Journal of Convex Analysis
Volume 22 (2015), No. 4, 939–962

Continuity and Selections of the Intersection Operator Applied to Nonconvex Sets

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Received: June 30, 2014
Revised manuscript received: November 28, 2014
Accepted: November 29, 2014

For a convex body C in a Banach space E we consider the class $\mathcal{S}(C)$ of closed sets $A \subset E$ satisfying the support condition with respect to C. If C is a ball with radius r, then $\mathcal{S}(C)$ is exactly the class of uniformly r-prox-regular sets. We prove that the intersection operator $(A, C) \mapsto A \cap C$ is uniformly Hausdorff continuous and has a uniformly continuous selection on the family of pairs (A, C) such that C is closed and uniformly convex, $rA \in \mathcal{S}(C)$ with $r \in (0, 1)$, and $A \cap C \neq \emptyset$. We also deduce some new sufficient condition for affirmative solution of the splitting problem for selections.

Keywords: Support condition, weak convexity, proximal regularity, quasiball, multifunction, selection

2010 Mathematics Subject Classification: 41A50, 41A65, 52A21

1. Introduction

The problems of the continuity and the existence of continuous selections for the intersection operator arises in many applications and such theories as the theory of dynamic systems with phase constraints, differential inclusions theory, set-valued analysis, optimal control theory, etc. Consider two Hausdorff continuous multifunctions $A : T \to 2^E$ and $C : T \to 2^E$ from the metric space (T, ϱ_T) to subsets of the Banach space E. What properties of the multifunctions are sufficient for the multifunction $t \mapsto F(t) := A(t) \cap C(t)$ to be Hausdorff continuous and to have a continuous selection on T? These and akin questions for multifunctions with convex values were investigated by Moreau [17], De Blasi and Pianigiani [7], Penot [18] and others. It was shown that to obtain the desired properties of $F(\cdot)$ one can assume that the multifunctions $A(\cdot)$ and $C(\cdot)$ are continuous, closed convex bounded valued and $F(t)$ has nonempty interior for all $t \in T$. The aim of our article is to obtain the desired properties of $F(\cdot)$ without assumptions about convexity of $A(t)$ and nonemptiness of the interior of $F(t)$.

*Supported by the Russian Foundation for basic research, project 13-01-00295-a.

ISSN 0944-6532 / $ 2.50

If $E = L_1$, the sufficient conditions for $F(\cdot)$ to have a continuous selection have been obtained by Fryszkowski [12] and Goncharov, Tolstonogov [13] in terms of decomposability. Balashov and Repovš in [3] showed that to obtain the desired properties of $F(\cdot)$ it suffices to assume that $A(t)$ is closed convex and $C(t)$ is closed and uniformly convex. The same authors in [4] reduced the convexity of $A(t)$ to some properties in terms of the modulus of nonconvexity. Instead of the modulus of nonconvexity we use the support condition. An advantage of the support condition approach is that it can be applied for any uniformly convex Banach space, while the modulus of nonconvexity approach demands that the convexity modulus of the Banach space is of the second order (see [4]).

The multifunction $t \mapsto F(t) = A(t) \cap C(t)$ can be considered as the superposition of the multifunctions $t \mapsto A(t)$, $t \mapsto C(t)$ and the intersection operator $(A, C) \mapsto A \cap C$. So, since the multifunctions $A(\cdot)$ and $C(\cdot)$ are continuous, to obtain the desired properties of $F(\cdot)$ we need to obtain the continuity and the existence of continuous selections for the intersection operator.

One more application of our results is the following splitting problem for selections, introduced by Repovš and Semenov [22]. Let A, B be closed convex sets (in original setting) or sets from some other class (in more general setting) and $X = A + B := \{a + b \mid a \in A,\ b \in B\}$. Do there necessary exist continuous functions $a : X \to A$ and $b : X \to B$ with the property that $a(x) + b(x) = x$ for all $x \in X$? It was shown in [2] that the answer in the original setting is negative. Balashov and Repovš [4, Example 4.1] showed that the answer to this question is affirmative provided that the set A is closed and weakly convex with some modulus $\gamma(\cdot)$ and the set B is closed and uniformly convex with modulus $\delta(\cdot)$ concerted with $\gamma(\cdot)$ by rather complicated relation. We show that instead of weak convexity of A with the modulus $\gamma(\cdot)$ one can assume that $rA := \{ra \mid a \in A\}$ satisfies the support condition with respect to $(-B)$ with some $r \in (0, 1)$.

The class $\mathcal{S}(C)$ of closed sets $A \subset E$ satisfying the support condition with respect to a convex body $C \subset E$ is sufficiently wide and includes both convex and smooth sets. This class is a distant descendant of Federer's class of sets with positive reach [11]. Recall that a set $A \subset \mathbb{R}^n$ is called a set with positive reach, if there exists some neighborhood of A such that its every point has a unique metric projection onto A. Akin properties of the metric projection in Hilbert space were investigated by Shapiro [23]. Clarke, Stern and Wolenski [8] introduced and studied the concept of the proximal smooth sets, which generalizes the concept of sets with positive reach to a Hilbert space. Poliquin, Rockafellar and Thibault [19] explored the concept of prox-regular sets and showed the link between prox-regularity, proximal smothness, Shapiro property and some other properties of sets in Hilbert space. Later concept of prox-regularity was extended on a Banach space by Bernard, Thibault and Zlateva in [5] and [6]. Colombo, Goncharov, Mordukhovich and Pereira [10], [14], [15] considered a time-minimum problem with constant dynamics. In fact, they generalized the concept of prox-regularity, substituting the norm of the space by a nonsymmetric norm. In other words, they substituted the unit ball by a bounded quasiball. We use term "quasiball" for a closed convex set with the origin in its interior. The following step of extension was made in [16], where weakly convex sets with respect to an unbounded quasiball were considered. The properties of these sets were used in [16] to investigate some minimization problems. It turned out that the notion of weak convexity with respect to a quasiball is invariant to translation of the quasiball in such a way that the origin remains in its interior. This observation leads to the class $\mathcal{S}(C)$ of closed sets, satisfying the support condition with respect to the convex body C. This class generalizes the classes of sets with positive reach, proximally smooth sets, uniformly r-prox-regular sets, and weakly convex sets with respect to a quasiball.

The rest of the paper is organized as follows. In Section 2 we introduce the support condition for sets and discuss elementary properties of such sets. Section 3 is devoted to the relations between the support condition, weak convexity and r-prox-regularity. The main results (Theorems 4.1–4.4) are formulated in Section 4. The proof of Theorem 4.1 is given in Section 5. Section 6 is devoted to the problem of continuity of the metric projection. Other main results are proved in Section 7. In Section 8 we discuss the relation between the support condition and the modulus of nonconvexity.

2. The support condition

Let E be a real Banach space. For a set $A \subset E$ by $\operatorname{int} A$ and ∂A we denote the interior and the boundary of A, respectively. We use $\langle p, x\rangle$ to denote the value of the functional $p \in E^*$ at the vector $x \in E$. For $R \geq 0$ and $c \in E$ we denote by $\mathfrak{B}_R(c)$ the closed ball of radius R with center c. We put $\mathfrak{B}_R = \mathfrak{B}_R(0)$. A set $C \subset E$ is called *convex body* if it is closed convex and $\operatorname{int} C \neq \emptyset$.

Definition 2.1. A set $A \subset E$ is said to satisfy the *support condition* with respect to a convex body $C \subset E$ if for all $t \in (0,1)$ and for all $a \in A$, $c \in C$ the relation

$$(A - a) \cap t(\operatorname{int} C - c) = \emptyset$$

implies that (see Fig. 2.1)

$$(A - a) \cap (\operatorname{int} C - c) = \emptyset. \tag{1}$$

We use $\mathcal{S}(C)$ to denote the class of all closed sets $A \subset E$, satisfying the support condition with respect to the convex body $C \subset E$. If $C \subset E$ is a singleton, we suppose that $\mathcal{S}(C)$ is the class of all closed sets $A \subset E$.

Remark 2.2. If we substitute $a \in A$, $c \in C$ with $a \in \partial A$, $c \in \partial C$ in Definition 2.1, we get an equivalent definition. Indeed, for any convex body $C \subset E$, closed set $A \subset E$, number $t \in (0,1)$, and points $a \in A$, $c \in C$ the equality $(A - a) \cap t(\operatorname{int} C - c) = \emptyset$ implies that $a \in \partial A$, $c \in \partial C$. On the other hand, by closedness of A and C the inclusions $a \in \partial A$, $c \in \partial C$ imply that $a \in A$, $c \in C$.

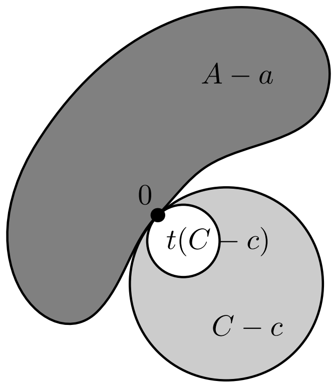

Figure 2.1

Lemma 2.3. *For any convex body $C \subset E$ the class $\mathcal{S}(C)$ includes all closed convex subsets of E.*

Proof. Let $C \subset E$ be a convex body, $A \subset E$ be a closed convex set. We claim that $A \in \mathcal{S}(C)$. Suppose that $t \in (0,1)$, $a \in A$, and $c \in C$ are such that $(A-a) \cap t(\operatorname{int} C - c) = \emptyset$. It suffices to deduce (1). By the Hahn–Banach separation theorem there exists a functional $f \in E^*$ such that

$$\langle f, a'-a\rangle < t\langle f, c'-c\rangle, \quad \forall a' \in A,\ \forall c' \in \operatorname{int} C. \tag{2}$$

Taking $a' = a$, we have

$$\langle f, c'-c\rangle > 0, \quad \forall c' \in \operatorname{int} C.$$

Using (2), we get

$$\langle f, a'-a\rangle < \langle f, c'-c\rangle, \quad \forall a' \in A,\ \forall c' \in \operatorname{int} C.$$

This implies (1). □

To clarify the geometrical meaning of the support condition, we compare it with a more simple and visual property for a convex body C to support a set A at a point a.

Definition 2.4. A convex body $C \subset E$ is said to *support* a set $A \subset E$ at a point $a \in \partial A$ if there exists a point $c \in \partial C$ such that (1) holds.

Lemma 2.5. *Let $C \subset E$ be a compact convex body, $A \in \mathcal{S}(C)$. Then C supports A at any point $a \in \partial A$.*

Proof. Fix a point $a \in \partial A$. There exists a sequence $\{x_k\} \subset E \setminus A$ such that

$$\lim_{k\to\infty} x_k = a. \tag{3}$$

Let us fix a point $c_0 \in \operatorname{int} C$ and for all $k \in \mathbb{N}$ let us denote

$$\varrho_k = \inf\{t > 0 \mid A \cap (x_k + t(C - c_0)) \neq \emptyset\}. \tag{4}$$

Since C is compact and A is closed, it follows that for any $k \in \mathbb{N}$ there exists $c_k \in C$ such that the point

$$a_k = x_k + \varrho_k(c_k - c_0) \tag{5}$$

belongs to A. Using once more the compactness of C and taking a proper subsequence $\{c_k\}$, we suppose that $\{c_k\}$ converges to some $\hat{c} \in C$. Bearing in mind (3), (4) and that $c_0 \in \operatorname{int} C$, we deduce that $\lim_{k\to\infty} \varrho_k = 0$. Hence, without loss of generality, we assume that $\varrho_k < 1$ for all $k \in \mathbb{N}$. Taking into account (3) and (5), we get

$$\lim_{k\to\infty} a_k = a. \tag{6}$$

The relations (5) and $a_k \in A$, $x_k \notin A$ imply that $\varrho_k > 0$. Thus, by (4) we have $A \cap (x_k + \varrho_k(\operatorname{int} C - c_0) = \emptyset$. Hence, due to (5) we get $(A - a_k) \cap \varrho_k(\operatorname{int} C - c_k) = \emptyset$, wherein $a_k \in \partial A$ and $c_k \in \partial C$. Since $A \in \mathcal{S}(C)$, it follows that $(A - a_k) \cap (\operatorname{int} C - c_k) = \emptyset$ for all $k \in \mathbb{N}$. Using the relations $\lim_{k\to\infty} c_k = \hat{c}$ and (6), we deduce that $(A - a) \cap (\operatorname{int} C - \hat{c}) = \emptyset$. So, C supports A at a. □

Remark 2.6. Under the assumptions of Lemma 2.5 the space E is finite dimensional, since the space includes a compact set C with nonempty interior.

Lemma 2.7. *Let A and C be convex bodies in a reflexive Banach space E, let C be bounded. Then C supports A at any point $a \in \partial A$.*

Proof. Fix a point $a \in \partial A$. By the Hahn–Banach separation theorem there exists a unit functional $f \in E^*$ such that $\langle f, a\rangle = \sup_{x\in A}\langle f, x\rangle$. Using boundness of C and reflexivity of E, one can find a point $c \in C$ such that $\langle f, c\rangle = \inf_{x\in C}\langle f, x\rangle$. So, (1) holds. □

Remark 2.8. The assumptions of Lemma 2.5 about the compactness of C and of Lemma 2.7 about the boundedness of C are essential. Really, let us consider the following convex bodies in $E = \mathbb{R}^2$:

$$A = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \le 0\},$$
$$C = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0,\ x_2 \ge \frac{1}{x_1}\right\}.$$

Lemma 2.3 implies that $A \in \mathcal{S}(C)$. However C doesn't support A at any $a \in \partial A$.

Remark 2.9. In infinite dimensional space E the condition $A \in \mathcal{S}(\mathfrak{B}_1)$ doesn't imply that the unit ball $\mathfrak{B}_1$ supports A at any point $a \in \partial A$. Indeed, consider the following set in the Hilbert space $E = \ell_2$:

$$A = \{x = (x_1, x_2, \dots) \in \ell_2 \mid x_k \ge 0,\ \forall k \in \mathbb{N}\}.$$

Lemma 2.3 implies that $A \in \mathcal{S}(\mathfrak{B}_1)$. Nevertheless, the unit ball $\mathfrak{B}_1$ doesn't support A at the point $a = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, \dots\right) \in \partial A$.

In the latter example the interior of A is empty. However, there is a more complicated example of a set $A \in \mathcal{S}(\mathfrak{B}_1)$ in ℓ_2 such that A is the closure of its interior, but the unit ball $\mathfrak{B}_1$ doesn't support A at the point $a = 0 \in \partial A$. This example is given by the formula

$$A = \left\{ x = (x_0, x_1, x_2, \ldots) \in \ell_2 \;\middle|\; 4 \cdot \sum_{k=1}^{\infty} \varphi_k^2(x_k) \leq x_0^4 \right\},$$

where

$$\varphi_k(t) = \begin{cases} 0, & |t| \leq \frac{1}{k}, \\ |t| - \frac{1}{k}, & |t| > \frac{1}{k}. \end{cases}$$

Remark 2.10. The fact that a convex compact body $C \subset E$ supports a set $A \subset E$ at any point $a \in \partial A$ doesn't imply that $A \in \mathcal{S}(C)$. Indeed, consider the closed set $A = \{(0,0),(1,0)\}$ in $E = \mathbb{R}^2$. The unit ball $\mathfrak{B}_1$ supports A at any point $a \in \partial A$, but $A \notin \mathcal{S}(\mathfrak{B}_1)$.

The following two lemmas state simple properties of the class $\mathcal{S}(C)$.

Lemma 2.11. *Let $C \subset E$ be a convex body, $x, y \in E$, $A \in \mathcal{S}(C)$. Then $(A + x) \in \mathcal{S}(C + y)$.*

Proof. Let us denote $A' = A + x$, $C' = C + y$. Assume that $t \in (0,1)$, $a' \in A'$, $c' \in C'$ are such that $(A' - a') \cap t(\operatorname{int} C' - c') = \emptyset$. It suffices to prove that

$$(A' - a') \cap (\operatorname{int} C' - c') = \emptyset. \tag{7}$$

Denote $a = a' - x$, $c = c' - y$. Then $a \in A$, $c \in C$ and $(A - a) \cap t(\operatorname{int} C - c) = (A' - a') \cap t(\operatorname{int} C' - c') = \emptyset$. Since $A \in \mathcal{S}(C)$, it follows that $(A - a) \cap (\operatorname{int} C - c) = \emptyset$. This yields (7). □

Lemma 2.12. *Let $C \subset E$ be a convex body, $0 < \lambda \leq \tau$, $A \in \mathcal{S}(C)$. Then $\tau A \in \mathcal{S}(\lambda C)$.*

Proof. Assume that $t \in (0,1)$, $a' \in \tau A$, $c' \in \lambda C$ satisfy the condition $(\tau A - a') \cap t(\lambda \operatorname{int} C - c') = \emptyset$. We need to prove that

$$(\tau A - a') \cap (\lambda \operatorname{int} C - c') = \emptyset. \tag{8}$$

Denote $a = \frac{1}{\tau} a'$, $c = \frac{1}{\lambda} c'$. We have $a \in A$, $c \in C$, $\tau(A - a) \cap t\lambda(\operatorname{int} C - c) = \emptyset$, $\frac{t\lambda}{\tau} \in (0,1)$. Bearing in mind that $A \in \mathcal{S}(C)$, we get $(A - a) \cap (\operatorname{int} C - c) = \emptyset$. By convexity of C and using the inclusion $c \in C$, we deduce that $\frac{\lambda}{\tau}(\operatorname{int} C - c) \subset \operatorname{int} C - c$. Consequently, $\tau(A - a) \cap \lambda(\operatorname{int} C - c) = \emptyset$. So, (8) holds. □

3. Weak convexity, r-prox-regularity and the support condition

We say that $M \subset E$ is a *quasiball* if M is closed convex and $0 \in \operatorname{int} M$.

Let $M \subset E$ be a quasiball. Recall that the *Minkowski functional* of M is defined by

$$\mu_M(x) = \inf \left\{ t > 0 \,\middle|\, x \in tM \right\}, \quad \forall x \in E. \tag{9}$$

Remark 3.1. The Minkowski functional of any quasiball $M \subset E$ is sublinear, i.e.

$$\mu_M(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1 \mu_M(x_1) + \lambda_2 \mu_M(x_2), \quad \forall x_1, x_2 \in E, \ \forall \lambda_1, \lambda_2 \ge 0.$$

Through $\varrho_M(x_0, A)$ we denote the *M-distance* from the point $x_0 \in E$ to the set $A \subset E$:

$$\varrho_M(x_0, A) = \inf_{a \in A} \mu_M(x_0 - a). \tag{10}$$

Remark 3.2. For any quasiball $M \subset E$ and any set $A \subset E$ the functions $\mu_M(\cdot)$ and $\varrho_M(\cdot, A)$ are Lipschitz continuous on E with the constant $\frac{1}{\sigma}$, where $\sigma = \sup\{r > 0 \mid \mathfrak{B}_r \subset M\} = \inf_{x \in \partial M} \|x\|$.

Remark 3.3. For any quasiball $M \subset E$, set $A \subset E$, and point $x_0 \in E$ we have

$$\varrho_M(x_0, A) = \inf\{t > 0 \mid A \cap (x_0 - tM) \ne \emptyset\}.$$

Given a set $A \subset E$ and a point $x_0 \in E$, the *M-projection* of x_0 onto A or, in other words, the *set of best approximations* is

$$P_M(x_0, A) = \left\{ a \in A \,\middle|\, \mu_M(x_0 - a) \le \varrho_M(x_0, A) \right\}. \tag{11}$$

For every $\eta > 0$ we also denote the *set of nearly best approximations*:

$$P_M^{\eta}(x_0, A) = \left\{ a \in A \,\middle|\, \mu_M(x_0 - a) \le \varrho_M(x_0, A) + \eta \right\}. \tag{12}$$

The set of *unit M-normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N_M^1(a, A) = \{z \in \partial M \mid \exists t > 0 : \ a \in P_M(a + tz, A)\}. \tag{13}$$

Remark 3.4. If $M = \mathfrak{B}_1$, then the Minkowski functional $\mu_M(\cdot)$ is the norm of E, the M-distance is the common distance, the M-projection is the metric projection, and the set of unit M-normals is the set of unit proximal normals.

Definition 3.5. A set $A \subset E$ is called *weakly convex* with respect to the quasiball $M \subset E$ if

$$a \in P_M(a + z, A), \quad \forall a \in A, \ \forall z \in N_M^1(a, A). \tag{14}$$

Lemma 3.6. *Let $M \subset E$ be a quasiball, $A \subset E$, $a \in A$, $z \in \partial M$, $t > 0$. Then the following assertions are equivalent:*

(*i*) $\varrho_M(a + tz, A) = t$;
(*ii*) $\varrho_M(a + tz, A) \geq t$;
(*iii*) $a \in P_M(a + tz, A)$;
(*iv*) $A \cap (a + tz - t \operatorname{int} M) = \emptyset$.

Proof. Since $z \in \partial M$, by (9) it follows that $\mu_M(z) = 1$. Using (10) and $a \in A$, we get $\varrho_M(a + tz, A) \leq \mu_M(tz) = t$. Thus, (*i*) $\Leftrightarrow$ (*ii*).

Using (11), $\mu_M(z) = 1$, and $a \in A$, we obtain (*ii*) $\Leftrightarrow$ (*iii*). Remark 3.3 implies that (*ii*) $\Leftrightarrow$ (*iv*). □

Lemma 3.7. *For any quasiball $M \subset E$ the following assertions are equivalent:*

(*i*) $A \in \mathcal{S}(-M)$;
(*ii*) *$A \subset E$ is closed and weakly convex with respect to M.*

Proof. (*i*) $\Rightarrow$ (*ii*). Assume that (*i*) holds. Then A is closed by Definition 2.1. We need to prove that A is weakly convex with respect to the quasiball M. Take any $a \in A$ and $z \in N^1_M(a, A)$. Due to (13) there exists $t > 0$ such that $a \in P_M(a + tz, A)$. Lemma 3.6 implies that $A \cap (a + tz - t \operatorname{int} M) = \emptyset$. Choose $\tau \in (0, 1)$ such that $\tau \leq t$. Since M is convex and $z \in M$, it follows that $\tau(z - \operatorname{int} M) \subset t(z - \operatorname{int} M)$. Hence, $(A - a) \cap \tau(z - \operatorname{int} M) = \emptyset$. Using (*i*), we get $(A - a) \cap (z - \operatorname{int} M) = \emptyset$. Lemma 3.6 implies that $a \in P_M(a + z, A)$. So, by Definition 3.5 (*ii*) holds.

(*ii*) $\Rightarrow$ (*i*). Assume that (*ii*) holds. It suffices to show that $A \in \mathcal{S}(-M)$. Suppose that $t \in (0, 1)$ and $a \in A$, $z \in M$ are such that $(A - a) \cap t(z - \operatorname{int} M) = \emptyset$. We need to prove that $(A - a) \cap (z - \operatorname{int} M) = \emptyset$. According to Remark 2.2, we have $z \in \partial M$. Lemma 3.6 implies that $a \in P_M(a + tz, A)$. Using (*ii*), we get $a \in P_M(a + z, A)$. Thus, by Lemma 3.6 we deduce that $(A - a) \cap (z - \operatorname{int} M) = \emptyset$. □

Remark 3.8. If E is a strongly convex space, then $\mathcal{S}(\mathfrak{B}_r)$ is exactly the class of r-prox-regular sets [5], [6] and is the same as the family of sets satisfying the support condition of weak convexity with constant r (see [1]). It can be easily seen by comparing Definition 3.5 with corresponding definitions in the cited works and using Lemma 3.7. For Hilbert space the class $\mathcal{S}(\mathfrak{B}_r)$ is the same as the class of proximally smooth sets [8]. In finite dimensional Euclidean setting this class is the same as the class of sets with positive reach [11]. In more general case, when M is a quasiball in a Banach space, the class $\mathcal{S}(M)$ was considered in [10], [14], [15], [16].

Remark 3.9. The class of closed and weakly convex sets with respect to a quasiball M is invariant to translation of M in such a way that the origin remains in its interior (i.e., translated M is a quasiball). This is a direct consequence of Lemmas 2.11 and 3.7.

The *modulus of convexity* of a set $M \subset E$ [20] is the following function for $\varepsilon \geq 0$:

$$\delta_M(\varepsilon) = \inf \left\{ \delta > 0 \;\middle|\; \exists a, b \in M : \; \mathfrak{B}_\delta \left(\frac{a+b}{2} \right) \not\subset M, \; \|a - b\| = \varepsilon \right\}. \tag{15}$$

As usual, the infimum of the empty set is $+\infty$. If $M = \mathfrak{B}_1$, then the function $\delta_M(\cdot)$ is the modulus of convexity of the space E, introduced by Clarkson [9].

A set $M \subset E$ is said to be *uniformly convex* if $\delta_M(\varepsilon) > 0$ for all $\varepsilon > 0$.

Remark 3.10. If a set $M \subset E$ is uniformly convex, then it is convex.

Proof. Let a and b be two different elements of a uniformly convex set M. Then there exists $\delta > 0$ such that $\mathfrak{B}_\delta\left(\frac{a+b}{2}\right) \subset M$. Fix any $z \in \left[a, \frac{a+b}{2}\right]$. One can find $c \in [a, b] \cap \mathfrak{B}_\delta\left(\frac{a+b}{2}\right)$ and $\lambda = \frac{m}{2^n} \in [0, 1]$ with $m, n \in \mathbb{N}$ such that $z = (1 - \lambda)a + \lambda c$. The property $x, y \in M \;\Rightarrow\; \frac{x+y}{2} \in M$ applied n times yields $z \in M$. So, $\left[a, \frac{a+b}{2}\right] \subset M$ and, analogically, $\left[\frac{a+b}{2}, b\right] \subset M$. Thus, $[a, b] \subset M$. $\square$

Remark 3.11. If there exists a uniformly convex body in E, then E admits an equivalent uniformly convex norm, i.e. E is superreflexive.

Proof. Theorem 2.1 in [3] implies that any uniformly convex body $M \subset E$ is bounded. Fix any $c \in \operatorname{int} M$. Since $B = (M - c) \cap (c - M)$ is uniformly convex, bounded and symmetric body, it is the unit ball of some uniformly convex norm, equivalent to the norm of E. $\square$

Proposition 3.12 ([16, Theorem 2.5.]). *Let $M \subset E$ be a uniformly convex quasiball and $A \subset E$ be a closed set. Then the following statements are equivalent:*

(*i*) *A is weakly convex with respect to M;*
(*ii*) *the M-projection mapping $x \mapsto P_M(x, A)$ is single-valued and continuous on $U_M(A) = \{x \in E \mid 0 < \varrho_M(x, A) < 1\}$;*
(*iii*) *for any point $x_0 \in U_M(A)$ and any sequence $\{a_k\} \subset A$ the condition $\lim_{k\to\infty} \mu_M(x_0 - a_k) = \varrho_M(x_0, A)$ implies that $\{a_k\}$ converges.*

4. Main results

The *Hausdorff excess* of $A_1 \subset E$ over $A_2 \subset E$ is

$$h^+(A_1, A_2) = \sup_{a_1 \in A_1} \inf_{a_2 \in A_2} \|a_1 - a_2\|. \tag{16}$$

The *Pompeiu–Hausdorff distance* between $A_1 \subset E$ and $A_2 \subset E$ is

$$h(A_1, A_2) = \max\{h^+(A_1, A_2), \; h^+(A_2, A_1)\}.$$

Let (T, ϱ_T) be a metric space. A multifunction $F : T \to 2^E$ is said to be *Hausdorff continuous* if for any $t_0 \in T$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $h(F(t), F(t_0)) < \varepsilon$ for all $t \in T$, satisfying the inequality $\varrho_T(t, t_0) < \delta$. A multifunction $F : T \to 2^E$ is said to be *uniformly Hausdorff continuous*, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $h(F(t_1), F(t_2)) < \varepsilon$ for all $t_1, t_2 \in T$, satisfying the inequality $\varrho_T(t_1, t_2) < \delta$.

Given a family Φ of pairs (A, C), where $A \subset E$ and $C \subset E$, we say that the *intersection operator* $(A, C) \mapsto A \cap C$ is *uniformly Hausdorff continuous* on Φ, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $h(A_1 \cap C_1, A_2 \cap C_2) < \varepsilon$ for all $(A_1, C_1) \in \Phi$ and $(A_2, C_2) \in \Phi$, satisfying the inequality $h(A_1, A_2) + h(C_1, C_2) < \delta$.

Theorem 4.1. *Let $r \in (0, 1)$ and let Φ be a family of pairs (A, C), where $C \subset E$ is closed and uniformly convex, $A \subset E$ satisfies the inclusion*

$$rA \in \mathcal{S}(C). \tag{17}$$

Suppose that

$$\inf_{(A,C)\in\Phi} \delta_C(\varepsilon) > 0, \quad \forall \varepsilon > 0 \tag{18}$$

and

$$A \cap C \neq \emptyset, \quad \forall (A, C) \in \Phi. \tag{19}$$

Then the intersection operator $(A, C) \mapsto A \cap C$ is uniformly Hausdorff continuous on Φ.

Recall that a function $f : T \to E$ is called a *selection* of a multifunction $F : T \to 2^E$ if

$$f(t) \in F(t), \quad \forall t \in T.$$

Given a family Φ of pairs (A, C), where $A \subset E$ and $C \subset E$, we say that a function $\xi : \Phi \to E$ is a *selection of the intersection operator* $(A, C) \mapsto A \cap C$ if

$$\xi(A, C) \in A \cap C, \quad \forall (A, C) \in \Phi. \tag{20}$$

Theorem 4.2. *Under the assumptions of Theorem* 4.1 *the intersection operator $\Phi \ni (A, C) \mapsto A \cap C$ has a uniformly continuous selection $\xi : \Phi \to E$.*

Theorems 4.1 and 4.2 immediately imply the following one.

Theorem 4.3. *Let (T, ϱ_T) be a metric space, $r \in (0, 1)$. Suppose that the multifunctions $A : T \to 2^E$ and $C : T \to 2^E$ are such that for any $t \in T$ the set $C(t)$ is closed and uniformly convex, $A(t)$ satisfies the inclusion*

$$rA(t) \in \mathcal{S}(C(t)), \quad \forall t \in T. \tag{21}$$

Suppose that

$$\inf_{t\in T} \delta_{C(t)}(\varepsilon) > 0, \quad \forall \varepsilon > 0$$

and

$$F(t) = A(t) \cap C(t) \neq \emptyset, \quad \forall t \in T.$$

Then the following statements hold:

(*i*) *if the multifunctions $A : T \to 2^E$ and $C : T \to 2^E$ are Hausdorff continuous, then the multifunction $F(\cdot)$ is Hausdorff continuous and has a continuous selection $f : T \to E$;*

(*ii*) *if the multifunctions $A : T \to 2^E$ and $C : T \to 2^E$ are uniformly Hausdorff continuous, then the multifunction $F(\cdot)$ is uniformly Hausdorff continuous and has a uniformly continuous selection $f : T \to E$.*

Theorem 4.4 (On the splitting problem). *Let $B \subset E$ be uniformly convex,*

$$rA \in \mathcal{S}(-B), \tag{22}$$

where $r \in (0, 1)$. Then for $X = A+B$ there exist uniformly continuous functions $a : X \to A$ and $b : X \to B$ such that $a(x) + b(x) = x$ for all $x \in X$.

5. Proof of Theorem 4.1

The *diameter* of $A \subset E$ is defined as

$$\operatorname{diam} A = \sup_{x,\, y \in A} \|x - y\|. \tag{23}$$

Remark 5.1. The modulus of convexity for any convex set $M \subset E$ is a nondecreasing function and

$$\delta_M(\varepsilon) \le \frac{\varepsilon}{2}, \quad \forall \varepsilon \in (0, \operatorname{diam} M), \tag{24}$$

$$\delta_M(\varepsilon) = +\infty, \quad \forall \varepsilon > \operatorname{diam} M. \tag{25}$$

Remark 5.2. For any $x \in E$ and $t \in \mathbb{R}$ the moduli of convexity of sets $M \subset E$ and $C = tM + x$ are related by the following equation:

$$\delta_C(|t|\varepsilon) = |t|\delta_M(\varepsilon), \quad \forall \varepsilon \ge 0. \tag{26}$$

Proposition 5.3. *The modulus of convexity of any convex set $M \subset E$ satisfies the following inequality:*

$$\delta_M(\lambda\varepsilon) \le \lambda\delta_M(\varepsilon), \quad \forall \lambda \in (0, 1),\ \forall \varepsilon > 0. \tag{27}$$

If $\varepsilon < \operatorname{diam} M$, the inequality (27) is proved in [3, Lemma 2.1]. If $\varepsilon = \operatorname{diam} M$ and $\delta_M(\varepsilon) < +\infty$, this inequality can be proved in the same way. In all other cases (25) implies that $\delta_M(\varepsilon) = +\infty$ and (27) holds trivially.

Theorem 2.1 in [3] implies the following proposition.

Proposition 5.4. *If a set $M \subset E$ is uniformly convex, then it is bounded and*

$$\operatorname{diam} M \le 1 + \frac{1}{\delta_M(1)},$$

where we take $\frac{1}{\delta_M(1)} = 0$ in the case of $\delta_M(1) = +\infty$.

Lemma 5.5. *Suppose that a uniformly convex quasiball $M \subset E$ is such that $M \subset \mathfrak{B}_R$, where $R > 0$. Then for any $x, y \in E \setminus \{0\}$ we have*

$$\delta_M\left(\left\|\frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)}\right\|\right) \le \frac{R \cdot (\mu_M(x) + \mu_M(y) - \mu_M(x+y))}{\min\{\mu_M(x), \mu_M(y)\}}.$$

Proof. Let us denote

$$a = \frac{x}{\mu_M(x)}, \qquad b = \frac{y}{\mu_M(y)}, \qquad c = \frac{a+b}{2}, \qquad \delta = \delta_M(\|a-b\|).$$

Since $a, b \in M \subset \mathfrak{B}_R$, it follows that $\|c\| \le R$. Bearing in mind (15) and closedness of M, we get $\mathfrak{B}_\delta(c) \subset M$. Hence, $\left(1 + \frac{\delta}{R}\right) c \in M$ and, therefore, $\mu_M(c) \le \frac{1}{1+\frac{\delta}{R}}$. Taking into account (24), we have $\delta = \delta_M(\|a-b\|) \le \frac{\|a-b\|}{2} \le R$ and, consequently,

$$\mu_M(c) \le 1 - \frac{\delta}{2R}. \tag{28}$$

Denote $\mu_1 = \mu_M(x)$, $\mu_2 = \mu_M(y)$. Without loss of generality we assume that $\mu_2 \le \mu_1$. Using Remark 3.1 and the equation $\mu_M(a) = 1$, we obtain

$$\begin{aligned}\mu_M(x+y) &= \mu_M((\mu_1 - \mu_2)a + \mu_2(a+b)) \\ &\le (\mu_1 - \mu_2) \cdot \mu_M(a) + \mu_2 \cdot \mu_M(a+b) = \mu_1 - \mu_2 + 2\mu_2 \cdot \mu_M(c).\end{aligned}$$

By (28), we get

$$\mu_M(x+y) \le \mu_1 + \mu_2 - \frac{\mu_2 \delta}{R} = \mu_1 + \mu_2 - \frac{\delta}{R}\min\{\mu_1, \mu_2\}. \qquad \square$$

Lemma 5.6. *Suppose that a uniformly convex quasiball $M \subset E$ is such that $M \subset \mathfrak{B}_R$ for some $R > 0$. Let $A \in \mathcal{S}(-M)$ and $x_0 \in E$ be such that $\varrho_M(x_0, A) = \varrho_0 < 1$; let $\varepsilon_0 > 0$ and $0 < \eta < \min\left\{\frac{1}{2}, \frac{(1-\varrho_0)\delta_M(\varepsilon_0)}{R}\right\}$. Suppose that $a_0 \in P_M(x_0, A)$, $a \in P_M^\eta(x_0, A)$ (see (12)). Then*

$$\|a - a_0\| < \frac{3}{2}\varepsilon_0.$$

Proof. By (11), (12) we have $\max\{\mu_M(x_0 - a), \mu_M(x_0 - a_0)\} \le \varrho_0 + \eta < 1 + \eta < \frac{3}{2}$. If $\varepsilon_0 \ge \operatorname{diam} M$, then $\|a - a_0\| < \frac{3}{2}\operatorname{diam} M \le \frac{3}{2}\varepsilon_0$ and the proof is completed. Therefore we assume that $\varepsilon_0 < \operatorname{diam} M$. Using (24), we get

$$R\eta < \delta_M(\varepsilon_0) \le \frac{\varepsilon_0}{2}. \tag{29}$$

If $\varrho_0 = 0$, then $x_0 \in A$, $a_0 = x_0$ and, using (29), we get $\|a - x_0\| \le R\eta < \frac{\varepsilon_0}{2}$. So, in this case the required inequality also holds true. Therefore, we suppose that $\varrho_0 > 0$. Let us denote $z = \frac{x_0 - a_0}{\varrho_0}$, $x = x_0 - a$, $y = (1 - \varrho_0)z$. Since

$A \in \mathcal{S}(-M)$, it follows by Lemma 3.7 that A is closed and weakly convex with respect to M. By (13) we get $z \in N^1_M(a_0, A)$. Using Definition 3.5, we have $a_0 \in P_M(a_0 + z, A)$. Lemma 3.6 implies that $\varrho_M(a_0 + z, A) = 1$. Taking into account that $\varrho_0 \le \mu_M(x) \le \varrho_0 + \eta$, we obtain $\mu_M(x + y) = \mu_M(a_0 + z - a) \ge \varrho_M(a_0 + z, A) = 1 \ge \mu_M(x) + \mu_M(y) - \eta$. Thus, Lemma 5.5 yields

$$\delta_M \left(\left\| \frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)} \right\| \right) \le \frac{R\eta}{\min\{\varrho_0, 1 - \varrho_0\}} < \frac{\delta_M(\varepsilon_0)}{\varrho_0}. \tag{30}$$

By Proposition 5.3 we have $\frac{\delta_M(\varepsilon_0)}{\varrho_0} \le \delta_M\left(\frac{\varepsilon_0}{\varrho_0}\right)$. Using (30) and Remark 5.1, we get $\left\| \frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)} \right\| < \frac{\varepsilon_0}{\varrho_0}$. Taking into account that $\left\| \frac{x}{\mu_M(x)} - \frac{x}{\varrho_0} \right\| = \frac{\|x\|}{\mu_M(x)} \frac{|\mu_M(x) - \varrho_0|}{\varrho_0} \le \frac{R\eta}{\varrho_0}$, $\left\| \frac{y}{\mu_M(y)} - \frac{x}{\varrho_0} \right\| = \frac{\|a - a_0\|}{\varrho_0}$, we deduce that

$$\begin{aligned} \|a - a_0\| &= \varrho_0 \left\| \frac{y}{\mu_M(y)} - \frac{x}{\varrho_0} \right\| \\ &\le \varrho_0 \left\| \frac{x}{\mu_M(x)} - \frac{x}{\varrho_0} \right\| + \varrho_0 \left\| \frac{x}{\mu_M(x)} - \frac{y}{\mu_M(y)} \right\| \le R\eta + \varepsilon_0. \end{aligned}$$

Using (29), we complete the proof. □

Lemma 5.7. *Let a set $C \subset E$ be uniformly convex and let a set $A \subset E$ be such that $rA \in \mathcal{S}(C)$ with $r \in (0, 1)$. Suppose that $A \cap C \ne \emptyset$, $a \in A \cap (C + \mathfrak{B}_\varepsilon)$, where ε satisfies the following inequalities:*

$$0 < \varepsilon < \min\left\{ \frac{\varepsilon_0}{2}, \frac{\delta_C(\varepsilon_0)}{2r}, \frac{(1 - r)\delta_C(\varepsilon_0)\delta_C(r\varepsilon_0)}{rR_1} \right\} \tag{31}$$

with $\varepsilon_0 > 0$ and

$$R_1 = 1 + \frac{1}{\delta_C(1)}. \tag{32}$$

Then there exists $a_0 \in A \cap C$ such that

$$\|a - a_0\| < \frac{3}{2}\varepsilon_0. \tag{33}$$

Proof. If $\varepsilon_0 \ge \operatorname{diam} C$, we take a_0 as an arbitrary element of $A \cap C$. Then $a \in A \cap (C + \mathfrak{B}_\varepsilon)$ implies that $\|a - a_0\| \le \operatorname{diam} C + \varepsilon \le \varepsilon_0 + \varepsilon$. By (31), in this case (33) holds. Therefore, we assume that

$$\varepsilon_0 < \operatorname{diam} C.$$

According to (23), there exist $c_1, c_2 \in C$ such that $\|c_1 - c_2\| > \varepsilon_0$. Using (15) and closedness of C, for $\sigma = \delta_C(\varepsilon_0)$ and $c_0 = \frac{c_1 + c_2}{2}$ we get $\mathfrak{B}_\sigma(c_0) \subset C$. By (32) and Proposition 5.4 we deduce that $C \subset \mathfrak{B}_{R_1}(c_0)$. So, $M = \frac{1}{r}(c_0 - C)$ is a quasiball and $\mathfrak{B}_{\sigma/r} \subset M \subset \mathfrak{B}_{R_1/r} = \mathfrak{B}_R$ with $R = \frac{R_1}{r}$. Lemmas 2.11 and 2.12 imply that $A \in \mathcal{S}(-M)$. The relation $\mathfrak{B}_{\sigma/r} \subset M$ yields $\mathfrak{B}_\varepsilon \subset -\frac{\varepsilon r}{\sigma} M$ and therefore $a \in C + \mathfrak{B}_\varepsilon = c_0 - rM + \mathfrak{B}_\varepsilon \subset c_0 - \left(r + \frac{\varepsilon r}{\sigma}\right) M$. Consequently, $\mu_M(c_0 - a) \le r + \eta$, where $\eta = \frac{\varepsilon r}{\sigma} = \frac{\varepsilon r}{\delta_C(\varepsilon_0)}$.

If $a \in C$, then we take $a_0 = a$ and (33) holds trivially. Therefore we assume that $a \notin C$. Since $C = c_0 - rM$, it follows that $\mu_M(c_0 - a) \ge r$. Denote $z = \frac{c_0 - a}{\mu_M(c_0 - a)}$. According to Remark 3.2, the function $\varphi(t) = \varrho_M(c_0 - tz, A) + t$ is continuous. Taking into account that $A \cap (c_0 - rM) = A \cap C \ne \emptyset$, we get $\varphi(0) = \varrho_M(c_0, A) \le r$. Due to $\varphi(r) \ge r$, there exists $t_0 \in [0, r]$ such that $\varphi(t_0) = r$, i.e., $\varrho_M(c_0 - t_0 z, A) = r - t_0$. According to Proposition 3.12, there exists $a_0 \in P_M(c_0 - t_0 z, A)$. Hence, $a_0 \in c_0 - t_0 z - (r - t_0)M \subset c_0 - rM = C$ and

$$\mu_M(c_0 - t_0 z - a) = \mu_M(c_0 - a) - t_0 \le r + \eta - t_0 = \varrho_M(c_0 - t_0 z, A) + \eta.$$

Consequently, $a \in P_M^\eta(c_0 - t_0 z, A)$. Let us denote $x_0 = c_0 - t_0 z$, $\varrho_0 = r - t_0$. The relationships (26), (31), $R = \frac{R_1}{r}$, $\eta = \frac{\varepsilon r}{\delta_C(\varepsilon_0)}$, $1 - r \le 1 - \varrho_0$ imply that $\eta < \min\left\{\frac{1}{2}, \frac{(1-\varrho_0)\delta_M(\varepsilon_0)}{R}\right\}$. Applying Lemma 5.6, we complete the proof. $\square$

Lemma 5.8. *Let $A_1, A_2, C_1, C_2 \subset E$ be such that C_2 is uniformly convex, $rA_2 \in \mathcal{S}(C_2)$, $r \in (0, 1)$ and $A_i \cap C_i \ne \emptyset$ for $i = 1, 2$. Suppose that for some $\varepsilon_0 > 0$ we have*

$$h^+(A_1, A_2) + h^+(C_1, C_2) < \min\left\{\frac{\varepsilon_0}{2}, \frac{\delta_{C_2}(\varepsilon_0)}{2r}, \frac{(1-r)\delta_{C_2}(\varepsilon_0)\delta_{C_2}(r\varepsilon_0)}{rR}\right\},$$

with

$$R = 1 + \frac{1}{\delta_{C_2}(1)}.$$

Then

$$h^+(A_1 \cap C_1, A_2 \cap C_2) < 2\varepsilon_0.$$

Proof. Fix a number ε such that

$$h^+(A_1, A_2) + h^+(C_1, C_2) < \varepsilon < \min\left\{\frac{\varepsilon_0}{2}, \frac{\delta_{C_2}(\varepsilon_0)}{2r}, \frac{(1-r)\delta_{C_2}(\varepsilon_0)\delta_{C_2}(r\varepsilon_0)}{rR}\right\}.$$

Let us denote $\gamma = \frac{1}{2}\left(\varepsilon - h^+(A_1, A_2) - h^+(C_1, C_2)\right)$. Fix an arbitrary point $x_1 \in A_1 \cap C_1$. According to (16), there exist $a_2 \in A_2$ and $c_2 \in C_2$ such that

$$\|a_2 - x_1\| < h^+(A_1, A_2) + \gamma, \qquad \|c_2 - x_1\| < h^+(C_1, C_2) + \gamma.$$

Consequently, $\|a_2 - c_2\| < h^+(A_1, A_2) + h^+(C_1, C_2) + 2\gamma = \varepsilon$ and therefore $a_2 \in A_2 \cap (C_2 + \mathfrak{B}_\varepsilon)$. Due to Lemma 5.7 there exists $x_2 \in A_2 \cap C_2$ such that $\|a_2 - x_2\| < \frac{3}{2}\varepsilon_0$. Since $\|a_2 - x_1\| < \varepsilon < \frac{\varepsilon_0}{2}$, it follows that $\|x_1 - x_2\| \le \|a_2 - x_2\| + \|a_2 - x_1\| < 2\varepsilon_0$. This completes the proof. $\square$

Lemma 5.8 implies Theorem 4.1.

6. The continuity of the metric projection

Remark 6.1. The definition of the Minkowski functional (9) implies that for any quasiball $M \subset E$ we have

$$t\mu_{tM}(z) = \mu_M(z), \quad \forall t \geq 0, \ \forall z \in E.$$

Remark 6.2. If the quasiballs $M_1, M_2 \subset E$ are such that $M_1 \subset M_2$, then

$$\mu_{M_2}(z) \leq \mu_{M_1}(z), \quad \forall z \in E.$$

Lemma 6.3. *Let $M_1 \subset E$ and $M_2 \subset E$ be quasiballs. Suppose that $\mathfrak{B}_\sigma \subset M_1$ for some $\sigma > 0$. Then*

$$\mu_{M_1}(z) \leq \left(1 + \frac{h^+(M_2, M_1)}{\sigma}\right) \mu_{M_2}(z), \quad \forall z \in E.$$

Proof. Let us fix any $h > h^+(M_2, M_1)$. Since $\mathfrak{B}_\sigma \subset M_1$, it follows that $\mathfrak{B}_h \subset \frac{h}{\sigma} M_1$. Therefore $M_2 \subset M_1 + \mathfrak{B}_h \subset \left(1 + \frac{h}{\sigma}\right) M_1$. So, using Remarks 6.1 and 6.2, we get $\mu_{M_1}(z) \leq \left(1 + \frac{h}{\sigma}\right) \mu_{M_2}(z)$ for any $z \in E$. Passing to the limit as $h \to h^+(M_2, M_1)$, we complete the proof. □

The definition of the M-distance (10) and the Lipschitz continuity of $\mu_M(\cdot)$ (see Remark 3.2) imply the following remark.

Remark 6.4. If a quasiball $M \subset E$ satisfies the inclusion $\mathfrak{B}_\sigma \subset M$ for some $\sigma > 0$, then for all $x \in E$ and $A_1, A_2 \subset E$ we have

$$|\varrho_M(x, A_1) - \varrho_M(x, A_2)| \leq \frac{h(A_1, A_2)}{\sigma}.$$

Lemma 6.3, Remark 3.2, and Remark 6.4 yield the following lemma.

Lemma 6.5. *Let $M_1, M_2 \subset E$ be quasiballs and $\mathfrak{B}_\sigma \subset M_1$ for some $\sigma > 0$. Then for all $x_1, x_2 \in E$ and $A_1, A_2 \subset E$ we have*

$$\varrho_{M_1}(x_1, A_1) \leq \left(1 + \frac{h^+(M_2, M_1)}{\sigma}\right) \varrho_{M_2}(x_2, A_2) + \frac{\|x_1 - x_2\| + h(A_1, A_2)}{\sigma}.$$

As a consequence of Theorem 4.1, we obtain the following theorem about continuity of the set of nearly best approximations (12).

Theorem 6.6. *Let Ψ be a family of quads (M, A, x, η), where $M \subset E$ is a uniformly convex quasiball, $A \in \mathcal{S}(-M)$, $x \in E$, $\eta \geq 0$. Suppose that there exist positive numbers r, R, σ such that*

$$\mathfrak{B}_\sigma \subset M \subset \mathfrak{B}_R, \qquad \varrho_M(x, A) + \eta \leq r < 1, \quad \forall (M, A, x, \eta) \in \Psi,$$

$$\inf_{(M,A,x,\eta) \in \Psi} \delta_M(\varepsilon) > 0, \quad \forall \varepsilon > 0.$$

Then the multifunction $(M, A, x, \eta) \mapsto P_M^\eta(x, A)$ is uniformly Hausdorff continuous on Ψ, i.e., for all $\beta > 0$ there exists $\tau > 0$ such that for all $(M_1, A_1, x_1, \eta_1) \in \Psi$, $(M_2, A_2, x_2, \eta_2) \in \Psi$ we have

$$\begin{aligned} & h(M_1, M_2) + h(A_1, A_2) + \|x_1 - x_2\| + \sigma|\eta_1 - \eta_2| < \tau \\ \Rightarrow \quad & h\left(P_{M_1}^{\eta_1}(x_1, A_1),\ P_{M_2}^{\eta_2}(x_2, A_2)\right) < \beta. \end{aligned}$$

Proof. For any $(M, A, x, \eta) \in \Psi$, we denote

$$\mathcal{C}(M, A, x, \eta) = x - (\varrho_M(x, A) + \eta)\, M. \tag{34}$$

Consider the family

$$\Phi = \{(A, C) \mid C = \mathcal{C}(M, A, x, \eta),\ (M, A, x, \eta) \in \Psi\}.$$

If $(A, C) \in \Phi$, then $C = x - tM$, where $t = \varrho_M(x, A) + \eta \le r < 1$, $(M, A, x, \eta) \in \Psi$. According to Lemmas 2.11 and 2.12, we have $rA \in \mathcal{S}(-tM) = \mathcal{S}(C)$. Using Remark 5.2 and Proposition 5.3, for any $\varepsilon > 0$ we get $\delta_C(\varepsilon) = t\delta_M\left(\frac{\varepsilon}{t}\right) \ge \delta_M(\varepsilon)$. Hence,

$$\inf_{(A,C)\in\Phi} \delta_C(\varepsilon) \ge \inf_{(M,A,x,\eta)\in\Phi} \delta_M(\varepsilon) > 0, \quad \forall \varepsilon > 0.$$

Fix an arbitrary $\beta > 0$. By Theorem 4.1 there exists $\gamma > 0$ such that for all $(A_1, C_1) \in \Phi$, $(A_2, C_2) \in \Phi$ we have

$$h(A_1, A_2) + h(C_1, C_2) < \gamma \quad \Rightarrow \quad h(A_1 \cap C_1, A_2 \cap C_2) < \beta. \tag{35}$$

Let us denote

$$\tau = \frac{\gamma\sigma}{R + \sigma}. \tag{36}$$

Fix two quads $(M_1, A_1, x_1, \eta_1) \in \Psi$ and $(M_2, A_2, x_2, \eta_2) \in \Psi$ such that

$$h(M_1, M_2) + h(A_1, A_2) + \|x_1 - x_2\| + \sigma|\eta_1 - \eta_2| < \tau. \tag{37}$$

It suffices to show that

$$h\left(P_{M_1}^{\eta_1}(x_1, A_1),\ P_{M_2}^{\eta_2}(x_2, A_2)\right) < \beta. \tag{38}$$

We denote

$$\varrho_i = \varrho_{M_i}(x_i, A_i), \qquad C_i = \mathcal{C}(M_i, A_i, x_i, \eta_i), \quad i = 1, 2.$$

Note that $\varrho_i + \eta_i \le r < 1$. Lemma 6.5 implies that

$$|\varrho_1 - \varrho_2| \le \frac{h(M_1, M_2) + h(A_1, A_2) + \|x_1 - x_2\|}{\sigma}.$$

Using (34), (36), (37), we get

$$\begin{aligned} & h(A_1, A_2) + h(C_1, C_2) \\ & \le h(A_1, A_2) + \|x_1 - x_2\| + h\left((\varrho_1 + \eta_1)M_1, (\varrho_2 + \eta_2)M_2\right) \\ & \le h(A_1, A_2) + \|x_1 - x_2\| + (|\varrho_1 - \varrho_2| + |\eta_1 - \eta_2|)R + (\varrho_1 + \eta_1)h(M_1, M_2) \\ & \le \tau\left(1 + \frac{R}{\sigma}\right) = \gamma. \end{aligned}$$

So, according to (35), we get $h(A_1 \cap C_1, A_2 \cap C_2) < \beta$. Taking in account that $P^{\eta_i}_{M_i}(x_i, A_i) = A_i \cap C_i$ for $i = 1, 2$, we obtain (38). □

The assumption of Theorem 6.6 about uniform convexity of quasiballs may be substituted by the assumption about uniform convexity of sets to be projected onto. Doing so, we obtain the following theorem.

Theorem 6.7. *Let Ψ be a family of quads (M, C, x, η), where $M \subset E$ is a quasiball, $C \subset E$ is closed and uniformly convex, $x \in E$, $\eta \ge 0$. Suppose that there exist positive numbers R, R_1, σ such that*

$$\mathfrak{B}_\sigma \subset M \subset \mathfrak{B}_R, \qquad \varrho_M(x, C) + \eta \le R_1, \quad \forall (M, C, x, \eta) \in \Psi,$$

$$\inf_{(M,C,x,\eta)\in\Psi} \delta_C(\varepsilon) > 0, \quad \forall \varepsilon > 0.$$

Then the multifunction $(M, C, x, \eta) \mapsto P^\eta_M(x, C)$ is uniformly Hausdorff continuous on Ψ.

Proof. For any quad $(M, C, x, \eta) \in \Psi$ we denote $\mathcal{A}(M, C, x, \eta) = x - (\varrho_M(x, C) + \eta)M$. Fix an arbitrary $r \in (0, 1)$. Lemma 2.3 implies that $r\mathcal{A}(M, C, x, \eta) \in \mathcal{S}(C)$ for any $(M, C, x, \eta) \in \Psi$. Bearing in mind that $P^\eta_M(x, C) = C \cap \mathcal{A}(M, C, x, \eta)$ and applying Theorem 4.1 for the family $\Phi = \{(A, C) \,|\, A = \mathcal{A}(M, C, x, \eta),\ (M, C, x, \eta) \in \Psi\}$ by the analogous reasoning as in the proof of Theorem 6.6 we get the desired statement. □

7. Proofs of Theorems 4.2, 4.4

The *Minkowski difference* of $A \subset E$ and $B \subset E$ is

$$A \overset{*}{-} B = \left\{x \in E \;\middle|\; x + B \subset A\right\}.$$

Lemma 7.1. *Let $A \subset E$ be closed convex and $B \subset E$ be bounded. Then $A + B \overset{*}{-} B = A$.*

Proof. By definition of the Minkowski difference the inclusion $a \in A + B \overset{*}{-} B$ is equivalent to $a + B \subset A + B$. According to Rådström's lemma [21, Lemma 1] the latter inclusion is equivalent to the inclusion $a \in A$. □

Lemma 7.2. *Let $X \subset E$ be closed convex, $x_0 \in E$, $\mathfrak{B}_\sigma(x_0) \subset X \subset \mathfrak{B}_R(x_0)$, $\sigma > 0$, $R > 0$, $\delta \in [0, \sigma]$, $\varepsilon = \frac{R\delta}{\sigma}$. Then $X \subset X \overset{*}{-} \mathfrak{B}_\delta + \mathfrak{B}_\varepsilon$.*

Proof. Let us denote $X_0 = X - x_0$. Thus, $\mathfrak{B}_\delta = \frac{\delta}{\sigma}\mathfrak{B}_\sigma \subset \frac{\delta}{\sigma}X_0$, $\mathfrak{B}_\varepsilon = \frac{\varepsilon}{R}\mathfrak{B}_R = \frac{\delta}{\sigma}\mathfrak{B}_R \supset \frac{\delta}{\sigma}X_0$. Due to convexity of X_0 and using the inequality $\delta \leq \sigma$, we get $X_0 = X_0 \overset{*}{-} \frac{\delta}{\sigma}X_0 + \frac{\delta}{\sigma}X_0 \subset X_0 \overset{*}{-} \mathfrak{B}_\delta + \mathfrak{B}_\varepsilon$. □

Lemma 7.3. *Let $C_1 \subset E$ be closed convex and $\mathfrak{B}_{2\sigma_1}(x_0) \subset C_1 \subset \mathfrak{B}_R(x_0)$ for some $x_0 \in E$, $\sigma_1 > 0$, $R > 0$. Let $C_2 \subset E$ satisfy the inclusion $C_2 \subset C_1 + \mathfrak{B}_h$ for some $h \geq 0$. Suppose that $\sigma_2 \geq \max\{0, \sigma_1 - \frac{h}{2}\}$. Then the sets $D_1 = C_1 \overset{*}{-} \mathfrak{B}_{\sigma_1}$ and $D_2 = C_2 \overset{*}{-} \mathfrak{B}_{\sigma_2}$ satisfy the inclusion $D_2 \subset D_1 + \mathfrak{B}_\varepsilon$ with $\varepsilon = \frac{3hR}{2\sigma_1}$.*

Proof. At first we suppose that $\sigma_1 \geq \frac{3}{2}h$. Since the set $X = C_1 \overset{*}{-} \mathfrak{B}_{\sigma_1 - 3h/2}$ is closed convex and $\mathfrak{B}_{\sigma_1}(x_0) \subset X \subset \mathfrak{B}_R(x_0)$, it follows by Lemma 7.2 that $X \subset X \overset{*}{-} \mathfrak{B}_{3h/2} + \mathfrak{B}_\varepsilon$. According to Lemma 7.1, we have $C_1 + \mathfrak{B}_h \overset{*}{-} \mathfrak{B}_{\sigma_1 - h/2} = C_1 + \mathfrak{B}_h \overset{*}{-} \mathfrak{B}_h \overset{*}{-} \mathfrak{B}_{\sigma_1 - 3h/2} = C_1 \overset{*}{-} \mathfrak{B}_{\sigma_1 - 3h/2}$. Consequently,

$$\begin{aligned} D_2 &\subset C_1 + \mathfrak{B}_h \overset{*}{-} \mathfrak{B}_{\sigma_2} \subset C_1 + \mathfrak{B}_h \overset{*}{-} \mathfrak{B}_{\sigma_1 - h/2} = C_1 \overset{*}{-} \mathfrak{B}_{\sigma_1 - 3h/2} \\ &= X \subset X \overset{*}{-} \mathfrak{B}_{3h/2} + \mathfrak{B}_\varepsilon = C_1 \overset{*}{-} \mathfrak{B}_{\sigma_1} + \mathfrak{B}_\varepsilon = D_1 + \mathfrak{B}_\varepsilon. \end{aligned}$$

Now we suppose that $\sigma_1 < \frac{3}{2}h$. We claim that

$$D_2 \subset C_1 + \mathfrak{B}_{3h/2 - \sigma_1}. \tag{39}$$

If $\sigma_1 \geq \frac{h}{2}$, then $D_2 \subset C_1 + \mathfrak{B}_h \overset{*}{-} \mathfrak{B}_{\sigma_2} \subset C_1 + \mathfrak{B}_h \overset{*}{-} \mathfrak{B}_{\sigma_1 - h/2} = C_1 + \mathfrak{B}_{3h/2 - \sigma_1}$. The latter equality follows from Lemma 7.1. So, in this case (39) holds true. If $\sigma_1 < \frac{h}{2}$, then $D_2 \subset C_1 + \mathfrak{B}_h \overset{*}{-} \mathfrak{B}_{\sigma_2} \subset C_1 + \mathfrak{B}_h \subset C_1 + \mathfrak{B}_{3h/2 - \sigma_1}$ and (39) also holds true.

Since $x_0 \in C_1 \overset{*}{-} \mathfrak{B}_{\sigma_1}$, $C_1 \subset \mathfrak{B}_R(x_0)$, it follows that $C_1 \subset x_0 + \mathfrak{B}_R \subset C_1 \overset{*}{-} \mathfrak{B}_{\sigma_1} + \mathfrak{B}_R$. Thus, by (39) we have

$$D_2 \subset C_1 \overset{*}{-} \mathfrak{B}_{\sigma_1} + \mathfrak{B}_R + \mathfrak{B}_{3h/2 - \sigma_1} = D_1 + \mathfrak{B}_{R + 3h/2 - \sigma_1}. \tag{40}$$

Using the inclusions $\mathfrak{B}_{2\sigma_1}(x_0) \subset C_1 \subset \mathfrak{B}_R(x_0)$, we get $R > \sigma_1$. Bearing in mind that $\sigma_1 < \frac{3}{2}h$, we deduce that $\left(\frac{3h}{2} - \sigma_1\right)\left(\frac{R}{\sigma_1} - 1\right) > 0$. Consequently, $R + \frac{3h}{2} - \sigma_1 < \frac{3hR}{2\sigma_1} = \varepsilon$. Using (40), we complete the proof. □

Lemma 7.4. *Let Ω be a family of closed and uniformly convex sets $C \subset E$ and*

$$\inf_{C \in \Omega} \delta_C(\varepsilon) > 0, \quad \forall \varepsilon > 0. \tag{41}$$

Then there exist uniformly continuous functions $z : \Omega \to E$ and $\sigma : \Omega \to [0, +\infty)$ such that

$$\mathfrak{B}_{\sigma(C)}(z(C)) \subset C, \quad \forall C \in \Omega \tag{42}$$

and

$$\inf_{C \in \Omega:\ \operatorname{diam} C > \varepsilon} \sigma(C) > 0, \quad \forall \varepsilon > 0. \tag{43}$$

Proof. For any $C \in \Omega$ we define

$$\sigma(C) = \frac{1}{2} \sup\{\sigma > 0 \mid \exists x \in E : \ \mathfrak{B}_\sigma(x) \subset C\}. \tag{44}$$

First we claim that

$$|\sigma(C_1) - \sigma(C_2)| \le \frac{1}{2} h(C_1, C_2), \quad \forall C_1, C_2 \in \Omega. \tag{45}$$

Suppose the contrary: there exist $C_1, C_2 \in \Omega$ such that $\sigma(C_1) - \sigma(C_2) > \frac{1}{2} h(C_1, C_2)$. Fix a number σ_0 such that $2\sigma(C_1) > \sigma_0 > 2\sigma(C_2) + h(C_1, C_2)$. Due to (44) there exists a vector $x \in E$ such that $\mathfrak{B}_{\sigma_0}(x) \subset C_1$. Hence, $\mathfrak{B}_{\sigma_0}(x) \subset C_2 + \mathfrak{B}_{h(C_1,C_2)}$. So, by Lemma 7.1 we deduce that $\mathfrak{B}_{\sigma_0 - h(C_1,C_2)}(x) \subset C_2$. Using (44), we get $2\sigma(C_2) \ge \sigma_0 - h(C_1, C_2)$. This contradicts the inequality $\sigma_0 > 2\sigma(C_2) + h(C_1, C_2)$. Thus, (45) holds true.

Now we claim that

$$C \overset{*}{-} \mathfrak{B}_{2\sigma(C)} \ne \emptyset, \quad \forall C \in \Omega. \tag{46}$$

Indeed, let us fix $C \in \Omega$. Consider the sequence of embedded closed convex bounded sets $D_k = C \overset{*}{-} \mathfrak{B}_{(2-1/k)\sigma(C)}$. Remark 3.11 implies that E is reflexive. Hence, $\bigcap_{k \in \mathbb{N}} D_k \ne \emptyset$ is nonempty. This yields (46).

Let us denote

$$\delta(\varepsilon, \Omega) = \inf_{C \in \Omega} \delta_C(\varepsilon), \quad \varepsilon > 0.$$

The assumption (41) means that $\delta(\varepsilon, \Omega) > 0$ for all $\varepsilon > 0$. If a set $C \in \Omega$ satisfies the inequality $\operatorname{diam} C > \varepsilon$ for some $\varepsilon > 0$, then there exist $c_1, c_2 \in C$ such that $\|c_1 - c_2\| > \varepsilon$. Using (15), (44), we get $2\sigma(C) \ge \delta_C(\varepsilon) \ge \delta(\varepsilon, \Omega)$. Thus, we have

$$\operatorname{diam} C > \varepsilon \quad \Rightarrow \quad 2\sigma(C) \ge \delta(\varepsilon, \Omega), \quad \forall C \in \Omega, \ \forall \varepsilon > 0. \tag{47}$$

For any $C \in \Omega$ let us define the closed and uniformly convex set

$$D(C) = C \overset{*}{-} \mathfrak{B}_{\sigma(C)}. \tag{48}$$

We claim that the multifunction $D(\cdot)$ is uniformly Hausdorff continuous on Ω, namely,

$$\begin{aligned} & h(C_1, C_2) < \min\left\{\varepsilon, \frac{2\varepsilon\delta(\varepsilon, \Omega)}{3R}\right\} \\ \Rightarrow \quad & h\left(D(C_1), D(C_2)\right) < 2\varepsilon, \quad \forall C_1, C_2 \in \Omega, \ \forall \varepsilon > 0, \end{aligned}$$

where

$$R = 1 + \frac{1}{\delta(1, \Omega)}.$$

Indeed, fix any $\varepsilon > 0$ and $C_1, C_2 \in \Omega$ such that $h(C_1, C_2) < \min\left\{\varepsilon, \frac{2\varepsilon\delta(\varepsilon,\Omega)}{3R}\right\}$. Suppose that $\operatorname{diam} C_1 < \varepsilon$. Fix any $x_1 \in D(C_1)$, $x_2 \in D(C_2)$. As $x_2 \in C_2$, there exists $x_1' \in C_1$ such that $\|x_1' - x_2\| \le h(C_1, C_2) < \varepsilon$. Hence, $\|x_1 - x_2\| < \varepsilon + \|x_1 - x_1'\| \le \varepsilon + \operatorname{diam} C_1 < 2\varepsilon$. So, in this case we get $h(D(C_1), D(C_2)) < 2\varepsilon$. If $\operatorname{diam} C_2 < \varepsilon$ we have the same inequality. Now suppose that $\min\{\operatorname{diam} C_1$ $\operatorname{diam} C_2\} \ge \varepsilon$. Using (47), we get $2\min\{\sigma(C_1), \sigma(C_2)\} \ge \delta(\varepsilon, \Omega)$. Bearing in mind (46), we fix $x_i \in C_i \overset{*}{-} \mathfrak{B}_{2\sigma(C_i)}$, $i = 1, 2$. Since by Proposition 5.4 we have $\operatorname{diam} C_i \le R$, it follows that $\mathfrak{B}_{2\sigma(C_i)}(x_i) \subset C_i \subset \mathfrak{B}_R(x_i)$ for $i = 1, 2$. Lemma 7.3 and (45) imply that $h(D(C_1), D(C_2)) \le \frac{3Rh(C_1, C_2)}{\delta(\varepsilon, \Omega)} < 2\varepsilon$. Thus, the uniform continuity of $D(\cdot)$ is proved.

Since for any $C \in \Omega$ the set $D(C)$ is nonempty, closed and uniformly convex, it follows that $P_{\mathfrak{B}_1}(0, D(C))$ is a singleton. Let us define $z(C) \in E$ by

$$\{z(C)\} = P_{\mathfrak{B}_1}(0, D(C)).$$

Theorem 6.7 and the uniform continuity of $D(\cdot)$ imply that function $z : \Omega \to E$ is uniformly continuous. According to (45), the function $\sigma : \Omega \to E$ is Lipschitz continuous. The inclusion $z(C) \in D(C)$ and the equality (48) imply (42). The inequality (43) follows from (47). $\square$

Proof of Theorem 4.2. According to Lemma 7.4 applied to the family $\Omega = \{C \subset E \mid \exists A \subset E : (A, C) \in \Phi\}$, there exist uniformly continuous functions $z : \Omega \to E$ and $\sigma : \Omega \to [0, +\infty)$ such that

$$\mathfrak{B}_{\sigma(C)}(z(C)) \subset C, \quad \forall C \in \Omega \tag{49}$$

and

$$\inf_{C \in \Omega:\ \operatorname{diam} C > \varepsilon} \sigma(C) > 0, \quad \forall \varepsilon > 0. \tag{50}$$

For any $C \in \Omega$ we put

$$M(C) = \frac{1}{r}(z(C) - C). \tag{51}$$

Since $z(\cdot)$ is uniformly continuous, it follows that the multifunction $M(\cdot)$ is uniformly continuous on Ω. The inclusion (17) implies that

$$A \in \mathcal{S}(-M(C)), \quad \forall (A, C) \in \Phi. \tag{52}$$

Using (49), (51), we obtain

$$\mathfrak{B}_{\sigma(C)/r} \subset M(C), \quad \forall C \in \Omega. \tag{53}$$

Since by (26) we have $\delta_{M(C)}(\varepsilon) = \frac{1}{r}\delta_C(r\varepsilon)$ for any $\varepsilon > 0$, $C \in \Omega$, according to (18) it follows that

$$\delta(\varepsilon, \Omega) := \inf_{C \in \Omega} \delta_{M(C)}(\varepsilon) > 0, \quad \forall \varepsilon > 0. \tag{54}$$

Let us denote

$$R = 1 + \frac{1}{\delta(1, \Omega)}. \tag{55}$$

According to Proposition 5.4, for any $C \in \Omega$ we have

$$\operatorname{diam} M(C) \leq R, \quad \forall C \in \Omega. \tag{56}$$

Let us define $\xi : \Phi \to E$. Fix any $(A, C) \in \Phi$.

If C is a singleton, then we define $\xi(A, C)$ by

$$\{\xi(A, C)\} = C. \tag{57}$$

Suppose that C is not a singleton. Then $M(C)$ is a uniformly convex quasiball. By (19), (51) we deduce that $A \cap (z(C) - rM(C)) = A \cap C \neq \emptyset$ for all $(A, C) \in \Phi$. Hence, due to Remark 3.3 we get

$$\varrho_{M(C)}(z(C), A) \leq r, \quad \forall (A, C) \in \Phi. \tag{58}$$

By (17), Lemma 3.7 and Proposition 3.12 it follows that $P_{M(C)}(z(C), A)$ is a singleton. We define $\xi(A, C)$ by

$$\{\xi(A, C)\} = P_{M(C)}(z(C), A). \tag{59}$$

We claim that $\xi : \Phi \to E$ is a selection of the intersection operator, i.e. (20) holds true. If $(A, C) \in \Phi$ is such that the set C is a singleton, then (20) is a consequence of (19) and (57). If the pair $(A, C) \in \Phi$ is such that C includes more than one element, then by (11) we have $\xi(A, C) \in z(C) - \varrho_{M(C)}(z(C), A)M(C)$. Bearing in mind (58) we get $\xi(A, C) \in z(C) - rM = C$. Using the inclusion $P_{M(C)}(z(C), A) \subset A$, we obtain (20).

To prove the uniform continuity of the function $\xi : \Phi \to E$ let us fix an arbitrary $\varepsilon > 0$. Using (49) – (59) and applying Theorem 6.6 to the family

$$\Psi = \{ (M(C), A, z(C), 0) \mid (A, C) \in \Phi, \ \operatorname{diam} C > \varepsilon \},$$

we deduce the existence of $\gamma \in (0, \varepsilon)$ such that for any $(A_1, C_1) \in \Phi$, $(A_2, C_2) \in \Phi$ we have

$$\begin{aligned} & (\min\{\operatorname{diam} C_1, \operatorname{diam} C_2\} > \varepsilon, \quad h(A_1, A_2) + h(C_1, C_2) < \gamma) \\ \Rightarrow \quad & \|\xi(A_1, C_1) - \xi(A_2, C_2)\| < \varepsilon. \end{aligned} \tag{60}$$

We claim that

$$\begin{aligned} & h(A_1, A_2) + h(C_1, C_2) < \gamma \\ \Rightarrow \quad & \|\xi(A_1, C_1) - \xi(A_2, C_2)\| < 2\varepsilon, \quad \forall (A_1, C_1) \in \Phi, \ \forall (A_2, C_2) \in \Phi. \end{aligned} \tag{61}$$

Indeed, fix any pairs $(A_1, C_1) \in \Phi$, $(A_2, C_2) \in \Phi$ such that $h(A_1, A_2) + h(C_1, C_2) < \gamma$. If $\min\{\operatorname{diam} C_1, \operatorname{diam} C_2\} > \varepsilon$, then by (60) we have $\|\xi(A_1, C_1) - \xi(A_2, C_2)\| < \varepsilon < 2\varepsilon$. Now suppose that $\min\{\operatorname{diam} C_1, \operatorname{diam} C_2\} \leq \varepsilon$. In this case, using the inclusions $\xi(A_i, C_i) \in C_i$ for $i = 1, 2$, we have $\|\xi(A_1, C_1) - \xi(A_2, C_2)\| \leq h(C_1, C_2) + \min\{\operatorname{diam} C_1, \operatorname{diam} C_2\} < \gamma + \varepsilon < 2\varepsilon$. Thus, (61) is proved. So, the function $\xi : \Phi \to E$ is uniformly continuous. □

Proof of Theorem 4.4. Let us consider uniformly continuous multifunctions $A(x) = A$ and $C(x) = x - B$. Lemma 2.11 implies that $rA(x) = rA \in \mathcal{S}(-C(x))$ for all $x \in X$. According to Remark 5.2, we have $\inf_{x \in X} \delta_{C(x)}(\varepsilon) = \delta_B(\varepsilon) > 0$ for all $\varepsilon > 0$. By Theorem 4.3, there exists a uniformly continuous selection $a : X \to E$ of the multifunction $X \ni x \mapsto A(x) \cap C(x)$. Putting $b(x) = x - a(x)$, we complete the proof. □

8. The support condition and the modulus of nonconvexity

The *modulus of nonconvexity* for a set $A \subset E$ (according to [4]) is

$$\gamma_A(\varepsilon)$$
$$= \inf\left\{\gamma \geq 0 \mid \mathfrak{B}_\gamma\left(\frac{x_1 + x_2}{2}\right) \cap A \neq \emptyset, \quad \forall x_1, x_2 \in A : \; \|x_1 - x_2\| \leq \varepsilon\right\}, \quad \varepsilon \geq 0.$$

Proposition 8.1. *Let $M \subset E$ be a uniformly convex quasiball. Suppose that the modulus of nonconvexity for a closed set $A \subset E$ and the modulus of convexity of M are related by the following inequality:*

$$\gamma_A(\varepsilon) \leq \delta_M(\varepsilon), \quad \forall \varepsilon > 0.$$

Then for any point $x_0 \in U_M(A) = \{x \in E \mid 0 < \varrho_M(x, A) < 1\}$ and any sequence $\{a_k\} \subset A$ the condition $\lim_{k \to \infty} \mu_M(x_0 - a_k) = \varrho_M(x_0, A)$ implies that $\{a_k\}$ converges.

If $M = \mathfrak{B}_d(0)$, $d > 0$, then Proposition 8.1 in fact is deduced in the proof of Theorem 1.1 in [4], though the statement of this theorem is weaker: for any $x_0 \in U_M(A)$ the set $P_M(x_0, A)$ is a singleton. If M is a uniformly convex quasiball, Proposition 8.1 can be proved in the same way.

Lemma 8.2. *Let $C \subset E$ be uniformly convex. Assume that the modulus of nonconvexity for a closed set $A \subset E$ and the modulus of convexity of C are related by the following inequality:*

$$\gamma_A(\varepsilon) \leq \delta_C(\varepsilon), \quad \forall \varepsilon > 0.$$

Then $A \in \mathcal{S}(C)$.

Proof. Fix a point $c_0 \in C$ and consider a quasiball $M = c_0 - C$. Using Remark 5.2 and Propositions 3.12 and 8.1, we deduce that A is weakly convex with respect to M. Applying Lemma 3.7, we complete the proof. □

Lemma 8.2 and Remark 5.2 imply that Theorems 4.3, 4.4 remain valid if we substitute (21) by

$$r\gamma_{A(t)}(\varepsilon) \leq \delta_{C(t)}(r\varepsilon), \quad \forall \varepsilon > 0, \; \forall t \in T$$

and (22) by

$$r\gamma_A(\varepsilon) \leq \delta_B(r\varepsilon), \quad \forall \varepsilon > 0.$$

Akin results were obtained in [4] (see Theorem 3.6 and Example 4.1) under some more complicated assumptions. Note that the notion of nonconvexity modulus is effective only in the spaces with modulus of convexity of the second order, as shown in [4]. While the support condition, considered here, is effective in any uniformly convex Banach space.

Acknowledgements. The author thanks the referee for very useful comments.

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