Journal of Convex Analysis
Volume 22 (2015), No. 4, 1041–1060

Quasi-Metrizability of Bornological Biuniverses in ZF

Artur Piękosz
Institute of Mathematics, Cracow University of Technology, Warszawska 24, 31-155 Cracow, Poland
pupiekos@cyfronet.pl

Eliza Wajch
Institute of Mathematics and Physics, Siedlce University of Natural Sciences and Humanities, 3 Maja 54, 08-110 Siedlce, Poland
eliza.wajch@wp.pl

Received: September 2, 2014
Revised manuscript received: December 4, 2014
Accepted: January 9, 2015

Hu's metrization theorem for bornological universes is shown to hold in **ZF** and it is adapted to a quasi-metrization theorem for bornologies in bitopological spaces. The problem of uniform quasi-metrization of quasi-metric bornological universes is investigated and several relevant results independent of **ZF** are obtained. In particular, some theorems of Garrido and Meroño that contain necessary and sufficient conditions for a bornology in a metric space to be uniformly metrizable are generalized to quasi-metric bornological universes in **ZF**, while some other theorems of Garrido and Meroño are shown to fail in a model for **ZF**.

Keywords: Bornology, bitopological space, quasi-metric, ZF

2010 Mathematics Subject Classification: Primary 54E55, 54A35; Secondary 54E35, 54E99

1. Introduction

A *bitopological space* is a triple (X, τ_1, τ_2) where X is a set and τ_1, τ_2 are topologies in X. A *quasi-pseudometric* in a set X is a function $d : X \times X \to [0; +\infty)$ such that, for all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$ and $d(x, x) = 0$. A quasi-pseudometric d in X is called a *quasi-metric* if, for all $x, y \in X$, the condition $d(x, y) = 0$ implies $x = y$ (cf. [6], [15]).

Let d be a quasi-pseudometric in X. *The conjugate* of d is the quasi-pseudometric d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ for $x, y \in X$. The d-ball with centre $x \in X$ and radius $r \in (0; +\infty)$ is the set $B_d(x, r) = \{y \in X : d(x, y) < r\}$. The collection $\tau(d) = \{V \subseteq X : \forall_{x \in V} \exists_{n \in \omega} B_d(x, \frac{1}{2^n}) \subseteq V\}$ is *the topology in* X *induced by* d. The triple $(X, \tau(d), \tau(d^{-1}))$ is *the bitopological space associated with* d.

ISSN 0944-6532 / \$ 2.50

Definition 1.1. A bitopological space (X, τ_1, τ_2) is *(quasi)-metrizable* if there exists a (quasi)-metric d in X such that $\tau_1 = \tau(d)$ and $\tau_2 = \tau(d^{-1})$ (cf. pp. 74–75 of [15]).

One can find a considerable number of quasi-metrization theorems in [2] and in other sources (cf. [6]).

We recall that, according to [1] and [12], a *boundedness* in a set X is a non-void ideal of subsets of X. A boundedness $\mathcal{B}$ in X is called a *bornology* in X if every singleton of X is a member of $\mathcal{B}$ (cf. 1.1.1 in [10]).

Definition 1.2 (cf. Definition 4.1 of [12]). If $\mathcal{B}$ is a boundedness in X, then a collection $\mathcal{A}$ is called *a base* for $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and every set of $\mathcal{B}$ is a subset of a member of $\mathcal{A}$. *A second-countable boundedness* is a boundedness which has a countable base.

Definition 1.3. Let (X, τ_1, τ_2) be a bitopological space. A boundedness $\mathcal{B}$ in X will be called (τ_1, τ_2)*-proper* if, for each $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $\mathrm{cl}_{\tau_2} A \subseteq \mathrm{int}_{\tau_1}(B)$. If $\tau = \tau_1 = \tau_2$ and the boundedness $\mathcal{B}$ is (τ, τ)-proper, we will say that $\mathcal{B}$ is τ*-proper.*

Let us notice that if (X, τ) is a topological space, then a boundedness $\mathcal{B}$ in X is τ-proper if and only if the universe $((X, \tau), \mathcal{B})$ is proper in the sense of Definition 3.4 of [12].

Definition 1.4.

(i) We say that *a bornological biuniverse* is an ordered pair $((X, \tau_1, \tau_2), \mathcal{B})$ where (X, τ_1, τ_2) is a bitopological space and $\mathcal{B}$ is a bornology in X.

(ii) A *bornological universe* is an ordered pair $((X, \tau), \mathcal{B})$ where (X, τ) is a topological space and $\mathcal{B}$ is a bornology in X (cf. Definition 1.2 of [12]).

Definition 1.5. Let d be a quasi-metric in X and let A be a subset of X. Then:

(i) A is called *d-bounded* if there exist $x \in X$ and $r \in (0; +\infty)$ such that $A \subseteq B_d(x, r)$;

(ii) if A is not d-bounded, we say that A is *d-unbounded*;

(iii) $\mathcal{B}_d(X)$ is the collection of all d-bounded subsets of X.

For a quasi-metric d in X, a set $A \subseteq X$ can be simultaneously d-bounded and d^{-1}-unbounded.

Example 1.6. For $x, y \in \omega$, let $d(x, y) = 0$ if $x = y$ and $d(x, y) = 2^x$ if $x \neq y$. Then $\omega = B_d(0, 2)$, so ω is d-bounded. However, for arbitrary $x \in \omega$ and $r \in (0; +\infty)$, if $y \in \omega$ is such that $2^y > r$, then $y \notin B_{d^{-1}}(x, r)$. Therefore, ω is d^{-1}-unbounded.

Definition 1.7. We say that a bornological biuniverse $((X, \tau_1, \tau_2), \mathcal{B})$ is *(quasi)-metrizable* if there exists a (quasi)-metric d in X such that $\tau_1 = \tau(d), \tau_2 = \tau(d^{-1})$

and $\mathcal{B} = \mathcal{B}_d(X)$.

It is obvious that if τ is a topology in X, then a bornological biuniverse $((X, \tau, \tau), \mathcal{B})$ is metrizable if and only if the bornological universe $((X, \tau), \mathcal{B})$ is metrizable in the sense of Definition 10.1 of [12]. Let us recall this definition.

Definition 1.8. Let $((X, \tau), \mathcal{B})$ be a bornological universe. We say that:

(i) $((X, \tau), \mathcal{B})$ is *metrizable (in the sense of Hu)* if there exists a metric d on X such that $\tau = \tau(d)$ and $\mathcal{B} = \{A \subseteq X : \mathrm{diam}_d(A) < +\infty\}$ where $\mathrm{diam}_d(A) = \sup\{d(x, y) : x, y \in A\}$;

(ii) $((X, \tau), \mathcal{B})$ is *quasi-metrizable* if there exists a quasi-metric d on X such that $\tau = \tau(d)$ and, moreover, $\mathcal{B}$ is the collection of all d-bounded sets.

We show in Section 4 that if a bornological biuniverse $((X, \tau, \tau), \mathcal{B})$ is quasi-metrizable, then the bornological universe $((X, \tau), \mathcal{B})$ is metrizable.

Of course, it is impossible to prove anything in mathematics without axioms. The basic set-theoretic system of axioms used in this paper is **ZF** (cf. [16]–[17]). If a relatively independent of **ZF** axiom **A** is added to **ZF**, we shall write **ZF** + **A** and clearly denote our theorems proved in **ZF** + **A** but not in **ZF**. As far as set-theoretic axioms are concerned, we use standard notation from [9] and [17]. In particular, we denote **ZF** + **AC** by **ZFC**. According to Theorem 1 of [20] and Theorem 13.2 of [12], the following theorem can be called *Hu's metrization theorem for bornological universes*:

Theorem 1.9. *It holds true in* **ZFC** *that a bornological universe* $((X, \tau), \mathcal{B})$ *is metrizable if and only if it is proper, while, simultaneously,* (X, τ) *is metrizable and* $\mathcal{B}$ *has a countable base.*

One of the main aims of our present work is to show that the proof to Hu's metrization theorem in [12] highly involves the axiom **CC** of countable choice and to prove in **ZF** the following generalization of Theorem 1.9:

Theorem 1.10. *It is true in* **ZF** *that a bornological biuniverse* $((X, \tau_1, \tau_2), \mathcal{B})$ *is quasi-metrizable if and only if* $\mathcal{B}$ *has a countable base and it is* (τ_1, τ_2)*- proper, while the bitopological space* (X, τ_1, τ_2) *is quasi-metrizable.*

We deduce Theorem 1.9 from 1.10 and we prove a stronger theorem than 1.10 in Section 4 (Theorem 4.7). We also give some other applications of Theorem 1.10. Especially in Sections 2, 3 and 7, we give examples of unprovable in **ZF** results on bornological universes that were obtained by other authors probably either in **ZFC** or in naive preaxiomatic set theory. Section 5 contains a generalization of Theorem 13.5 of [12]. We pay a special attention to [8] and, in Section 6, we modify the basic theorem of [8] to get necessary and sufficient conditions for a bornological quasi-metric universe to be uniformly quasi-metrizable in **ZF** (Theorem 6.5). Finally, in Section 8, we modify a theorem about compact bornologies from [8].

We recommend [5] as a monograph on topology that we use. Models of set theory applied by us are described in [9], [11], [13] and [14].

2. Countability

The axiom of countable choice is usually denoted by **CC**, **ACC** or **CAC**.

We shall use the following standard notions of finiteness and infinity:

Definition 2.1. A set X is called:

(i) *finite* or *T-finite* (truly finite) if there exists $n \in \omega$ such that X is equipollent with n;

(ii) *D-finite* or *Dedekind-finite* if no proper subset of X is equipollent with X;

(iii) *infinite* or *T-infinite* if it is not finite, and *D-infinite* if it is not D-finite.

A set is T-finite if and only if it is finite in Tarski's sense (cf. Definition 4.4 of [9]). Other notions relevant to finiteness were studied, for example, in [4]. The term *truly finite* was suggested by K. Kunen in a private communication with E. Wajch.

Let us establish three distinct notions of countability.

Definition 2.2. A set X is called:

(i) *countable* or *at most countable*, or *T-countable* (truly countable) if X is equipollent with a subset of ω;

(ii) *D-countable* if every D-infinite subset of X is equipollent with X;

(iii) *W-countable* if every well-orderable subset of X is D-countable.

To each notion of countability Q corresponds a notion of uncountability.

Definition 2.3. We say that a set is *Q-uncountable* if it is not Q-countable where Q stands for T, D or W. Sets that are T-uncountable are called *uncountable.*

Let us denote by **CC**(D-fin) the following statement: every non-void countable collection of pairwise disjoint non-void D-finite sets has a choice function. As usual, **CC**(fin) is the statement: every non-void countable collection of pairwise disjoint non-void finite sets has a choice function.

Proposition 2.4. *The following conditions are equivalent:*

(i) **CC**(*D-fin*)*;*

(ii) *every D-countable set is countable.*

Proof. Let X be a set. Assume that X is D-countable. If X is D-infinite, then X is countable (cf. [21], p. 48). Assume that X D-finite. Then if (i) holds, it follows from E13 of Section 4.1 of [9] that the set X is finite, so countable. Hence (i) implies (ii). Now, assume that (ii) holds and that X is D-finite. Then X is D-countable, so countable. This implies that X is equipollent with a finite subset of ω and, in consequence, X is finite. By E13 of Section 4.1 of [9], (ii) implies (i). $\square$

Fact 2.5. *For every D-finite set X, the following conditions are equivalent:*

(i) *X is finite;*
(ii) *$X \cup \omega$ is D-countable.*

Corollary 2.6. *If X is an infinite D-finite set, then the set $X \cup \omega$ is D-uncountable.*

Fact 2.7. *A set X is countable if and only if $X \cup \omega$ is D-countable.*

Corollary 2.8. *In every model $\mathbf{M}$ for $\mathbf{ZF}$ such that there is in $\mathbf{M}$ an infinite D-finite subset of $\mathbb{R}$, the collection of all D-countable subsets of $\mathbb{R}$ is not a bornology.*

Fact 2.9. *In every set X, the following collections are bornologies:*

(i) *the collection $\mathbf{FB}(X)$ of all finite subsets of X;*
(ii) *the collection of all D-finite subsets of X;*
(iii) *the collection of all countable subsets of X;*
(iv) *the collection of all W-countable subsets of X.*

Several remarks on D-countability can be found in [21].

3. Second-countable bornological biuniverses

One may deduce wrongly from Theorem 5.5 of [12] that every base of a second-countable boundedness $\mathcal{B}$ certainly contains a countable base for $\mathcal{B}$. However, we are going to prove that Theorem 5.5 of [12] is relatively independent of $\mathbf{ZF}$. To do this, let us consider the following bornologies in $\mathbb{R}$:

$$\mathbf{UB}(\mathbb{R}) = \{A \subseteq \mathbb{R} : \exists_{r \in \mathbb{R}} A \subseteq (-\infty; r)\},$$

$$\mathbf{LB}(\mathbb{R}) = \{A \subseteq \mathbb{R} : \exists_{r \in \mathbb{R}} A \subseteq (r; +\infty)\}.$$

Of course, $\mathbf{UB}(\mathbb{R})$ and $\mathbf{LB}(\mathbb{R})$ are second-countable.

Proposition 3.1. *Equivalent are:*

(i) $\mathbf{CC}(\mathbb{R})$*;*
(ii) *for every unbounded to the right subset D of $\mathbb{R}$, the collection $\mathcal{A}(D) = \{(-\infty; d) : d \in D\}$ contains a countable base for $\mathbf{UB}(\mathbb{R})$;*
(iii) *for every unbounded to the left subset D of $\mathbb{R}$, the collection $\mathcal{A}(D) = \{(d; +\infty) : d \in D\}$ contains a countable base for $\mathbf{LB}(\mathbb{R})$.*

Proof. First, assume that $\mathbf{CC}(\mathbb{R})$ holds and that D is an unbounded to the right subset of $\mathbb{R}$. It follows from Theorem 3.8 of [9] that there exists an unbounded sequence $(d_n)_{n\in\omega}$ of elements of $D \cap [0; +\infty)$. Then $\{(-\infty; d_n) : n \in \omega\}$ is a countable base for $\mathbf{UB}(\mathbb{R})$.

Now, suppose that $\mathbf{CC}(\mathbb{R})$ does not hold. By Theorem 3.8 of [9], there exists an unbounded subset B of $\mathbb{R}$ which does not contain any unbounded sequence. Then the set $D = B \cup \{-x : x \in B\}$ does not contain any unbounded sequence. The collection $\mathcal{A}(D)$ is a base for $\mathbf{UB}(\mathbb{R})$ such that $\mathcal{A}(D)$ does not contain any countable base for $\mathbf{UB}(\mathbb{R})$. Hence (i) is equivalent to (ii).

To show that (ii) and (iii) are equivalent, it suffices to make a suitable use of the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -x$ for $x \in \mathbb{R}$. $\square$

Corollary 3.2. *Let* $\mathbf{M}$ *be any model for* $\mathbf{ZF}$ *such that* $\mathbf{CC}(\mathbb{R})$ *fails in* $\mathbf{M}$*. Then the bornology* $\mathbf{UB}(\mathbb{R})$ *has a base which does not contain any countable base for* $\mathbf{UB}(\mathbb{R})$*. In consequence, Theorem* 5.5 *of* [12] *is false in* $\mathbf{M}$*.*

Proposition 3.3 ($\mathbf{ZF} + \mathbf{CC}$). *If a boundedness* $\mathcal{B}$ *in* X *has a countable base, then every base for* $\mathcal{B}$ *contains a countable base for* $\mathcal{B}$*.*

Proof. Let $\mathcal{A}$ be a base for $\mathcal{B}$. Consider an arbitrary countable base $\mathcal{C}$ for $\mathcal{B}$. Then $\mathcal{C} \neq \emptyset$. For $C \in \mathcal{C}$, let $\mathcal{A}(C) = \{A \in \mathcal{A} : C \subseteq A\}$. Since $\mathcal{A}$ is a base for $\mathcal{B}$, we have $\mathcal{A}(C) \neq \emptyset$ whenever $C \in \mathcal{C}$. Using $\mathbf{CC}$, we deduce that there exists $x \in \prod_{C\in\mathcal{C}} \mathcal{A}(C)$. Then $\mathcal{A}_0 = \{x(C) : C \in \mathcal{C}\} \subseteq \mathcal{A}$ and $\mathcal{A}_0$ is a countable base for $\mathcal{B}$. $\square$

We can get the following correct modification in $\mathbf{ZF}$ of Theorem 5.5 of [12]:

Proposition 3.4. *Let* $\mathcal{C}$ *be a countable base for a boundedness* $\mathcal{B}$ *in* X *such that* $\mathcal{B}$ *does not have a maximal set with respect to inclusion. Then there exists a strictly increasing sequence* (A_n) *of members of* $\mathcal{C}$ *such that the collection* $\{A_n : n \in \omega\}$ *is a base for* $\mathcal{B}$*.*

Proof. It follows from the countability of $\mathcal{C}$ that we can write $\mathcal{C} = \{C_n : n \in \omega\}$. Let $A_0 = C_0$. Since $\mathcal{B}$ does not contain maximal bounded sets, there exists $B \in \mathcal{B}$ such that B is not a subset of $A_0 \cup C_1$ and there exists $C \in \mathcal{C}$ such that $A_0 \cup B \cup C_1 \subseteq C$. This proves that there exists $C \in \mathcal{C}$ such that $A_0 \cup C_1 \neq C$ and $A_0 \cup C_1 \subseteq C$. Let $n_1 = \min\{n \in \omega : A_0 \cup C_1 \subset C_n\}$ and $A_1 = C_{n_1}$. Of course, we use the symbol $\subset$ for strict inclusion. Suppose that, for $m \in \omega \setminus \{0\}$, we have already defined the set $A_m \in \mathcal{C}$. In much the same way as above, we take $n_{m+1} = \min\{n \in \omega : A_m \cup C_{m+1} \subset C_n\}$ and $A_{m+1} = C_{n_{m+1}}$. The sequence (A_n) has the required properties. $\square$

Although Theorem 5.5 of [12] is unprovable in $\mathbf{ZF}$, the following proposition about bornological biuniverses clearly shows that Theorem 5.6 of [12] holds true in $\mathbf{ZF}$; however, in the proof of Theorem 5.6 in [12], an illegal in $\mathbf{ZF}$ countable choice was involved. Therefore, we offer its more careful proof in $\mathbf{ZF}$.

Proposition 3.5. *Let us suppose that (X, τ_1, τ_2) is a bitopological space, while $\mathcal{B}$ is a second-countable (τ_1, τ_2)-proper boundedness in X such that $\mathcal{B}$ does not have maximal sets with respect to inclusion. Then there exists a strictly increasing sequence (A_n) of τ_1-open sets such that $\mathcal{A} = \{A_n : n \in \omega\}$ is a base for $\mathcal{B}$ such that $\mathrm{cl}_{\tau_2} A_n \subset A_{n+1}$ for each $n \in \omega$.*

Proof. Take, by Proposition 3.4, a strictly increasing countable base $\mathcal{C} = \{C_n : n \in \omega\}$ for $\mathcal{B}$. Let $A_0 = \mathrm{int}_{\tau_1} C_0$. Suppose that, for $m \in \omega$, we have already defined a τ_1-open set $A_m \in \mathcal{B}$. We use similar arguments to the ones given in the proof to Proposition 3.4 with the exception that, since $\mathcal{B}$ is (τ_1, τ_2)-proper, we may define $n_{m+1} = \min\{n \in \omega : \mathrm{cl}_{\tau_2}(A_m \cup C_{m+1}) \subset \mathrm{int}_{\tau_1} C_n\}$ and $A_{m+1} = \mathrm{int}_{\tau_1} C_{n_{m+1}}$. □

4. Quasi-metrization theorems for bornological biuniverses

If τ is a topology on X and if $A \subseteq X$, we denote $\tau \mid_A = \{A \cap V : V \in \tau\}$. For the real line $\mathbb{R}$, the topology $u = \{\emptyset, \mathbb{R}\} \cup \{(-\infty; a) : a \in \mathbb{R}\}$ is called *the upper topology* on $\mathbb{R}$, while the topology $l = \{\emptyset, \mathbb{R}\} \cup \{(a; +\infty) : a \in \mathbb{R}\}$ is called *the lower topology* on $\mathbb{R}$ (cf. [6], [19]). If $A \subseteq \mathbb{R}$, then we use (A, u, l) as an abbreviation of $(A, u \mid_A, l \mid_A)$ where $u = u \mid_A$ and $l = l \mid_A$.

Definition 4.1. Suppose that (X, τ_1^X, τ_2^X) and (Y, τ_1^Y, τ_2^Y) are bitopological spaces. A mapping $f : X \to Y$ is called *bicontinuous with respect to* $(\tau_1^X, \tau_2^X, \tau_1^Y, \tau_2^Y)$ (in abbreviation: bicontinuous) if

$$\{f^{-1}(V) : V \in \tau_i^Y\} \subseteq \tau_i^X$$

for each $i \in \{1, 2\}$.

A crucial role in the study of bornologies is played by a concept of a characteristic function of a bornology which is also called a forcing function (cf. [3] and [12]). We need to extend this concept to bornological biuniverses.

Definition 4.2. Let (X, τ_1, τ_2) be a bitopological space. Then a (τ_1, τ_2)-*characteristic function* for a bornology $\mathcal{B}$ in X, is a bicontinuous function $f : (X, \tau_1, \tau_2) \to ([0; +\infty), u, l)$ such that

$$\mathcal{B} = \{A \subseteq X : \sup\{f(x) : x \in A\} < +\infty\}.$$

Fact 4.3 (cf. 4.1 of [15]). *Let d be a quasi-metric on X and let $x_0 \in X$. Define $f(x) = d(x_0, x)$ for $x \in X$. Then the function $f : (X, \tau(d), \tau(d^{-1})) \to ([0; +\infty), u, l)$ is bicontinuous.*

Definition 4.4. We say that a (quasi)-metric d *induces a bornological biuniverse* $((X, \tau_1, \tau_2), \mathcal{B})$ if $\tau_1 = \tau(d), \tau_2 = \tau(d^{-1})$ and $\mathcal{B} = \mathcal{B}_d(X)$.

Proposition 4.5. *Suppose that a bornological biuniverse $((X, \tau_1, \tau_2), \mathcal{B})$ is (quasi)-metrizable. Then there exists a (τ_1, τ_2)-characteristic function for the bornology $\mathcal{B}$.*

Proof. Let us consider an arbitrary point $x_0 \in X$ and any (quasi)-metric d such that d induces the bornological biuniverse $((X, \tau_1, \tau_2), \mathcal{B})$. Then, by Fact 4.3, a (τ_1, τ_2)-characteristic function for $\mathcal{B}$ is the function $f : X \to \mathbb{R}$ where $f(x) = d(x_0, x)$ for $x \in X$. $\square$

Proposition 4.6. *Suppose that a bornological biuniverse $((X, \tau_1, \tau_2), \mathcal{B})$ is such that $\mathcal{B}$ has a (τ_1, τ_2)-characteristic function. Then $\mathcal{B}$ is both second-countable and (τ_1, τ_2)-proper.*

Proof. Let f be any (τ_1, τ_2)-characteristic function for $\mathcal{B}$. For $n \in \omega$, let $A_n = f^{-1}((-\infty, n])$. Then the collection $\{A_n : n \in \omega\}$ is a countable base for $\mathcal{B}$ such that $\mathrm{cl}_{\tau_2} A_n \subseteq \mathrm{int}_{\tau_1} A_{n+1}$. $\square$

Theorem 4.7. *Let us suppose that (X, τ_1, τ_2) is a (quasi)-metrizable bitopological space and that $\mathcal{B}$ is a bornology in X. Then the following conditions are all equivalent:*

(*i*) *the bornological biuniverse $((X, \tau_1, \tau_2), \mathcal{B})$ is (quasi)-metrizable;*
(*ii*) *there exists a (τ_1, τ_2)-characteristic function for $\mathcal{B}$;*
(*iii*) *the bornology $\mathcal{B}$ is (τ_1, τ_2)-proper and it has a countable base.*

Proof. Let us consider any quasi-metric σ on X such that $\tau_1 = \tau(\sigma)$ and $\tau_2 = \tau(\sigma^{-1})$. Put $d(x, y) = \min\{\sigma(x, y), 1\}$ for $x, y \in X$. It is easy to observe that if $X \in \mathcal{B}$, then all conditions (*i*)–(*iii*) are fulfilled. Assume that $X \notin \mathcal{B}$. It follows from Proposition 4.5 that (*i*) implies (*ii*). Assume (*ii*) and suppose that f is a (τ_1, τ_2)-characteristic function for $\mathcal{B}$. For $x, y \in X$, let $\rho(x, y) = d(x, y) + \max\{f(y) - f(x), 0\}$. Then the quasi-metric ρ induces the bornological biuniverse $((X, \tau_1, \tau_2), \mathcal{B})$. In the case when σ is a metric, we can put $\rho(x, y) = d(x, y) + \mid f(y) - f(x) \mid$ to obtain a metric that induces $((X, \tau_1, \tau_2), \mathcal{B})$. Hence (*ii*) implies (*i*).

Now, assume that (*iii*) holds. Since $X \notin \mathcal{B}$, it follows from Proposition 3.5 that there exists a base $\{A_n : n \in \omega\}$ for $\mathcal{B}$ such that $\mathrm{cl}_{\tau_2} A_n$ is a proper subset of $\mathrm{int}_{\tau_1} A_{n+1}$ for each $n \in \omega$. We may assume that $A_0 = \emptyset$. For $n \in \omega \setminus \{0\}$ and $x \in X$, let $f_n(x) = d(\mathrm{cl}_{\tau_2} A_n, x)$ and $g_n(x) = d(x, X \setminus \mathrm{int}_{\tau_1} A_{n+1})$. Then $f_n : (X, \tau_1, \tau_2) \to ([0; +\infty), u, l)$ and $g_n : (X, \tau_1, \tau_2) \to ([0; +\infty), l, u)$ are bicontinuous. For each $x \in X$, we have $f_n(x) + g_n(x) \neq 0$, so we can put $h_n(x) = \frac{f_n(x)}{f_n(x)+g_n(x)}$. Moreover, we define $h_0(x) = 1$ for each $x \in X$. It is easy to check that the function $h_n : (X, \tau_1, \tau_2) \to ([0; 1], u, l)$ is bicontinuous for each $n \in \omega$ (cf. the proof to Corollary 2.2.16 in [19]). Let $\psi(x) = h_n(x) + n$ when $x \in \mathrm{int}_{\tau_1} A_{n+1} \setminus \mathrm{int}_{\tau_1} A_n$. We are going to prove that the function ψ is bicontinuous with respect to (τ_1, τ_2, u, l).

Let $x \in \mathrm{int}_{\tau_1} A_{n+1} \setminus \mathrm{int}_{\tau_1} A_n$ and $y \in \mathrm{int}_{\tau_1} A_{m+1} \setminus \mathrm{int}_{\tau_1} A_m$. Consider any real numbers r, s such that $r < \psi(x) < s$. We assume that $n \neq 0$. There exists $U_s \in \tau_1$ such that $x \in U_s \subseteq \mathrm{int}_{\tau_1} A_{n+1}$ and if $y \in U_s$, then $h_n(y) + n < s$. There exists $V_r \in \tau_2$ such that $x \in V_r \subseteq X \setminus \mathrm{cl}_{\tau_2} A_{n-1}$ and if $y \in V_r$, then

$h_n(y) + n > r$. Of course, if $m = n$, then $\psi(y) < s$ when $y \in U_s$, while $\psi(y) > r$ when $y \in V_r$. Let us assume that $m \neq n$. Suppose that $y \in U_s$. Then $m < n$, so $\psi(y) \leq 1 + m \leq n \leq \psi(x) < s$.

Suppose that $y \in V_r$. If $m > n$, we have $\psi(y) \geq m \geq 1 + n \geq \psi(x) > r$. Let $m < n$. Since $y \notin \mathrm{int}_{\tau_1} A_{n-1}$, we have $m + 1 \geq n$. As $m + 1 \leq n$, we have $m + 1 = n$. If $x \notin \mathrm{cl}_{\tau_2} A_n$ we could take $V_r^* = V_r \cap (X \setminus \mathrm{cl}_{\tau_2} A_n) \in \tau_2$ and observe that if $y \in V_r^*$, then $m \geq n$ and $\psi(y) > r$. Let us consider the case when $m < n$ and $x \in \mathrm{cl}_{\tau_2} A_n$. Then $\psi(x) = n$. We take a positive real number ϵ such that $n - \epsilon > r$. Since $h_{n-1}(x) = 1$, there exists $W_\epsilon \in \tau_2$ such that $x \in W_\epsilon$ and $h_{n-1}(t) > 1 - \epsilon$ for each $t \in W_\epsilon$. If $y \in W_\epsilon \cap V_r$ and $m + 1 = n$, then $\psi(y) = h_{n-1}(y) + n - 1 > 1 - \epsilon + n - 1 > r$. The case when $n = 0$ is also obvious now. This completes the proof that ψ is bicontinuous with respect to (τ_1, τ_2, u, l). It is easy to check that $\mathcal{B} = \{A \subseteq X : \sup \psi(A) < +\infty\}$, so ψ is a (τ_1, τ_2)-characteristic function for $\mathcal{B}$. Hence *(ii)* follows from *(iii)*. To complete the proof, it suffices to apply Proposition 4.6. □

Corollary 4.8. *Theorem* 1.10 *is true.*

Corollary 4.9. *The assumption of* **ZFC** *can be weakened to* **ZF** *in Theorem* 1.9*.*

Corollary 4.10. *Let us suppose that* (X, τ) *is a topological space and* $\mathcal{B}$ *is a bornology in* X*. Then the bornological biuniverse* $((X, \tau, \tau), \mathcal{B})$ *is quasi-metrizable if and only if the bornological universe* $((X, \tau), \mathcal{B})$ *is metrizable.*

Proof. It suffices to prove that if there exists a quasi-metric which induces $((X, \tau, \tau), \mathcal{B})$, then $((X, \tau), \mathcal{B})$ is metrizable. Let d be a quasi-metric in X such that $\tau = \tau(d) = \tau(d^{-1})$ and $\mathcal{B} = \mathcal{B}_d(X)$. Define $\rho = \max\{d, d^{-1}\}$. Then ρ is a metric in X such that $\tau(\rho) = \tau$. Moreover, by Theorem 4.7, the bornology $\mathcal{B}$ is second-countable and τ-proper; hence, the bornological universe $((X, \tau), \mathcal{B})$ is metrizable by Theorem 4.7. □

Example 4.11. Let d be the quasi-metric from Example 1.6. Then $\tau(d) = \tau(d^{-1}) = \mathcal{P}(\omega)$. Moreover, $\mathcal{B}_d(\omega) = \mathcal{P}(\omega)$ and $\mathcal{B}_{d^{-1}}(\omega) = \mathbf{FB}(\omega)$. The metric $\rho = \max\{d, d^{-1}\}$ does not induce $((\omega, \mathcal{P}(\omega), \mathcal{P}(\omega)), \mathcal{B}_d(\omega))$; however, ρ induces $((\omega, \mathcal{P}(\omega), \mathcal{P}(\omega)), \mathbf{FB}(\omega))$.

Example 4.12. Let $\tau_{S,r}$ be the right half-open interval topology in $\mathbb{R}$ and let $\tau_{S,l}$ be the left half-open interval topology in $\mathbb{R}$. Then $(\mathbb{R}, \tau_{S,r})$ is the Sorgenfrey line.

(i) The bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathbf{UB}(\mathbb{R}))$ is not metrizable but it is quasi-metrizable by the following quasi-metric ρ_S:

$$\rho_S(x, y) = \begin{cases} y - x, & x \leq y \\ 1, & x > y. \end{cases}$$

Let us notice that $\mathcal{B}_{\rho_S^{-1}}(\mathbb{R}) = \mathbf{LB}(\mathbb{R})$ and the quasi-metric ρ_S does not induce the bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathbf{LB}(\mathbb{R}))$. However, the bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathbf{LB}(\mathbb{R}))$ is induced by the quasi-metric ρ_L defined as follows:

$$\rho_L(x,y) = \begin{cases} \min\{y-x, 1\}, & x \leq y \\ 1 + x - y, & x > y. \end{cases}$$

(ii) The non-metrizable bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathcal{P}(\mathbb{R}))$ is quasi-metrizable by the quasi-metric $\rho_{S,1}$ defined as follows:

$$\rho_{S,1}(x,y) = \begin{cases} \min\{1, y-x\}, & x \leq y \\ 1, & x > y. \end{cases}$$

Example 4.13. We consider the following *hedgehog-like scheme*. Let (X, d) be a quasi-metric space such that X has at least two distinct points. Let S be a non-empty set. We fix $x_0 \in X$ and put $Y_s = (X \setminus \{x_0\}) \times \{s\}$ for $s \in S$. Let us fix $p \notin \bigcup_{s \in S} Y_s$ and put $Y = \{p\} \cup \bigcup_{s \in S} Y_s$. Let $x, y \in X \setminus \{x_0\}$ and $s, s' \in S$. We define $\rho(p,p) = 0, \rho((x,s),p) = d(x, x_0), \rho(p,(x,s)) = d(x_0, x)$ and $\rho((x,s),(y,s)) = d(x,y)$. If $s \neq s'$, we put $\rho((x,s),(y,s')) = d(x,x_0) + d(x_0,y)$. Let us consider the collection $\mathcal{B}$ of all sets $A \subseteq Y$ such that there are finite $S(A) \subseteq S$ such that $A \subseteq \{p\} \cup \bigcup_{s \in S(A)} Y_s$. Then $\mathcal{B}$ is a bornology in Y. If S is countable, then $\mathcal{B}$ is second-countable. If S is infinite and, simultaneously, x_0 is an accumulation point of $(X, \tau(d))$, then the bornology $\mathcal{B}$ is not $(\tau(\rho), \tau(\rho^{-1}))$-proper. Let us denote the bornological biuniverse $((Y, \tau(\rho), \tau(\rho^{-1})), \mathcal{B})$ by $J(X, d, x_0, S)$ and let $Y(X, d, x_0, S) = (Y, \tau(\rho))$. We can apply $J(X, d, x_0, S)$ as follows.

(i) If $X = [0;1]$ and $d(x,y) =\mid x - y \mid$ for $x, y \in X$, then the bornological universe $J(X, d, 0, \omega)$ is not quasi-metrizable although its bornology is second-countable. In this case, $Y(X, d, x_0, \omega)$ is the hedgehog space of spininess ω (cf. 4.1.5 of [5]), so we can call $((Y, \tau(\rho)), \mathcal{B})$ *the bornological hedgehog space of spininess* ω.

(ii) If ρ_S is the quasi-metric defined in Example 4.12 (i), then the bornological biuniverse $J(\mathbb{R}, \rho_S, 0, \omega)$ is not quasi-metrizable but its bornology has a countable base.

(iii) Let C be the unit circle in $\mathbb{R}^2$. We fix $x_0 \in C$ and we consider the Euclidean metric d_e in C. The bornological biuniverse $J(C, d_e, x_0, \omega)$ is not quasi-metrizable although its bornology has a countable base. We can call $J(C, d_e, x_0, \omega)$ *the bornological metric wedge sum of* ω *circles*. In this case, the topological space $Y(C, d_e, x_0, \omega)$ is not compact.

(iv) It is worthwhile to compare $J(C, d_e, x_0, \omega)$ with *the bornological Hawaiian earring* $(H, \mathcal{B}_H)$ where $H = \bigcup_{n \in \omega \setminus \{0\}} H_n$ is considered with its natural topology inherited from $\mathbb{R}^2$ and, for each $n \in \omega \setminus \{0\}$, the set H_n is the circle with centre $(\frac{1}{n}, 0)$ and radius $\frac{1}{n}$, while $\mathcal{B}_H$ is the collection of all sets $A \subseteq H$ such that there exist sets $n(A) \in \omega$ such that $A \subseteq \bigcup_{n \in n(A) \setminus \{0\}} H_n$.

Then H is compact and the bornology $\mathcal{B}_H$ has a countable base. Since there does not exist $A \in \mathcal{B}_H$ such that $(0,0) \in \mathrm{int}_{d_e} A$, it follows from Theorem 4.7 that the bornological universe $(H, \mathcal{B}_H)$ is not quasi-metrizable.

In view of the examples above, when d is a quasi-metric in X and $\mathcal{B}$ is a bornology in X but d does not induce the bornological biuniverse $(X, \mathcal{B}) = ((X, \tau(d), \tau(d^{-1})), \mathcal{B})$, it might be interesting to find, in terms of d, necessary and sufficient conditions for $(X, \mathcal{B})$ to be quasi-metrizable. To do this, we need the following concept:

Definition 4.14. Let d be a quasi-pseudometric in a set X and let $\delta \in (0; +\infty)$. For a set $A \subseteq X$, *the δ-neighbourhood of A with respect to d* is the set $[A]_d^\delta = \bigcup_{a \in A} B_d(a, \delta)$.

Let us notice that if $\emptyset \neq A \subseteq X$, then $[A]_d^\delta = \{x \in X : d(A, x) < \delta\}$.

Theorem 4.15. *For every bornological biuniverse $((X, \tau_1, \tau_2), \mathcal{B})$, the following conditions are equivalent:*

(*i*) $((X, \tau_1, \tau_2), \mathcal{B})$ *is (quasi)-metrizable;*

(*ii*) *there exists a (quasi)-metric d in X such that $\tau_1 = \tau(d), \tau_2 = \tau(d^{-1})$ and $\mathcal{B}$ has a base $\{B_n : n \in \omega\}$ with the following property:*

$$\forall_{n \in \omega} \exists_{\delta \in (0; +\infty)} [B_n]_d^\delta \subseteq B_{n+1}.$$

Proof. Let $((X, \tau_1, \tau_2), \mathcal{B})$ be a bornological biuniverse. Suppose that (i) holds and that d is a (quasi)-metric in X such that d induces $((X, \tau_1, \tau_2), \mathcal{B})$. We consider an arbitrary $x_0 \in X$ and, for $n \in \omega$, we define $B_n = B_d(x_0, n+1)$. Since $[B_n]_d^{\frac{1}{2}} \subseteq B_{n+1}$, we infer that (ii) follows from (i).

Assume that (ii) is satisfied. Let $C \subseteq X$ and $D \subseteq X$ be such that, for some $\delta \in (0; +\infty)$, the inclusion $[C]_d^\delta \subseteq D$ holds. Let $x \in \mathrm{cl}_{\tau_2} C$. There exists $y \in C \cap B_{d^{-1}}(x, \delta)$. Then $d(y, x) < \delta$, so $x \in [C]_d^\delta$. Therefore $\mathrm{cl}_{\tau_2} C \subseteq [C]_d^\delta$. Of course, since $[C]_d^\delta \subseteq D$, we have $[C]_d^\delta \subseteq \mathrm{int}_{\tau_1} D$. In consequence, $\mathrm{cl}_{\tau_2} C \subseteq \mathrm{int}_{\tau_1} D$. Now, we deduce from Theorem 4.7 that (ii) implies (i). $\square$

Corollary 4.16. *For every bornological universe $((X, \tau), \mathcal{B})$, the following conditions are equivalent:*

(*i*) *the universe $((X, \tau), \mathcal{B})$ is (quasi)-metrizable;*

(*ii*) *there exists a (quasi)-metric d in X such that $\tau = \tau(d)$ and, simultaneously, $\mathcal{B}$ has a base $\{B_n : n \in \omega\}$ with the following property:*

$$\forall_{n \in \omega} \exists_{\delta \in (0; +\infty)} [B_n]_d^\delta \subseteq B_{n+1}.$$

The following example shows that the sets B_n can be d-unbounded in Theorem 4.15 and Corollary 4.16.

Example 4.17. For the bornological biuniverse $((\mathbb{R}, \tau_{S,r}, \tau_{S,l}), \mathbf{LB}(\mathbb{R}))$ and for the quasi-metric ρ_S from Example 4.12, the sets $B_n = [-n; +\infty)$ with $n \in \omega$ satisfy condition (ii) of Theorem 4.15 if we put $d = \rho_S$ and $\delta = 1$. However, the sets B_n are all ρ_S-unbounded.

5. The kernel of a boundedness

If X is a topological space and $\mathcal{B}$ is a boundedness in X, a notion of a kernel of the universe $(X, \mathcal{B})$ was introduced in Definition 6.3 in [12]. We adapt this notion to our needs.

Definition 5.1. Let τ be a topology in a set X. If $\mathcal{B}$ is a boundedness in X, then the *τ-kernel* of $\mathcal{B}$ is the set

$$\Lambda_\tau(\mathcal{B}) = \bigcup \{\mathrm{int}_\tau A : A \in \mathcal{B}\}.$$

Definition 5.2. Let (X, τ_1, τ_2) be a bitopological space. Suppose that $\mathcal{B}$ is a boundedness in X and put $\Lambda = \Lambda_{\tau_1}(\mathcal{B})$. Let $\mathcal{B}_\Lambda = \{A \cap \Lambda : A \in \mathcal{B}\}$. Then the ordered pair $((\Lambda, \tau_1 |_\Lambda, \tau_2 |_\Lambda), \mathcal{B}_\Lambda)$ will be called *the bornological biuniverse induced by $\mathcal{B}$*.

Fact 5.3. *Suppose that (X, τ_1, τ_2) is a bitopological space. If $\mathcal{B}$ is a (τ_1, τ_2)-proper boundedness in X and if $\Lambda = \Lambda_{\tau_1}(\mathcal{B})$, then the bornology $\mathcal{B}_\Lambda$ in Λ is $(\tau_1 |_\Lambda, \tau_2 |_\Lambda)$-proper.*

For a topological space $X = (X, \tau)$ and a boundedness $\mathcal{B}$ in X, when $\Lambda = \Lambda_\tau(\mathcal{B})$, Theorem 13.5 of [12] concerns the problem of the metrizability of the bornological universe $(\Lambda, \mathcal{B}_\Lambda)$ under the assumption that Λ is a separable metrizable subspace of X. However, the proof to Theorem 13.5 in [12] is not in **ZF**. We give a generalization to bornological universes of Theorem 13.5 of [12] and show its proof in **ZF**. We also show that the assumption of separability is needless in Theorem 13.5 of [12].

Theorem 5.4. *Assume that (X, τ_1, τ_2) is a bitopological space and that $\mathcal{B}$ is a second-countable (τ_1, τ_2)-proper boundedness in X. Let Λ be the τ_1-kernel of $\mathcal{B}$ and suppose that the bitopological space $(\Lambda, \tau_1 |_\Lambda, \tau_2 |_\Lambda)$ is quasi-metrizable. Then there exists a quasi-metric ρ on Λ such that the following conditions are satisfied:*

- (i) *$\tau_1 |_\Lambda = \tau(\rho)$ and $\tau_2 |_\Lambda = \tau(\rho^{-1})$;*
- (ii) *$\mathcal{B}$ is the collection of all ρ-bounded subsets of Λ;*
- (iii) *for each pair of points $x_0 \in \Lambda$, $x_* \in X \setminus \Lambda$ and for each positive real number b, there exists $G \in \tau_2$ such that $x_* \in G$ and $\rho(x_0, x) > b$ whenever $x \in G \cap \Lambda$.*

Proof. Since $\mathcal{B}$ is (τ_1, τ_2)-proper, we have $\mathcal{B} = \mathcal{B}_\Lambda$. In view of Fact 5.3, $\mathcal{B}$ is $(\tau_1 |_\Lambda, \tau_2 |_\Lambda)$-proper. In the light of Theorem 4.7, there exists in **ZF** a quasi-metric ρ in Λ such that both conditions (i) and (ii) are satisfied. Let $x_0 \in \Lambda$

and $x_* \in X \setminus \Lambda$. Consider an arbitrary positive real number b. Put $B = \{x \in \Lambda : \rho(x_0, x) \leq b\}$. Of course, $B \in \mathcal{B}$. Since $\mathcal{B}$ is (τ_1, τ_2)-proper, there exists $U \in \tau_1 \cap \mathcal{B}$ such that $B \subseteq U$. Using the assumption that $\mathcal{B}$ is (τ_1, τ_2)-proper once again, we deduce that $\mathrm{cl}_{\tau_2} U \subseteq \Lambda$. Let $G = X \setminus \mathrm{cl}_{\tau_2} U$. Then $G \in \tau_2$, $G \cap \Lambda \subseteq \Lambda \setminus B$ and $x_* \in G$. It is evident that if $x \in G \cap \Lambda$, then $\rho(x_0, x) > b$. $\square$

Now, we can immediately deduce in **ZF** the following improvement of Theorem 13.5 of [12]:

Corollary 5.5. *If $\mathcal{B}$ is a second-countable proper boundedness in a topological space X such that the set $\Lambda = \bigcup \mathcal{B}$ is a metrizable subspace of X, then there exists a metric ρ on Λ such that the following conditions are satisfied:*

(*i*) *the topology of Λ as a subspace of X is induced by ρ;*

(*ii*) $\mathcal{B} = \{A \subseteq \Lambda : \mathrm{diam}_\rho(A) < +\infty\}$*;*

(*iii*) *for each pair of points $x_0 \in \Lambda$, $x_* \in X \setminus \Lambda$ and for each positive real number b, there exists an open set G in X such that $x_* \in G$ and $\rho(x_0, x) > b$ whenever $x \in G \cap \Lambda$.*

6. Uniformly quasi-metrizable bornologies

This section has been inspired by the necessary and sufficient conditions for a bornology to be uniformly metrizable given in [8]. We adapt the conditions of Theorem 2.4 of [8] to bornologies in quasi-metric spaces.

For $x, y \in \mathbb{R}$, let us put

$$\rho_u(x, y) = \max\{y - x, 0\}, \qquad \rho_l(x, y) = \max\{x - y, 0\}.$$

Then ρ_u, ρ_l are quasi-pseudometrics in $\mathbb{R}$ such that $\rho_u^{-1} = \rho_l$; moreover, $\tau(\rho_u)$ is the upper topology u in $\mathbb{R}$, while $\tau(\rho_l)$ is the lower topology l in $\mathbb{R}$.

Definition 6.1. Let d_X, d_Y be quasi-pseudometrics in sets X and Y, respectively. We say that a mapping $f : X \to Y$ is (d_X, d_Y)-*uniformly continuous* if the following condition is satisfied:

$$\forall_{\epsilon \in (0;+\infty)} \exists_{\delta \in (0;+\infty)} \forall_{x_1, x_2 \in X} [d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon].$$

Definition 6.2. Quasi-pseudometrics d_0, d_1 in a set X are called *uniformly equivalent* if the following condition holds:

$$\forall_{\epsilon \in (0;+\infty)} \exists_{\delta_0, \delta_1 \in (0;+\infty)} \forall_{x, y \in X} \forall_{i \in \{0,1\}} [d_i(x, y) < \delta_i \Rightarrow d_{1-i}(x, y) < \epsilon].$$

Definition 6.3. Suppose that (X, d) is a quasi-metric space and that $\mathcal{B}$ is a bornology in X. We say that $\mathcal{B}$ is *uniformly quasi-metrizable with respect to d* if there exists a quasi-metric ρ in X such that d and ρ are uniformly equivalent, while $\mathcal{B}$ is the collection of all ρ-bounded sets.

Definition 6.4. We say that quasi-metrics d, ρ in X are *uniformly locally identical* if they are uniformly equivalent and there exists $\delta \in (0; +\infty)$ such that, for all $x, y \in X$, we have $\rho(x, y) = d(x, y)$ whenever $d(x, y) < \delta$ (cf. [22] and Remark 2.5 of [8]).

Theorem 6.5. *Suppose that (X, d) is a quasi-metric space and that $\mathcal{B}$ is a bornology in X. Then the following conditions are all equivalent:*

(*i*) *$\mathcal{B}$ is uniformly quasi-metrizable with respect to d;*
(*ii*) *$\mathcal{B}$ has a base $\{B_n : n \in \omega\}$ such that, for some $\delta \in (0; +\infty)$ and for each $n \in \omega$, the inclusion $[B_n]_d^\delta \subseteq B_{n+1}$ holds;*
(*iii*) *there exists a quasi-metric ρ in X such that d, ρ are uniformly locally identical and $\mathcal{B}$ is the collection of all ρ-bounded sets;*
(*iv*) *there exists a (d, ρ_u)-uniformly continuous $(\tau(d), \tau(d^{-1}))$-characteristic function for $\mathcal{B}$.*

Proof. Assume (*i*) and suppose that ρ is a uniformly equivalent with d quasi-metric in X such that $\mathcal{B}$ is the collection of all ρ-bounded sets. Let $x_0 \in X$ and, for $n \in \omega$, let $B_n = B_\rho(x_0, n+1)$. We choose $\delta \in (0; +\infty)$ such that $\rho(x, y) < \frac{1}{2}$ whenever $d(x, y) < \delta$. Then $[B_n]_d^\delta \subseteq [B_n]_\rho^{\frac{1}{2}} \subseteq B_{n+1}$ for each $n \in \omega$, so (*ii*) follows from (*i*).

Now, let us suppose that (*ii*) holds. We may assume that $\delta \in (0; +\infty)$ and that $\{B_n : n \in \omega\}$ is a base for $\mathcal{B}$ such that $B_0 = \emptyset, B_1 \neq \emptyset$ and $[B_n]_d^\delta \subseteq B_{n+1}$ for each $n \in \omega$. We shall mimic the proof to Proposition 2.2 in [8] and change parts of it to show that (*iii*) follows from (*ii*). We define $\phi_0(x) = 1$ for each $x \in X$. If $n \in \omega \setminus \{0\}$, we define $\phi_n(x) = \min\{1, \frac{1}{\delta} d(B_n, x)\}$ for each $x \in X$. It is easy to check that the function ϕ_n is (d, ρ_u)-uniformly continuous; moreover, $\phi_n(B_n) \subseteq \{0\}$ and $\phi_n(X \setminus B_{n+1}) \subseteq \{1\}$. Let us consider the function $\chi : X \to [0; +\infty)$ defined by

$$\chi(x) = n - 2 + \phi_{n-1}(x)$$

for each $x \in B_n \setminus B_{n-1}$ and for each $n \in \omega \setminus \{0\}$. To prove that χ is (d, ρ_u)-uniformly continuous, let us consider an arbitrary pair x, y of points of X such that $d(x, y) < \delta$. Let $n \in \omega$ be the unique natural number such that $x \in B_n \setminus B_{n-1}$. If $z \in X \setminus B_{n+1}$, then $d(x, z) \geq \delta$. This implies that $y \in B_{n+1}$. Let $m \in \omega \setminus \{0\}$ be the unique natural number such that $y \in B_m \setminus B_{m-1}$. Then $m \leq n+1$. We have $\chi(y) - \chi(x) = m - n + \phi_{m-1}(y) - \phi_{n-1}(x)$. If $m = n+1$, then $\chi(y) - \chi(x) = 1 + \phi_n(y) - \phi_{n-1}(x) = \phi_n(y) - \phi_n(x) + \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{2}{\delta} d(x, y)$. If $m = n$, then $\chi(y) - \chi(x) = \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{1}{\delta} d(x, y)$. Suppose that $m < n$. Then $m - n + 1 \leq 0, x \notin B_m, y \in B_{n-1}$ and $\chi(y) - \chi(x) = m - n + 1 + \phi_{m-1}(y) - \phi_{m-1}(x) + \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{2}{\delta} d(x, y)$. In consequence, χ is (d, ρ_u)-uniformly continuous. Therefore, $\chi : (X, \tau(d), \tau(d^{-1})) \to (\mathbb{R}, u, l)$ is bicontinuous. Since, for $A \subseteq X$, we have that $A \in \mathcal{B}$ if and only if χ is bounded on A, the function χ is a $(\tau(d), \tau(d^{-1}))$-characteristic function for $\mathcal{B}$. In much the same way as in Remark 2.5 of [8], we can define $\rho(x, y) = \max\{\min\{d(x, y), 1\}, \frac{\delta}{2} \max\{\chi(y) -$

$\chi(x), 0\}\}$ to get a quasi-metric ρ uniformly locally identical with d such that $\mathcal{B}$ is the collection of all ρ-bounded sets. Thus (ii) implies (iii).

Let us assume that (iii) holds. We take a quasi-metric ρ in X such that d and ρ are uniformly locally identical and $\mathcal{B} = \mathcal{B}_\rho(X)$. We fix $x_0 \in X$ and define $f(x) = \rho(x_0, x)$ for $x \in X$ to get a $(\tau(\rho), \tau(\rho^{-1}))$-characteristic function f for $\mathcal{B}$ such that f is (d, ρ_u)-uniformly continuous. Hence (iii) implies (iv).

Finally, we suppose that (iv) holds and we consider an arbitrary function g such that g is a $(\tau(d), \tau(d^{-1}))$-characteristic function for $\mathcal{B}$ and g is (d, ρ_u)-uniformly continuous. For $x, y \in X$, we can define $d_g(x, y) = \min\{d(x, y), 1\} + \max\{g(y) - g(x), 0\}$ to see that (iv) implies (i). $\square$

Corollary 6.6. *Theorem* 2.4 *and Remark* 2.5 *of* [8] *hold true in* **ZF**.

The following example shows that, for a quasi-metric d in X and a bornology $\mathcal{B}$ in X, it may happen that the bornological universe $((X, \tau(d)), \mathcal{B})$ is quasi-metrizable or even metrizable, while $\mathcal{B}$ is not uniformly quasi-metrizable with respect to d.

Example 6.7. Let $\mathcal{B} = \mathbf{UB}(\mathbb{R}) \cap \mathbf{LB}(\mathbb{R})$. We put $d(x, y) =\mid 2^x - 2^y \mid$ for all $x, y \in \mathbb{R}$. We observe that, for a fixed $\delta \in (0; +\infty)$, there exists $n(\delta) \in \omega$ such that if $C_m = [-m; m]$ for $m \in \omega$ with $m > n(\delta)$, then $(-\infty; m) \subseteq [C_m]_d^\delta$. This, together with Theorem 6.5, implies that $\mathcal{B}$ is not uniformly quasi-metrizable with respect to d. Of course, if $\rho(x, y) =\mid x - y \mid$ for all $x, y \in \mathbb{R}$, then $\tau(d) = \tau(\rho)$ and $\mathcal{B} = \mathcal{B}_\rho(\mathbb{R})$, so the bornological universe $((\mathbb{R}, \tau(d)), \mathcal{B})$ is metrizable.

Remark 6.8. One can easily reformulate Theorems 4.7 and 4.15 to get simultaneously necessary and sufficient conditions for a bornological biuniverse to be quasi-pseudometrizable. One can also use quasi-pseudometrics instead of quasi-metrics in Theorem 6.5 to obtain conditions equivalent with the uniform quasi-pseudometrizability of a bornology with respect to a given quasi-pseudometric.

7. Applications to independence from ZF

Mimicking [8], let us consider the following bornologies in a metric space (X, d): the bornology $\mathbf{FB}(X)$ of all finite subsets of X, the bornology $\mathbf{CB}_d(X)$ generated by the compact subsets of (X, d), the bornology $\mathbf{TB}_d(X)$ of all totally bounded subspaces of (X, d), as well as the bornology $\mathbf{BB}_d(X)$ of all Bourbaki-bounded sets. Several theorems about equivalents of the uniform metrizability of the bornologies $\mathbf{FB}(X)$, $\mathbf{CB}_d(X)$, $\mathbf{TB}_d(X)$ and $\mathbf{BB}_d(X)$ in **ZFC** were proved in [8]. We are going to show that some of the above-mentioned theorems of [8] are independent of **ZF**, while other theorems of [8] can be proved in **ZF**. Clearly, we have already shown in the previous section that both Proposition 2.2 and Theorem 2.4 of [8] hold true in **ZF**.

The following theorem will be helpful:

Theorem 7.1. *Equivalent are:*

(i) $\mathbf{CC}$(*fin*);

(ii) *for every discrete space X, the bornological universe $(X, \mathbf{FB}(X))$ is metrizable (in the sense of Hu) if and only if X is countable.*

Proof. Assume (i) and let X be any discrete space such that the bornological universe $(X, \mathbf{FB}(X))$ is metrizable. It follows from Theorem 4.7 that $\mathbf{FB}(X)$ has a countable base. If $\mathcal{A}$ is a countable base for $\mathbf{FB}(X)$, then $X = \bigcup \mathcal{A}$, so, by Proposition 3.5 of [9], X is countable if $\mathbf{CC}$(fin) holds. It is obvious that if X is a countable discrete space, then the bornological universe $(X, \mathbf{FB}(X))$ is metrizable in $\mathbf{ZF}$ by Theorem 4.7.

Now, assume that $\mathbf{CC}$(fin) fails. Then, in view of Proposition 3.5 of [9], there exists a sequence $(A_n)_{n\in\omega}$ of pairwise disjoint non-void finite sets such that the set $Z = \bigcup_{n\in\omega} A_n$ is uncountable. Let us equip Z with its discrete topology. Then the collection $\{\bigcup_{m\in n} A_m : n \in \omega\}$ is a countable base for $\mathbf{FB}(Z)$. Then, by Theorem 4.7, the bornological universe $(Z, \mathbf{FB}(Z))$ is metrizable. This contradicts (ii). □

For a set X and a cardinal number κ, let us use the notation $[X]^{\leq\kappa}$ for the collection of all subsets A of X such that A is of cardinality at most κ and the notation $[X]^{<\kappa}$ for the collection of all subsets of X that are of cardinality $< \kappa$. (cf. Definition I.13.19 of [17]). Then $[X]^{<\omega} = \mathbf{FB}(X)$, while $[X]^{\leq\omega}$ is the bornology of all at most countable subsets of X.

The proof to the following interesting theorem is somewhat more complicated than to Theorem 7.1.

Theorem 7.2. *Equivalent are:*

(i) *for every sequence $(X_n)_{n\in\omega}$ of non-void at most countable sets X_n, the product $\prod_{n\in\omega} X_n$ is non-void;*

(ii) *for every discrete space X, the bornological universe $(X, [X]^{\leq\omega})$ is metrizable if and only if X is countable.*

Proof. Of course, if X is a countable discrete space, the bornological universe $(X, [X]^{\leq\omega}) = (X, \mathcal{P}(X))$ is metrizable.

Assume (i). Let X be a discrete space such that the bornological universe $(X, [X]^{\leq\omega})$ is metrizable. Then, by Theorem 4.7, there exists a countable base $\mathcal{B} = \{X_n : n \in \omega\}$ for the bornology $[X]^{\leq\omega}$. Suppose that X is uncountable. We may assume that $X_n \subseteq X_{n+1}$ and that $X_n \neq X_{n+1}$ for each $n \in \omega$. By (i), there exists $x \in \prod_{n\in\omega}(X_{n+1} \setminus X_n)$. Then, for such an x, if $A = \{x(n) : n \in \omega\}$, then $A \in [X]^{\leq\omega}$, while there does not exist $n \in \omega$ such that $A \subseteq X_n$. This is impossible because $\mathcal{B}$ is a base for $[X]^{\leq\omega}$. Therefore, (i) implies (ii).

Now, let us suppose that (i) is false. Consider any sequence $(X_n)_{n\in\omega}$ of non-empty countable sets such that $\prod_{n\in\omega} X_n = \emptyset$. For each $n \in \omega$, the set $Y_n = \prod_{i\in n+1} X_i$ is countable and non-empty. In much the same way as in the proof to Theorem 2.12 of [9], we can show that there does not exist an infinite set $M \subseteq \omega$ such that $\prod_{n\in M} Y_n \neq \emptyset$. Let $Y = \bigcup_{n\in\omega} Y_n$ and let $f : \omega \to Y$ be an injection. Then the set $M_f = \{n \in \omega : f(\omega) \cap Y_n \neq \emptyset\}$ is finite. This proves that Y is uncountable and if $A_n = \bigcup_{m\in n+1} Y_m$ for $n \in \omega$, then the collection $\mathcal{A} = \{A_n : n \in \omega\}$ is a countable base for $[Y]^{\leq\omega}$. Let us equip Y with its discrete topology. Then, by Theorem 4.7, the bornological universe $(Y, [Y]^{\leq\omega})$ is metrizable and, in consequence, (ii) is false. Hence (ii) implies (i). $\square$

Remark 7.3. It is unknown to us whether there is a model for **ZF** in which **CUT** fails (cf. [9]), while condition (i) of Theorem 7.2 is satisfied.

Remark 7.4. It is evident that conditions (1) and (2) of Theorem 2.6 of [8] are equivalent in **ZF**. In view of our Theorem 6.5 and the proof of (3) $\Rightarrow$ (1) of Theorem 2.6 given in [8], we have that (3) $\Rightarrow$ (1) of Theorem 2.6 of [8] holds true in **ZF**. Since **CC**(fin) is relatively independent of **ZF**, it follows from our Theorem 7.1 that Theorem 2.6 of [8] is relatively independent of **ZF**. If **M** is a model for **ZF** + $\neg$**CC**(fin), then Theorems 7.1 and 6.5 show that there exists in **M** an uncountable metric space (X, d) such that $\mathbf{FB}(X)$ is uniformly metrizable with respect to d, so Theorem 2.6 of [8] fails in **M**. Now, we can deduce from Proposition 3.5 of [9] that Theorem 2.6 of [8] is equivalent with **CC**(fin).

Remark 7.5. Let us notice that both (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (1) of Theorem 3.1 of [8] hold true in **ZF**. Unfortunately, Theorem 3.1 of [8] is relatively independent of **ZF**. Namely, in much the same way as in Remark 7.4, we can show that in every model **M** for **ZF** + $\neg$**CC**(fin), there exists an uncountable set X such that, for the discrete metric d in X, the bornology $\mathbf{CB}_d(X)$ is uniformly metrizable with respect to d, while (X, d) is not Lindelöf but it is obviously uniformly locally compact.

Remark 7.6. As Gutierres showed in [7], while working with completions of metric spaces, one must be more careful in **ZF** than in **ZF** + **CC**. Let us observe that if **M** is a model for **ZF** such that there is in **M** an uncountable set X such that $\mathbf{FB}(X)$ is uniformly metrizable with respect to the discrete metric d in X, then $\mathbf{TB}_d(X) = \mathbf{FB}(X) = \mathbf{BB}_d(X)$ is uniformly metrizable, while (X, d) is neither Lindelöf nor Bourbaki-separable. Therefore, Theorems 4.2 and 5.8 of [8] fail in **M**. In the light of our Theorem 7.1 and of the fact that **CC**(fin) is relatively independent of **ZF**, Theorems 4.2 and 5.8 of [8] are relatively independent of **ZF**.

Since many articles about bornologies have been published so far, it may take a lot of time to investigate which of the theorems in the articles can fail in some models for **ZF**. There are theorems about connections between bornologies and realcompactifications that have already appeared in print (cf. [20]) and they seem to be unprovable in **ZF**. In view of Theorem 10.12 of [18], perhaps, some of them can be proved in **ZF** + **UFT** where **UFT** stands for the Ultrafilter Theorem (cf. [9]). Let us leave it as an open problem which of the theorems about bornologies that have been proved by other authors in **ZFC** may fail in models for **ZF** and which of them can be proved under weaker assumptions than **ZFC**. We have given only a partial solution to this problem.

8. Compact bornologies in quasi-metric spaces

In the light of Remark 7.5, Theorem 3.1 of [8] may fail in a model for **ZF**. We are going to prove in **ZF** its modified version for compact bornologies in quasi-metric spaces.

For a topological space (X, τ), let $\mathbf{CB}_\tau(X)$ be the bornology in X generated by the collection of all compact subsets of (X, τ). If it is useful, we shall use the notation $\mathbf{CB}((X, \tau))$ for $\mathbf{CB}_\tau(X)$.

Definition 8.1. Let d be a quasi-metric in X.

(i) We denote by $\mathbf{CB}_d(X)$ the bornology $\mathbf{CB}_{\tau(d)}(X)$.

(ii) We say that X is *uniformly locally compact with respect to* d if there exists $\delta \in (0; +\infty)$ such that $B_d(x, \delta) \in \mathbf{CB}_d(X)$ for each $x \in X$.

The following example shows that, contrary to compact bornologies in metric spaces, it may happen that, for a quasi-metric d in X, there is a set $A \in \mathbf{CB}_d(X)$ such that $\mathrm{cl}_{\tau(d)} A \notin \mathbf{CB}_d(X)$.

Example 8.2. Let us consider the set $X = X_1 \cup X_2$ where $X_1 = \{\frac{1}{2^{2n}} : n \in \omega\}$ and $X_2 = \{\frac{1}{2^{2n+1}} : n \in \omega\}$. Let $x, y \in X$. If $x = y$, we put $d(x, y) = 0$. When $x \neq y$, we put $d(x, y) = 1$ if either $x, y \in X_1$ or $x, y \in X_2$, or $x \in X_1, y \in X_2$; moreover, we put $d(x, y) = y$ if $x \in X_2, y \in X_1$. In this way, we have defined a quasi-metric on X such that, for each $y \in X_2$, the set $A_y = \{y\} \cup X_1$ is compact in $(X, \tau(d))$, while $\mathrm{cl}_{\tau(d)} A_y = X \notin \mathbf{CB}_d(X)$.

Definition 8.3. We say that a topological space (X, τ) is σ-**CB** if there exists a countable collection $\mathcal{A} \subseteq \mathbf{CB}_\tau(X)$ such that $X = \bigcup \mathcal{A}$.

Remark 8.4. Clearly, it holds true in **ZF** that every σ-compact space is σ-**CB** and every σ-**CB** Hausdorff space is σ-compact. In every model for **ZF** + **CC**, a topological space is σ-compact if and only if it is σ-**CB**. We do not know whether there is a model for **ZF** + $\neg$**CC** in which a topological space can be simultaneously σ-**CB** and not σ-compact.

Theorem 8.5. *Let d be a (quasi)-metric in X. Then the following conditions are equivalent:*

(i) $\mathbf{CB}_d(X)$ *is uniformly (quasi)-metrizable with respect to d;*

(ii) *X is uniformly locally compact with respect to d and $(X, \tau(d))$ is σ-***CB***.*

Proof. Assume (i). Let ρ be a uniformly equivalent with d quasi-metric in X such that $\mathbf{CB}_d(X)$ is the collection of all ρ-bounded sets. There exists $\delta \in (0; +\infty)$ such that $\rho(x, y) < 1$ whenever $d(x, y) < \delta$. Then, for each $x \in X$, we have $B_d(x,\delta) \subseteq B_\rho(x,1) \in \mathbf{CB}_d(X)$, so X is uniformly locally compact with respect to d. Moreover, since, by Theorem 4.7, $\mathbf{CB}_d(X)$ has a countable base, we deduce that $(X,\tau(d))$ is σ-$\mathbf{CB}$.

Now, assume (ii). Let $\delta \in (0;+\infty)$ be such that, for each $x \in X$, we have $B_d(x,\delta) \in \mathbf{CB}_d(X)$. Let C be compact in $(X,\tau(d))$. It follows from the compactness of C that there exists a finite set $K \subseteq C$ such that $C \subseteq \bigcup_{x\in K} B_d(x,\frac{\delta}{2})$. Then $[C]_d^{\frac{\delta}{2}} \subseteq \bigcup_{x\in K}[B_d(x,\frac{\delta}{2})]_d^{\frac{\delta}{2}} \subseteq \bigcup_{x\in K} B_d(x,\delta) \in \mathbf{CB}_d(X)$. Therefore, $[C]_d^{\frac{\delta}{2}} \in \mathbf{CB}_d(X)$ (cf. proof to 3.1 in [8]). This implies that $[C]_d^{\frac{\delta}{2}} \in \mathbf{CB}_d(X)$ whenever $C \in \mathbf{CB}_d(X)$.

Let $\mathcal{A} = \{A_n : n \in \omega\} \subseteq \mathbf{CB}_d(X)$ be such that $X = \bigcup \mathcal{A}$. We may assume that $A_n \subseteq A_{n+1}$ for each $n \in \omega$. If C is compact in $(X,\tau(d))$ then, since $C \subseteq \bigcup_{n\in\omega}[A_n]_d^{\frac{\delta}{2}}$, there exists $m \in \omega$ such that $C \subseteq \bigcup_{n\in m+1}[A_n]_d^{\frac{\delta}{2}} = [A_m]_d^{\frac{\delta}{2}} \in \mathbf{CB}_d(X)$. Therefore, the collection $\{[A_n]_d^{\frac{\delta}{2}} : n \in \omega\}$, is a countable base for $\mathbf{CB}_d(X)$. This, together with the fact that $[C]_d^{\frac{\delta}{2}} \in \mathbf{CB}_d(X)$ whenever $C \in \mathbf{CB}_d(X)$, implies that there exists a subsequence $(B_n)_{n\in\omega}$ of the sequence $([A_n]_d^{\frac{\delta}{2}})_{n\in\omega}$ such that $[B_n]_d^{\frac{\delta}{2}} \subseteq B_{n+1}$ for each $n \in \omega$. Then $\{B_n : n \in \omega\}$ is a base for $\mathbf{CB}_d(X)$. This, together with Theorem 6.5, implies that (i) follows from (ii). $\square$

Example 8.6. Let (X,d) be the quasi-metric space from Example 8.2. Then condition (ii) of Theorem 8.5 is satisfied; hence, the bornology $\mathbf{CB}_d(X)$ is uniformly quasi-metrizable with respect to d. We can also define a uniformly locally identical with d quasi-metric ρ in X such that $\mathbf{CB}_d(X) = \mathcal{B}_\rho(X)$. To do this, let us consider $x,y \in X$. We put $\rho(x,y) = 0$ if $x = y$. Now, suppose that $x \neq y$. Then $\rho(x,y) = 1$ if $x,y \in X_1$. For $x \in X_2$ and $y \in X_1$, we define $\rho(x,y) = y$. Finally, for $x \in X$ and $y \in X_2$, we put $\rho(x,y) = \frac{1}{y}$.

Example 8.7. If d is the metric in $\mathbb{R}$ defined in Example 6.7, then $(\mathbb{R},\tau(d))$ is σ-$\mathbf{CB}$, while $\mathbb{R}$ is not uniformly locally compact with respect to d.

Finally, using similar arguments to the ones of the proof to Theorem 8.5, we deduce the following corollary:

Corollary 8.8 (cf. [22], Theorem 3.1 of [8] and Corollary 3.3 of [8]). *For every metric space (X,d), it holds true in* $\mathbf{ZF}$ *that $\mathbf{CB}_d(X)$ is uniformly metrizable with respect to d if and only if (X,d) is both σ-compact and uniformly locally compact.*

Of course, one can modify Theorem 8.5 and Corollary 8.8 to get their appropriate versions for quasi-pseudometrizability and pseudometrizability, respectively.

References

[1] J. W. Alexander: On the concept of a topological space, Proc. Natl. Acad. Sci. USA 25 (1939) 52–54.

[2] A. Andrikopoulos: The quasimetrization problem in the (bi)topological spaces, Int. J. Math. Math. Sci. 2007 (2007), Article ID 76904, 19 pp.

[3] G. Beer: On metric boundedness structures, Set-Valued Anal. 7 (1999) 195–208.

[4] O. De la Cruz: Finiteness and choice, Fundam. Math. 173 (2002) 57–76.

[5] R. Engelking: General Topology, Sigma Series in Pure Mathematics 6, Heldermann, Berlin (1989).

[6] P. Fletcher, W. F. Lindgren: Quasi-Uniform Spaces, Marcel Dekker, New York (1982).

[7] G. Gutierres: The Axiom of Countable Choice in Topology, Ph.D. Thesis, Dep. of Math, Coimbra Univ., Coimbra (2004).

[8] I. Garrido, A. S. Meroño: Uniformly metrizable bornologies, J. Convex Analysis 20(1) (2013) 285–299.

[9] H. Herrlich: Axiom of Choice, Springer, Berlin (2006).

[10] H. Hogbe-Nlend: Bornologies and Functional Analysis, North-Holland, Amsterdam (1977).

[11] P. Howard, J. E. Rubin: Consequences of the Axiom of Choice, Math. Surveys Monogr. 59, Amer. Math. Soc., Providence (1998).

[12] S. T. Hu: Boundedness in a topological space, J. Math. Pures. Appl. 28 (1949) 287–320.

[13] T. Jech: The Axiom of Choice, North-Holland, Amsterdam (1973).

[14] T. Jech: Set Theory, Springer, Berlin (2003).

[15] J. C. Kelly: Bitopological spaces, Proc. Lond. Math. Soc. 13 (1963) 71–89.

[16] K. Kunen: Set Theory, North-Holland, Amsterdam (1980).

[17] K. Kunen: The Foundations of Mathematics, College Publications, London (2009).

[18] A. Piękosz, E. Wajch: Compactness and compactifications in generalized topology, Topol. Appl. 194 (2015) 241–268.

[19] S. Salbany: Bitopological Spaces, Compactifications and Completions, Math. Monographs, Univ. of Cape Town, Rondesbosch (1974).

[20] T. Vroegrijk: On realcompactifications defined by bornologies, Acta Math. Hung. 133(4) (2011) 387–395.

[21] E. Wajch: Conditions of separation for quasi-pseudometrics, Folia Math. 16(1) (2009) 45–55.

[22] R. Williamson, L. Janos: Constructing metrics with the Heine-Borel property, Proc. Amer. Math. Soc. 100 (1987) 567–573.