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Strongly Convergent Iterative Methods for Generalized Split Feasibility Problems in Hilbert Spaces

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In this paper, motivated by the idea of the split feasibility problem and results for solving the problem, we consider generalized split feasibility problems and then establish two Halpern type strong convergence theorems which are related to the problems. Furthermore, we prove strong convergence of an iterative scheme generated by the shrinking projection method. As applications, we get new and well-known strong convergence theorems which are connected with fixed point problem, split feasibility problem and equilibrium problem.

Keywords: Maximal monotone operator, inverse-strongly monotone mapping, fixed point, strong convergence theorem, equilibrium problem, split feasibility problem

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1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists

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$\alpha > 0$ such that

$$\langle x - y, Ux - Uy \rangle \geq \alpha \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [7] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [6] considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [6] is to find a point $z \in H_1$ such that

$$z \in \left(\bigcap_{i=1}^{m} A_i^{-1} 0 \right) \cap \left(\bigcap_{j=1}^{n} T_j^{-1}(B_j^{-1} 0) \right),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator, where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and generalized split feasibility problems including the split common null point problem; see, for instance, [6, 8, 15, 31]. In the study, the authors used established results for solving the problems. In particular, established convergence theorems have been used for finding solutions of the problems. On the other hand, we know many existence and convergence theorems for inverse strongly monotone mappings in Hilbert spaces; see, for instance, [10, 16, 18, 20, 24, 26].

In this paper, motivated by the ideas of these problems and results, we consider generalized split feasibility problems and then obtain strong convergence theorems which are related to the problems. We first obtain some fundamental properties for inverse strongly monotone mappings and resolvents of maximal monotone operators in Hilbert spaces. For example, we extend the result of (1) from metric projections to nonexpansive mappings. Then using these properties, we establish two Halpern type strong convergence theorems for finding a solution to the generalized split feasibility problem. Furthermore, we prove strong convergence of an iterative scheme generated by the shrinking projection method. The results are generalizations of strong convergence theorems which have already been obtained. As applications, we get new and well-known strong convergence theorems which are connected with the split feasibility problem and an equilibrium problem.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot\,,\,\cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [23] that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle; \tag{2}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{3}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{4}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle \tag{5}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [21]. Let $\alpha > 0$ be a given constant. A mapping $A\colon C \to H$ is said to be α-inverse strongly monotone if $\langle x-y, Ax-Ay\rangle \geq \alpha \|Ax-Ay\|^2$ for all $x, y \in C$. It is known that $\|Ax - Ay\| \leq (1/\alpha)\|x-y\|$ for all $x, y \in C$ if A is α-inverse strongly monotone. Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B on H is said to be monotone if $\langle x-y, u-v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}\colon H \to \mathrm{dom}(B)$, which is called the resolvent of B for r. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of fixed points of J_r. It is also known that $\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu|/\lambda)\|x - J_\lambda x\|$ holds for all $\lambda, \mu > 0$ and $x \in H$; see [21, 11] for more details. As a matter of fact, we know the following lemma [20].

Lemma 2.1. *Let H be a real Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

Furthermore, for a mapping A of C into H, we know that $F(J_\lambda(I - \lambda A)) = (A+B)^{-1}0$ for all $\lambda > 0$; see [4]. We also know the following lemmas:

Lemma 2.2 ([3], [30]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty}\beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty}\gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 ([13]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \cdots$ *and* $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

Let H be a Hilbert space and let S be a firmly nonexpansive mapping of H into itself with $F(S) \neq \emptyset$. Then we have that

$$\langle x - Sx, Sx - y\rangle \geq 0 \tag{6}$$

for all $x \in H$ and $y \in F(S)$. In fact, we have that for all $x \in H$ and $y \in F(S)$

$$\begin{aligned}\langle x - Sx, Sx - y\rangle &= \langle x - y + y - Sx, Sx - y\rangle \\ &= \langle x - y, Sx - y\rangle + \langle y - Sx, Sx - y\rangle \\ &\geq \|Sx - y\|^2 - \|Sx - y\|^2 = 0.\end{aligned}$$

From [27], we also have the following lemmas.

Lemma 2.4. *Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then a mapping $A^*(I-T)A : H_1 \to H_1$ is $\frac{1}{2\|AA^*\|}$-inverse strongly monotone.*

Lemma 2.5. *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rA^*(I-T)A)z$;

(ii) $0 \in A^*(I-T)Az + Bz$;

(iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Hilbert space H, define $\text{s-Li}_n\, C_n$ and $\text{w-Ls}_n\, C_n$ as follows: $x \in \text{s-Li}_n\, C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$ w-$\mathrm{Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \text{s-}\mathop{\mathrm{Li}}_n C_n = \text{w-}\mathop{\mathrm{Ls}}_n C_n, \tag{7}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [14] and we write $C_0 = \text{M-}\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [14]. The following lemma is easily deduced from the theorem for a strictly convex reflexive Banach space with the Kadec-Klee property proved by Tsukada; see [28].

Lemma 2.6. *Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a Hilbert space H. If $C_0 = \bigcap_{n=1}^{\infty} C_n$ is nonempty, then $P_{C_n}u \to P_{C_0}u$ for any $u \in H$.*

3. Main results

In this section, we first prove a strong convergence theorem which is related to the split feasibility problem and generalizes Wittmann's theorem [29] in a Hilbert space.

Theorem 3.1. *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \alpha_n u_n + (1 - \alpha_n) J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \le \lambda_n \le \frac{1}{\|A^*A\|}, \qquad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in B^{-1}0 \cap A^{-1}F(T)$, where $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$.

Proof. Put $y_n = J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n$ and let $z \in B^{-1}0 \cap A^{-1}F(T)$. We have $z = J_{\lambda_n}(z - \lambda_n A^*(I - T)Az)$. Since J_{λ_n} is nonexpansive and $(I - T)$ is

$\frac{1}{2}$-inverse strongly monotone, we have that

$$\begin{aligned}
&\|y_n - z\|^2 \qquad\qquad (8)\\
&= \|J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n - J_{\lambda_n} z\|^2\\
&\le \|x_n - \lambda_n A^*(I-T)Ax_n - z\|^2\\
&= \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, A^*(I-T)Ax_n\rangle + (\lambda_n)^2\|A^*(I-T)Ax_n\|^2\\
&= \|x_n - z\|^2 - 2\lambda_n\langle Ax_n - Az, (I-T)Ax_n\rangle + (\lambda_n)^2\|A^*(I-T)Ax_n\|^2\\
&\le \|x_n - z\|^2 - \lambda_n\|(I-T)Ax_n\|^2 + (\lambda_n)^2\langle A^*(I-T)Ax_n, A^*(I-T)Ax_n\rangle\\
&= \|x_n - z\|^2 - \lambda_n\|(I-T)Ax_n\|^2 + (\lambda_n)^2\langle AA^*(I-T)Ax_n, (I-T)Ax_n\rangle\\
&\le \|x_n - z\|^2 - \lambda_n\|(I-T)Ax_n\|^2 + (\lambda_n)^2\|AA^*\|\,\|(I-T)Ax_n\|^2\\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n\|AA^*\| - 1)\|(I-T)Ax_n\|^2.
\end{aligned}$$

From $0 < a \le \lambda_n \le \frac{1}{\|AA^*\|}$ we have that $\|y_n - z\| \le \|x_n - z\|$ for all $n \in \mathbb{N}$. Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}}\|u_n - z\| \le M$. From $x_{n+1} = \alpha_n u_n + (1-\alpha_n)y_n$, we have that

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(u_n - z) + (1-\alpha_n)(y_n - z)\|\\
&\le \alpha_n\|u_n - z\| + (1-\alpha_n)\|x_n - z\|.
\end{aligned}$$

Putting $K = \max\{M, \|x_1 - z\|\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \le K$. Suppose that $\|x_k - z\| \le K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned}
\|x_{k+1} - z\| &\le \alpha_k\|u_k - z\| + (1-\alpha_k)\|x_k - z\|\\
&\le \alpha_k K + (1-\alpha_k)K = K.
\end{aligned}$$

By induction, we obtain that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$ and $\{y_n\}$ are bounded. Since

$$\|(I-T)Ax_n\|^2 \le 2\langle (I-T)Ax_n, Ax_n - Az\rangle \le 2\|(I-T)Ax_n\|\|Ax_n - Az\|,$$

$\{(I-T)Ax_n\}$ is bounded. Then we have that $\{A^*(I-T)Ax_n\}$ is bounded. Putting $v_n = x_n - \lambda_n A^*(I-T)Ax_n$, we have that

$$\begin{aligned}
x_{n+2} - x_{n+1} = {}& \alpha_{n+1}u_{n+1} - \alpha_n u_n\\
&+ (1-\alpha_{n+1})J_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}A^*(I-T)Ax_{n+1})\\
&- (1-\alpha_n)J_{\lambda_n}(x_n - \lambda_n A^*(I-T)Ax_n)\\
= {}& (\alpha_{n+1} - \alpha_n)u_n + \alpha_{n+1}(u_{n+1} - u_n)\\
&+ (1-\alpha_{n+1})\{J_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}A^*(I-T)Ax_{n+1})\\
&- J_{\lambda_{n+1}}v_n + J_{\lambda_{n+1}}v_n - J_{\lambda_n}v_n + J_{\lambda_n}v_n\} - (1-\alpha_n)J_{\lambda_n}v_n.
\end{aligned}$$

Then we have from Lemma 2.1 that

$$
\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&\leq |\alpha_{n+1} - \alpha_n| \ \|u_n\| + \alpha_{n+1}\|u_{n+1} - u_n\| \\
&\qquad + (1 - \alpha_{n+1})\|x_{n+1} - \lambda_{n+1}A^*(I - T)Ax_{n+1} - (x_n - \lambda_n A^*(I - T)Ax_n)\| \\
&\qquad + (1 - \alpha_{n+1})\|J_{\lambda_{n+1}}v_n - J_{\lambda_n}v_n\| + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}v_n\| \\
&\leq |\alpha_{n+1} - \alpha_n| \ \|u_n\| + \alpha_{n+1}\|u_{n+1} - u_n\| \\
&\qquad + (1 - \alpha_{n+1})\|(I - \lambda_{n+1}A^*(I - T)A)x_{n+1} - (I - \lambda_{n+1}A^*(I - T)A)x_n \\
&\qquad + (I - \lambda_{n+1}A^*(I - T)A)x_n - (x_n - \lambda_n A^*(I - T)Ax_n)\| \\
&\qquad + (1 - \alpha_{n+1})\|J_{\lambda_{n+1}}v_n - J_{\lambda_n}v_n\| + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}v_n\| \\
&\leq |\alpha_{n+1} - \alpha_n| \ \|u_n\| + \alpha_{n+1}\|u_{n+1} - u_n\| \\
&\qquad + (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|A^*(I - T)Ax_n\| \\
&\qquad + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}v_n\| + \|J_{\lambda_{n+1}}v_n - J_{\lambda_n}v_n\| \\
&\leq |\alpha_{n+1} - \alpha_n| \ \|u_n\| + \alpha_{n+1}\|u_{n+1} - u_n\| \\
&\qquad + (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|A^*(I - T)Ax_n\| \\
&\qquad + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}v_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\|J_{\lambda_{n+1}}v_n - v_n\| \\
&\leq |\alpha_{n+1} - \alpha_n| \ \|u_n\| + \alpha_{n+1}\|u_{n+1} - u_n\| \\
&\qquad + (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|A^*(I - T)Ax_n\| \\
&\qquad + |\alpha_{n+1} - \alpha_n|\|J_{\lambda_n}v_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{a}\|J_{\lambda_{n+1}}v_n - v_n\|.
\end{aligned}
$$

Using Lemma 2.2, we obtain that

$$\|x_{n+2} - x_{n+1}\| \to 0. \tag{9}$$

We also have from (3) that

$$
\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|\alpha_n(u_n - x_n) + (1 - \alpha_n)(y_n - x_n)\|^2 \\
&= \alpha_n\|u_n - x_n\|^2 + (1 - \alpha_n)\|y_n - x_n\|^2 - \alpha_n(1 - \alpha_n)\|u_n - y_n\|^2
\end{aligned}
$$

and hence

$$(1 - \alpha_n)\|y_n - x_n\|^2 = \|x_{n+1} - x_n\|^2 - \alpha_n\|u_n - x_n\|^2 + \alpha_n(1 - \alpha_n)\|u_n - y_n\|^2$$

From $\alpha_n \to 0$, we get

$$y_n - x_n \to 0. \tag{10}$$

From $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$, we have that $\{\lambda_n\}$ is a Cauchy sequence. Then we have $\lambda_n \to \lambda_0 \in [a, 1/\|AA^*\|]$. Using $v_n = x_n - \lambda_n A^*(I - T)Ax_n$ and

$y_n = J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n$, we have from Lemma 2.1 that

$$\begin{aligned} &\|J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n - y_n\| \qquad (11)\\ &= \|J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n - J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n\|\\ &= \|J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n - J_{\lambda_0}(I - \lambda_n A^*(I-T)A)x_n\\ &\qquad + J_{\lambda_0}(I - \lambda_n A^*(I-T)A)x_n - J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n\|\\ &\le \|(I - \lambda_0 A^*(I-T)A)x_n - (I - \lambda_n A^*(I-T)A)x_n\| + \|J_{\lambda_0}v_n - J_{\lambda_n}v_n\|\\ &\le |\lambda_0 - \lambda_n|\|A^*(I-T)Ax_n\| + \frac{|\lambda_0 - \lambda_n|}{\lambda_0}\|J_{\lambda_0}v_n - v_n\| \to 0. \end{aligned}$$

We also have from (10) and (11) that

$$\begin{aligned} &\|x_n - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n\| \qquad (12)\\ &\le \|x_n - y_n\| + \|y_n - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n\| \to 0. \end{aligned}$$

We will use (11) and (12) later.

Put $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$. Let us show that $\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle \le 0$. Put $l = \limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle$. Then without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $l = \lim_{i\to\infty}\langle u - z_0, y_{n_i} - z_0\rangle$ and $\{y_{n_i}\}$ converges weakly to some point $w \in H_1$. From $\|x_n - y_n\| \to 0$, we also have that $\{x_{n_i}\}$ converges weakly to $w \in H_1$. On the other hand, from $\lambda_n \to \lambda_0 \in [a, \frac{1}{\|AA^*\|}]$, we have $\lambda_{n_i} \to \lambda_0 \in [a, \frac{1}{\|AA^*\|}]$. Using (12), we have that

$$\|x_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_{n_i}\| \to 0.$$

Since $J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)$ is nonexpansive, we have $w = J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)w$. We have from Lemma 2.5 that $w \in B^{-1}0 \cap A^{-1}F(T)$. Thus we have

$$l = \lim_{i\to\infty}\langle u - z_0, y_{n_i} - z_0\rangle = \langle u - z_0, w - z_0\rangle \le 0.$$

Since $x_{n+1} - z_0 = \alpha_n(u_n - z_0) + (1-\alpha_n)(y_n - z_0)$, we have

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &\le (1-\alpha_n)^2\|y_n - z_0\|^2 + 2\alpha_n\langle u_n - z_0, x_{n+1} - z_0\rangle \qquad (13)\\ &\le (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u_n - u, x_{n+1} - z_0\rangle\\ &\qquad + 2\alpha_n\langle u - z_0, x_{n+1} - x_n + x_n - y_n + y_n - z_0\rangle. \end{aligned}$$

From Lemma 2.2, we have that $x_n \to z_0$. This completes the proof. □

Next, we prove another strong convergence theorem.

Theorem 3.2. *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \ne \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n u_n + (1-\alpha_n)J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n)$$

for all $n \in \mathbb{N}$, *where* $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ *and* $\{\alpha_n\} \subset (0, 1)$ *satisfy*

$$0 < a \leq \lambda_n \leq \frac{1}{\|A^*A\|}, \qquad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $z_0 \in B^{-1}0 \cap A^{-1}F(T)$, *where* $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$.

Proof. Let $z \in B^{-1}0 \cap A^{-1}F(T)$. As in the proof of Theorem 3.1, we obtain that

$$\begin{aligned} & \|J_{\lambda_n}(x_n - \lambda_n A^*(I - T)Ax_n) - z\|^2 \\ \leq\ & \|x_n - z\|^2 + \lambda_n(\lambda_n \|AA^*\| - 1) \|(I - T)Ax_n\|^2 \\ \leq\ & \|x_n - z\|^2 . \end{aligned} \tag{14}$$

Let $y_n = \alpha_n u_n + (1 - \alpha_n)J_{\lambda_n}(x_n - \lambda_n A^*(I - T)Ax_n)$. We have that

$$\begin{aligned} \|y_n - z\| =\ & \|\alpha_n(u_n - z) + (1 - \alpha_n)(J_{\lambda_n}(x_n - \lambda_n A^*(I - T)Ax_n) - z)\| \\ & \leq \alpha_n \|u_n - z\| + (1 - \alpha_n) \|x_n - z\| . \end{aligned}$$

Using this, we get that

$$\begin{aligned} \|x_{n+1} - z\| =\ & \|\beta_n(x_n - z) + (1 - \beta_n)(y_n - z)\| \\ & \leq \beta_n \|x_n - z\| + (1 - \beta_n) \|y_n - z\| \\ & \leq \beta_n \|x_n - z\| + (1 - \beta_n)(\alpha_n \|u_n - z\| + (1 - \alpha_n) \|x_n - z\|) \\ & = (1 - \alpha_n(1 - \beta_n))\|x_n - z\| + \alpha_n(1 - \beta_n)\|u_n - z\|. \end{aligned}$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}} \|u_n - z\| \leq M$. Putting $K = \max\{\|x_1 - z\|, M\}$, we have that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \leq K$. Suppose that $\|x_k - z\| \leq K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned} \|x_{k+1} - z\| & \leq (1 - \alpha_k(1 - \beta_k))\|x_k - z\| + \alpha_k(1 - \beta_k)\|u_k - z\| \\ & \leq (1 - \alpha_k(1 - \beta_k))K + \alpha_k(1 - \beta_k)K = K. \end{aligned}$$

By induction, we obtain that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$, $\{y_n\}$ and $\{J_{\lambda_n}(x_n - \lambda_n A^*(I - T)Ax_n)\}$ are bounded. Take $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$. Putting $z_n = J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n$, from the definition of $\{x_n\}$ we have that

$$x_{n+1} - x_n = \beta_n x_n + (1 - \beta_n)\{\alpha_n u_n + (1 - \alpha_n)z_n\} - x_n$$

and hence

$$\begin{aligned} x_{n+1} - x_n - (1 - \beta_n)\alpha_n u_n & = \beta_n x_n + (1 - \beta_n)(1 - \alpha_n)z_n - x_n \\ & = (1 - \beta_n)\{(1 - \alpha_n)z_n - x_n\} \\ & = (1 - \beta_n)\{z_n - x_n - \alpha_n z_n\}. \end{aligned}$$

Thus we have that

$$\begin{aligned}
&\langle x_{n+1} - x_n - (1-\beta_n)\alpha_n u_n, x_n - z_0\rangle \\
&= (1-\beta_n)\langle z_n - x_n, x_n - z_0\rangle - (1-\beta_n)\langle \alpha_n z_n, x_n - z_0\rangle \\
&= -(1-\beta_n)\langle x_n - z_n, x_n - z_0\rangle - (1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle.
\end{aligned} \tag{15}$$

From (4) and (14), we have that

$$\begin{aligned}
&2\langle x_n - z_n, x_n - z_0\rangle \\
&= \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|z_n - z_0\|^2 \\
&\geq \|x_n - z_0\|^2 + \|z_n - x_n\|^2 - \|x_n - z_0\|^2 \\
&= \|z_n - x_n\|^2.
\end{aligned} \tag{16}$$

From (15) and (16), we have that

$$\begin{aligned}
&-2\langle x_n - x_{n+1}, x_n - z_0\rangle \\
&= 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\qquad - 2(1-\beta_n)\langle x_n - z_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle \\
&\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\qquad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle.
\end{aligned} \tag{17}$$

Furthermore, using (4) and (17), we have that

$$\begin{aligned}
&\|x_{n+1} - z_0\|^2 - \|x_n - x_{n+1}\|^2 - \|x_n - z_0\|^2 \\
&\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\qquad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle.
\end{aligned}$$

Setting $\Gamma_n = \|x_n - z_0\|^2$, we have that

$$\begin{aligned}
&\Gamma_{n+1} - \Gamma_n - \|x_n - x_{n+1}\|^2 \\
&\leq 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\
&\qquad - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle.
\end{aligned} \tag{18}$$

Noting that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\beta_n)\alpha_n(u_n - z_n) + (1-\beta_n)(z_n - x_n)\| \\
&\leq (1-\beta_n)\big(\|z_n - x_n\| + \alpha_n\|u_n - z_n\|\big)
\end{aligned} \tag{19}$$

and hence

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &\leq (1-\beta_n)^2\big(\|z_n - x_n\| + \alpha_n\|u_n - z_n\|\big)^2 \\
&= (1-\beta_n)^2\|z_n - x_n\|^2 \\
&\qquad + (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|u_n - z_n\| + \alpha_n^2\|u_n - z_n\|^2\big).
\end{aligned} \tag{20}$$

Thus we have from (18) and (20) that

$$\begin{aligned}\Gamma_{n+1} - \Gamma_n \le {} & \|x_n - x_{n+1}\|^2 + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle \\ & - (1-\beta_n)\|z_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle \\ \le {} & (1-\beta_n)^2\|z_n - x_n\|^2 \\ & + (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|u_n - z_n\| + \alpha_n^2\|u_n - z_n\|^2\big) \\ & + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle - (1-\beta_n)\|z_n - x_n\|^2 \\ & - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle\end{aligned}$$

and hence

$$\begin{aligned} & \Gamma_{n+1} - \Gamma_n + \beta_n(1-\beta_n)\|z_n - x_n\|^2 \\ & \le (1-\beta_n)^2\big(2\alpha_n\|z_n - x_n\|\|u_n - z_n\| + \alpha_n^2\|u_n - z_n\|^2\big) \\ & \quad + 2(1-\beta_n)\alpha_n\langle u_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle z_n, x_n - z_0\rangle. \end{aligned} \tag{21}$$

We will divide the proof into two cases.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge N$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we have from (21) that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{22}$$

From (19) we have that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{23}$$

We also have that

$$\begin{aligned}\|y_n - z_n\| &= \|\alpha_n u_n + (1-\alpha_n)z_n - z_n\| \\ &= \alpha_n\|u_n - z_n\| \to 0.\end{aligned} \tag{24}$$

Furthermore, from $\|y_n - x_n\| \le \|y_n - z_n\| + \|z_n - x_n\|$, we have that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{25}$$

Take $\lambda_0 \in [a, \frac{1}{\|AA^*\|}]$. Putting $v_n = x_n - \lambda_n A^*(I-T)Ax_n$, we have from Lemma 2.1 that

$$\begin{aligned} & \|\alpha_n u_n + (1-\alpha_n)J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n - y_n\| \\ & = (1-\alpha_n)\|J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n - J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n\| \\ & = (1-\alpha_n)\|J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n - J_{\lambda_0}(I - \lambda_n A^*(I-T)A)x_n \\ & \qquad + J_{\lambda_0}(I - \lambda_n A^*(I-T)A)x_n - J_{\lambda_n}(I - \lambda_n A^*(I-T)A)x_n\| \\ & \le (1-\alpha_n)\{\|(I - \lambda_0 A^*(I-T)A)x_n - v_n\| + \|J_{\lambda_0}v_n - J_{\lambda_n}v_n\|\} \\ & \le (1-\alpha_n)\left\{|\lambda_0 - \lambda_n|\|A^*(I-T)Ax_n\| + \frac{|\lambda_0 - \lambda_n|}{\lambda_0}\|J_{\lambda_0}v_n - v_n\|\right\}. \end{aligned} \tag{26}$$

We also have

$$\begin{aligned}
&\|x_n - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n\| \qquad (27)\\
&\le \|x_n - y_n\| + \|y_n - (\alpha_n u_n + (1-\alpha_n)J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n)\|\\
&\qquad + \|\alpha_n u_n + (1-\alpha_n)J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n\\
&\qquad - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n\|\\
&= \|x_n - y_n\| + \|y_n - (\alpha_n u_n + (1-\alpha_n)J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n)\|\\
&\qquad + \alpha_n\|u_n - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_n\|.
\end{aligned}$$

We will use (26) and (27) later.

We show that $\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle \le 0$, where $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$. Put $l = \limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle$. Then without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $l = \lim_{i\to\infty}\langle u - z_0, y_{n_i} - z_0\rangle$ and $\{y_{n_i}\}$ converges weakly to some point $w \in H_1$. From $\|x_n - y_n\| \to 0$, we also have that $\{x_{n_i}\}$ converges weakly to $w \in H_1$. On the other hand, since $\{\lambda_n\} \subset (0,\infty)$ satisfies $0 < a \le \lambda_n \le \frac{1}{\|AA^*\|}$, there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\{\lambda_{n_{i_j}}\}$ converges to a number $\lambda_0 \in [a, \frac{1}{\|AA^*\|}]$. Using (26), we have that

$$\|\alpha_{n_{i_j}} u_{n_{i_j}} + (1-\alpha_{n_{i_j}})J_{\lambda_0}(I - \lambda_0 A)x_{n_{i_j}} - y_{n_{i_j}}\| \to 0.$$

Furthermore, using (27), we have that

$$\begin{aligned}
&\|x_{n_{i_j}} - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_{n_{i_j}}\|\\
&\le \|x_{n_{i_j}} - y_{n_{i_j}}\| + \|y_{n_{i_j}} - \{\alpha_{n_{i_j}} u_{n_{i_j}} + (1-\alpha_{n_{i_j}})J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_{n_{i_j}}\}\|\\
&\qquad + \alpha_{n_{i_j}}\|u_{n_{i_j}} - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_{n_{i_j}}\| \to 0.
\end{aligned}$$

Since $J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)$ is nonexpansive, we have $w = J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)w$. From Lemma 2.5, we have that $w \in B^{-1}0 \cap A^{-1}F(T)$. Then we have that

$$l = \lim_{j\to\infty}\langle u - z_0, y_{n_{i_j}} - z_0\rangle = \langle u - z_0, w - z_0\rangle \le 0.$$

Since $y_n - z_0 = \alpha_n(u_n - z_0) + (1-\alpha_n)(J_{\lambda_n}(x_n - \lambda_n A^*(I-T)Ax_n) - z_0)$, we have from (2) that

$$\begin{aligned}
\|y_n - z_0\|^2 \le (1-\alpha_n)^2 \|J_{\lambda_n}(x_n - \lambda_n A^*(I-T)Ax_n) - z_0\|^2\\
+ 2\alpha_n \langle u_n - z_0, y_n - z_0\rangle.
\end{aligned}$$

From (14), we have

$$\|y_n - z_0\|^2 \le (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n \langle u_n - z_0, y_n - z_0\rangle.$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - z_0\|^2 \\
\leq\ & \beta_n \|x_n - z_0\|^2 + (1 - \beta_n) \|y_n - z_0\|^2 \\
\leq\ & \beta_n \|x_n - z_0\|^2 + (1 - \beta_n) \left((1 - \alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n \langle u_n - z_0, y_n - z_0\rangle\right) \\
=\ & \left(\beta_n + (1 - \beta_n)(1 - \alpha_n)^2\right) \|x_n - z_0\|^2 + 2(1 - \beta_n)\alpha_n \langle u_n - z_0, y_n - z_0\rangle \\
\leq\ & (1 - (1 - \beta_n)\alpha_n) \|x_n - z_0\|^2 + 2(1 - \beta_n)\alpha_n \langle u_n - z_0, y_n - z_0\rangle \\
=\ & (1 - (1 - \beta_n)\alpha_n) \|x_n - z_0\|^2 + 2(1 - \beta_n)\alpha_n \langle u_n - u, y_n - z_0\rangle \\
& \quad + 2(1 - \beta_n)\alpha_n \langle u - z_0, y_n - z_0\rangle .
\end{aligned}$$

By Lemma 2.2, we obtain that $x_n \to z_0$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 2.3 that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. Thus we have from (21) that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\beta_{\tau(n)}(1 - \beta_{\tau(n)})\|z_{\tau(n)} - x_{\tau(n)}\|^2 \\
\leq\ & (1 - \beta_{\tau(n)})^2 2\alpha_{\tau(n)}\|z_{\tau(n)} - x_{\tau(n)}\|\|u_{\tau(n)} - z_{\tau(n)}\| \\
& + (1 - \beta_{\tau(n)})^2\alpha_{\tau(n)}^2\|u_{\tau(n)} - z_{\tau(n)}\|^2 \\
& + 2(1 - \beta_{\tau(n)})\alpha_{\tau(n)}\langle u_{\tau(n)}, x_{\tau(n)} - z_0\rangle \\
& - 2(1 - \beta_{\tau(n)})\alpha_{\tau(n)}\langle z_{\tau(n)}, x_{\tau(n)} - z_0\rangle.
\end{aligned} \tag{28}$$

Using $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, we have from (28) and Lemma 2.3 that

$$\lim_{n\to\infty} \|z_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{29}$$

As in the proof of Case 1 we have that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{30}$$

and

$$\lim_{n\to\infty} \|y_{\tau(n)} - z_{\tau(n)}\| = 0. \tag{31}$$

Since $\|y_{\tau(n)} - x_{\tau(n)}\| \leq \|y_{\tau(n)} - z_{\tau(n)}\| + \|z_{\tau(n)} - x_{\tau(n)}\|$, we have that

$$\lim_{n\to\infty} \|y_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{32}$$

For $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$, let us show that $\limsup_{n\to\infty} \left\langle z_0 - u, y_{\tau(n)} - z_0\right\rangle \geq 0$. Put

$$l = \limsup_{n\to\infty} \left\langle z_0 - u, y_{\tau(n)} - z_0\right\rangle .$$

Without loss of generality, there exists a subsequence $\{y_{\tau(n_i)}\}$ of $\{y_{\tau(n)}\}$ such that $l = \lim_{i\to\infty} \langle u - z_0, y_{\tau(n_i)} - z_0 \rangle$ and $\{y_{\tau(n_i)}\}$ converges weakly to some point $w \in H_1$. From $\|y_n - x_n\| \to 0$, we also have that $\{x_{\tau(n_i)}\}$ converges weakly to $w \in H_1$. As in the proof of Case 1 we have that $w \in B^{-1}0 \cap A^{-1}F(T)$. Then we have

$$l = \lim_{i\to\infty} \langle z_0 - u, y_{\tau(n_i)} - z_0\rangle = \langle z_0 - u, w - z_0\rangle \geq 0.$$

As in the proof of Case 1, we also have that

$$\left\|y_{\tau(n)} - z_0\right\|^2 \leq (1 - \alpha_{\tau(n)})^2 \left\|x_{\tau(n)} - z_0\right\|^2 + 2\alpha_{\tau(n)} \left\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0 \right\rangle$$

and then

$$\begin{aligned}\|x_{\tau(n)+1} - z_0\|^2 &\leq \beta_{\tau(n)} \left\|x_{\tau(n)} - z_0\right\|^2 + (1 - \beta_{\tau(n)}) \left\|y_{\tau(n)} - z_0\right\|^2 \\ &\leq \left(1 - (1 - \beta_{\tau(n)})\alpha_{\tau(n)}\right) \left\|x_{\tau(n)} - z_0\right\|^2 \\ &\qquad + 2(1 - \beta_{\tau(n)})\alpha_{\tau(n)} \left\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0 \right\rangle.\end{aligned}$$

From $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have that

$$(1 - \beta_{\tau(n)})\alpha_{\tau(n)}) \left\|x_{\tau(n)} - z_0\right\|^2 \leq 2(1 - \beta_{\tau(n)})\alpha_{\tau(n)} \langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle.$$

Since $(1 - \beta_{\tau(n)})\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}\left\|x_{\tau(n)} - z_0\right\|^2 &\leq 2\langle u_{\tau(n)} - z_0, y_{\tau(n)} - z_0\rangle \\ &= 2\langle u_{\tau(n)} - u, y_{\tau(n)} - z_0\rangle + 2\langle u - z_0, y_{\tau(n)} - z_0\rangle\end{aligned}$$

Thus we have that

$$\limsup_{n\to\infty} \left\|x_{\tau(n)} - z_0\right\|^2 \leq 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$ as $n \to \infty$. From (30), we have also that $x_{\tau(n)} - x_{\tau(n)+1} \to 0$. Thus $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 2.3 again, we obtain that

$$\|x_n - z_0\| \leq \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

In the end of this section, we prove strong convergence of an iterative scheme generated by the shrinking projection method, which was first proposed by Takahashi, Takeuchi, and Kubota [25].

Theorem 3.3. *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 \in H_1$, $C_1 = H_1$, and $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} y_n &= J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n, \\ C_{n+1} &= \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, *where* $\{\lambda_n\} \subset (0, \infty)$ *satisfies*

$$0 < \lambda_n \leq \frac{1}{\|A^*A\|} \quad \textit{and} \quad \limsup_{n\to\infty} \lambda_n > 0.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $z_0 \in B^{-1}0 \cap A^{-1}F(T)$, *where* $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$.

Proof. We first show that the sequence $\{x_n\}$ is well defined. Let $x \in H_1$ and $y = J_\lambda(I - \lambda A^*(I - T)A)x$ with $0 < \lambda \leq 1/\|A^*A\|$. As in the proof of Theorem 3.1, we have $\|y - z\|^2 \leq \|x - z\|^2$ for $z \in B^{-1}0 \cap A^{-1}F(T)$ and thus

$$\emptyset \neq B^{-1}0 \cap A^{-1}F(T) \subset \{z \in H_1 : \|y - z\| \leq \|x - z\|\}.$$

Moreover, since

$$\begin{aligned}\{z \in H_1 : \|y - z\| \leq \|x - z\|\} &= \{z \in H_1 : \|y - z\|^2 \leq \|x - z\|^2\} \\ &= \{z \in H_1 : \|y\|^2 - \|x\|^2 \leq 2\langle y - x, z\rangle\},\end{aligned}$$

it is closed and convex. Applying these facts inductively, we obtain that C_n is nonempty, closed, and convex for every $n \in \mathbb{N}$, and hence $\{x_n\}$ is well defined.

Let $C_0 = \bigcap_{n=1}^\infty C_n$. Then since $C_0 \supset B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$, C_0 is also nonempty. Let $z_n = P_{C_n}u$ for every $n \in \mathbb{N}$. Then, by Lemma 2.6, we have $z_n \to z_0 = P_{C_0}u$. Since a metric projection is nonexpansive, it follows that

$$\begin{aligned}\|x_n - z_0\| &\leq \|x_n - z_n\| + \|z_n - z_0\| \\ &= \|P_{C_n}u_n - P_{C_n}u\| + \|z_n - z_0\| \\ &\leq \|u_n - u\| + \|z_n - z_0\| \to 0,\end{aligned}$$

and hence $x_n \to z_0$.

Since $z_0 \in C_0 = \bigcap_{n=1}^\infty C_n$, we have $\|y_n - z_0\| \leq \|x_n - z_0\|$ for all $n \in \mathbb{N}$. Tending $n \to \infty$, we get that $y_n \to z_0$. By the assumption of $\{\lambda_n\}$, there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ converging to $\lambda_0 > 0$. We know that J_{λ_0}, A^*, T, and A are all continuous, thus $\{J_{\lambda_0}(I-\lambda_0 A^*(I-T)A)x_n\}$ and $\{(I-\lambda_0 A^*(I-T)A)x_n\}$ are convergent sequences and therefore bounded. Using Lemma 2.1, we have

$$\begin{aligned}&\|y_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*(I - T)A)x_{n_i}\| \\ =\ &\left\|J_{\lambda_{n_i}}(I - \lambda_{n_i} A^*(I - T)A)x_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*(I - T)A)x_{n_i}\right\| \\ =\ &\|J_{\lambda_{n_i}}(I - \lambda_{n_i} A^*(I - T)A)x_{n_i} - J_{\lambda_0}(I - \lambda_{n_i} A^*(I - T)A)x_{n_i} \\ &\quad + J_{\lambda_0}(I - \lambda_{n_i} A^*(I - T)A)x_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*(I - T)A)x_{n_i}\| \\ \leq\ &\frac{|\lambda_{n_i} - \lambda_0|}{\lambda_0}\|J_{\lambda_0}(I - \lambda_{n_i} A^*(I - T)A)x_{n_i} - (I - \lambda_{n_i} A^*(I - T)A)x_{n_i}\| \\ &\quad + |\lambda_{n_i} - \lambda_0|\,\|A^*(I - T)Ax_{n_i}\| \to 0.\end{aligned}$$

On the other hand, by the continuity of $J_{\lambda_0}(I - \lambda_0 A^*(I - T)A)$, we have

$$\|J_{\lambda_0}(I - \lambda_0 A^*(I - T)A)x_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*(I - T)A)z_0\| \to 0.$$

Hence we have

$$\begin{aligned}
&\|z_0 - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)z_0\| \\
&\leq \|z_0 - y_{n_i}\| + \|y_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_{n_i}\| \\
&\qquad + \|J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)x_{n_i} - J_{\lambda_0}(I - \lambda_0 A^*(I-T)A)z_0\| \to 0.
\end{aligned}$$

This implies $z_0 \in B^{-1}0 \cap A^{-1}F(T)$ by Lemma 2.5. Since $z_0 = P_{C_0}u \in B^{-1}0 \cap A^{-1}F(T)$ and $B^{-1}0 \cap A^{-1}F(T) \subset C_0$, we have $z_0 = P_{B^{-1}0\cap A^{-1}F(T)}u$, which completes the proof. □

4. Applications

Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty, \infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y),\ \forall y \in H\}$$

for all $x \in H$. By Rockafellar [17], it is shown that ∂f is maximal monotone. Let C be a nonempty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Then $i_C : H \to (-\infty, \infty]$ is a proper, lower semicontinuous and convex function on H and hence ∂i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$ as follows:

$$J_\lambda x = (I + \lambda \partial i_C)^{-1}x, \quad \forall x \in H,\ \lambda > 0.$$

On the other hand, for any $u \in C$, we also define the normal cone $N_C(u)$ of C at u as follows:

$$N_C(u) = \{z \in H : \langle z, y - u\rangle \leq 0,\ \forall y \in C\}.$$

Then we have that for any $x \in C$

$$\begin{aligned}
\partial i_C(x) &= \{z \in H : i_C(x) + \langle z, y - x\rangle \leq i_C(y),\ \forall y \in H\} \\
&= \{z \in H : \langle z, y - x\rangle \leq 0,\ \forall y \in C\} = N_C(x).
\end{aligned}$$

Thus we have that

$$\begin{aligned}
u = J_\lambda x &\Leftrightarrow (I + \lambda\partial i_C)^{-1}x = u \Leftrightarrow x \in u + \lambda \partial i_C(u) \\
&\Leftrightarrow x \in u + \lambda N_C(u) \Leftrightarrow x - u \in \lambda N_C(u) \\
&\Leftrightarrow \langle x - u, y - u\rangle \leq 0, \quad \forall y \in C \\
&\Leftrightarrow P_C(x) = u.
\end{aligned}$$

Putting $B = \partial i_C$ in Theorems 3.1 and 3.2, we have $J_{\lambda_n} = P_C$ for any $n \in \mathbb{N}$. Using this fact, we can obtain strong convergence theorems in a Hilbert space. We first obtain the following theorem of Wittmann [29] from Theorem 3.1; see also Halpern [12] and Takahashi [22].

Theorem 4.1 ([29]). *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let U be a nonexpansive mapping of C into itself such that $F(U) \neq \emptyset$. Let $u \in C$, $x_1 = x \in C$ and let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Ux_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $F(U)$, where $z_0 = P_{F(U)}u$.

Proof. Set $H_1 = H_2 = H$, $B = \partial i_C$, $T = UP_C$, $A = I$ and $u_n = u$ for all $n \in \mathbb{N}$ in Theorem 3.1. Then we have that $F(T) = F(U)$ and $J_\lambda = P_C$ for all $\lambda > 0$. Furthermore, putting $\lambda_n = 1$ for all $n \in \mathbb{N}$, we have that for $x_1 = x \in C$ and $n \in \mathbb{N}$

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1 - \alpha_n)Ux_n \\ &= \alpha_n u + (1 - \alpha_n)P_C UP_C x_n \\ &= \alpha_n u + (1 - \alpha_n)P_C T x_n \\ &= \alpha_n u + (1 - \alpha_n)P_C(x_n - (I - T)x_n) \\ &= \alpha_n u + (1 - \alpha_n)P_C(I - 1 \cdot I^*(I - T)I)x_n. \end{aligned}$$

Thus we have the desired result from Theorem 3.1. □

We also get the following theorem from Theorem 3.2.

Theorem 4.2. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap A^{-1}F(T) \neq \emptyset$. Let $u \in H_1$, $x_1 = x \in H_1$ and let $\{x_n\}$ be a sequence in H_1 generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n u + (1 - \alpha_n)P_C(x_n - \lambda_n A^*(I - T)Ax_n)\}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \le \lambda_n \le \frac{1}{\|A^*A\|}, \qquad 0 < c \le \beta_n \le d < 1,$$
$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $C \cap A^{-1}F(T)$, where $z_0 = P_{C\cap A^{-1}F(T)}u$.

Furthermore, we obtain the following theorem from Theorem 3.3.

Theorem 4.3. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap A^{-1}F(T) \neq \emptyset$. Let $u \in H_1$, $x_1 \in H_1$, $C_1 = H_1$, and $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} y_n &= P_C(I - \lambda_n A^*(I-T)A)x_n, \\ C_{n+1} &= \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}}u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$ satisfies

$$0 < \lambda_n \leq \frac{1}{\|A^*A\|} \qquad \text{and} \qquad \limsup_{n\to\infty} \lambda_n > 0.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}F(T)$, where $z_0 = P_{C\cap A^{-1}F(T)}u$.

Let C be a nonempty, closed and convex subset of a real Hilbert space H, let $f : C \times C \to \mathbb{R}$ be a bifunction. Then we consider the following equilibrium problem: Find $z \in C$ such that

$$f(z,y) \geq 0, \quad \forall y \in C. \tag{33}$$

The set of such $z \in C$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{z \in C : f(z,y) \geq 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
(A3) $\limsup_{t\to 0} f(tz + (1-t)x, y) \leq f(x,y)$ for all $x, y, z \in C$;
(A4) $f(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$.

We know the following lemmas; see, for instance, [5] and [9].

Lemma 4.4 ([5]). *Let C be a nonempty closed convex subset of H, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in C$.

Lemma 4.5 ([9]). *Define the resolvent $T_r : H \to C$ of f for $r > 0$ as follows:*

$$T_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

(i) T_r *is single-valued;*
(ii) T_r *is firmly nonexpansive, i.e., for all* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(iii) $F(T_r) = EP(f)$;
(iv) $EP(f)$ *is closed and convex.*

Takahashi, Takahashi and Toyoda [20] showed the following. See [2] for a more general result.

Lemma 4.6 ([20]). *Let C be a nonempty, closed and convex subset of a Hibert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A1)–(A4). *Define A_f as follows:*

$$A_f(x) = \begin{cases} \{z \in H : f(x,y) \geq \langle y - x, z \rangle, \ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then $EP(f) = A_f^{-1}(0)$ and A_f is maximal monotone with the domain of A_f in C. Furthermore,

$$T_r(x) = (I + rA_f)^{-1}(x), \quad \forall r > 0.$$

We obtain the following theorem from Theorem 3.1.

Theorem 4.7. *Let H_1 and H_2 be Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions* (A1)–(A4) *and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 4.6. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}F(T) \neq \emptyset$. Let $u \in H_1$, $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq \frac{1}{\|A^*A\|}, \qquad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $EP(f) \cap A^{-1}F(T)$, where $z_0 = P_{EP(f)\cap A^{-1}F(T)}u$.

Proof. Define A_f for the bifunction f and set $B = A_f$ and set $u_n = u$ for all $n \in \mathbb{N}$ in Theorem 3.1. Thus we have the desired result from Theorem 3.1. $\square$

As in the proof of Theorem 4.7, we obtain the following result from Theorem 3.2. See also [1].

Theorem 4.8. *Let H_1 and H_2 be Hilbert spaces. Let C be a nonempty, closed and convex subset of a real Hilbert space H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions* (A1)–(A4) *and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma* 4.6*. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}F(T) \neq \emptyset$. Let $u \in H_1$, $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)T_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq \frac{1}{\|A^*A\|}, \qquad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point z_0 of $EP(f) \cap A^{-1}F(T)$, where $z_0 = P_{EP(f)\cap A^{-1}F(T)}u$.

Using Theorem 3.3, we finally get the following theorem.

Theorem 4.9. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions* (A1)–(A4) *and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma* 4.6*. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $EP(f) \cap A^{-1}F(T) \neq \emptyset$. Let $u \in H_1$, $x_1 \in H_1$, $C_1 = H_1$, and $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} y_n &= T_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n, \\ C_{n+1} &= \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}}u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$ satisfies

$$0 < \lambda_n \leq \frac{1}{\|A^*A\|} \quad \text{and} \quad \limsup_{n\to\infty} \lambda_n > 0.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in EP(f) \cap A^{-1}F(T)$, where $z_0 = P_{EP(f)\cap A^{-1}F(T)}u$.

References

[1] K. Aoyama, Y. Kimura: Strong convergence theorems for strongly nonexpansive sequences, Appl. Math. Comput. 217 (2011) 7537–7545.

[2] K. Aoyama, Y. Kimura, W. Takahashi: Maximal monotone operators and maximal monotone functions for equilibrium problems, J. Convex Analysis 15 (2008) 395–409.

[3] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda: Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007) 2350–2360.

[4] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda: On a strongly nonexpansive sequence in Hilbert spaces, J. Nonlinear Convex Anal. 8 (2007) 471–489.

[5] E. Blum, W. Oettli: From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123–145.

[6] C. Byrne, Y. Censor, A. Gibali, S. Reich: The split common null point problem, J. Nonlinear Convex Anal. 13 (2012) 759–775.

[7] Y. Censor, T. Elfving: A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8 (1994) 221–239.

[8] Y. Censor, A. Segal: The split common fixed-point problem for directed operators, J. Convex Analysis 16 (2009) 587–600.

[9] P. L. Combettes, S. A. Hirstoaga: Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.

[10] H. Cui, F. Wang: Strong convergence of the gradient-projection algorithm in Hilbert spaces, J. Nonlinear Convex Anal. 14 (2013) 245–251.

[11] K. Eshita, W. Takahashi: Approximating zero points of accretive operators in general Banach spaces, JP J. Fixed Point Theory Appl. 2 (2007) 105–116.

[12] B. Halpern: Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 957–961.

[13] P. E. Maingé: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Appl. 16 (2008) 899–912.

[14] U. Mosco: Convergence of convex sets and of solutions of variational inequalities, Adv. Math. 3 (1969) 510–585.

[15] A. Moudafi: The split common fixed point problem for demicontractive mappings, Inverse Probl. 26 (2010), Article ID 055007, 6 pp.

[16] N. Nadezhkina, W. Takahashi: Strong convergence theorem by hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings, SIAM J. Optim. 16 (2006) 1230–1241,

[17] R. T. Rockafellar: On the maximal monotonicity of subdifferential mappings, Pacific J. Math. 33 (1970) 209–216.

[18] S. Takahashi, W. Takahashi: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506–515.

[19] S. Takahashi, W. Takahashi: Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008) 1025–1033.

[20] S. Takahashi, W. Takahashi, M. Toyoda: Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces, J. Optim. Theory Appl. 147 (2010) 27–41.

[21] W. Takahashi: Nonlinear Functional Analysis, Yokohama Publishers, Yokohama (2000).

[22] W. Takahashi: Convex Analysis and Approximation of Fixed Points, Yokohama Publishers, Yokohama (2000) (in Japanese).

[23] W. Takahashi: Introduction to Nonlinear and Convex Analysis, Yokohama Publishers, Yokohama (2009).

[24] W. Takahashi: Strong convergence theorems for maximal and inverse-strongly monotone mappings in Hilbert spaces and applications, J. Optim. Theory Appl. 157 (2013) 781–802.

[25] W. Takahashi, Y. Takeuchi, R. Kubota: Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276–286.

[26] W. Takahashi, N.-C. Wong, J.-C. Yao: Weak and strong mean convergence theorems for extended hybrid mappings in Hilbert spaces, J. Nonlinear Convex Anal. 12 (2011) 553–575.

[27] W. Takahashi, H.-K. Xu, J.-C. Yao: Iterative methods for generalized split feasibility problems in Hilbert spaces, Set-Valued Var. Anal. 23 (2015) 205–221.

[28] M. Tsukada: Convergence of best approximations in a smooth Banach space, J. Approx. Theory 40 (1984) 301–309.

[29] R. Wittmann: Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992) 486–491.

[30] H.-K. Xu: Another control condition in an iterative method for nonexpansive mappings, Bull. Aust. Math. Soc. 65 (2002) 109–113.

[31] H.-K. Xu: A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem, Inverse Probl. 22 (2006) 2021–2034.