

Rotund Renormings in Spaces of Bochner Integrable Functions

Marián Fabian*

*Institute of Mathematics, Czech Academy of Sciences,
Žitná 25, 115 67 Praha 1, Czech Republic
fabian@math.cas.cz*

Sebastián Lajara†

*Departamento de Matemáticas, Escuela de Ingenieros Industriales,
Universidad de Castilla-La Mancha, Campus Universitario, 02071 Albacete, Spain
sebastian.lajara@uclm.es*

Dedicated to Václav Zizler on the occasion of his 70th birthday.

Received: December 30, 2013

Revised manuscript received: December 5, 2014

Accepted: December 5, 2014

We show that if μ is a probability measure and X is a Banach space, then the Lebesgue-Bochner space $L^1(\mu, X)$ admits an equivalent norm which is rotund (uniformly rotund in every direction, locally uniformly rotund, or midpoint locally uniformly rotund) if X does. We also prove that if X admits a uniformly rotund norm, then the space $L^1(\mu, X)$ has an equivalent norm whose restriction to every reflexive subspace is uniformly rotund. This is done via the Luxemburg norm associated to a suitable Orlicz function.

Keywords: Lebesgue-Bochner space $L^1(\mu, X)$, rotund norm, URED norm, LUR norm, MLUR norm, UR norm, Luxemburg norm, Orlicz function

2010 Mathematics Subject Classification: 46B03, 46B20, 46E40

1. Introduction and statement of the results

In the paper [14], Smith and Turett proved that if μ is a probability measure and X is a Banach space, then the canonical norm of the Lebesgue-Bochner function space $L^p(\mu, X)$ inherits the classical properties of convexity of the norm of X whenever $1 < p < +\infty$. Some results of the same nature for various smoothness properties were previously established in [9] and [10]. We notice that, since the canonical norm of the space $L^1(\mu)$ is neither rotund nor smooth, and this space

*M. Fabian was supported by grant P201/12/0290 and by Institutional Research Plan of the Academy of Sciences of Czech Republic No. AVOZ 101 905 03.

†S. Lajara was supported by MTM 2008-05396/MTM/Fondos FEDER and by JCCM PEII11-0132-7661.

embeds isometrically into $L^1(\mu, X)$, the analogues of the aforementioned results for $p = 1$ do not make sense. However, it is possible to lift some geometrical properties of the space X to $L^1(\mu, X)$ under a renorming. In [5], we constructed an equivalent norm (namely, an Orlicz-Bochner norm) on $L^1(\mu, X)$ which preserves several properties of smoothness of the space X . The purpose of this work is to show that a renorming of this type also permits to transfer the relevant properties of rotundity of the norm of X to the space $L^1(\mu, X)$.

Our notation is standard and can be found, for instance, in [2] and [3]. From now on, let $(X, \|\cdot\|)$ be a real Banach space. The symbols S_X and B_X stand for the unit sphere and the closed unit ball of X , respectively. If (Ω, Σ, μ) is a probability space, we denote by $L^1(\Omega, \Sigma, \mu; X)$ or simply by $L^1(\mu, X)$ the Banach space made up of all (equivalence classes of) Bochner integrable functions $f : \Omega \rightarrow X$, endowed with the norm $\|f\|_{L^1(\mu, X)} = \int_{\Omega} \|f(\omega)\| d\mu(\omega)$. The norm $\|\cdot\|$ is called *rotund* (*R*) if the unit sphere S_X does not contain any non-degenerate segment, that is, if $x = y$ whenever $x, y \in X$ and $\|(x + y)/2\| = \|x\| = \|y\|$. The norm $\|\cdot\|$ is called *uniformly rotund* (*UR*) if, for any two sequences (x_n) and (y_n) in S_X such that

$$\lim_{n \rightarrow \infty} \|x_n + y_n\| = 2$$

we have $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$. If $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ for every $x \in S_X$ and every sequence (x_n) in S_X such that

$$\lim_{n \rightarrow \infty} \|x_n + x\| = 2,$$

we say that $\|\cdot\|$ is *locally uniformly rotund* (*LUR*). The norm $\|\cdot\|$ is called *midpoint locally uniformly rotund* (*MLUR*) if for any two sequences (x_n) and (y_n) in S_X and each $z \in S_X$ such that

$$\lim_{n \rightarrow \infty} \|(x_n + y_n)/2 - z\| = 0,$$

we have $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$. Finally, we say that $\|\cdot\|$ is *uniformly rotund in every direction* (*URED*) if, for any two sequences (x_n) and (y_n) in X and every $z \in X$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n\| = 1, \quad \lim_{n \rightarrow \infty} \|y_n\| = 1, \\ \lim_{n \rightarrow \infty} \|x_n + y_n\| = 2 \quad \text{and} \quad x_n - y_n \in \mathbb{R}z \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

we have $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

It is easy to see that $UR \Rightarrow LUR \Rightarrow MLUR \Rightarrow R$ and that $UR \Rightarrow URED \Rightarrow R$. It is well known (see e.g. [2, Section IV.4]) that a Banach space admits an equivalent UR norm if, and only if, X is superreflexive. The class of spaces that admit an equivalent LUR norm is very large. It includes in particular weakly countably determined spaces (see e.g. [2, Section VII.1]). Therefore, separable spaces, reflexive spaces, and spaces of the types $c_0(\Gamma)$ and $L^1(\mu)$, where Γ is any

set and μ is any probability measure, are LUR renormable. An easy convexity argument (see e.g. [2, Theorem II.6.8]) yields that a Banach space X has an equivalent URED norm if there exist a Hilbert space H and a bounded, linear and one-to-one operator $T : X \rightarrow H$. Thus, every separable space, the dual of every separable space and the long James space $J(\eta)$ (for every ordinal η) admit renormings of this kind. Another important example of a Banach space with an equivalent URED norm is $L^1(\mu)$, where μ is a probability measure (cf. [2, Theorem II.7.16]).

The converses of the implications above do not hold in general, even up to renorming. The first example of a MLUR space with no equivalent LUR renorming (a space of continuous functions on a scattered compact space) was provided in [6]. In [1] it was shown that the space ℓ^∞ , which admits an equivalent URED norm (being the dual of a separable space), does not have any equivalent MLUR renorming. A LUR renormable space with no equivalent URED norm is $c_0(\Gamma)$, where Γ is any uncountable set (cf. [2, Prop. II.7.9]). Another example of LUR space that admits no equivalent URED renorming (a nonseparable reflexive space with an unconditional basis) was constructed in [7] (cf. [2, Theorem IV.6.5]). For a detailed information on rotundity properties, and the applications of these notions to the Geometry of Banach spaces we refer to the monographs [2] and [4].

In this paper, we show that the properties of rotundity, uniform rotundity in every direction, locally uniform rotundity, and midpoint locally uniform rotundity of the norm on X lift to $L^1(\mu, X)$ under a suitable renorming. More precisely, we have the following result.

Theorem 1.1. *Let (Ω, Σ, μ) be a probability space and let $(X, \|\cdot\|)$ be a Banach space. Then, there exists an equivalent norm $|||\cdot|||$ on $L^1(\mu, X)$ (dependent on $\|\cdot\|$), such that, if $\|\cdot\|$ is rotund, URED, LUR, or MLUR, then so is $|||\cdot|||$, respectively.*

Since the (non-reflexive) space $L^1([0, 1])$ does not admit any uniformly rotund norm, an analogue of the theorem above for the property of uniform rotundity can not be achieved. We have, however, the following result.

Theorem 1.2. *Let (Ω, Σ, μ) be a probability space and let $(X, \|\cdot\|)$ be a (superreflexive) Banach space. If $\|\cdot\|$ is uniformly rotund, then the norm $|||\cdot|||$ on $L^1(\mu, X)$ constructed in Theorem 1.1 is such that its restriction to every reflexive subspace of it is uniformly rotund.*

We notice that the LUR case of Theorem 1.1 was previously established in [13] by a different method. The approach employed there was based on the technique of Troyanski and Zizler about LUR renorming in Banach spaces with projectional resolutions of the identity; cf. [2, Chapter VII]. Our method provides an explicit formula for the new norm, namely, an Orlicz-Bochner norm. Moreover, this construction preserves the lattice structure whenever X is a Banach lattice (see

[5, Remark 2.6]). Further, from the former theorems and our previous results in [5] we can deduce that, if the norm $\|\cdot\|$ of the space X has simultaneously any of the rotundity properties considered above and some property of smoothness, then the Orlicz-Bochner norm that we construct on $L^1(\mu, X)$, or its restriction to the reflexive subspaces of $L^1(\mu, X)$, shares the same properties of rotundity and smoothness, and no extra averaging process is needed to get the combination of such properties. This fact applies in particular if X is superreflexive or if X is Asplund and simultaneously weakly compactly generated.

2. Proofs

As we mentioned before, the renorming in the theorems above is the Orlicz-Bochner norm associated to a suitable Orlicz function and the norm on X with the relevant properties of rotundity. In what follows, we shall consider an Orlicz function $M : \mathbb{R} \rightarrow [0, +\infty)$, that is, M is a convex, even function such that M increases on $[0, +\infty)$, $M(t) \rightarrow +\infty$ as $t \rightarrow +\infty$ and $M(0) = 0$. Further, we shall assume that M is Lipschitz and twice differentiable, and that M'' is decreasing on $[0, +\infty)$ and $\lim_{t \rightarrow +\infty} t^2 M''(t) \in (0, +\infty)$. Examples of such functions are

$$M_1(t) = |t| - \log(1 + |t|), \quad t \in \mathbb{R}$$

and

$$M_2(t) = \int_0^{|t|} \arctan(s) \, ds, \quad t \in \mathbb{R}.$$

An argument like in [8] (inequality (7) and Lemma 6) yields that, if M is an Orlicz function satisfying the properties above, then

$$M(\alpha t) \geq \alpha^2 M(t) \quad \text{for all } 0 < \alpha < 1 \text{ and all } t \in \mathbb{R}, \quad (1)$$

and

$$M(a) + M(b) - 2M\left(\frac{a+b}{2}\right) \geq \frac{1}{4}M''(\max\{a, b\})(b-a)^2 \quad \text{for all } a, b \geq 0. \quad (2)$$

Now, consider the set $B = \{f \in L^1(\mu, X) : \int_{\Omega} M(\|f(\omega)\|) \, d\mu(\omega) \leq 1\}$. From the properties of M it follows that B is convex, closed, bounded, symmetric and has non-empty interior. In particular, the Minkowski functional of B , that is, the function

$$|||f||| = \inf \left\{ \rho > 0 : \int_{\Omega} M(\|f(\omega)/\rho\|) \, d\mu(\omega) \leq 1 \right\}, \quad f \in L^1(\mu, X) \quad (3)$$

defines a norm on $L^1(\mu, X)$, the Luxemburg norm associated to M and $\|\cdot\|$. In [5, page 250], it is shown that, if C is the Lipschitz constant of M , then

$$\frac{1}{C}|||f||| \leq \|f\|_{L^1(\mu, X)} \leq \frac{2M'(1) + 1}{M'(1)}|||f||| \quad \text{for all } h \in L^1(\mu, X). \quad (4)$$

In particular, $||| \cdot |||$ is an equivalent norm on $L^1(\mu, X)$. We notice that if $f \in L^1(\mu, X)$, then $|||f||| = 1$ if, and only if, $\int_{\Omega} M(\|f(\omega)\|) d\mu(\omega) = 1$.

We are ready to prove our theorems.

Proof of Theorem 1.1. We shall prove that the norm given by equality (3) satisfies the required properties. First, assume that the norm $\| \cdot \|$ on X is rotund. We shall show that $||| \cdot |||$ is rotund as well. Pick any $f, g \in L^1(\mu, X)$ such that $|||(f + g)/2||| = |||f||| = |||g||| = 1$. The convexity and monotonicity of M , together with inequality (2) yield

$$\begin{aligned} 0 &= \int_{\Omega} M(\|f(\omega)\|) d\mu(\omega) + \int_{\Omega} M(\|g(\omega)\|) d\mu(\omega) \\ &\quad - 2 \int_{\Omega} M\left(\frac{\|f(\omega) + g(\omega)\|}{2}\right) d\mu(\omega) \\ &= \int_{\Omega} \left[M(\|f(\omega)\|) + M(\|g(\omega)\|) - 2M\left(\frac{\|f(\omega)\| + \|g(\omega)\|}{2}\right) \right] d\mu(\omega) \\ &\quad + 2 \int_{\Omega} \left[M\left(\frac{\|f(\omega)\| + \|g(\omega)\|}{2}\right) - M\left(\frac{\|f(\omega) + g(\omega)\|}{2}\right) \right] d\mu(\omega) \\ &\geq \frac{1}{4} \int_{\Omega} M''(\max\{\|f(\omega)\|, \|g(\omega)\|\}) \cdot (\|f(\omega)\| - \|g(\omega)\|)^2 d\mu(\omega) \\ &\quad + 2 \int_{\Omega} \left[M\left(\frac{\|f(\omega)\| + \|g(\omega)\|}{2}\right) - M\left(\frac{\|f(\omega) + g(\omega)\|}{2}\right) \right] d\mu(\omega). \end{aligned}$$

Since the integrands in the last expression in the inequality above are both non-negative we have

$$M''(\max\{\|f(\omega)\|, \|g(\omega)\|\}) \cdot (\|f(\omega)\| - \|g(\omega)\|)^2 = 0$$

and

$$M\left(\frac{\|f(\omega)\| + \|g(\omega)\|}{2}\right) - M\left(\frac{\|f(\omega) + g(\omega)\|}{2}\right) = 0$$

for almost all $\omega \in \Omega$. Bearing in mind that M'' does not vanish and that M is strictly increasing on $[0, +\infty)$, we can conclude that

$$\|f(\omega)\| = \|g(\omega)\| \quad \text{and} \quad \|f(\omega)\| + \|g(\omega)\| = \|f(\omega) + g(\omega)\|$$

for almost all $\omega \in \Omega$. These equalities and the rotundity of the norm $\| \cdot \|$ guarantee that $f(\omega) = g(\omega)$ for almost all $\omega \in \Omega$. Thus, the norm $||| \cdot |||$ is rotund.

Now, suppose that the norm $\| \cdot \|$ on X is URED. Assume that $h \in L^1(\mu, X)$ is given and that $(f_n), (g_n)$ are two sequences in $L^1(\mu, X)$ such that $\lim_{n \rightarrow \infty} |||f_n||| = \lim_{n \rightarrow \infty} |||g_n||| = 1$, $\lim_{n \rightarrow \infty} |||f_n + g_n||| = 2$, and that $f_n - g_n \in \mathbb{R}h$ for every $n \in \mathbb{N}$. Since M is Lipschitz, from (4) it easily follows that $\int_{\Omega} M(\|f_n(\omega)\|) d\mu(\omega) \rightarrow 1$,

$\int_{\Omega} M(\|g_n(\omega)\|) d\mu(\omega) \longrightarrow 1$ and $\int_{\Omega} M\left(\frac{\|f_n(\omega)+g_n(\omega)\|}{2}\right) d\mu(\omega) \longrightarrow 1$ as $n \rightarrow \infty$. If $h(\cdot) = 0$ almost everywhere, there is nothing to prove. From now on assume that $\mu(\{\omega \in \Omega : h(\omega) \neq 0\}) > 0$; then, necessarily, there is $\eta > 0$ such that the set $\{\omega \in \Omega : \|h(\omega)\| > \eta\}$ has positive measure; denote this set by Ω_0 . For every $n \in \mathbb{N}$ find a suitable $\lambda_n \in \mathbb{R}$ such that $f_n - g_n = \lambda_n h$. We have to show that $\lim_{n \rightarrow \infty} \lambda_n = 0$. Assume this is not the case. Then there exists $\varepsilon > 0$ such that $|\lambda_n| > \varepsilon$ (for simplicity, by taking subsequences if necessary) for every $n \in \mathbb{N}$.

Let us write, for each $n \in \mathbb{N}$,

$$\varepsilon_n := \int_{\Omega} \left[M(\|f_n(\omega)\|) + M(\|g_n(\omega)\|) - 2M\left(\frac{\|f_n(\omega) + g_n(\omega)\|}{2}\right) \right] d\mu(\omega).$$

Then $\varepsilon_n \longrightarrow 0$ as $n \rightarrow \infty$. On the other hand,

$$\begin{aligned} \varepsilon_n &= \int_{\Omega} a_n(\omega) d\mu(\omega) \\ &\quad + \int_{\Omega} \left[M\left(\frac{\|f_n(\omega)\| + \|g_n(\omega)\|}{2}\right) - M\left(\frac{\|f_n(\omega) + g_n(\omega)\|}{2}\right) \right] d\mu(\omega), \end{aligned}$$

where

$$a_n(\omega) := M(\|f_n(\omega)\|) + M(\|g_n(\omega)\|) - 2M\left(\frac{\|f_n(\omega)\| + \|g_n(\omega)\|}{2}\right).$$

According to inequality (2), for every $n \in \mathbb{N}$ we have

$$a_n(\omega) \geq \frac{1}{4} M''(\max\{\|f_n(\omega)\|, \|g_n(\omega)\|\})(\|f_n(\omega)\| - \|g_n(\omega)\|)^2 \quad \text{for all } \omega \in \Omega,$$

and integrating we get

$$\begin{aligned} &\int_{\Omega} a_n(\omega) d\mu(\omega) \\ &\geq \frac{1}{4} \int_{\Omega} M''(\max\{\|f_n(\omega)\|, \|g_n(\omega)\|\}) \cdot (\|f_n(\omega)\| - \|g_n(\omega)\|)^2 d\mu(\omega). \end{aligned}$$

Thus,

$$\begin{aligned} \varepsilon_n &\geq \frac{1}{4} \int_{\Omega} M''(\max\{\|f_n(\omega)\|, \|g_n(\omega)\|\})(\|f_n(\omega)\| - \|g_n(\omega)\|)^2 d\mu(\omega) \\ &\quad + 2 \int_{\Omega} \left[M\left(\frac{\|f_n(\omega)\| + \|g_n(\omega)\|}{2}\right) - M\left(\frac{\|f_n(\omega) + g_n(\omega)\|}{2}\right) \right] d\mu(\omega). \end{aligned}$$

Taking into account that the two integrals in this inequality are non-negative, and that $\varepsilon_n \longrightarrow 0$ as $n \rightarrow \infty$, it follows that

$$\begin{aligned} &\int_{\Omega} M''(\max\{\|f_n(\omega)\|, \|g_n(\omega)\|\}) \cdot (\|f_n(\omega)\| - \|g_n(\omega)\|)^2 d\mu(\omega) \longrightarrow 0 \\ &\text{as } n \rightarrow \infty \end{aligned} \tag{5}$$

and

$$\int_{\Omega} \left[M \left(\frac{\|f_n(\omega)\| + \|g_n(\omega)\|}{2} \right) - M \left(\frac{\|f_n(\omega) + g_n(\omega)\|}{2} \right) \right] d\mu(\omega) \longrightarrow 0 \quad (6)$$

as $n \rightarrow \infty$.

Now, a standard trick applied to the convergences above yields an infinite set $N \subset \mathbb{N}$ such that for almost all $\omega \in \Omega$,

$$M''(\max\{\|f_n(\omega)\|, \|g_n(\omega)\|\}) \cdot (\|f_n(\omega)\| - \|g_n(\omega)\|)^2 \longrightarrow 0 \quad (7)$$

as $N \ni n \rightarrow \infty$

and

$$M \left(\frac{\|f_n(\omega)\| + \|g_n(\omega)\|}{2} \right) - M \left(\frac{\|f_n(\omega) + g_n(\omega)\|}{2} \right) \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty. \quad (8)$$

Further, using Fatou's lemma, we have

$$\begin{aligned} & \int_{\Omega} \liminf_{N \ni n \rightarrow \infty} (M(\|f_n(\omega)\|) + M(\|g_n(\omega)\|)) d\mu(\omega) \\ & \leq \liminf_{N \ni n \rightarrow \infty} \int_{\Omega} (M(\|f_n(\omega)\|) + M(\|g_n(\omega)\|)) d\mu(\omega) = 2. \end{aligned}$$

Thus, for almost all $\omega \in \Omega$ we have $\liminf_{N \ni n \rightarrow \infty} [M(\|f_n(\omega)\|) + M(\|g_n(\omega)\|)] < +\infty$, and hence also

$$\liminf_{N \ni n \rightarrow \infty} (\|f_n(\omega)\| + \|g_n(\omega)\|) < +\infty. \quad (9)$$

Now, fix an $\omega \in \Omega_0$ such that (7), (8) and (9) hold. From (9) we can find an infinite set $N_1 \subset N$ such that the limit $\lim_{N_1 \ni n \rightarrow \infty} (\|f_n(\omega)\| + \|g_n(\omega)\|)$ exists and is equal to $\liminf_{N \ni n \rightarrow \infty} (\|f_n(\omega)\| + \|g_n(\omega)\|)$. Thus, $\sup_{n \in N_1} (\|f_n(\omega)\| + \|g_n(\omega)\|) =: c < +\infty$. Then (7) implies that

$$M''(c) (\|f_n(\omega)\| - \|g_n(\omega)\|)^2 \longrightarrow 0 \quad \text{as } N_1 \ni n \rightarrow \infty,$$

and hence, bearing in mind that $M''(c) > 0$, we get $\lim_{N_1 \ni n \rightarrow \infty} (\|f_n(\omega)\| - \|g_n(\omega)\|) = 0$. Find now an infinite set $N_2 \subset N_1$ such that the limit $\lim_{N_2 \ni n \rightarrow \infty} \|f_n(\omega)\| =: a$ exists; note that $a < +\infty$ and that $\lim_{N_2 \ni n \rightarrow \infty} \|g_n(\omega)\| = a$. Then, (8) yields that $M(\frac{1}{2}\|f_n(\omega) + g_n(\omega)\|) \longrightarrow M(a)$ as $N_2 \ni n \rightarrow \infty$, and since M is increasing and continuous on the interval $[0, +\infty)$ it follows that $\lim_{N_2 \ni n \rightarrow \infty} \|f_n(\omega) + g_n(\omega)\| = 2a$. If $a = 0$, we would have that $0 = \lim_{n \rightarrow \infty} \|f_n(\omega) - g_n(\omega)\| = \lim_{n \rightarrow \infty} |\lambda_n| \|h(\omega)\| \geq \eta \varepsilon > 0$, a contradiction; hence $a > 0$. Now, putting $x_n = f_n(\omega)/a$, $y_n = g_n(\omega)/a$ for $n \in N_2$, we get that $\|x_n\| \longrightarrow 1$, $\|y_n\| \longrightarrow 1$ and $\|x_n + y_n\| \longrightarrow 2$ as $N_2 \ni n \rightarrow \infty$, and

$x_n - y_n \in \mathbb{R}h(\omega)$ for every $n \in \mathbb{N}$. Then the URED property of the norm $\|\cdot\|$ implies that

$$\|x_n - y_n\| \longrightarrow 0, \quad \text{and so} \quad \|f_n(\omega) - g_n(\omega)\| \longrightarrow 0 \quad \text{as } N_2 \ni n \rightarrow \infty.$$

But, simultaneously, we know that $\|f_n(\omega) - g_n(\omega)\| = |\lambda_n| \|h(\omega)\| > \varepsilon \eta > 0$ for every $n \in \mathbb{N}$; again a contradiction. Consequently, the norm $\|\cdot\|$ is URED.

Next, assume that the norm $\|\cdot\|$ on X is LUR. Consider any $f \in L^1(\mu, X)$ and any sequence $(f_n) \subset L^1(\mu, X)$ such that $\|f_n\| = \|f\| = 1$ for all $n \in \mathbb{N}$ and $\|f + f_n\| \longrightarrow 2$ as $n \rightarrow \infty$. We have to show that $\|f_n - f\| \longrightarrow 0$ as $n \rightarrow \infty$. Assume that this is not so. Then there exists $\varepsilon > 0$ so that $\|f_n - f\| > \varepsilon$ (for simplicity) for every $n \in \mathbb{N}$. Then, using inequality (1) we obtain

$$\begin{aligned} & \int_{\Omega} M\left(\frac{\|f_n(\omega) - f(\omega)\|}{2}\right) d\mu(\omega) \\ &= \int_{\Omega} M\left(\frac{\|f_n - f\|}{2} \cdot \frac{\|f_n(\omega) - f(\omega)\|}{\|f_n - f\|}\right) d\mu(\omega) \\ &\geq \int_{\Omega} \frac{1}{4} \|f_n - f\|^2 M\left(\frac{\|f_n(\omega) - f(\omega)\|}{\|f_n - f\|}\right) d\mu(\omega) \\ &= \frac{1}{4} \|f_n - f\|^2 > \frac{\varepsilon^2}{4} \end{aligned} \tag{10}$$

for each $n \in \mathbb{N}$. On the other hand, using the general assertions (7) and (8) for the URED case, where g_n is replaced by f , we have for almost all $\omega \in \Omega$,

$$M''(\max\{\|f(\omega)\|, \|f_n(\omega)\|\}) \cdot (\|f(\omega)\| - \|f_n(\omega)\|)^2 \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty$$

and

$$M\left(\frac{\|f(\omega)\| + \|f_n(\omega)\|}{2}\right) - M\left(\frac{\|f(\omega) + f_n(\omega)\|}{2}\right) \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty \tag{11}$$

where N is a suitable fixed infinite subset of $\mathbb{N}$. Fix for a while any $\omega \in \Omega$ for which the convergences above occur. Let $\lambda(\omega) \in [0, +\infty]$ be an accumulation point of the sequence $(\|f_n(\omega)\|)_{n \in N}$. Assume that $\lambda(\omega) = +\infty$. Then, as $\lim_{t \rightarrow +\infty} M''(t) t^2 \in (0, +\infty)$, we have

$$\begin{aligned} 0 &= \lim_{N_1 \ni n \rightarrow \infty} M''(\max\{\|f(\omega)\|, \|f_n(\omega)\|\}) \cdot (\|f(\omega)\| - \|f_n(\omega)\|)^2 \\ &= \lim_{N_1 \ni n \rightarrow \infty} M''(\|f_n(\omega)\|) \cdot (\|f(\omega)\| - \|f_n(\omega)\|)^2 \\ &= \lim_{t \rightarrow +\infty} M''(t) (\|f(\omega)\| - t)^2 > 0, \end{aligned}$$

where N_1 is a suitable infinite subset of N . We obtained a contradiction. Therefore $\lambda(\omega)$ must be a finite number. Then we get $M''(\max\{\|f(\omega)\|, \lambda(\omega)\}) (\|f(\omega)\| - \lambda(\omega))^2 = 0$, and so $\lambda(\omega) = \|f(\omega)\|$. Consequently,

$$\|f_n(\omega)\| \longrightarrow \|f(\omega)\| \quad \text{as } N \ni n \rightarrow \infty \tag{12}$$

for almost all $\omega \in \Omega$. Thus, using (11), we obtain

$$M\left(\frac{\|f(\omega) + f_n(\omega)\|}{2}\right) \longrightarrow M(\|f(\omega)\|) \quad \text{as } N \ni n \rightarrow \infty.$$

But M is continuous and strictly increasing on $[0, +\infty)$, so

$$\|f(\omega) + f_n(\omega)\| \longrightarrow 2\|f(\omega)\| \quad \text{as } N \ni n \rightarrow \infty \text{ for almost every } \omega \in \Omega.$$

Since the norm $\|\cdot\|$ on X is LUR, from the latter convergence and (12) we deduce that

$$\|f_n(\omega) - f(\omega)\| \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty \text{ for almost every } \omega \in \Omega. \quad (13)$$

It remains to perform a final attack. In what follows we shall use ideas from the proof of Vitali's convergence theorem (see e.g. [11, Exercise 12.9]). Put

$$\varphi_n(\omega) := M(\|f(\omega)\|) + M(\|f_n(\omega)\|) - 2M\left(\frac{\|f_n(\omega) - f(\omega)\|}{2}\right), \quad \omega \in \Omega, \quad n \in N.$$

The convexity and monotonicity of M yields that

$$\varphi_n(\omega) \geq M(\|f(\omega)\|) + M(\|f_n(\omega)\|) - 2M\left(\frac{\|f_n(\omega)\| + \|f(\omega)\|}{2}\right) \geq 0$$

for every $\omega \in \Omega$ and every $n \in N$. Further, (12) and (13) imply that $\varphi_n(\omega) \longrightarrow 2M(\|f(\omega)\|)$ as $N \ni n \rightarrow \infty$ for almost all $\omega \in \Omega$. Hence, by Fatou's lemma,

$$\begin{aligned} 2 &= \int_{\Omega} 2M(\|f(\omega)\|) \, d\mu(\omega) \leq \liminf_{N \ni n \rightarrow \infty} \int_{\Omega} \varphi_n(\omega) \, d\mu(\omega) \\ &= 2 - \limsup_{N \ni n \rightarrow \infty} \int_{\Omega} 2M\left(\frac{\|f_n(\omega) - f(\omega)\|}{2}\right) \, d\mu(\omega), \end{aligned}$$

and therefore

$$\lim_{N \ni n \rightarrow \infty} \int_{\Omega} M\left(\frac{\|f_n(\omega) - f(\omega)\|}{2}\right) \, d\mu(\omega) = 0.$$

But this contradicts inequality (10). We thus proved that the norm $|||\cdot|||$ is LUR.

It remains to show that if the norm $\|\cdot\|$ on X is MLUR, then so is $|||\cdot|||$. Pick $h \in L^1(\mu, X)$ and two sequences $(f_n), (g_n)$ in $L^1(\mu, X)$ such that $|||f_n||| = |||g_n||| = |||h||| = 1$ for all $n \in \mathbb{N}$ and $|||(f_n + g_n)/2 - h||| \longrightarrow 0$ as $n \rightarrow \infty$.

Arguing by contradiction, suppose that there is $\varepsilon > 0$ such that $|||f_n - g_n||| > \varepsilon$ (for simplicity) for every $n \in \mathbb{N}$. Then, thanks to inequality (1), we have

$$\begin{aligned} & \int_{\Omega} M\left(\frac{\|f_n(\omega) - g_n(\omega)\|}{2}\right) d\mu(\omega) \\ &= \int_{\Omega} M\left(|||(f_n - g_n)/2||| \frac{\|f_n(\omega) - g_n(\omega)\|}{|||f_n - g_n|||}\right) d\mu(\omega) \\ &\geq |||(f_n - g_n)/2|||^2 > \varepsilon^2/4 \end{aligned} \quad (14)$$

for all $n \in \mathbb{N}$.

Now, for $n \in \mathbb{N}$ set $h_n = \frac{1}{2}(f_n + g_n)$. We already know that the norm $|||\cdot|||$ is rotund. Hence $|||h - h_n||| \geq 1 - |||h_n||| > 0$ for all $n \in \mathbb{N}$. We may and do assume that $|||h_n - h||| < 1$ for every $n \in \mathbb{N}$. Bearing in mind that M is convex we deduce that

$$\begin{aligned} & \int_{\Omega} M(|h_n(\omega) - h(\omega)|) d\mu(\omega) \\ &\leq |||h_n - h||| \int_{\Omega} M\left(\frac{|h_n(\omega) - h(\omega)|}{|||h_n - h|||}\right) d\mu(\omega) = |||h_n - h||| \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, using the fact that $|||h_n - h||| \rightarrow 0$ as $n \rightarrow \infty$, it follows that

$$\int_{\Omega} M(|h_n(\omega) - h(\omega)|) d\mu(\omega) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, there is an infinite set $N \subset \mathbb{N}$ such that

$$M(|h_n(\omega) - h(\omega)|) \rightarrow 0 \text{ as } N \ni n \rightarrow \infty,$$

for almost all $\omega \in \Omega$. Consequently

$$|(f_n(\omega) + g_n(\omega))/2 - h(\omega)| \rightarrow 0 \text{ as } N \ni n \rightarrow \infty, \quad (15)$$

for almost all $\omega \in \Omega$. For each $n \in N$ and each $\omega \in \Omega$ put

$$a_n(\omega) := M(\|f_n(\omega)\|) + M(\|h(\omega)\|) - 2M\left(\frac{\|f_n(\omega)\| + \|h(\omega)\|}{2}\right)$$

and

$$b_n(\omega) := M(\|g_n(\omega)\|) + M(\|h(\omega)\|) - 2M\left(\frac{\|g_n(\omega)\| + \|h(\omega)\|}{2}\right).$$

The convexity and monotonicity of M guarantees that $a_n(\omega) \geq 0$, $b_n(\omega) \geq 0$, and

$$\begin{aligned} & M\left(\frac{\|(f_n(\omega) + g_n(\omega))/2\| + \|h(\omega)\|}{2}\right) + M\left(\frac{\|g_n(\omega)\| + \|h(\omega)\|}{2}\right) \\ &\geq 2M\left(\frac{\|(f_n(\omega) + g_n(\omega))/2\| + 2\|h(\omega)\|}{4}\right) \end{aligned}$$

for all $n \in N$ and all ω . Thus

$$\begin{aligned} 0 &\leq \int_{\Omega} a_n(\omega) d\mu(\omega) + \int_{\Omega} b_n(\omega) d\mu(\omega) \\ &\leq 4 - 4 \int_{\Omega} M \left(\frac{\|(f_n(\omega) + g_n(\omega))\| + 2\|h(\omega)\|}{4} \right) d\mu(\omega) \end{aligned} \quad (16)$$

for all $n \in N$. On the other hand, from (15) we get

$$M \left(\frac{\|(f_n(\omega) + g_n(\omega))\| + 2\|h(\omega)\|}{4} \right) \longrightarrow M(\|h(\omega)\|) \text{ as } N \ni n \rightarrow \infty$$

for almost all $\omega \in \Omega$. Fatou's Lemma and (16) then yield that

$$\begin{aligned} (1 =) &\int_{\Omega} M(\|h(\omega)\|) d\mu(\omega) \\ &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} M \left(\frac{\|f_n(\omega) + g_n(\omega)\| + 2\|h(\omega)\|}{4} \right) d\mu(\omega). \end{aligned}$$

This inequality and (16) imply that $\lim_{N \ni n \rightarrow \infty} \int_{\Omega} a_n(\omega) d\mu(\omega) = \lim_{N \ni n \rightarrow \infty} \int_{\Omega} b_n(\omega) d\mu(\omega) = 0$. Thus, there exists an infinite set $N_1 \subset N$ such that $a_n(\omega) \rightarrow 0$ and $b_n(\omega) \rightarrow 0$ as $N_1 \ni n \rightarrow \infty$ for almost all $\omega \in \Omega$, and by (2),

$$\lim_{N_1 \ni n \rightarrow \infty} M''(\max\{\|f_n(\omega)\|, \|h(\omega)\|\}) (\|f_n(\omega)\| - \|h(\omega)\|)^2 = 0$$

and

$$\lim_{N_1 \ni n \rightarrow \infty} M''(\max\{\|g_n(\omega)\|, \|h(\omega)\|\}) (\|g_n(\omega)\| - \|h(\omega)\|)^2 = 0.$$

Using the same argument as in the LUR case, we deduce that

$$\lim_{N_1 \ni n \rightarrow \infty} \|f_n(\omega)\| = \lim_{N_1 \ni n \rightarrow \infty} \|g_n(\omega)\| = \|h(\omega)\| \text{ for almost all } \omega \in \Omega.$$

Since the norm $\|\cdot\|$ is MLUR, from these equalities and (15) we easily deduce that

$$\lim_{N_1 \ni n \rightarrow \infty} \|f_n(\omega) - g_n(\omega)\| = 0 \text{ for almost all } \omega \in \Omega.$$

Now, imitating again Vitali's argument, for each $n \in N_1$ and each $\omega \in \Omega$ we put

$$\varphi_n(\omega) := M(\|f_n(\omega)\|) + M(\|g_n(\omega)\|) - 2M(\|f_n(\omega) - g_n(\omega)\|/2).$$

Then, $\varphi_n(\omega) \geq 0$ for all $n \in N_1$ and $\lim_{N_1 \ni n \rightarrow \infty} \varphi_n(\omega) = 2M(\|h(\omega)\|)$ for almost all $\omega \in \Omega$. Therefore, according to Fatou's lemma, we have

$$\begin{aligned} (2 =) \int_{\Omega} 2M(\|h(\omega)\|) d\mu(\omega) &\leq \liminf_{N_1 \ni n \rightarrow \infty} \int_{\Omega} \varphi_n(\omega) d\mu(\omega) \\ &= 2 - 2 \limsup_{N_1 \ni n \rightarrow \infty} \int_{\Omega} M(\|f_n(\omega) - g_n(\omega)\|/2) d\mu(\omega). \end{aligned}$$

We thus arrived at the equality

$$\lim_{N_1 \ni n \rightarrow \infty} \int_{\Omega} M(\|f_n(\omega) - g_n(\omega)\|/2) d\mu(\omega) = 0,$$

which contradicts (14). Therefore, the norm $|||\cdot|||$ is MLUR, as we wanted to show. $\square$

Now, we proceed with the proof of Theorem 1.2.

Proof of Theorem 1.2. On $L^1(\mu, X)$, we still consider the norm $|||\cdot|||$ constructed in the proof of Theorem 1.1. Let Y be a reflexive subspace of $L^1(\mu, X)$. Pick any two sequences (f_n) and (g_n) in $L^1(\mu, X)$ such that $|||f_n||| = |||g_n||| = 1$, $|||f_n + g_n||| \rightarrow 2$ as $n \rightarrow \infty$, and that $f_n, g_n \in Y$ (or more generally that just $f_n - g_n \in Y$) for all $n \in \mathbb{N}$. Arguing by contradiction, assume that there is $\varepsilon > 0$ such that $|||f_n - g_n||| > \varepsilon$ (for simplicity) for all $n \in \mathbb{N}$. This inequality and (4) imply that

$$\int_{\Omega} \|f_n(\omega) - g_n(\omega)\| d\mu(\omega) > \frac{\varepsilon}{C}. \quad (17)$$

On the other hand, because of the reflexivity of Y it follows that the set $\{f_1 - g_1, f_2 - g_2, \dots\}$ is uniformly integrable (see e.g. [3, p. 104]), that is, there is a $\delta > 0$ so small that

$$\int_A \|f_n(\omega) - g_n(\omega)\| d\mu(\omega) < \frac{\varepsilon}{2C}$$

whenever $n \in \mathbb{N}$, $A \in \Sigma$, and $\mu(A) \leq \frac{2(2M'(1) + 1)}{M'(1)}\delta$.

For $n \in \mathbb{N}$ put

$$A_n := \left\{ \omega \in \Omega : \|f_n(\omega)\| + \|g_n(\omega)\| > \frac{1}{\delta} \right\}.$$

According to Chebyshev's inequality, from (4) it follows that

$$\begin{aligned} \mu(A_n) &\leq \delta \int_{A_n} (\|f_n(\omega)\| + \|g_n(\omega)\|) d\mu(\omega) \leq \frac{(2M'(1) + 1)}{M'(1)} (|||f_n||| + |||g_n|||) \delta \\ &= \frac{2(2M'(1) + 1)}{M'(1)} \delta. \end{aligned}$$

Thus,

$$\int_{A_n} \|f_n(\omega) - g_n(\omega)\| d\mu(\omega) < \frac{\varepsilon}{2C} \quad \text{for each } n \in \mathbb{N}. \quad (18)$$

Now, define

$$\begin{aligned} \tilde{f}_n(\omega) &= f_n(\omega), \quad \tilde{g}_n(\omega) = g_n(\omega) \quad \text{if } \omega \in \Omega \setminus A_n, \\ \text{and } \tilde{f}_n(\omega) &= \tilde{g}_n(\omega) = 0 \quad \text{if } \omega \in A_n. \end{aligned}$$

It is clear that $\tilde{f}_n, \tilde{g}_n \in L^1(\mu, X)$ for each $n \in \mathbb{N}$. Further, we observe that for all $n \in \mathbb{N}$ and for all $\omega \in \Omega$ we have

$$\begin{aligned} 0 &\leq M\left(\frac{1}{2}(\|\tilde{f}_n(\omega)\| + \|\tilde{g}_n(\omega)\|)\right) - M\left(\frac{1}{2}\|\tilde{f}_n(\omega) + \tilde{g}_n(\omega)\|\right) \\ &\leq M\left(\frac{1}{2}(\|f_n(\omega)\| + \|g_n(\omega)\|)\right) - M\left(\frac{1}{2}\|f_n(\omega) + g_n(\omega)\|\right) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq M''(\tfrac{1}{\delta})\left(\|\tilde{f}_n(\omega)\| - \|\tilde{g}_n(\omega)\|\right)^2 \\ &\leq M''(\max\{\|f_n(\omega)\|, \|g_n(\omega)\|\})\left(\|f_n(\omega)\| - \|g_n(\omega)\|\right)^2. \end{aligned}$$

Hence, taking into account (5) and (6) (which are at hand also in the UR case), we can find an infinite set $N \subset \mathbb{N}$ such that for almost all $\omega \in \Omega$ we have

$$M\left(\frac{1}{2}(\|\tilde{f}_n(\omega)\| + \|\tilde{g}_n(\omega)\|)\right) - M\left(\frac{1}{2}\|\tilde{f}_n(\omega) + \tilde{g}_n(\omega)\|\right) \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty$$

and

$$M''(\tfrac{1}{\delta})\left(\|\tilde{f}_n(\omega)\| - \|\tilde{g}_n(\omega)\|\right)^2 \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty,$$

and so

$$\|\tilde{f}_n(\omega)\| - \|\tilde{g}_n(\omega)\| \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty.$$

Thus, for almost all $\omega \in \Omega$ we have $M\left(\|\tilde{f}_n(\omega)\|\right) - M\left(\frac{1}{2}\|\tilde{f}_n(\omega) + \tilde{g}_n(\omega)\|\right) \longrightarrow 0$ as $N \ni n \rightarrow \infty$, and, since $\|\tilde{f}_n(\omega)\| + \|\tilde{g}_n(\omega)\| \leq \frac{1}{\delta}$ and M is increasing and continuous on $[0, +\infty)$, we get

$$\begin{aligned} &\|\tilde{f}_n(\omega) + \tilde{g}_n(\omega)\| - 2\|\tilde{f}_n(\omega)\| \longrightarrow 0 \\ \text{and } &\|\tilde{f}_n(\omega)\| - \|\tilde{g}_n(\omega)\| \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty. \end{aligned} \tag{19}$$

On the other hand, it is easy to see that if $(x_n), (y_n)$ are bounded sequences in a Banach space with UR norm $\|\cdot\|$ and $\|x_n + y_n\| - 2\|x_n\| \longrightarrow 0$ and $\|x_n\| - \|y_n\| \longrightarrow 0$ as $n \rightarrow \infty$, then $\|x_n - y_n\| \longrightarrow 0$ as $n \rightarrow \infty$. Substituting here $x_n := \tilde{f}_n(\omega)$ and $y_n := \tilde{g}_n(\omega)$, $n \in N$, and using (19) we can conclude that

$$\|\tilde{f}_n(\omega) - \tilde{g}_n(\omega)\| \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty$$

for almost all $\omega \in \Omega$. Then the Lebesgue's dominated convergence theorem guarantees that

$$\int_{\Omega} \|\tilde{f}_n(\omega) - \tilde{g}_n(\omega)\| d\mu(\omega) \longrightarrow 0 \quad \text{as } N \ni n \rightarrow \infty. \tag{20}$$

But, because of (17) and (18) we have

$$\begin{aligned} \frac{\varepsilon}{C} &< \int_{\Omega} \|f_n(\omega) - g_n(\omega)\| d\mu(\omega) \\ &= \int_{A_n} \|f_n(\omega) - g_n(\omega)\| d\mu(\omega) + \int_{\Omega} \|\tilde{f}_n(\omega) - \tilde{g}_n(\omega)\| d\mu(\omega) \\ &< \frac{\varepsilon}{2C} + \int_{\Omega} \|\tilde{f}_n(\omega) - \tilde{g}_n(\omega)\| d\mu(\omega) \end{aligned}$$

for all $n \in N$. And this is in contradiction with (20). $\square$

We end this paper with some comments about the possibility to achieve a combination of properties of rotundity and smoothness on the space $L^1(\mu, X)$.

Remarks.

- (1) In [5, Theorem 2.1] it is shown that if M is the function M_2 considered at page 1028 and the norm $\|\cdot\|$ on the Banach space X is Gâteaux (uniformly Gâteaux) smooth, then so is the corresponding Orlicz-Bochner norm $|||\cdot|||$ on $L^1(\mu, X)$. Thus, as an immediate consequence of this fact and Theorem 1.1, we deduce that *if $(X, \|\cdot\|)$ is a Banach space whose norm is Gâteaux smooth or uniformly Gâteaux smooth, and simultaneously rotund or URED or LUR or MLUR, then so is the norm $|||\cdot|||$ on $L^1(\mu, X)$ constructed in Theorem 1.1.*
- (2) From Theorem 3.2 in [5] we have that if the norm $\|\cdot\|$ on X is Fréchet (uniformly Fréchet) smooth, then the restriction of $|||\cdot|||$ to every reflexive subspace of $L^1(\mu, X)$ is Fréchet (uniformly Fréchet) smooth. Combining the parenthetic part of this assertion with Theorem 1.2 we get that, *if X is superreflexive, then the space $L^1(\mu, X)$ admits an equivalent norm whose restriction to every reflexive subspace is simultaneously UR and uniformly Fréchet smooth.* This fact provides a (new) proof of the Lebesgue-Bochner counterpart of Rosenthal's result in [12] that, if X is superreflexive, then every reflexive subspace of $L^1(\mu, X)$ is superreflexive.
- (3) Taking into account the fact that the norm $|||\cdot|||$ preserves the LUR property, the Fréchet version of Theorem 3.2 in [5] and [2, Theorem VII.1.14] yield that, *if the space X is both Asplund and weakly compactly generated (in particular, if X is reflexive), then $L^1(\mu, X)$ has an equivalent LUR norm whose restriction to every reflexive subspace is Fréchet smooth.*

References

- [1] G. Aleksandrov, I. Dimitrov: On equivalent weakly midpoint locally uniformly rotund renormings of the space ℓ_{∞} , in: Math. and Math. Education (Sunny Beach, 1985), Bulg. Acad. of Sciences, Sofia (1985) 189–191 (in Russian).
- [2] R. Deville, G. Godefroy, V. Zizler: Smoothness and Renormings in Banach Spaces. Pitman Monographs and Surveys in Pure and Appl. Math. 64, Longman, Harlow (1993).

- [3] J. Diestel, J. J. Uhl: Vector Measures, Math. Surveys 15, Amer. Math. Soc., Providence (1977).
- [4] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler: Banach Space Theory: The Basis for Linear and Nonlinear Analysis, CMS Books in Mathematics, Springer, New York (2011).
- [5] M. Fabian, S. Lajara: Smooth renormings of the Lebesgue-Bochner function space $L^1(\mu, X)$, Stud. Math. 209 (2012) 247–265.
- [6] R. Haydon: Trees in renorming theory, Proc. Lond. Math. Soc. 78 (1999) 541–585.
- [7] D. Kutzarova, S. Troyanski: Reflexive Banach spaces without equivalent norms which are uniformly convex or uniformly differentiable in every direction, Stud. Math. 72 (1982) 91–95.
- [8] S. Lajara, A. Pallarés, S. Troyanski: Moduli of convexity and smoothness of reflexive subspaces of L^1 , J. Funct. Anal. 261 (2011) 3211–3225.
- [9] I. E. Leonard, K. Sundaresan: Smoothness and duality in $L_p(E, \mu)$, J. Math. Anal. Appl. 46 (1974) 513–522.
- [10] I. E. Leonard, K. Sundaresan: Geometry of Lebesgue-Bochner function spaces - smoothness, Trans. Amer. Math. Soc. 198 (1974) 229–251.
- [11] J. Lukeš, J. Malý: Measure and Integral, Matfyzpress, Prague (1995).
- [12] H. P. Rosenthal: On subspaces of L^p , Ann. Math. 97 (1973) 344–373.
- [13] G. Schluchtermann: Renorming in the space of Bochner integrable functions $L_1(\mu, X)$, Manuscr. Math. 73(4) (1991) 397–409.
- [14] M. A. Smith, B. Turett: Rotundity in Lebesgue-Bochner function spaces, Trans. Amer. Math. Soc. 257 (1980) 105–118.