class of inductive locally convex spaces, and showed that a Mackey space is inductive if (and only if) it is primitive. This result, the definitions of *inductive* and *primitive* (omitted here), and [5, Theorem 6] motivate characterizations, below, previously unknown to us.

A Tichonov space X is the *inductive limit* of an increasing sequence $\{X_n\}_n$ of subsets covering X if subsets S must be open in X whenever each $S \cap X_n$ is relatively open in X_n . In the context of utility functions, mathematical economists call X the *inclusive* (or *inclusion*) *inductive limit* of $\{X_n\}_n$ [3].

If $\mathcal{M} = \{X_\alpha\}_{\alpha \in A}$ is a family of subsets covering X , we say that a real-valued function f on X is [$\mathcal{M}$ -continuous] $\langle$ strictly $\mathcal{M}$ -continuous $\rangle$ if, for each $\alpha \in A$, [the restriction $f|_{X_\alpha}$ is continuous] $\langle$ there is some $g_\alpha \in C(X)$ with $f|_{X_\alpha} = g_\alpha|_{X_\alpha}$ $\rangle$. Our definition of $\langle$ strict $\rangle$ $\mathcal{M}$ -continuity coincides with Arkhangel'skii's $\langle$ strict $\rangle$ τ -continuity (τ an infinite cardinal) [1] when $\mathcal{M}$ consists of all subsets of X of cardinality τ .

In [5, Theorem 6] we proved $[(a) \Leftrightarrow (d)]$ of Theorem 2.1, below. Half with some

effort. The other half with ease: $[(a) \Rightarrow (1) \Rightarrow (4) \Rightarrow (d)]$ is clear. Part (c) had eluded us altogether. The whole follows very easily given $[(6) \Rightarrow (1)]$, *le point délicat*.

Theorem 2.1. *The following four statements are equivalent.*

- (a) X is a P-space.
- (b) X is the inductive limit of each increasing sequence $\mathcal{M} = \{X_n\}_n$ of subsets covering X .
- (c) For each $\mathcal{M}$ as in (b), every $\mathcal{M}$ -continuous function is continuous.
- (d) For each $\mathcal{M}$ as in (b), every strictly $\mathcal{M}$ -continuous function is continuous.

Proof. $(a) \Rightarrow (b)$. Suppose each $S \cap X_n$ is relatively open and $x \in S \cap X_m$ for some m . For each $k \in \mathbb{N}$ let $\mathcal{O}_k$ be an open set in X such that $\mathcal{O}_k \cap X_{m+k} = S \cap X_{m+k}$. By (a), $\bigcap_k \mathcal{O}_k$ is a neighborhood of x , clearly contained in S . Therefore S is open in X .

$(b) \Rightarrow (c)$. If f is $\mathcal{M}$ -continuous and U is open in $\mathbb{R}$, then $f^{-1}(U) \cap X_n = (f|_{X_n})^{-1}(U)$ is open in X_n for each n . Hence (b) implies $f^{-1}(U)$ is open in X .

$(c) \Rightarrow (d)$. Trivially.

$(d) \Rightarrow (a)$. We demonstrate (6). Let $\{f_k\}_k$ be a pointwise eventually zero sequence in $C_p(X)$. Set $X_n = \bigcap_{k>n} f_k^{-1}(0)$ for each $n \in \mathbb{N}$, and let f be the pointwise limit of the series $\sum_k f_k$. Clearly, $\mathcal{M} := \{X_n\}_n$ is as in (b), and $f|_{X_n} = (\sum_{k \leq n} f_k)|_{X_n}$ for each n , so that f is strictly $\mathcal{M}$ -continuous. By (d), f is continuous, and (6) holds. $\square$

Corollary 2.2. *X is a P-space if and only if for every increasing sequence $\mathcal{M} = \{X_n\}_n$ of subsets covering X , the canonical embedding of $C_p(X)$ into the product $\prod_{n=1}^{\infty} C_p(X_n)$ has closed range.*

Proof. *Necessity:* Given $\mathcal{M} = \{X_n\}_n$ as above, the canonical embedding T is defined by $T(f) = (f|_{X_1}, f|_{X_2}, \dots)$ for each $f \in C(X)$. Assume X is a P-space, so that (c) holds. Let $u = (g_1, g_2, \dots)$ be a point in the product space. If there exist $\varepsilon > 0$, $n < m$ and $x \in X_n$ such that $|g_n(x) - g_m(x)| = \varepsilon$, then an open neighborhood of u misses each $T(f)$, since $|g_n(x) - f(x)|, |g_m(x) - f(x)| < \varepsilon/2$ would imply $|g_n(x) - g_m(x)| < \varepsilon$. Thus if u is in the closure of $T(C_p(X))$, then $g_m|_{X_n} = g_n$ for all $n < m$. We assume this holds, so that a function f on X is defined by positing $f|_{X_n} = g_n$ for all n . Now (c) implies f is continuous and $u = T(f)$, as desired.

Sufficiency: We show the closed range condition implies (d). If $\mathcal{M} = \{X_n\}_n$ determines a canonical embedding T and f is strictly $\mathcal{M}$ -continuous, there exists $\{h_n\}_n \subset C(X)$ such that each $f|_{X_n} = h_n|_{X_n}$. Because $\{X_n\}_n$ is increasing,

$$T(h_n) = (f|_{X_1}, \dots, f|_{X_n}, h_n|_{X_{n+1}}, h_n|_{X_{n+2}}, \dots).$$

As $n \rightarrow \infty$, the images converge in the product space to $(f|_{X_1}, \dots, f|_{X_k}, \dots)$,

which equals $T(g)$ for some g in $C(X)$ since the range of T is assumed closed. Therefore $f|_{X_k} = g|_{X_k}$ for all k , so that $f = g$ is continuous, and (d) holds. $\square$

The $C_p(X)$ view of weak barrelledness [9] partitions the sixteen distinct properties listed in [13] into just two equivalence classes represented by *barrelled* and *inductive*. The dual of $C_p(X)$ is denoted by $L(X)$, and $L_m(X)$ denotes $L(X)$ given its Mackey topology $\mu(L(X), C(X))$. Following Buchwalter and Schmets, one easily sees, as in [5], that $L_m(X)$ is *Baire-like if and only if X is finite; is barrelled if and only if X is discrete; is ℓ^∞ -barrelled if and only if X is a P -space; and has none of the sixteen properties if X is not a P -space*. In [13], Saxon completed the $L_m(X)$ view of weak barrelledness: $L_m(X)$ spaces trichotomize the sixteen properties into three equivalence classes represented by Baire-like, barrelled, and ℓ^∞ -barrelled, answering the question in [5] by proving that $L_m(X)$ is barrelled if (and only if) it is $\aleph_0$ -barrelled. His statement and proof involve convexity and the Saxon-Tweddle splitting theorem for $\aleph_0$ -barrelled Mackey spaces, unlike the direct topological approach below.

Theorem 2.3. *The following three statements are equivalent.*

- (i) X is discrete.
- (ii) Every bounded set in $C_p(X)$ is relatively compact (Buchwalter/Schmets).
- (iii) Every bounded σ -compact set in $C_p(X)$ is relatively compact.

Proof. If (i) holds, then $C_p(X) = \mathbb{R}^X$ and (ii) holds by the Tichonov theorem. Obviously, (ii) implies (iii).

Now assume (iii) holds and X is not discrete. To complete the proof, we argue to a contradiction. Since singleton sets are compact, (iii) $\Rightarrow$ (2) $\Rightarrow$ (1), so X is a P -space and therefore its topology has a base of open-and-closed sets [7, 4K 8]; i.e., a base of (open) sets U whose characteristic functions $\mathbf{1}_U$ are continuous. Let y be a non-isolated point in X . Zorn's lemma produces a maximal family $\mathcal{F}$ of non-empty pairwise disjoint open sets U which miss y and have continuous characteristic functions $\mathbf{1}_U$ ($U \in \mathcal{F}$), where $\mathbf{1}_U$ is defined to be identically 1 on U and 0 on $X \setminus U$. Since $\mathcal{F}$ is maximal and y is not isolated, every neighborhood of y meets $\bigcup \mathcal{F}$. Thus the characteristic function χ of $\bigcup \mathcal{F}$ is not continuous at y .

For each $n \in \mathbb{N}$ let

$$K_n = \{\mathbf{1}_W : W \text{ is the union of } \leq n \text{ members of } \mathcal{F}\}.$$

Each K_n is in $C_p(X)$, closed and bounded in $\mathbb{R}^X$, hence compact in $C_p(X)$, so the pointwise bounded countable union $\bigcup_n K_n$ is relatively compact in $C_p(X)$ by (iii). Therefore the closure K in $\mathbb{R}^X$ of the union coincides with the closure in $C_p(X)$. But, transparently, χ is in K , thus is continuous, a contradiction. $\square$

Remarkably, all these results hide the weak barrelledness that led us to them.

3. Applications

A sequence $\{f_n\}_n$ in $C(X)$ is *pointwise eventually constant* if for each $x \in X$ there exists a constant $f(x)$ such that $f_n(x) = f(x)$ for all but finitely many n . Now $\{f_n\}_n$ is pointwise eventually [zero] (constant) if and only if [the sequence $\{\sum_{k \leq n} f_k\}_n$ of partial sums is pointwise eventually constant] (the sequence $\{f_n - f_{n+1}\}_n$ is eventually zero), and thus (6) is equivalent to

(6') Every pointwise eventually constant sequence converges in $C_p(X)$.

Similarly, (7) is equivalent to

(7') Every uniformly bounded pointwise eventually constant sequence converges in $C_p(X)$.

Conditions (6'), (7') appear to dramatically relax (4). One may also observe that a function f in $\mathbb{R}^X$ is strictly $\mathcal{M}$ -continuous for some $\mathcal{M}$ as in (b) if and only if f is the limit (in $\mathbb{R}^X$) of a pointwise eventually constant sequence in $C_p(X)$, and thus $[(a) \Leftrightarrow (d)]$ follows immediately from $[(1) \Leftrightarrow (6')]$. A subset A of $C_p(X)$ is *eventually complete* if every pointwise eventually constant sequence in A has a (pointwise) limit in A . A subset A of a topological vector space E is *relatively sequentially complete* if every Cauchy sequence in A converges in E ; thus every relatively countably compact set is relatively sequentially complete. If E is a countable union of relatively sequentially complete sets, it is σ -relatively sequentially complete. Let $C^b(X)$ denote the linear subspace of functions in $C(X)$ having bounded range. With the uniform norm, $C^b(X)$ becomes a Banach space, denoted $C_u^b(X)$, having unit ball $\{f \in C(X) : |f(x)| \leq 1 \text{ for all } x \in X\}$.

Theorem 3.1. *The following three statements are equivalent.*

- (A) X is a P -space.
- (B) $C_p(X)$ is σ -relatively sequentially complete.
- (C) $C_p(X)$ is a countable union of closed eventually complete sets.

Proof. From $[(1) \Rightarrow (4), (6')]$ immediately follows $[(A) \Rightarrow (B), (C)]$.

$(B) \Rightarrow (A)$. We demonstrate (7'), assuming $C_p(X)$ is covered by relatively sequentially complete subsets K_n ($n \in \mathbb{N}$). Let $\{f_n\}_n$ be a uniformly bounded eventually constant sequence in $C(X)$ with limit f in $\mathbb{R}^X$. The Baire category theorem provides $q \in \mathbb{N}$ such that, in the Banach space $C_u^b(X)$, the closure B of $A := K_q \cap C_u^b(X)$ has an interior point h . Thus there is some $\varepsilon > 0$ such that

$$h + \varepsilon D \subset B,$$

where D is the unit ball in $C_u^b(X)$. Since $\{f_n\}_n$ is uniformly bounded, there is some $\delta > 0$ such that $\delta f_n \in \varepsilon D$ for all n . Therefore $\{h + \delta f_n\}_n$ is a pointwise eventually constant sequence in B whose limit in $\mathbb{R}^X$ is $h + \delta f$. Since A is norm-dense in B , for each n there is some $g_n \in K_q$ with $|g_n(x) - (h + \delta f_n)(x)| < 1/n$ for all $x \in X$. One readily sees that $\{g_n\}_n$ converges pointwisely to $h + \delta f$,

which must be in $C(X)$ since K_q is relatively sequentially complete. Therefore f is in $C(X)$; i.e., (γ') holds.

$(C) \Rightarrow (A)$. To demonstrate (γ') again under the assumption that the covering sets K_n are closed and eventually complete, we repeat the first part of the above proof, quickly obtaining $A = B$ and $h + \delta f \in K_q$ without need of the intermediate g_n . $\square$

[Velichko] (Tkachuk and Shakhmatov) proved that if $C_p(X)$ is $[\sigma\text{-compact}]$ ($\sigma\text{-countably compact}$), then X is finite. (The converses are trivial.) Via angelicity, Ferrando/Kąkol equated both hypotheses to the formally weaker condition that $C_p(X)$ be σ -relatively countably compact [4], [8, Proposition 9.6, Theorem 9.13]. We relax the hypothesis yet further, requiring only that $C_p(X)$ be the countable union of (pointwise) bounded relatively sequentially complete sets K_n ($n \in \mathbb{N}$). The key step: X is a P-space $[(B) \Rightarrow (A)]$. If we now assume $\{x_n\}_n$ is a sequence of distinct points in X then, as K_n is pointwise bounded, there exists a positive number M_n such that for each $n \in \mathbb{N}$,

$$|f(x_n)| < M_n \text{ for every } f \in K_n.$$

One well knows that, since X is a P-space, there exists $h \in C(X)$ with $h(x_n) = M_n$ for each n [8, Lemma 9.5]. But this means that $h \notin \bigcup_n K_n = C(X)$, a contradiction, and we must conclude that X is finite. We have proved

Corollary 3.2. *If $C_p(X)$ is a countable union of bounded relatively sequentially complete sets, then X is finite.*

Again, the converse is trivial. Unlike ours, Arkhangel'skii's more complicated treatment centers on a beautiful result that, as he notes, does not have a converse, to wit: *If Y is a dense subset of X and $C_p(Y|X)$ is σ -countably compact, then Y is a P-space and X is pseudocompact* [1, I.2.2]. (In a slight abuse of notation, $C_p(Y|X)$ is that subspace of $C_p(Y)$ consisting of all functions in $C(Y)$ that can be extended to a function in $C(X)$.) We generalize this result *a la* Theorem 3.1:

Theorem 3.3. *Let Y be a nonempty subset of the Tichonov space X . Then Y is a P-space if $C_p(Y|X)$ is a countable union of sets K_n that are (α) relatively sequentially complete, or (β) closed and eventually complete. In either case (α) or (β) , if Y is dense in X and each K_n is bounded, then X is pseudocompact.*

Proof. The last statement follows from the fact that, in any case, if Y is dense and each K_n is bounded, there cannot exist a discrete sequence $\{U_n\}_n$ of nonempty open sets in X . Otherwise, we could choose $x_n \in U_n \cap Y$ and $M_n > 0$ and $h \in C(X)$ as above with $h|_Y \notin \bigcup_n K_n = C_p(Y|X)$, a contradiction. This is just Arkhangel'skii's argument in the first few lines of the proof of [1, I.2.2].

Suppose $y \in Y$ and U is the intersection of any decreasing sequence $\{U_n\}_n$ of neighborhoods in X of y . We must show that $U \cap Y$ is a relative neighborhood of y in both cases (α) and (β) . As in Theorem 1.1, we inductively choose a decreasing sequence $\{V_n\}_n$ of neighborhoods of y in X and a sequence $\{h_n\}_n \subset C(X)$ such that, for all $n \in \mathbb{N}$, we have $|h_n| \leq 1$, $V_n \subset U_n$, $h_n(y) = 1$, h_n vanishes off V_n , and $|h_k(x) - 1| < 1/n$ for all $x \in V_{n+1}$ when $k \leq n$.

Now $\{h_n\}_n$ is a uniformly bounded pointwise eventually constant sequence in $C(X)$ whose limit h in $\mathbb{R}^X$ is identically 1 on $V := \bigcap_k V_k$ and vanishes elsewhere. For each n let $J_n = \{f \in C(X) : f|_Y \in K_n\}$. Since $\{K_n\}_n$ covers $C_p(Y|X)$, it is clear that $\{J_n\}_n$ covers $C(X)$. The Baire category theorem provides $q \in \mathbb{N}$ such that, in the Banach space $C_u^b(X)$, the closure B of $A := J_q \cap C^b(X)$ has an interior point g . Thus there exists a fixed $\delta > 0$ such that each $g + \delta h_n \in B$.

In case (β) , continuity of the restriction map $R : C_p(X) \rightarrow C_p(Y|X)$ ensures that $R^{-1}(K_q) = J_q$ is closed in $C_p(X)$, so that $B = A$ and

$$K_q = R(J_q) \supset R(A) = R(B) \supset \{(g + \delta h_n)|_Y\}_n.$$

The sequence $\{(g + \delta h_n)|_Y\}_n$ must converge in the eventually complete K_q , necessarily to $(g + \delta h)|_Y = g|_Y + \delta h|_Y$, so that $h|_Y \in C_p(Y|X)$. By the definition of $C_p(Y|X)$ there exists $f \in C(X)$ such that $f|_Y = h|_Y$, and

$$V \cap Y = h^{-1}((1/2, 3/2)) \cap Y = f^{-1}((1/2, 3/2)) \cap Y$$

is a relative neighborhood of y contained in $U \cap Y$, as desired.

Assume (α) holds. The proof of Theorem 3.1 shows that $(g + \delta h)|_Y$ is the limit in $\mathbb{R}^Y$ of a sequence in K_q and so is in $C_p(Y|X)$. Thus $h|_Y \in C_p(Y|X)$, etc. $\square$

Corollary 3.2 follows immediately from Theorem 3.3: With $Y = X$, invoke the standard fact that pseudocompact P-spaces are finite.

Although case (α) relaxes Arkhangel'skii's hypothesis, neither (α) nor (β) admits the converse.

Example 3.4. Let X be the one-point compactification of $\mathbb{N}$; i.e., $X = \mathbb{N} \cup \{\omega_0\}$, $Y := \mathbb{N}$ is discrete, and $\{X \setminus \{1, \dots, n\} : n \in \mathbb{N}\}$ is a base of neighborhoods of ω_0 . Then Y is a P-space dense in X , but neither case (α) nor (β) holds.

Indeed, $C_p(Y|X)$ is just the Banach space c of convergent real sequences with the coarser topology of coordinate-wise convergence. Thus $C_p(Y|X)$ contains, for each $n \in \mathbb{N}$, the point f_n , where

$$f_n(i) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \leq i \leq n; \\ 0 & \text{if } i \text{ is even and } 1 \leq i \leq n; \\ 0 & \text{if } i > n. \end{cases}$$

Each f_n is in the sup-norm unit ball of c . Assuming case (β) , Baire provides $g \in c$ and $\delta > 0$ such that the eventually constant sequence $\{g + \delta f_n\}_n$ converges in $C_p(Y|X)$, necessarily to the point u whose i^{th} coordinate is $g(i) + \delta$ for i odd, $g(i)$ for i even. Since $g \in c$, it is clear that u is not in $c = C_p(Y|X)$, a contradiction. If we assume case (α) , we obtain, as above, a possibly different sequence which still must converge in $C_p(Y|X)$ to some u of the previous form, still a contradiction. $\diamond$

However, we may obtain converses by considering subsets between Y and X . Trivially, X is a P -space if and only if every subset containing Y is a P -space. Less trivially,

Theorem 3.5. *Let Y denote a fixed dense subset of the Tichonov space X and let Z denote sets of the form $Y \cup \{x\}$, where x is an arbitrary point in X . The following seven statements are equivalent.*

- I. X is a P -space.
- II. Each Z (of the above form) is a P -space.
- III. Each $C_p(Z|X)$ is sequentially complete.
- IV. Each $C_p(Z|X)$ is σ -relatively sequentially complete.
- V. Each $C_p(Z|X)$ is eventually complete.
- VI. Each $C_p(Z|X)$ is a countable union of closed eventually complete sets.
- VII. $C_p(Y|X)$ is eventually complete and each sequence in $C(X)$ which is eventually zero at the points of Y is also eventually zero at the points of X .

Proof. $I \Rightarrow II$. Trivially; every subset of a P -space is a P -space.

$II \Rightarrow I$. Let f be the limit in $\mathbb{R}^X$ of a sequence in $C(X)$. From $[(4) \Leftrightarrow (1)]$, and assuming II , we have each $f|_Z \in C(Z)$, and it suffices to show that $f \in C(X)$. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Set $Z_0 = Y \cup \{x_0\}$. Since $f|_{Z_0} \in C(Z_0)$, in X there exists an open neighborhood U of x_0 such that $|f(y) - f(x_0)| < \varepsilon/2$ for all $y \in U \cap Y$. Let $u_1 \in U$ and set $Z_1 = Y \cup \{u_1\}$. Since $f|_{Z_1} \in C(Z_1)$, in X there is an open neighborhood V of u_1 such that $|f(y) - f(u_1)| < \varepsilon/2$ for all $y \in V \cap Y$. By density there exists $y_0 \in U \cap V \cap Y$. Therefore

$$|f(u_1) - f(x_0)| \leq |f(u_1) - f(y_0)| + |f(y_0) - f(x_0)| < \varepsilon.$$

Since $x_0 \in X$ and $u_1 \in U$ were arbitrarily chosen, we have $f \in C(X)$.

$I \Rightarrow III$. Given Z and a sequence $\{f_n\}_n \subset C(X)$ such that $\{f_n(z)\}_n$ is a real Cauchy sequence for every $z \in Z$, it suffices to show that $\{f_n(x)\}_n$ is Cauchy for each $x \in X$, since $C_p(X)$ is sequentially complete by I . Suppose to the contrary, that $\{f_n(x_0)\}_n$ is not Cauchy for some $x_0 \in X$. Then there exists $\varepsilon > 0$ and a subsequence $\{g_n\}_n$ of $\{f_n\}_n$ such that $|g_n(x_0) - g_{n+1}(x_0)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Set

$$h_n = \frac{g_n - g_{n+1}}{g_n(x_0) - g_{n+1}(x_0)}.$$

Clearly, $h_n(x_0) = 1$ for all n , and $\lim_n h_n(y) = 0$ for all $y \in Y \subset Z$. Since X is a P-space, so is the subset $Z_0 := Y \cup \{x_0\}$. By $[(1) \Rightarrow (4)]$, the Cauchy sequence $\{h_n|_{Z_0}\}_n$ must converge in $C_p(Z_0)$ to the function h , where $h(x_0) = 1$ and $h(Y) = \{0\}$, contradicting continuity of h .

Obviously, $III \Rightarrow IV$, and Theorem 3.3 ensures $IV \Rightarrow II$. Thus $I-IV$ are equivalent. Also, $III \Rightarrow V \Rightarrow VI$, and Theorem 3.3 ensures $VI \Rightarrow II$. Thus $I-VI$ are equivalent. We conclude the proof by showing $V \Leftrightarrow VII$.

$VII \Rightarrow V$. Let $\{f_n\}_n$ be a sequence in $C(X)$ which is eventually constant at the points of a given Z . Now $\{f_n - f_{n+1}\}_n$ is eventually zero at the points of Z and thus, by VII , at the points of X , so that $\{f_n\}_n$ is eventually constant at the points of X and has a limit f in $\mathbb{R}^X$. It is enough to show that $f \in C(X)$. Since $C_p(Y|X)$ is assumed to be eventually complete, there exists $g \in C(X)$ with $g|_Y = f|_Y$. In fact, $\{g - f_n\}_n$ is eventually zero at the points of Y , hence at the points of X . Therefore $f = g \in C(X)$; i.e., V holds.

$V \Rightarrow VII$. The first part of VII follows trivially. Suppose $\{f_n\}_n$ is a sequence in $C(X)$ which is eventually zero at the points of Y but not at some point $x \in X$. There is a subsequence $\{h_n\}_n$ such that $h_n(x) \neq 0$ for each $n \in \mathbb{N}$. If $g_n = h_n/h_n(x)$ and $Z = Y \cup \{x\}$, then $\{g_n|_Z\}_n$ is a pointwise eventually constant sequence in $C_p(Z|X)$ whose pointwise limit g is in $C(Z)$ by V . But $g(x) = 1$ while g vanishes on the dense subset Y of Z , contradicting continuity; VII follows. $\square$

In our Example, the fact that $X \setminus Y$ is a singleton now seems prototypical. Also, we have the characterization that Arkhangel'skii may have sought:

Corollary 3.6. *X is a P-space if and only if $C_p(Y|X)$ is σ -relatively sequentially complete for each dense subset Y of X .*

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