Journal of Convex Analysis
Volume 22 (2015), No. 4, 1107–1124

Critical Points on Closed Convex Sets vs. Critical Points and Applications

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Received: June 11, 2014
Accepted: February 2, 2015

The existence of multiple critical points for a locally Lipschitz continuous functional Φ on a closed convex subset C of a Banach space X is investigated. The problem of finding extra conditions under which critical points for Φ on C turn out to be critical on X is also addressed. Two applications concerning elliptic variational-hemivariational inequalities are then worked out.

Keywords: Multiple critical points, functional defined on closed convex set, Schauder invariance condition, elliptic variational-hemivariational inequality

1991 Mathematics Subject Classification: 58E05, 49J40, 49J52

1. Introduction

The very recent paper [18] deals with the critical point theory for a family of non-differentiable functionals defined on a convex closed subset of a Banach space. This work and the next one [19] will be mainly devoted to applications.

Our abstract framework is the following. Let $(X, \|\cdot\|)$ be a real Banach space, let C be a nonempty, convex, closed subset of X, and let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz continuous function. Denote by $\partial\Phi$ the generalized sub-differential [6] of Φ. We say that $x \in C$ is a critical point for Φ on C if

$$\inf_{x^*\in\partial\Phi(x)} \sup_{z\in C,\|z-x\|<1} \langle x^*, x - z\rangle = 0.$$

Theorem 3.6 below establishes the existence of at least two critical points for Φ on C. It extends [3, Corollaries 1–2] and [10, Corollary 5.12] to non-smooth

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ISSN 0944-6532 / \$ 2.50

functionals on closed convex sets; cf. also [12, Theorem 17.2]. Its proof chiefly exploits a technical result (Lemma 3.5) and an appropriate version, previously obtained in [18], of the Brézis-Nirenberg critical point theorem.

Remember that $x \in X$ is called critical for Φ provided $0 \in \partial\Phi(x)$. Evidently, critical points of Φ on C are not necessarily critical for Φ. However, sometimes, finding critical points on convex closed sets turns out to be easier and of applicative interest; see for instance [3, 5, 17, 23, 24, 25] as well as [8, Section 2.2]. A known case is the following.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N < 6$, with a smooth boundary $\partial\Omega$, and let $g \in L^{2^{*\prime}}(\Omega) \setminus \{0\}$ be nonnegative on Ω, where $2^{*\prime}$ denotes the conjugate of the critical Sobolev exponent for the embedding $H_0^1(\Omega) \subseteq L^q(\Omega)$. Consider the problem

$$\begin{cases} \Delta u + u^2 = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

L. Nirenberg, in [22, Theorem 2], proved that it has two solutions, one of which is negative, belonging to $H_0^1(\Omega)$; cf. also [4, pp. 147–148]. So, the question whether (1) possesses positive solutions naturally arises. K.-C. Chang, in [3], established the existence of at least two critical points for the energy functional associated with (1), namely

$$\Phi(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \left(\frac{1}{3}u^3 - g(x)u\right) dx, \quad u \in H_0^1(\Omega), \tag{2}$$

on the *positive cone*

$$C := \{u \in H_0^1(\Omega) : u \geq 0\}. \tag{3}$$

Nevertheless, none of them solves (1). In fact, Example 4.1 below shows that *Problem* (1) *may have no positive solution for* $g > 0$. This actually means that Chang's answer cannot be improved in general.

Consequently, the research of further hypotheses under which critical points for Φ on C turn out to be critical points of Φ takes interest. We shall prove that if X is reflexive, a locally Lipschitz continuous version of the so-called Schauder invariance condition [7, 11, 15, 20, 26] fits the purpose; see Theorems 4.4–4.5.

Whereas these results will be thoroughly employed in [19], Section 5 of the present paper contains applications of Theorems 3.3–3.6. The existence of one or two nontrivial solutions to the elliptic variational-hemivariational inequality *Find* $u \in C$ *such that*

$$\int_\Omega \nabla u \cdot \nabla(u - v)dx \leq \int_\Omega J^0(x, u(x); u(x) - v(x))dx \quad \forall\, v \in C \tag{4}$$

is investigated. Here, $(x, z) \mapsto J(x, z)$ denotes a function measurable in $x \in \Omega$ and locally Lipschitz continuous with respect to $z \in \mathbb{R}$, while $J^0(x, u(x); u(x) - v(x))$ stands for the generalized directional derivative [6] of $J(x, \cdot)$ at $u(x)$ along

$u(x) - v(x)$. Theorem 5.3 treats the case when C is the positive cone (3). Theorem 5.7 requires that

$$C := \{u \in H_0^1(\Omega) : 0 \le u \le \psi\}$$

for some nonnegative $\psi \in H^1(\Omega)$. Both of them extend previous results in [3, 13, 17]. Although (4) naturally falls within applications of the critical point theory developed by Motreanu-Panagiotopoulos [21] (see also [16]), the approach taken here looks slightly easier and fruitful. For a deep account on these topics we refer to the monographs [1, 8, 21].

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. Given a nonempty set $V \subseteq X$, write $\mathrm{int}(V)$ for the interior of V, ∂V for the boundary of V, and $\delta_V : X \to \mathbb{R} \cup \{+\infty\}$ for the indicator (function) of V, namely

$$\delta_V(x) := \begin{cases} 0 & \text{if } x \in V, \\ +\infty & \text{otherwise.} \end{cases}$$

If $x \in X$ and $\rho > 0$ then

$$B_\rho(x) := \{z \in X : \|z - x\| < \rho\}, \qquad \overline{B}_\rho(x) := \{z \in X : \|z - x\| \le \rho\}.$$

The symbol $(X^*, \|\cdot\|_{X^*})$ indicates the dual space of X while $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X. A function $\Phi : X \to \mathbb{R}$ is called locally Lipschitz continuous provided to every $x \in X$ there correspond a neighborhood V_x of x and a constant $L_x \ge 0$ such that

$$|\Phi(z) - \Phi(w)| \le L_x \|z - w\| \quad \forall\, z, w \in V_x.$$

In this case $\Phi^0(x; z)$ stands for the generalized directional derivative of Φ at the point x along the direction z, i.e.,

$$\Phi^0(x; z) := \limsup_{w \to x,\, t \to 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t}.$$

It is known [6, Proposition 2.1.1] that $\Phi^0(x; \cdot)$ is convex on X while Φ^0 turns out to be upper semi-continuous on $X \times X$. The generalized sub-differential of the function Φ in x, denoted by $\partial\Phi(x)$, is the set

$$\partial\Phi(x) := \left\{x^* \in X^* : \langle x^*, z \rangle \le \Phi^0(x; z)\ \forall\, z \in X\right\}.$$

Proposition 2.1.2 of [6] ensures that $\partial\Phi(x)$ turns out to be nonempty, convex, in addition to weak* compact, and that

$$\Phi^0(x; z) = \max_{x^* \in \partial\Phi(x)} \langle x^*, z \rangle, \quad x, z \in X. \tag{5}$$

Moreover, $x \in X$ is a critical point of Φ when $0 \in \partial\Phi(x)$.

Let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be convex, proper, and lower semi-continuous. The function ψ turns out to be continuous on $\mathrm{int}(D_\psi)$, where, as usual

$$D_\psi := \{x \in X : \ \psi(x) < +\infty\};$$

see for instance [8, Theorem 1.3.2]. If $\partial\psi(x)$ indicates the sub-differential of ψ at the point $x \in X$, namely

$$\partial\psi(x) := \{x^* \in X^* : \psi(z) - \psi(x) \geq \langle x^*, z - x\rangle \ \forall\, z \in X\},$$

and

$$D_{\partial\psi} := \{x \in X : \ \partial\psi(x) \neq \emptyset\}$$

then $D_{\partial\psi} \subseteq D_\psi$. A meaningful special case occurs when $\psi := \delta_C$, being C a nonempty, convex, closed subset of X. Evidently (see, e.g., [13, p. 296]),

$$\partial\delta_C(x) = \{x^* \in X^* : \ \langle x^*, z - x\rangle \leq 0 \ \forall\, z \in C\}, \tag{6}$$

which, in particular, implies $D_{\partial\delta_C} = D_{\delta_C} = C$. The set $\partial\delta_C(x)$ is usually called normal cone to C at x. If

$$(-\partial\delta_C(x)) \cap \partial\Phi(x) \subseteq \{0\} \quad \forall\, x \in \partial C$$

then we say that $\partial\Phi$ turns out to be outwardly directed at ∂C. This clearly rewrites as

$$\forall\, x \in \partial C,\ z^* \in \partial\Phi(x) \setminus \{0\} \quad \text{there exists } z \in C \text{ fulfilling } \langle z^*, z - x\rangle < 0. \tag{7}$$

Throughout the paper, Ω denotes a bounded domain of the real euclidean N-space $(\mathbb{R}^N, |\cdot|)$ with a smooth boundary and $m(\Omega)$ is the N-dimensional Lebesgue measure of Ω. If $q \in [1, +\infty)$ then q' indicates the conjugate exponent of q while $\|\cdot\|_q$ is the usual norm of $L^q(\Omega)$. On $H^1_0(\Omega)$ we introduce the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{1/2}, \quad u \in H^1_0(\Omega).$$

In view of inequality (p$_2$) below, it turns out to be equivalent to the standard one. Write 2^* for the critical exponent of Sobolev's embedding $H^1_0(\Omega) \subseteq L^q(\Omega)$. One has

$$2^* := \begin{cases} 2N/(N-2) & \text{for } N \geq 3, \\ \text{any } \sigma > 1 & \text{otherwise,} \end{cases}$$

and the embedding is compact whenever $1 \leq q < 2^*$. We denote by λ_1 the first eigenvalue of the operator $-\Delta$ in $H^1_0(\Omega)$. The following facts are known.

(p$_1$) $\lambda_1 > 0$.
(p$_2$) $\|u\|_2^2 \leq \frac{1}{\lambda_1}\|u\|^2 \ \ \forall\, u \in H^1_0(\Omega)$ (Poincaré's inequality).
(p$_3$) There exists a positive eigenfunction ϕ_1 corresponding to λ_1 such that $\|\phi_1\|_2 = 1$.

Finally, given $u, v : \Omega \to \mathbb{R}$, the symbol $u \leq v$ means $u(x) \leq v(x)$ a.e. in Ω.

3. A multiple critical point result

From now on, $(X, \|\cdot\|)$ is a real Banach space while C is a non-singleton, convex, closed subset of X. Suppose $\Phi : C \to \mathbb{R}$ admits a locally Lipschitz continuous extension on the whole X. Define, provided $x \in C$,

$$m_C(x) := \inf_{x^* \in \partial\Phi(x)} \sup_{z \in C \cap B_1(x)} \langle x^*, x - z \rangle. \tag{8}$$

Obviously, $m_C(x) \geq 0$ and the function $m_C : C \to [0, +\infty)$ is lower semi-continuous; see [13, Lemma 1]. Moreover, it coincides with the function

$$m(x) := \inf\{\|x^*\|_{X^*} : \; x^* \in \partial\Phi(x)\}, \quad x \in X,$$

introduced by Chang in [2], when $C := X$. According to [13], we say that $x \in C$ is a critical point of Φ on C if $m_C(x) = 0$. The following result is kwnown; see for instance [18, Theorem 5.3].

Lemma 3.1. *Let $x \in C$ be a local minimizer of $\Phi|_C$. Then $m_C(x) = 0$.*

Lemma 3.2. *Assume $x \in C$ and $\varepsilon > 0$ satisfy $\Phi^0(x; z - x) \geq -\varepsilon\|z - x\|$ for all $z \in C$. Then $m_C(x) \leq \varepsilon$.*

Proof. Set, provided $z \in X$,

$$\eta(z) := \Phi^0(x; z - x) + \delta_C(z) + \varepsilon\|z - x\|.$$

The function $\eta : X \to \mathbb{R} \cup \{+\infty\}$ turns out to be nonnegative, convex, and lower semi-continuous. This entails $0 \in \partial\eta(x)$, because $\eta(x) = 0$. Theorem 1.3.6 of [8] ensures that

$$\partial\eta(x) = [\partial_z \Phi^0(x; z - x)]_{z=x} + \partial\delta_C(x) + \varepsilon[\partial_z\|z - x\|]_{z=x}\,,$$

while a simple computation gives

$$[\partial_z \Phi^0(x; z - x)]_{z=x} = \partial\Phi(x)\,, \qquad [\partial_z\|z - x\|]_{z=x} = \{x^* \in X^* : \; \|x^*\|_{X^*} \leq 1\}.$$

Through $0 \in \partial\eta(x)$ we thus obtain $x^* + z^* + \varepsilon w^* = 0$ for some $x^* \in \partial\Phi(x)$, $z^* \in \partial\delta_C(x)$, and $w^* \in X^*$ with $\|w^*\|_{X^*} \leq 1$. So, by (6),

$$\begin{aligned}\langle x^*, x - z\rangle &= -\langle z^*, x - z\rangle - \varepsilon\langle w^*, x - z\rangle \\ &\leq -\varepsilon\langle w^*, x - z\rangle \leq \varepsilon \quad \forall\, z \in C \cap B_1(x),\end{aligned}$$

from which the conclusion follows at once. □

Given a non-decreasing continuous function $h : [0, +\infty) \to [0, +\infty)$ such that

$$\int_0^{+\infty} \frac{1}{1 + h(r)}\, dr = +\infty, \tag{9}$$

we say that Φ fulfills the Palais-Smale condition on C with respect to h when

$(\mathrm{PS})^h_{\Phi,C}$ *Every sequence* $\{x_n\} \subseteq C$ *such that* $\{\Phi(x_n)\}$ *is bounded and*

$$\lim_{n\to+\infty}(1+h(\|x_n\|))m_C(x_n)=0$$

contains a convergent subsequence.

If $h(r) := r$ in $[0, +\infty)$, then $(\mathrm{PS})^h_{\Phi,C}$ reduces to the so-called non-smooth Cerami condition on C; cf. [13, p. 377].

Let K be a compact metric space, let K_0 be a nonempty closed subset of K, and let $\gamma_0 \in C^0(K_0, C)$. Define

$$\Gamma_C := \{\gamma \in C^0(K, C) : \gamma|_{K_0} = \gamma_0\}. \tag{10}$$

Since C is convex, by the generalized Tietze's theorem one has $\Gamma_C \neq \emptyset$. Put

$$c := \inf_{\gamma\in\Gamma_C}\sup_{t\in K}\Phi(\gamma(t)).$$

The following result has been established in [18]; see [18, Theorem 6.3].

Theorem 3.3. *If* $(\mathrm{PS})^h_{\Phi,C}$ *is satisfied and, moreover,* $c > \max_{t\in K_0}\Phi(\gamma_0(t))$ *then* Φ *possesses a critical point* $\hat{x}$ *on* C *such that* $\Phi(\hat{x}) = c$.

Remark 3.4. Likewise the classical setting, i.e., $C := X$ and $\Phi \in C^1(X)$, the question whether there exists a critical point for Φ on C at the level c, which is not a local minimizer of $\Phi|_C$, looks both useful and interesting. Unfortunately, at present, we do not know the answer.

Lemma 3.5. *Suppose* $(\mathrm{PS})^h_{\Phi,C}$ *holds true,* $x_0 \in C$, $\hat{\rho} > 0$, *as well as* $\Phi(x_0) \leq \Phi(x)$ *in* $C \cap B_{\hat{\rho}}(x_0)$. *Then, either*

(i_1) $\Phi(x_0) < \inf\{\Phi(x) : x \in \partial B_\rho(x_0) \cap C\}$ *for some* $\rho \in (0, \hat{\rho})$, *or*

(i_2) *to every* $\rho \in (0, \hat{\rho})$ *there corresponds a local minimizer* $x_\rho \in \partial B_\rho(x_0) \cap C$ *of* $\Phi|_C$ *fulfilling* $\Phi(x_\rho) = \Phi(x_0)$.

Proof. If (i_1) were false then

$$\Phi(x_0) = \inf\{\Phi(x) : x \in \partial B_\rho(x_0) \cap C\} \quad \forall\, \rho \in (0, \hat{\rho}). \tag{11}$$

Pick $\rho \in (0, \hat{\rho})$, $\delta \in (0, \min\{\rho, \hat{\rho} - \rho\})$, and define

$$R := \{x \in X : \rho - \delta \leq \|x - x_0\| \leq \rho + \delta\} \cap C.$$

From (11) it clearly follows

$$\Phi(x_0) = \inf\{\Phi(x) : x \in \partial B_\rho(x_0) \cap C\} = \inf_{x\in R}\Phi(x).$$

Thus, for every $n \in \mathbb{N}$ we can find $z_n \in R$ enjoying the properties

$$\|z_n - x_0\| = \rho, \qquad \Phi(z_n) < \Phi(x_0) + \frac{1}{n^2}. \tag{12}$$

Set $\hat{r} := \rho + \|x_0\|$ as well as

$$\hat{h}(r) := h(r + \hat{r}), \quad r \in [0, +\infty).$$

Theorem 2.1 of [27] provides two sequences $\{r_n\} \subseteq (0, +\infty)$ and $\{x_n\} \subseteq R$ such that $r_n \to 0^+$,

$$\Phi(x_n) \leq \Phi(z_n), \quad \|x_n - z_n\| \leq r_n \quad \forall n \in \mathbb{N}, \tag{13}$$

$$\Phi(x) \geq \Phi(x_n) - \frac{1}{n(1 + \hat{h}(\|x_n - z_n\|))}\|x - x_n\| \quad \forall x \in R,\ n \in \mathbb{N}. \tag{14}$$

Since $\|x_n - z_n\| \geq \|x_n\| - \hat{r}$, inequality (14) easily yields

$$\Phi(x) \geq \Phi(x_n) - \frac{1}{n(1 + h(\|x_n\|))}\|x - x_n\|, \quad x \in R,\ n \in \mathbb{N}. \tag{15}$$

Fix $z \in C$. On account of (12)–(13) one has $x_n + t(z - x_n) \in R$ for any sufficiently small $t > 0$. Hence, through (15) we get

$$\frac{\Phi(x_n + t(z - x_n)) - \Phi(x_n)}{t} \geq -\frac{1}{n(1 + h(\|x_n\|))}\|z - x_n\|.$$

Letting $t \to 0^+$ leads to

$$\Phi^0(x_n; z - x_n) \geq -\frac{1}{n(1 + h(\|x_n\|))}\|z - x_n\| \quad \forall n \in \mathbb{N},\ z \in C. \tag{16}$$

By Lemma 3.2 inequality (16) becomes

$$(1 + h(\|x_n\|))m_C(x_n) \leq \frac{1}{n}, \quad n \in \mathbb{N},$$

while (12)–(13) ensure that the sequence $\{\Phi(x_n)\}$ is bounded. Now, because of $(\mathrm{PS})^h_{\Phi,C}$, we may suppose $x_n \to x_\rho$ for some $x_\rho \in C$. From (12)–(13) again it immediately follows $\|x_\rho - x_0\| = \rho$ as well as $\Phi(x_\rho) = \Phi(x_0)$. Since $\rho < \hat{\rho}$, there exists $\sigma > 0$ such that $B_\sigma(x_\rho) \subseteq B_{\hat{\rho}}(x_0)$, whence $\Phi(x_\rho) \leq \Phi(x)$ for all $x \in C \cap B_\sigma(x_\rho)$. □

We are in a position now to establish the next multiple critical point theorem. It extends [3, Corollaries 1–2] and [10, Corollary 5.12] to non-smooth functions on closed convex sets; see also [12, Theorem 17.2] besides [25, Theorem 11.8].

Theorem 3.6. *Let $(\mathrm{PS})^h_{\Phi,C}$ be satisfied, let $x_0 \in C$ be a local minimizer for $\Phi|_C$, and let $x_1 \in C$ be such that $\Phi(x_1) \leq \Phi(x_0)$. Then, either*

(i_3) *there exists a critical point $\hat{x} \in C$ of Φ on C fulfilling $\Phi(\hat{x}) > \Phi(x_0)$, or*

(i_4) *to every $\rho > 0$ small enough there corresponds a local minimizer $x_\rho \in \partial B_\rho(x_0) \cap C$ for $\Phi|_C$ such that $\Phi(x_\rho) = \Phi(x_0)$.*

Proof. If (i_4) were false then, by Lemma 3.5, we could find a sufficiently small $\rho > 0$ satisfying $\|x_1 - x_0\| > \rho$ as well as

$$\Phi(x_0) < \inf\{\Phi(x) : x \in \partial B_\rho(x_0) \cap C\}.$$

Put $K := [0,1]$, $K_0 := \{0,1\}$, and $\gamma_0(0) := x_0$, $\gamma_0(1) := x_1$. With Γ_C given by (10), observe next that

$$\begin{aligned}\max\{\Phi(x_0), \Phi(x_1)\} &= \Phi(x_0) < \inf\{\Phi(x) : x \in \partial B_\rho(x_0) \cap C\} \\ &\le \max_{t \in K} \Phi(\gamma(t)) \quad \forall\, \gamma \in \Gamma_C,\end{aligned}$$

because $\partial B_\rho(x_0) \cap \gamma(K) \neq \emptyset$, $\gamma \in \Gamma_C$. Hence,

$$\max\{\Phi(x_0), \Phi(x_1)\} < c := \inf_{\gamma \in \Gamma_C} \max_{t \in K} \Phi(\gamma(t)),$$

and Theorem 3.3 can be applied. Thus, there exists a critical point $\hat{x}$ of Φ on C such that $\Phi(\hat{x}) = c$, which shows (i_3). □

4. Critical points on C vs. critical points

Keep the same notation of Section 3. As already pointed out in the Introduction, critical points of Φ on C are not necessarily critical for Φ. Nevertheless, finding critical points on convex closed sets might be sometimes easier, besides of interest concerning applications; see, e.g., [3, 5, 17, 23, 24, 25] and [8, Section 2.2].

A known case is that of Problem (1). The existence of at least two critical points for the energy functional (2) associated with (1) on the positive cone (3) was established in [3]. Is some nontrivial critical point of Φ on C a positive solution to (1)? Without further assumptions on g the answer is no, as the next example shows.

Example 4.1. The Dirichlet problem

$$\begin{cases} u'' + u^2 = 1 & \text{in } (-T, T), \\ u(-T) = u(T) = 0 \end{cases} \tag{17}$$

has no positive solution for any $T > 0$ large enough.
To verify this, let $u \in H_0^1(-T, T)$ be a positive solution of (17) and let

$$M := \sup_{x \in (-T,T)} u(x).$$

By the local uniqueness theorem for ordinary differential equations, the function u turns out to be even around any of its critical points. Therefore, $M = u(0)$, u is increasing in $(-T, 0)$, decreasing in $(0, T)$, and

$$\frac{1}{2} u'(x)^2 + F(u(x)) = F(M) \quad \forall\, x \in (-T, T), \tag{18}$$

where $F(z) := \frac{1}{3}z^3 - z$. From (18) it follows $F(M) \geq F(z)$ in $[0, M]$, whence $M \geq \sqrt{3}$. Moreover, after integration over $[-T, 0]$, one has

$$\int_0^M \frac{1}{\sqrt{F(M) - F(z)}}\, dz = \sqrt{2}T. \tag{19}$$

We next claim that the function

$$\psi(M) := \int_0^M \frac{1}{\sqrt{F(M) - F(z)}}\, dz, \quad M \in [\sqrt{3}, +\infty),$$

turns out to be continuous and bounded in $[\sqrt{3}, +\infty)$. Indeed, pick any $M \geq 2\sqrt{3}$. Since F is increasing in $[1, M/2]$, there exists $\delta > 0$ fulfilling

$$F(M) - F(z) \geq F(M) - F(M/2) = \frac{7}{24}M^3 - \frac{1}{2}M \geq \delta M^3 \quad \forall\, z \in [1, M/2].$$

Consequently,

$$\int_1^{M/2} \frac{1}{\sqrt{F(M) - F(z)}}\, dz \leq \int_1^{M/2} \frac{1}{\sqrt{\delta M^3}}\, dz \leq \frac{1}{2\sqrt{\delta M}}. \tag{20}$$

Through the convexity of F in $[0, +\infty)$ we then get

$$F(M) - F(z) \geq F'(z)(M - z) = (z^2 - 1)(M - z), \quad z \geq M/2,$$

which implies

$$\begin{aligned} \int_{M/2}^M \frac{1}{\sqrt{F(M) - F(z)}}\, dz &\leq \int_{M/2}^M \frac{1}{\sqrt{(z^2 - 1)(M - z)}}\, dz \\ &\leq \frac{1}{\sqrt{(M/2)^2 - 1}} \int_{M/2}^M \frac{1}{\sqrt{M - z}}\, dz = \frac{\sqrt{2M}}{\sqrt{(M/2)^2 - 1}}. \end{aligned} \tag{21}$$

Gathering (20)–(21) together provides

$$\psi(M) \leq \int_0^1 \frac{1}{\sqrt{F(M) - F(z)}}\, dz + \frac{1}{2\sqrt{\delta M}} + \frac{\sqrt{2M}}{\sqrt{(M/2)^2 - 1}} \quad \forall\, M \geq 2\sqrt{3}.$$

Hence, $\psi(M) \to 0$ as $M \to +\infty$, and the assertion follows.
However, the boundedness of ψ contradicts (19) because $T > 0$ was arbitrary.

Remark 4.2. This example also shows that the conclusion of [17, Theorem 7] is false.

Now, the question of finding extra conditions under which critical points for Φ on C turn out to be critical points of Φ spontaneously arises. Theorem 4.5 below shows that a locally Lipschitz continuous version of the so-called Schauder invariance condition fits the purpose when X is reflexive.

Lemma 4.3. *$m_C(x_0) = 0$ if and only if $0 \in \partial\Phi(x_0) + \partial\delta_C(x_0)$.*

Proof. Pick $x_0 \in C$ such that $m_C(x_0) = 0$. Since $\partial\Phi(x_0)$ is weakly* compact in X^* while $x^* \mapsto \sup_{z\in C\cap B_1(x_0)}\langle x^*, x_0 - z\rangle$, $x^* \in X^*$, is obviously weakly* lower semi-continuous, there exists $x_0^* \in \partial\Phi(x_0)$ satisfying

$$m_C(x_0) = \sup_{z\in C\cap B_1(x_0)} \langle x_0^*, x_0 - z\rangle.$$

Therefore,

$$\langle x_0^*, x_0 - z\rangle \le 0 \quad \forall\, z \in C \cap B_1(x_0). \tag{22}$$

Pick $y \in C \setminus B_1(x_0)$ and define

$$z_y := x_0 + \frac{1}{2\|y - x_0\|}(y - x_0).$$

Since $z_y \in C\cap B_1(x_0)$, through (22) we obtain $\langle x_0^*, x_0 - y\rangle \le 0$. As $y \in C\setminus B_1(x_0)$ was arbitrary, one actually has

$$\langle x_0^*, x_0 - z\rangle \le 0 \quad \forall\, z \in C. \tag{23}$$

Therefore, $-x_0^* \in \partial\delta_C(x_0)$, and $0 \in \partial\Phi(x_0) + \partial\delta_C(x_0)$. The vice-versa is analogous. □

Theorem 4.4. *Let $\partial\Phi$ be outwardly directed at ∂C according to* (7). *If $x_0 \in C$ and $m_C(x_0) = 0$ then x_0 is a critical point of Φ.*

Proof. Pick any $x_0^* \in \partial\Phi(x_0) \cap (-\partial\delta_C(x_0))$. Such a point exists by Lemma 4.3. We claim that $x_0^* = 0$. Indeed, from $-x_0^* \in \partial\delta_C(x_0)$ it follows (23), which immediately yields $x_0^* = 0$ when $x_0 \in \operatorname{int}(C)$. Otherwise, $x_0 \in \partial C$. The condition on $\partial\Phi$ forces

$$\partial\Phi(x_0) \cap (-\partial\delta_C(x_0)) \subseteq \{0\},$$

whence $x_0^* = 0$ again. This completes the proof, because $x_0^* \in \partial\Phi(x_0)$. □

Now, let X be reflexive. As usual, we will identify X^{**} with X while $\mathcal{F} : X^* \to 2^X$ will denote the duality map, given by

$$\mathcal{F}(x^*) := \{x \in X : \langle x^*, x\rangle = \|x^*\|_{X^*}^2 = \|x\|^2\} \quad \forall\, x^* \in X^*.$$

The set $\mathcal{F}(x^*)$ turns out to be nonempty, convex, and closed; see, e.g. [9, pp. 311–319]. Define

$$\nabla\Phi(x) := \mathcal{F}(\partial\Phi(x)), \quad x \in X. \tag{24}$$

Clearly, $\nabla\Phi(x)$ depends on the choice of the duality pairing between X and X^* whenever it is compatible with the topology of X (thus, up to equivalent norms). If X is a Hilbert space, the duality paring becomes the scalar product and (24) gives the usual gradient. Write I for the identity operator on X.

Theorem 4.5. *Suppose X is reflexive and, moreover,*

$$(I - \nabla\Phi)(\partial C) \subseteq C. \tag{25}$$

Then every critical point of Φ on C turns out to be critical for Φ.

Proof. Fix $x \in \partial C$, $z^* \in \partial\Phi(x) \setminus \{0\}$, and $y \in \mathcal{F}(z^*)$. Due to (25) one has $x - y = z$ for some $z \in C$. Hence,

$$\langle z^*, z - x\rangle = -\langle z^*, y\rangle = -\|z^*\|^2_{X^*} < 0.$$

As $x \in \partial C$ was arbitrary, this means that $\partial\Phi$ is outwardly directed to ∂C. The conclusion follows from Theorem 4.4. $\square$

Remark 4.6. The well known Schauder invariance condition for a C^1-functional Φ on a Hilbert space X reads as $(I - \Phi')(C) \subseteq C$; see [11, 26, 20, 7, 15]. It has been extended to Banach spaces in [14].

Remark 4.7. The geometric condition on the sets $\partial\Phi(x)$, $x \in \partial C$, required by Theorem 4.4 is more general than (25). Indeed, if $\|\cdot\|$ is differentiable, $\Phi(x) := 2\|x\|^2$, and $C := \overline{B}_1(0)$ then $\partial\Phi(x) = \{4x^*\}$, where $x^* \in X^*$ fulfills

$$\langle x^*, x\rangle = \sup_{\|z^*\|_{X^*} \le 1} \langle z^*, x\rangle = \|x\|.$$

Therefore, $\partial\Phi$ is outwardly directed at ∂C, because

$$\langle 4x^*, 0 - x\rangle = -4\|x\| = -4 < 0.$$

Nevertheless, $\nabla\Phi(x) = \{4x\}$ and so (25) does not hold.

Combining Theorem 4.5 with critical point results on closed convex sets produces new critical point theorems. For instance, through Lemma 3.1 we obtain the following

Theorem 4.8. *Let X be reflexive, let $x \in C$ be a local minimizer for $\Phi|_C$, and let* (25) *be satisfied. Then x is a critical point of Φ.*

5. Solving Problem (4)

To avoid unnecessary technicalities, *for every $x \in \Omega$* will take the place of *for almost every $x \in \Omega$* and the variable x will be omitted when no confusion can arise.

Given a function $J : \Omega \times \mathbb{R} \to \mathbb{R}$, we shall posit the following assumptions.

(h_1) $J(\cdot, z)$ is measurable for all $z \in \mathbb{R}$, $J(x, 0) = 0$ and $J(x, \cdot)$ is locally Lipschitz continuous for every $x \in \Omega$.

Denote by $\partial J(x, z)$ the generalized sub-differential of $J(x, \cdot)$ at the point $z \in \mathbb{R}$.

(h_2) There exist $a_0 > 0$, $p \in [2, 2^*)$ fulfilling

$$\sup_{y \in \partial J(x,z)} |y| \le a_0(1 + |z|^{p-1}) \quad \text{in } \Omega \times \mathbb{R}.$$

A simple computation based on [8, Proposition 1.3.14] shows that (h_1)–(h_2) immediately force

$$|J(x, z)| \le \hat{a}_0(1 + |z|^p), \quad (x, z) \in \Omega \times \mathbb{R}.$$

(h_3) For appropriate $\mu > N(p/2 - 1)$ one has

$$\liminf_{z \to +\infty} \frac{z \inf \partial J(x, z) - 2J(x, z)}{z^\mu} > 0 \quad \text{uniformly in } x \in \Omega.$$

(h_4) $\limsup_{z \to 0^+} \frac{2J(x,z)}{z^2} < \lambda_1$ uniformly with respect to $x \in \Omega$.

(h_5) there exists $t_1 > 0$ such that

$$\int_\Omega J(x, t_1\phi_1(x))dx > \frac{\lambda_1}{2} t_1^2,$$

ϕ_1 being as in (p_3) of Section 2.

Under (h_1)–(h_2), the functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_\Omega J(x, u(x))dx \quad \forall\, u \in H_0^1(\Omega) \tag{26}$$

turns out to be well defined and locally Lipschitz continuous. Put

$$C := \{u \in H_0^1(\Omega) : u \ge 0\}. \tag{27}$$

Lemma 5.1. *If* (h_1)–(h_3) *hold true then* Φ *satisfies* $(\mathrm{PS})^h_{\Phi,C}$ *with* $h(r) := r$ *in* $[0, +\infty)$.

Proof. Pick any sequence $\{u_n\} \subseteq C$ such that $\{\Phi(u_n)\}$ is bounded and

$$\lim_{n \to +\infty} (1 + \|u_n\|) m_C(u_n) = 0.$$

Along a subsequence when necessary, for every $n \in \mathbb{N}$ there exists $u_n^* \in \partial\Phi(u_n)$ fulfilling

$$(1 + \|u_n\|)\langle u_n^*, u_n - v\rangle < \frac{1}{n}, \quad v \in C \cap B_1(u_n). \tag{28}$$

Theorem 1.3.10 of [8] provides $w_n \in L^{p'}(\Omega)$ with the properties

$$w_n(x) \in \partial J(x, u_n(x)) \quad \text{a.e. in } \Omega, \tag{29}$$

$$\langle u_n^*, u_n - v\rangle = \int_\Omega \nabla u_n \cdot \nabla(u_n - v)dx - \int_\Omega w_n(u_n - v)dx. \tag{30}$$

Through (28) and (30) one thus has

$$\int_\Omega \nabla u_n \cdot \nabla(u_n - v)dx - \int_\Omega w_n(u_n - v)dx < \frac{1}{n(1+\|u_n\|)}, \quad v \in C \cap B_1(u_n). \tag{31}$$

We next claim that $\{u_n\} \subseteq H_0^1(\Omega)$ is bounded. Indeed, there is no loss of generality in assuming $\|u_n\| > 0$ for all $n \in \mathbb{N}$. If

$$v := u_n + \frac{1}{2\|u_n\|} u_n$$

then (31) becomes

$$-\|u_n\|^2 + \int_\Omega w_n u_n dx < \varepsilon'_n, \quad \text{where } \varepsilon'_n := \frac{2\|u_n\|}{n(1+\|u_n\|)}.$$

The boundedness of $\{\Phi(u_n)\}$ entails

$$\|u_n\|^2 - 2\int_\Omega J(x, u_n(x))dx \le M_1 \quad \forall\, n \in \mathbb{N}. \tag{32}$$

Hence,

$$\int_\Omega (w_n(x)u_n(x) - 2J(x, u_n(x)))dx < M_1 + \varepsilon'_n, \quad n \in \mathbb{N}. \tag{33}$$

Now, (h_3) leads to

$$z \inf \partial J(x,z) - 2J(x,z) > a_1 z^\mu \quad \forall\, (x,z) \in \Omega \times [\delta, +\infty),$$

with suitable $a_1, \delta > 0$, while from (h_1)–(h_2) we easily infer

$$|yz - 2J(x,z)| \le a_2, \quad (x,z) \in \Omega \times [0,\delta],\ y \in \partial J(x,z).$$

Gathering the above inequalities together yields

$$z \inf \partial J(x,z) - 2J(x,z) > a_1 z^\mu - a_2 \quad \text{in } \Omega \times [0, +\infty).$$

Because of (33) this implies

$$a_1 \int_\Omega |u_n|^\mu dx - a_2\, m(\Omega) \le M_1 + \varepsilon'_n, \quad n \in \mathbb{N},$$

namely

$$\int_\Omega |u_n|^\mu dx \le M_2 \quad \forall\, n \in \mathbb{N}.$$

Suppose $\mu \le p$, which clearly involves no restriction. Fix $k \in (p, 2(1+\mu/N))$ and choose

$$\theta := \frac{2^*(k-\mu)}{k(2^*-\mu)}.$$

Since $0 < \theta < 1$ while

$$\frac{1}{k} = \frac{1-\theta}{\mu} + \frac{\theta}{2^*},$$

the Hölder inequality provides

$$\|u_n\|_k \leq \left(\int_\Omega |u_n|^\mu dx\right)^{\frac{1-\theta}{\mu}} \|u_n\|_{2^*}^\theta \leq M_2^{\frac{1-\theta}{\mu}} a_3 \|u_n\|^\theta \quad \forall\, n \in \mathbb{N}, \tag{34}$$

where the continuous embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$ has been also used. By (h_2) we get

$$|J(x,z)| \leq a_4(1 + |z|^k) \quad \text{in } \Omega \times \mathbb{R}$$

as $k > p$. Therefore, due to (32) and (34),

$$\|u_n\|^2 \leq 2\int_\Omega J(x, u_n(x))dx + M_1 \leq a_5\|u_n\|^{\theta k} + a_6, \quad n \in \mathbb{N},$$

with appropriate $a_5, a_6 > 0$. An elementary computation, which exploits the inequality $Nk < 2(N + \mu)$, shows that $\theta k < 2$, and the claim follows.

Passing to a subsequence if necessary, one has

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega), \qquad u_n \to u \quad \text{in } L^p(\Omega) \tag{35}$$

for some $u \in C$. The point

$$v_n := \begin{cases} u & \text{when } \|u_n - u\| < 1, \\ u_n - \dfrac{u_n - u}{2\|u_n - u\|} & \text{otherwise} \end{cases}$$

belongs to $C \cap B_1(u_n)$, whence, because of (31) and (35),

$$\int_\Omega \nabla u_n \cdot \nabla(u_n - u)dx - \int_\Omega w_n(u_n - u)dx < \frac{a_7}{n} \quad \forall\, n \in \mathbb{N}. \tag{36}$$

Through (h_2), (29), and (35) again, we finally achieve

$$\lim_{n\to+\infty} \int_\Omega w_n(u_n - u)dx = 0.$$

Consequently, by (36),

$$\limsup_{n\to+\infty} \|u_n\|^2 \leq \|u\|^2,$$

which leads to the conclusion. □

Remark 5.2. The above proof has been chiefly adapted from that of [13, Proposition 7].

Theorem 5.3. *Let* (h_1)–(h_5) *be satisfied and let* C *be as in* (27). *Then* (4) *possesses a nontrivial solution.*

Proof. Set $u_1 := t_1\phi_1$. On account of (h_5) one has $\Phi(u_1) < 0$, with Φ given by (26). Hence, owing to (h_1),

$$\max\{\Phi(0), \Phi(u_1)\} = 0.$$

We will show that

$$c := \inf_{\gamma\in\Gamma_C} \sup_{t\in[0,1]} \Phi(\gamma(t)) > 0, \tag{37}$$

where $\Gamma_C := \{\gamma \in C^0([0,1], C) : \gamma(0) = 0,\ \gamma(1) = u_1\}$. Pick any $q \in (p, 2^*)$. From (h_4) it follows

$$\frac{J(x,z)}{z^2} < a_8 < \frac{\lambda_1}{2}, \quad (x,z) \in \Omega \times (0,\delta], \tag{38}$$

with suitable $a_8, \delta > 0$, while (h_2) easily yields

$$|J(x,z)| \le a_9 z^p \le a_{10} z^q \quad \text{in } \Omega \times [\delta, +\infty). \tag{39}$$

Gathering (38)–(39) together we obtain

$$J(x,z) \le a_8 z^2 + a_{10} z^q, \quad (x,z) \in \Omega \times [0,+\infty).$$

Therefore, thanks to (p_2) and Sobolev's inequality,

$$\begin{aligned}
\Phi(u) &\ge \frac{1}{2}\|u\|^2 - a_8\|u\|_2^2 - a_{10}\|u\|_q^q \\
&\ge \left(\frac{1}{2} - \frac{a_8}{\lambda_1}\right)\|u\|^2 - a_{10}a_{11}\|u\|^q \\
&= \|u\|^2\left[\left(\frac{1}{2} - \frac{a_8}{\lambda_1}\right) - a_{10}a_{11}\|u\|^{q-2}\right] \quad \forall\, u \in C.
\end{aligned}$$

The choice of a_8 evidently forces

$$\left(\frac{1}{2} - \frac{a_8}{\lambda_1}\right) - a_{10}a_{11}\rho^{q-2} > 0$$

for any $\rho > 0$ small enough, which actually means

$$\inf_{u\in\partial B_\rho(0)\cap C} \Phi(u) > 0.$$

This proves (37). Now, Theorem 3.3 can be applied, and there exists $u \in C$ such that $\Phi(u) = c$ and $m_C(u) = 0$. One clearly has $u \ne 0$, because $c > 0$. Using [18, Theorem 4.1] gives

$$\Phi^0(u; v-u) \ge 0 \quad \forall\, v \in C.$$

Since, due to inequality (2) at p. 77 of [6] and Property 4 in [2],

$$\begin{aligned}\Phi^0(u; v-u) &\leq \int_\Omega \nabla u \cdot \nabla(v-u)dx + \int_\Omega (-J)^0(x, u(x); v(x)-u(x))dx \\ &= \int_\Omega \nabla u \cdot \nabla(v-u)dx + \int_\Omega J^0(x, u(x); u(x)-v(x))dx, \quad v \in C,\end{aligned}$$

the conclusion follows. □

Example 5.4. The nonlinearity J arising from Problem (1) is

$$J(x, z) := \frac{1}{3}z^3 - g(x)z, \quad (x, z) \in \Omega \times \mathbb{R}.$$

An elementary verification shows that it fulfills (h_1)–(h_5). So, by the above result, there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that $u \geq 0$ and

$$\int_\Omega (u^2 - g)(v-u)dx \leq \int_\Omega \nabla u \cdot \nabla(v-u)dx \quad \forall\, v \in H_0^1(\Omega),\ v \geq 0;$$

see also [3, Theorem 2]. However, without further conditions on the data, (1) may not have positive solutions; cf. Example 4.1.

Let $\psi \in H^1(\Omega)$ be such that $\psi \geq 0$ and let

$$C := \{u \in H_0^1(\Omega) : 0 \leq u \leq \psi\}. \tag{40}$$

It is evident that

Lemma 5.5. *Under assumptions* (h_1)–(h_2) *the functional* Φ *given by* (26) *turns out to be bounded below on* C. *If, moreover,* (h_5) *holds and* $t_1\phi_1 \leq \psi$ *then* $\inf_{u\in C} \Phi(u) < 0$.

Lemma 5.6. *Let* (h_1)–(h_2) *be satisfied. Then* Φ *complies with* $(\mathrm{PS})^h_{\Phi,C}$, *where* $h(r) := r$ *in* $[0, +\infty)$.

Proof. Pick any sequence $\{u_n\} \subseteq C$ such that $\{\Phi(u_n)\}$ is bounded and

$$\lim_{n\to+\infty} (1 + \|u_n\|)m_C(u_n) = 0.$$

The boundedness of $\{\Phi(u_n)\}$, besides (h_2), entail

$$\|u_n\|^2 \leq M_3 + 2\int_\Omega J(x, u_n(x))dx \leq M_3 + 2\hat{a}_0 \int_\Omega (1 + \psi^p)dx \quad \forall\, n \in \mathbb{N}.$$

Hence, $\{u_n\} \subseteq H_0^1(\Omega)$ turns out to be bounded. Now, the proof goes on exactly like that of Lemma 5.1. □

Theorem 5.7. *If* (h_1), (h_2), (h_4), (h_5) *hold true,* C *is as in* (40), *and* $t_1\phi_1 \leq \psi$ *then* (4) *possesses at least two nontrivial solutions.*

Proof. Thanks to Lemmas 5.5–5.6, Theorem 5.6 of [18] provides $u_1 \in C \setminus \{0\}$ fulfilling

$$\Phi(u_1) = \inf_{u \in C} \Phi(u) < 0.$$

The same arguments exploited in the proof of Theorem 5.3 ensure here that zero turns out to be a local minimizer for $\Phi|_C$. Since $\Phi(u_1) < 0 = \Phi(0)$, Theorem 3.6 can be applied, and we obtain a critical point $u_2 \in C \setminus \{0, u_1\}$ of Φ on C. At this point, reasoning as in the above-mentioned proof, it is easily seen that u_i satisfies (4), $i = 1, 2$. □

Remark 5.8. Theorem 5.7 generalizes Theorem 3 of [3].

Acknowledgements. Work performed under the auspices of GNAMPA of INDAM.

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