Journal of Convex Analysis
Volume 22 (2015), No. 4, 1091–1106

A Creation Principle of Some Fish Type Skeletons in the Sense of Leibniz

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Dedicated to Gottfried's Leibniz Euclidean monad.

Received: August 7, 2013
Accepted: February 2, 2015

We create some fish type skeletons in the two dimensional Euclidean Space which is composed of seven evolutionary schemes and they are derived by applying a plasticity principle of convex pentagons and quadrilaterals, the generalized plasticity of polygons which is obtained by splitting the weights along the main bone of the skeleton and the plasticity of some generalized Gauss trees via the creation principle of a Fermat-Torricelli tree structure of degree four. Leibniz perception of a fish type structure is accomplished via a parallel translation of the branches (bones) of the skeleton between the weighted Fermat-Torricelli points of degree at most four which have been created by the generalized geometric plasticity along the main bone of the skeleton.

Keywords: Weighted Fermat-Torricelli tree structure, Steiner minimal tree, convex pentagons, plasticity principle, degree of plasticity

1991 Mathematics Subject Classification: 51E12, 52A10, 52A55, 51E10

1. Gottfried Leibniz's geometric monad

Gotfreid Leibniz' work on philosophical studies on metaphysics and the Monadology recognized him as one of the most important followers of Rene Descartes' dualism (see in [14]).

We begin with the definition of Leibniz' monad.

Definition 1.1 ([14, Page 67]). A monad is a simple substance which goes to make up composites.

We note that a study on the categorization of substance is also given by Rene Descartes in his work [6]. Rene Descartes is not only considered as one of the greatest mathematicians but also as the father of modern philosophy.

Definition 1.2. We call a geometric Monad a superior reference point which goes to make up a tree structure in the two dimensional Euclidean space.

ISSN 0944-6532 / $ 2.50

There is an internal principle which is inherited at time zero from a superior reference point and we name it the plasticity principle of polygons. The plasticity principle of polygons is responsible for the creation of the geometric design of polygons and the flow of information on the derived tree structure from the initial superior reference point, the development of the intermediate evolutionary steps (composites) until a termination tree structure takes place.

In 1638, Rene Descartes posed the following isoperimetric problem to Pierre de Fermat:

Problem 1.3 ([1, Rene Descartes problem, page 235]). *Considering four given points A_1, A_2, A_3 and A_4 in $\mathbb{R}^2$ study the curves of the form*

$$\sum_{i=1}^{4} \|A_i - X\| = c = constant,$$

where X is a point in $\mathbb{R}^2$.

In 1643, Pierre de Fermat communicated the following problem to the Italian scientific community:

Problem 1.4 (Fermat Problem). *Given A_1, A_2, A_3 in $\mathbb{R}^2$ find a fourth point A_0 such that:*

$$\sum_{i=1}^{3} \|A_i - A_0\| = minimum.$$

The solution A_0 of the Fermat Problem is named as "the Fermat-Torricelli point" thanks to the main contributor E. Torricelli.

We denote by a_{ij} the Euclidean distance A_iA_j for $i, j = 0, 1, 2, 3, ..., n$, $i \neq j$.

We state the inverse weighted Fermat-Torricelli problem for n given points in $\mathbb{R}^2$.

Problem 1.5 ([8, Inverse weighted Fermat-Torricelli problem]). *Given a point A_0 which belongs to the convex hull of $A_1A_2A_3...A_n$ excluding the points $\{A_1A_2A_3...A_n\}$ in $\mathbb{R}^2$, does there exist a unique set of positive numbers (weights) $B_1, B_2, B_3, ...B_n$ such that*

$$B_1 + B_2 + B_3 + ... + B_n = c = const.,$$

for which A_0 minimizes

$$f(A_0) = B_1a_{01} + B_2a_{02} + B_3a_{03} + ... + B_na_{0n}.$$

We note that Y. Kupitz and H. Martini gave a characterization of the solutions of the weighted Fermat-Torricelli problem for m given points in $\mathbb{R}^n$ ([13], [1, pp. 250]) by proving the following theorem:

Theorem 1.6 ([13], [1, pp. 250]). *Let $A_1A_2\cdots A_m$ be m non-collinear points in $\mathbb{R}^n$, with non-negative weights $B_1, B_2, ..., B_m$.*

(i) *The weighted Fermat-Torricelli point of $A_1A_2\cdots A_m$ exists and is unique.*

(ii) *If for every $A_i \in \{A_1A_2...A_m\}$*

$$\left\|\sum_{j=1}^{m} B_j\vec{u}(A_iA_j)\right\| > B_i, \quad i \neq j,$$

for $i, j = 1, 2, 3, ..., n$, then

(a) *the weighted Fermat-Torricelli point A_0 does not belong to $\{A_1A_2A_3 ...A_m\}$.*

(b)

$$\sum_{i=1}^{m} B_i\vec{u}(A_0A_i) = \vec{0}$$

(weighted floating case).

(iii) *If there is a vertex A_i that satisfies the inequality*

$$\left\|\sum_{j=1}^{m} B_j\vec{u}(A_iA_j)\right\| \leq B_i, \quad i \neq j,$$

for $i, j = 1, 2, 3, ..., n$, then the weighted Fermat-Torricelli point A_0 coincides with A_i, where $\vec{u}(A_iA_j)$ is the unit vector with direction from A_i to A_j, for $i, j = 0, 1, 2, 3, .., n$ and $i \neq j$ (weighted absorbed case).

In 2002, S. Gueron and R. Tessler were the first that posed and solved the inverse weighted Fermat-Torricelli problem for three given points in $\mathbb{R}^2$.

In 2008, we distinguish the plasticity of quadrilaterals to the geometric plasticity principle and the dynamic plasticity principle in $\mathbb{R}^2$ (see in [22] and [20]) by giving a negative answer to the inverse weighted Fermat-Torricelli problem for four given points in $\mathbb{R}^2$.

Definition 1.7 ([20]). We call *geometric plasticity* of a weighted network for n given values of the weights in $\mathbb{R}^2$ which are formed by n linear segments with variable length and given weight meeting at a weighted Fermat-Torricelli point the set of solutions of the n variable lengths of the linear segments which correspond to a family of weighted networks that preserve the weighted Fermat-Torricelli point which is formed by the endpoints of the n linear segments, such that it always remains an interior point of the convex hull that corresponds to the weighted n-gons.

Definition 1.8 ([20]). We call *dynamic plasticity* of a weighted network in $\mathbb{R}^2$, which is formulated by four weighted linear segments meeting at the weighted Fermat-Torricelli point, the set of solutions of the four variable weights with respect to the 4-inverse weighted Fermat-Torricelli problem for a given constant value c which correspond to a family of weighted networks that preserve the weighted Fermat-Torricelli point and the boundary of the quadrilateral such that the three variable weights depend on a fourth variable weight and the value of c.

In 2011, we introduce the generalized geometric plasticity principle in $\mathbb{R}^3$ (see in [23, Proposition 4.1, pp. 848–850]). Therefore, by using a specific discretization of the weights along the n prescribed linear segments which connect the vertices A_i with the weighted Fermat-Torricelli point A_0 of a given n-gon in $\mathbb{R}^2$, we introduce the generalized geometric plasticity of n-gons.

Definition 1.9. We call *generalized geometric plasticity* of a weighted network for $\sum_{i=1}^{n} m_i$ given values of the weights in $\mathbb{R}^2$ which are formulated by $\sum_{i=1}^{n} m_i$ linear segments with variable length and given weights meeting at a weighted Fermat-Torricelli point, such that m_i weights correspond to the linear segment that connects the weighted Fermat-Torricelli point of the initial n-weighted network with A_i, for $i = 1, 2, 3, ..., n$, the set of solutions of the $\sum_{i=1}^{n} m_i$ variable lengths of the linear segments which correspond to a family of weighted networks that preserve the weighted Fermat-Torricelli point which is formed by the endpoints of the n linear segments and it always remains an interior point of the convex hull that corresponds to the weighted $\sum_{i=1}^{n} m_i$-gons.

In 2013, we prove a (dynamic) plasticity principle of convex quadrilaterals on a convex (or concave) surface with bounded specific curvature in $\mathbb{R}^3$ and we also derive some evolutionary structures of convex pentagons which is based on the plasticity principle of convex quadrilaterals or as a limiting case of the plasticity of closed hexahedra in $\mathbb{R}^3$ (see in [20], [19] and [18]).

Thus, we obtain that:

Theorem 1.10 ([20, Plasticity Principle of Quadrilalaterals, Theorem 2]). *Given four prescribed rays which meet at the weighted Fermat-Torricelli point A_0 and their endpoints form a convex quadrilateral in $\mathbb{R}^2$ and the weighted Fermat-Torricelli point A_0 belongs to the interior of this convex quadrilateral, an increase of the weight that corresponds to a prescribed ray causes a decrease to the two weights that correspond to the two neighboring prescribed rays and an increase to the weight that corresponds to the opposite prescribed ray.*

Theorem 1.11 ([20, Plasticity Principle of Pentagons, Theorem 2], [19, Theorem 7, pp. 25]). *Given five prescribed rays which meet at the weighted Fermat-Torricelli point A_0 and their endpoints form a convex pentagon in $\mathbb{R}^2$ and the weighted Fermat-Torricelli point belongs to the interior of this convex pentagon, an increase of the weight that corresponds to the fourth and fifth ray causes a decrease to the two weights that correspond to the first and third ray and an increase to the weight that corresponds to the second ray.*

We proceed by giving the necessary definitions of Euclidean minimal trees structures (Steiner trees, generalized Gauss trees) in $\mathbb{R}^2$ ([7], [11], [10]).

We denote by A_i a vertex (or a boundary vertex) of the Euclidean minimal tree structures and by $A_{0,j}$ a mobile vertex of the Euclidean minimal tree structures in $\mathbb{R}^2$, for $i = 1, 2, 3, ...n$ and $j = 1, 2, 3, ..., n-2$.

Definition 1.12 ([7]). A tree topology is a connection matrix specifying which pairs of points from the list $A_1, A_2 A_3, ..., A_n, A_{0,1}, A_{0,2}, ..., A_{0,n-2}$ have a connecting linear segment (edge).

Definition 1.13 ([10]). The degree of a vertex corresponds to the number of connections of the vertex with linear segments.

Definition 1.14 ([7], [18]). A Gauss tree topology of degree at most four is a tree topology with all boundary vertices of a pentagon and two mobile vertices having at most degree four.

Definition 1.15. A weighted Gauss tree of weighted minimum length with a Gauss tree topology of degree at most four in $\mathbb{R}^2$ is called a generalized Gauss tree of degree at most four.

Definition 1.16. A degenerate Gauss tree (weighted Fermat-Torricelli tree) with respect to a boundary pentagon is a weighted Fermat-Torricelli tree having one mobile vertex at the convex hull of the pentagon.

Definition 1.17 ([7], [16]). A Steiner topology of a tree is a tree topology with all vertices (boundary and mobile) having at most degree three.

Definition 1.18 ([7], [16]). A tree of minimum length with a Steiner topology is called a Steiner tree.

Definition 1.19 ([7], [10]). A Steiner tree is a full Steiner tree if all boundary vertices are of degree one.

Definition 1.20 ([7], [9], [16]). The number of full Steiner topologies for m boundary vertices is given by the function:

$$S_{top}(m) = \frac{2^{-(m-2)}(2m-4)!}{(m-2)!}.$$

The Steiner problem states that(see also [4, pp. 360], [12]):

"Given n points $A_1 A_2 ... A_n$ to find a connected system of straight line segments of shortest total length such that any two of the given points can be joined by a polygon consisting of segments of the system."

In 1968, E. Gilbert and H. Pollak found the solutions of the Steiner problem ([7], [1, pp. 328–329]).

Theorem 1.21 ([7], [1, pp. 328-329]). *A solution of the Steiner problem is a*

Steiner tree with at most $n-2$ *Fermat-Torricelli (or Steiner) points* $A_{0,i}$ *which are vertices of the polygonal tree which do not belong in* $\{A_1A_2...A_n\}$*, where each Fermat-Torricelli point has degree* 3*, and the angle between any two edges incident with a Fermat-Torricelli point is of* $120°$.

In 2013, we introduce the degree of plasticity of Euclidean minimal tree structures which extends the Steiner tree topologies in $\mathbb{R}^2$ ([24], [18]).

Definition 1.22 ([24]). The degree of plasticity (Degplasticity$\{A_1A_2...A_n\}$) of an Euclidean weighted minimal tree of a set with boundary vertices $\{A_1A_2...A_n\}$ which form a convex polygon in $\mathbb{R}^2$ that consists of exactly m ($m \leq n-2$) weighted Fermat-Torricelli points at the interior of the polygon is $n-2-m$.

Definition 1.23 ([18], [24]). The degree of plasticity (Degplasticity$\{A_1A_2...A_n\}$) of the weighted (full) Steiner minimal tree of a set with boundary vertices $\{A_1A_2...A_n\}$ which form a convex polygon in $\mathbb{R}^2$ that consists of exactly $n-2$ weighted Fermat-Torricelli points at the interior of the polygon is zero.

Definition 1.24 ([18], [24]). The degree of plasticity (Degplasticity$\{A_1A_2...A_n\}$) of a Fermat-Torricelli tree of a set with boundary vertices $\{A_1A_2...A_n\}$ which form a convex polygon in $\mathbb{R}^2$ that consists of exactly one weighted Fermat-Torricelli points at the interior of the polygon is $n-3$.

In 1859, it is well known that Darwin developed a theory on the origin of species via natural selection in his work [5]. In 2008, one of the most recent followers of Darwins theory of evolution N. Shubin argues that a human structure comes from a fish type structure in his book [15]. Recently, we introduce a quadrilateral monad in the sense of Leibniz in five evolutionary phases ([20, Section 9]).

We shall give the properties of a geometric Monad in the two dimensional Euclidean space, which create a fish type skeleton starting from a superior reference point (Monad) at time zero that consists of seven evolutionary steps in the spirit of Leibniz, by applying the geometric plasticity principle of polygons, the dynamic plasticity principle of pentagons and quadrilaterals, the generalized plasticity principle of polygons and the plasticity of generalized Gauss trees of degree at most four in the two dimensional Euclidean space.

2. Descartes-Fermat evolution of fish type skeletons in the Euclidean plane

Suppose that two evolutionary structures preexist at the geometric monad having two constant quantities C_{HT} and C_{HMB} at time zero. The positive constant quantity C_{HT} corresponds to the evolutionary structure of the Head Bones H_1F_1, H_2F_1 (or Head-Mouth Structure), the main bone F_1T of the skeleton and the tail bones TT_1 and TT_2. The positive constant quantity C_{HMB} corresponds to the evolutionary structure of the Head Bones H_1F_1, H_2F_1 and a leaf type structure which exist at the main bone of the skeleton and consists of the points F_i which

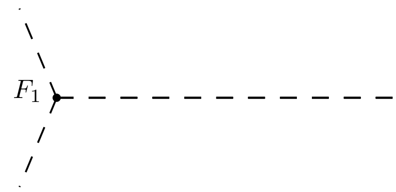

Figure 2.1: First Phase

exist at the main bone (main branch) of the skeleton and the bones (branches) $S_{i1}F_i$ and $S_{i2}F_{i2}$ for $i = 1, 2, 3, ..., k$. The branch $S_{ij}F_i$ is parallel to the branch $S_{1j}F_1$, for $j = 1, 2$, and $i = 2, 3, 4, ..., k$ (Fig. 2.7).

Definition 2.1. A fish type skeleton is a tree structure which consists of two leaf tree type structures that are located with respect to the main branch of the skeleton.

Definition 2.2. We call a Descartes-Fermat evolution of a geometric monad a structure of evolutionary schemes which creates an evolutionary tree structure with respect to the solutions of the n-inverse weighted Fermat-Torricelli problem in the two dimensional Euclidean Space.

We give the seven evolutionary schemes of a fish type skeleton in the Euclidean plane by using a Descartes-Fermat evolution which generalizes the Descartes Fermat evolution of a quadrilateral monad given in [20, Section 9].

2.1. First evolutionary structure of a fish type skeleton

Phase 1 starts at time zero from a reference point F_1 which possesses a positive quantity (information) $C_{HT} = B_{H_1} + B_{H_2} + B_T$, where B_{H_1}, B_{H_2} and B_T are three positive real numbers (weights). This reference point has inherited a tendency to split in three different directions and a symmetry defined by the one prescribed (main) direction. The main direction is responsible for the creation of the main bone of a fish type skeleton (Fig. 2.1).

2.2. Second evolutionary structure of a fish type skeleton

In phase 2, three different branches are growing towards the three directions with respect to the lines defined by H_1F_1, H_2F_1 and TF_1, with weights B_{H_1}, B_{H_2}, and B_T, respectively, until they form a boundary isosceles triangle $\triangle H_1H_2T$, such that F_1 is the weighted Fermat-Torricelli point of $\triangle T_1T_2T$ (Fig. 2.2).

We note that the solution of the inverse weighted Fermat-Torricelli problem for the boundary triangle $\triangle T_1T_2T$ with respect to B_{H_1}, B_{H_2}, and B_T which corresponds to the vertices H_1, H_2 and T. (Problem 1.5, $n = 3$) remains exactly the same by assigning the weights B_{H_1}, B_{H_2} and B_T to the linear segments H_1F_1, H_2F_1 and TF_1, respectively.

Thus, the solution of the modified inverse weighted Fermat-Torricelli problem for

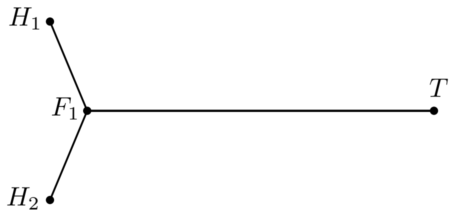

Figure 2.2: Second Phase

a boundary isosceles triangle yields ([8], [21, Proposition 3.2, 3.5, Corollary 3.4], [2, Proposition 5], [3]):

$$B_{H_1} = \frac{C_{HT}}{1 + \frac{\sin\angle H_1F_1T}{\sin\angle H_2F_1T} + \frac{\sin\angle H_1F_1H_2}{\sin\angle H_2F_1T}}, \tag{1}$$

$$B_{H_2} = \frac{C_{HT}}{1 + \frac{\sin\angle H_2F_1T}{\sin\angle H_1F_1T} + \frac{\sin\angle H_2F_1H_1}{\sin\angle H_1F_1T}}, \tag{2}$$

$$B_T = \frac{C_{HT}}{1 + \frac{\sin\angle H_1F_1T}{\sin\angle H_2F_1H_1} + \frac{\sin\angle H_2F_1T}{\sin\angle H_2F_1H_1}}, \tag{3}$$

By replacing $\angle H_2F_1T = \angle H_1F_1T$ in (1) and (2), we obtain that:

$$B_{H_1} = B_{H_2}.$$

2.3. Third evolutionary structure of a fish type skeleton

In phase 3, the internal principles of dynamic and geometric plasticity of quadrilaterals creates two evolutionary structures of convex quadrilaterals with respect to the main bone F_1T in the sense of action and reaction that exist in Nature.

More specifically, two branches start to grow simultaneously from F_1 until they reach the vertices T_{01} and T_{02}, such that $\angle T_{01}F_1T = \angle T_{02}F_1T$ (Fig. 2.3).

From the geometric plasticity principle of $H_2TT_{01}H_1$ and $H_1TT_{02}H_2$ the corresponding weighted Fermat-Torricelli point F_1 remains invariant ([8], [21, Proposition 3.2, 3.5, Corollary 3.4]).

From the dynamic plasticity principle of the boundary quadrilateral $H_2TT_{01}H_1$, we derive that an increase (decrease) in the weight $B_{T_{01}}$ causes a decrease (increase) in the weights B_{H_1} and B_T and an increase (decrease) in the weight B_{H_2}, such that the isoperimetric condition of the inverse weighted Fermat-Torricelli problem for four given points in $\mathbb{R}^2$ hold ([20, Plasticity Principle of Quadrilalaterals, Theorem 2], [22]):

$$B_{H_2} + B_T + B_{T_{01}} + B_{H_1} = C_{HT}.$$

Similarly, from the dynamic plasticity principle of the boundary quadrilateral $H_1TT_{02}H_2$, we derive that an increase (decrease) in the weight $B_{T_{02}}$ causes a

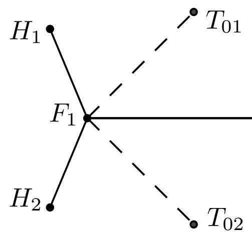

Figure 2.3: Third Phase

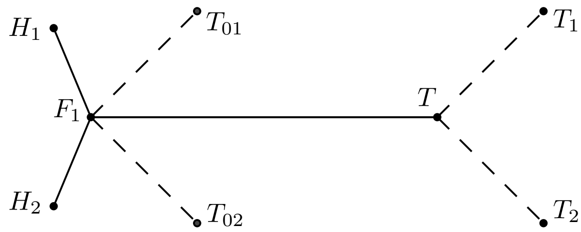

Figure 2.4: Fourth Phase

decrease (increase) in the weights B_{H_2} and B_T and an increase (decrease) in the weight B_{H_1}, such that the isoperimetric condition of the inverse weighted Fermat-Torricelli problem for four given points in $\mathbb{R}^2$ hold:

$$B_{H_1} + B_T + B_{T_{02}} + B_{H_2} = C_{HT}.$$

Thus, the symmetrical points T_{01} and T_{02} with respect to the line defined by F_1T creates two dynamic plasticity principles of the boundary isosceles trapezium $H_2TT_{01}H_1$ and $H_1TT_{02}H_2$ that act and react each other.

Therefore, an increase in $B_{T_{01}}$ and $B_{T_{02}}$ cancels any variation in B_{H_1} and B_{H_2} and creates a decrease only in B_T.

2.4. Fourth evolutionary structure of a fish type skeleton

In Phase 4 Leibniz perception of the plasticity of the quadrilalateral monad $H_1H_2T_1T_2$ (Fig. 2.4) is accomplished via parallel translation of $T_{01}F_1$ and $T_{02}F_1$ to T_1T and T_2T along the main bone of the skeleton F_1T ([24], [20, Section 9]).

Thus, we have:

$$B_{MS}u_{F_1T} = B_{T_{01}}u_{F_1T_{01}} + B_{T_{02}}u_{F_1T_{02}}$$

where u_{XY} is the unit vector in the direction of the linear segment XY in $\mathbb{R}^2$ ([20, Section 9]). The specific values for $B_{T_{01}}$ and $B_{T_{02}}$ are the "perception values" of the variables weights.

2.5. Fifth evolutionary structure of a fish type skeleton

We denote by B_0 the weight that corresponds to F_1, $B_{0'}$ the weight that corresponds to T and by $B_{F_1T} = \frac{B_0+B_{0'}}{2}$ the variable weight which correspond to the linear segment F_1T.

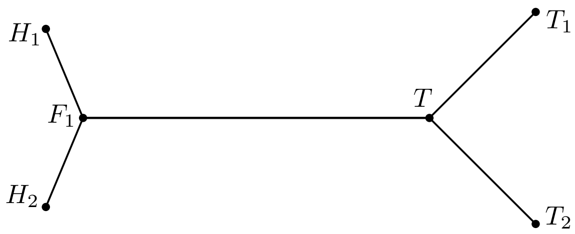

Figure 2.5: Fifth Phase

In the fifth phase Leibniz annihilation deals with the cancellation of the dynamic plasticity with respect to the quadrilateral monad $H_1T_1T_2H_2$ (isosceles trapezium) in $\mathbb{R}^2$ (Fig. 2.5).

The weighted full Steiner minimal tree of the boundary quadrilateral $H_1T_1T_2H_2$ is a generalized Gauss tree having a Steiner topology (Gauss topology) with degree at most three which minimizes the objective function

$$B_{H_1}\|H_1-F_1\|+B_{H_2}\|H_2-F_1\|+B_{T_1}\|T_1-T\|+B_{T_2}\|T_2-T\|+B_{F_1T}\|F_1-T\|. \quad (4)$$

By applying a method of cyclical differentiation given in [17] of (4) with respect to four variable angles, we derive two weighted Fermat-Torricelli tree structures of degree three with respect to $\triangle H_1H_2T$ and $\triangle T_1T_2F_1$ such that F_1 and T are the corresponding weighted Fermat-Torricelli points. The solutions of the inverse weighted Fermat-Torricelli problem with respect to $\triangle H_1H_2T$ and $\triangle T_1T_2F_1$ determine uniquely the values of the weights B_{H_1}, B_{H_2}, B_{T_1} and B_{T_2} and cancel the dynamic plasticity of $H_1T_2T_1H_1$ (Leibniz annihilation, Fig. 2.5).

2.6. Sixth evolutionary structure of a fish type skeleton

The cancellation of the dynamic plasticity (annihilation of fifth phase) of the Head-Main Bone-Tail structure (full weighted Steiner tree) leaves a residual absorbing positive quantity (residual weight) in the linear segment F_1T of the main bone which is responsible for unlocking the quantity C_{HMB} which shall create the bones (branches) $S_{i1}F_i$ and $S_{i2}F_{i2}$ for $i=1,2,3,...,k$, such that $S_{ij}F_i$ is parallel to the branch $S_{1j}F_1$, for $j=1,2$, and $i=2,3,4,...,k$

Definition 2.3. The residual absorbing rate $\tilde{B}_{00'}$ of a generalized Gauss tree of degree at most four with respect to the boundary convex quadrilateral $H_1T_1T_2H_2$ is $B_{H_1}+B_{H_2}+B_{T_1}+B_{T_2}-\frac{B_0+B_{0'}}{2}$.

The discretization of the residual rate of $H_1T_1T_2H_2$ along the main bone F_1T provides a tendency to create some knots F_2, F_3, ..., F_k, with weights $B'_{F_2}, ...,$ B'_{F_k} such that:

$$B'_{F_1}+B'_{F_2}+B'_{F_3}+B'_{F_4}+...+B'_{F_k}=\tilde{B}_{00'}. \quad (5)$$

In phase 6, the internal principles of dynamic and geometric plasticity of quadrilaterals creates two evolutionary structures of convex quadrilaterals with respect to the main bone F_1T which are similar with the structures which have been obtained in phase 3.

Two branches start to grow simultaneously from F_1 until they reach the vertices S_{11} and S_{12}, such that $\angle S_{11}F_1T = \angle S_{12}F_1T$ (Fig. 2.6).

From the geometric plasticity principle of $H_2TS_{11}H_1$ and $H_1TS_{12}H_2$ the corresponding weighted Fermat-Torricelli point F_1 remains invariant ([8], [21, Proposition 3.2, 3.5, Corollary 3.4]).

From the dynamic plasticity principle of the boundary quadrilateral $H_2TT_{01}H_1$, we derive that an increase (decrease) in the weight $B'_{S_{11}}$ causes a decrease (increase) in the weights B'_{H_1} and B'_T and an increase (decrease) in the weight B'_{H_2}, such that the isoperimetric condition of the inverse weighted Fermat-Torricelli problem for four given points in $\mathbb{R}^2$ hold ([20, Plasticity Principle of Quadrilalaterals, Theorem 2], [22]):

$$B'_{H_2} + B'_T + B'_{S_{11}} + B'_{H_1} = C_{HMB}.$$

Similarly, from the dynamic plasticity principle of the boundary quadrilateral $H_1TS_{12}H_2$, we derive that an increase (decrease) in the weight $B'_{S_{12}}$ causes a decrease (increase) in the weights B'_{H_2} and B'_T and an increase (decrease) in the weight B'_{H_1}, such that the isoperimetric condition of the inverse weighted Fermat-Torricelli problem for four given points in $\mathbb{R}^2$ hold:

$$B'_{H_1} + B'_T + B'_{S_{12}} + B'_{H_2} = C_{HT}.$$

Thus, we derive that an increase in $B'_{S_{11}}$ and $B'_{S_{12}}$ cancels any variation in B'_{H_1} and B'_{H_2} and creates a decrease only in B'_T.

A decrease in the weight B'_T causes a decrease in the residual absorbing weight $\tilde{B}_{00'}$ which was left from the annihilation phase of Head-Tail structure on the main bone of the skeleton. This result may also be interpreted as a consuming factor of the discretization of $\tilde{B}_{00'}$ which was split into k points B_{F_i} for $i = 1, 2, ..., k$ along F_1T.

The generalized geometric plasticity principle in $\mathbb{R}^2$ states that:

Proposition 2.4 ([18], [23]). *Let $A_1A_2A_3A_4A_5$ be a convex pentagon in $\mathbb{R}^2$ with non-negative weights B_i that correspond to each vertex A_i, respectively, which satisfy the weighted inequalities of the weighted floating case of Theorem* 1.6 *and A_0 is the corresponding weighted Fermat-Torricelli point. Assume that every non-negative weight B_i is split into n_i non-negative weights B_{ik} :*

$$\sum_{k=1}^{n_i} B_{ik} = B_i,$$

for $i = 1, 2, 3, 4, 5$. The weight $B_{i,k}$ corresponds to every vertex $A_{i,k}$ which belongs to the line segment A_0A_i, for every $k \neq n_i$ and the weight B_{i,n_i} corresponds

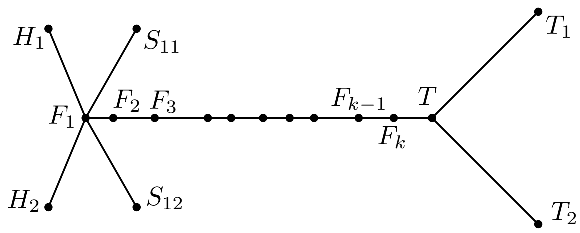

Figure 2.6: Sixth Phase

to the vertex $A_i = A_{i,n_i}$*. Then the weighted Fermat-Torricelli point of* $\{A_{i,k}\}$ *coincides with the generalized Fermat-Torricelli point of* $\{A_1A_2A_3A_4A_5\}$*, for* $i = 1, 2, 3, 4, 5$.

By applying Proposition 2.4 in the weighted convex pentagon $H_1S_{11}TS_{12}H_2$ and by splitting the weight B_T that corresponds to the edge F_1T (or at the vertex T) to F_{k-1} points such that:

$$B'_{F_1} + B'_{F_2} + B'_{F_3} + B'_{F_4} + ... + B'_{F_k} = B'_T,$$

we obtain that the corresponding weighted Fermat-Torricelli point F_1 remains invariant in $\mathbb{R}^2$.

2.7. Seventh evolutionary structure of a fish type skeleton

In phase 7, Leibniz perception via parallel translation creates a leaf tree type structure along the main bone which compose the bones of the skeleton (Fig. 2.7).

Specifically, the parallel translation of the branch $S_{ij}F_i$ to the branch $S_{1j}F_1$, for $j = 1, 2$, and $i = 2, 3, 4, ..., k$ shall create at the knots F_i for $i = 2, 3, ..., k$ a sequence of weighted Fermat-Torricelli tree structures of degree four.

We start with a weighted Steiner minimal tree of degree four (or generalized Gauss tree) of the boundary convex pentagon $H_2S_{22}TS_{21}H_1$ which consists of a weighted Fermat-Torricelli tree structure of degree three at the weighted Fermat-Torricelli point F_1 and a weighted Fermat-Torricelli tree structure of degree four at the weighted Fermat-Torricelli point F_2. ([18, Proposition 4]). The degree of plasticity $\deg(H_2S_{22}TS_{21}H_1)$ of the corresponding generalized Gauss tree of degree at most four is one. Therefore, we may derive the plasticity of the generalized Gauss tree of degree at most four by considering the following inverse (distorted) weighted Steiner minimal tree of a convex pentagon $A_1A_2A_3A_4A_5$, by replacing H_1 with A_4, H_2 with A_5 S_{22} with A_1, T with A_2, S_{21} with A_3, A_0 with F_2, $A_{0'}$ with F_1 and by assigning the weight B_i to each A_i for $i = 0', 0, 1, 2, 3, 4, 5$:

Problem 2.5 ([18, 5-Inverse weighted distorted Steiner minimal tree]). *Given two points* A_0 *and* $A_{0'}$ *which belongs to the interior of* $A_1A_2A_3A_4A_5$ *in* $\mathbb{R}^2$*, does there exist a unique set of positive weights* B_i*,* B_0*,* $B_{0'}$ *such that*

$$B_1 + B_2 + B_3 + B_{0'} = c = const.,$$

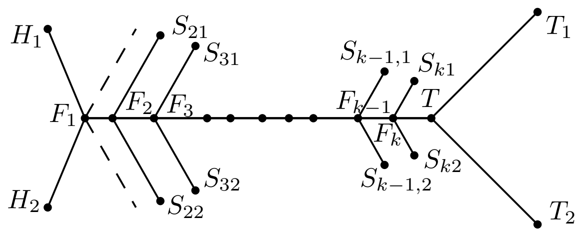

Figure 2.7: Seventh Phase

$$B_0 = (B_{0'}),$$

for which A_0 *and* $A_{0'}$ *minimizes*

$$f(A_0, A_{0'}) = B_1 a_{10'} + B_2 a_{20} + B_3 a_{30} + B_4 a_{40} + B_5 a_{50'} + B_0 a_{00'}$$

where a_{ik} *is the length of the linear segment* $A_k A_i$ *for* $i = 1, 2, 3, 4, 5,$ *and* $k \in \{0, 0'\}$.

We denote by $(B_i)_{1230'}$ the weight that corresponds to the vertex A_i that lie on $A_0 A_i$, for $i \in \{1, 2, 3, 0'\}$ and the weight $(B_i)_{ijk}$ corresponds to the vertex A_i that lie on $A_0 A_i$ with respect to $\triangle A_i A_j A_k$ $i, j, k \in \{1, 2, 3, 0'\}$, $i \neq j \neq k$ and by α_{lmn} the angle $\angle A_l A_m A_n$, for $l, m, n = 0', 0, 1, 2, 3, 4, 5$.

Theorem 2.6 ([18]). *The plasticity of the 5 distorted weighted Steiner minimal tree with respect to the boundary pentagon* $A_1 A_2 A_3 A_4 A_5$ *in* $\mathbb{R}^2$ *is given by the following equations:*

$$(B_4)_{045} = \frac{B_0 \sin(\alpha_{50'0})}{\sin(\alpha_{50'4})} \tag{6}$$

$$(B_5)_{045} = \frac{B_0 \sin(\alpha_{00'4})}{\sin(\alpha_{50'0})} \tag{7}$$

$$\left(\frac{B_2}{B_1}\right)_{1230'} = \left(\frac{B_2}{B_1}\right)_{123} \left[1 - \left(\frac{B_0'}{B_1}\right)_{1230'} \left(\frac{B_1}{B_0'}\right)_{130'}\right], \tag{8}$$

$$\left(\frac{B_3}{B_1}\right)_{1230'} = \left(\frac{B_3}{B_1}\right)_{123} \left[1 - \left(\frac{B_0'}{B_1}\right)_{1230'} \left(\frac{B_1}{B_0'}\right)_{120'}\right], \tag{9}$$

$$(B_0)_{045} = (B_{0'})_{1230'} \tag{10}$$

and

$$(B_1)_{1230'} + (B_2)_{1230'} + (B_3)_{1230'} + (B_{0'})_{1230'} = c_1 = const., \tag{11}$$

We denote by $S_{A_1 A_2 A_3 A_0 A_4 A_{0'} A_5}$ the (distorted) Steiner minimal tree of degree at most four with respect to the boundary convex pentagons $A_1 A_2 A_3 A_4 A_5$, by $A_{0'0i}$ the knot F_i, by A_{1i} the point S_{i1} and by A_{3i} the point S_{i3}, for $i = 3, 4, ..., k$, such that a parallel translation of the linear segments $A_1 A_0 \to A_{1i} A_{0'0i}$ and $A_3 A_0 \to A_{3i} A_{0'0i}$, for $i = 3, .., k$.

Following a similar process with the derivation of a weighted Steiner leaf tree type given in Proposition 7 in [18], we obtain a weighted Steiner leaf tree type in $\mathbb{R}^2$ which minimizes the objective function

$$B_4 a_{40'} + B_5 a_{50'} + \sum_{i=2}^{k} B_{1i} a_{(1i)(0'0i)} + \sum_{i=2}^{k} B_{2i} a_{(2i)(0'0i)} + \sum_{i=2}^{k} B_{0'0i} a_{0(00'i)}, \tag{12}$$

such that:

$$\left\| B_j \vec{U}_{A_i A_j} + B_p \vec{U}_{A_i A_p} \right\| > B_i, \tag{13}$$

$$\left\| B_m \vec{U}_{A_l A_m} + B_3 \vec{U}_{A_l A_n} + B_4 \vec{U}_{A_l A_q} \right\| > B_l, \tag{14}$$

for $i, j, p \in \{0r, 1r, 3r\}$, and $r = 3, .., k$, $l, m, n, q \in \{0', 1, 2, 3\}$ and $i \neq j \neq p$, $l \neq m \neq n \neq q$,

$$\sum_{q=2}^{k} \lambda_q = 1, \tag{15}$$

$$\sum_{i=2}^{k} B_{0'0i} = B_0 = B_{0'} = \frac{B_0 + B_{0'}}{2} \tag{16}$$

$$\sum_{i=2}^{k} B_{1i} = B_1 \tag{17}$$

$$\sum_{i=2}^{k} B_{3i} = B_3 \tag{18}$$

A_0, $A_{0'}$ remains the same with $S_{A_1 A_2 A_3 A_0 A_4 A_{0'} A_5}$ where $A_{00'i}$ belongs to $A_{0'}A_2$ with corresponding weight $B_{0'0i} = \lambda_i B_0$ and the vertex A_{ji} has a corresponding weight $B_{ji} = \lambda_i B_j$ for $j = 1, 3$ and $i = 2, 3, .., n$.

The weighted Steiner leaf tree type of the boundary polygon

$A_5 A_1 A_{13} A_{14} ... A_{1k} A_2 A_{3k} A_{3(k-1)} A_{33} A_3 A_4$ consists of the weighted Fermat-Torricelli point $A_{0'}$ (F_1) of degree three and the weighted Fermat-Torricelli points of degree four $A_{0'02}$ (or A_0 or F_2) $A_{0'03}$ (F_3)...,$A_{0'0k}$ (F_k).

From the dynamic plasticity principle of the boundary quadrilaterals $A_{0'0j} A_{1(j+1)} A_2 A_{3(j+1)}$ the consumption (decrease) of the weight $B_{0'0}(j+1)$ causes an increase to the weights $B_{1(j+1)}$ and $B_{3(j+1)}$ and a decrease to the weight $B_{0'0j}$.

Thus, this sequence of the dynamic plasticity of quadrilaterals such that

$$\sum_{i=2}^{k} B_{0'0i} = B_0 = B_{0'} = \frac{B_0 + B_{0'}}{2},$$

causes a bigger increase in the value of the weights B_{1j} and B_{3j} than the value of the weights B_{1j+1} and B_{3j+1} for $j = 2, 3, ..., k$ (Fig. 2.7).

We note that the annihilation of a fish type skeleton occurs when the residual absorbing rate that exists on the main bone of the skeleton has been fully consumed by the Steiner leaf tree type structure which has been created on the main bone of the fish type skeleton from the Head-Tail structure.

We conclude that the work of philosophers of the 17th and 18th century like Rene Descartes and his followers need to be reexamined and the process of evolution of species may be taken into consideration and may be developed again. N. Shubin's theory provided empirical evidence that even a human structure of a skeleton may have some common points with a fish type skeleton but the Descartes-Fermat evolution of a fish type skeleton provides some evidence that a fish type skeleton is governed by a geometric Euclidean Monad in the sense of Leibniz. As a future work, we may consider the creation of a chronomonad depending on time.

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