Journal of Convex Analysis
Volume 22 (2015), No. 4, 1125–1134

Brunn-Minkowski Inequality for the 1-Riesz Capacity and Level Set Convexity for the 1/2-Laplacian

M. Novaga
Dipartmento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
novaga@dm.unipi.it

B. Ruffini
Institut Fourier, 100 rue des maths, BP 74, 38402 St Martin d'Hères cedex, France
berardo.ruffini@ujf-grenoble.fr

Received: September 15, 2014
Revised manuscript received: December 16, 2014
Accepted: February 2, 2015

We prove that that the 1-Riesz capacity satisfies a Brunn-Minkowski inequality, and that the capacitary function of the 1/2-Laplacian is level set convex.

Keywords: Fractional Laplacian, Brunn-Minkowski inequality, level set convexity, Riesz capacity

1. Introduction

In this paper we consider the following problem

$$\begin{cases} (-\Delta)^s u = 0 & \text{on } \mathbb{R}^N \setminus E \\ u = 1 & \text{on } E \\ \lim_{|x|\to+\infty} u(x) = 0 \end{cases} \tag{1}$$

where $N \geq 2$, $s \in (0, N/2)$, and $(-\Delta)^s$ stands for the s-fractional Laplacian, defined as the unique pseudo-differential operator $(-\Delta)^s : \mathcal{S} \mapsto L^2(\mathbb{R}^N)$, being $\mathcal{S}$ the Schwartz space of functions with fast decay to 0 at infinity, such that

$$\mathcal{F}\left((-\Delta)^s f\right)(\xi) = |\xi|^{2s}\mathcal{F}(f)(\xi) \quad \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ denotes the Fourier transform. We refer to the guide [12, Section 3] for more details on the subject. A quantity strictly related to Problem (1) is the so-called *Riesz potential energy* of a set E, defined as

$$I_\alpha(E) = \inf_{\mu(E)=1} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}} \quad \alpha \in (0, N). \tag{2}$$

ISSN 0944-6532 / $ 2.50

It is possible to prove (see [19]) that if E is a compact set, then the infimum in the definition of $I_\alpha(E)$ is achieved by a unique Radon measure μ supported on the boundary of E if $\alpha \leq N-2$, and with support equal to the whole E if $\alpha \in (N-2, N)$. If μ is the optimal measure for the set E, we define the *Riesz potential* of E as

$$u(x) = \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-\alpha}}, \tag{3}$$

so that

$$I_\alpha(E) = \int_{\mathbb{R}^N} u(x) d\mu(x).$$

It is not difficult to check (see [19, 16]) that the potential u satisfies, in distributional sense, the equation

$$(-\Delta)^{\frac{\alpha}{2}} u = c(\alpha, N)\, \mu,$$

where $c(\alpha, N)$ is a positive constant, and that $u = I_\alpha(E)$ on E. In particular, if $s = \alpha/2$, then $u_E = u/I_{2s}(E)$ is the unique solution of Problem (1).

Following [19], we define the *α-Riesz capacity* of a set E as

$$\mathrm{Cap}_\alpha(E) = \frac{1}{I_\alpha(E)}. \tag{4}$$

We point out that this is not the only concept of capacity present in literature. Indeed, another one is given by the p-capacity of a set E, which for $N \geq 3$ and $p > 1$ is defined as

$$\mathcal{C}_p(E) = \min \left\{ \int_{\mathbb{R}^N} |\nabla \varphi|^p : \varphi \in C^1_c(\mathbb{R}^N, [0,1]),\ \varphi \geq \chi_E \right\} \tag{5}$$

where χ_E is the characteristic function of the set E. It is possible to prove that, if E is a compact set, then the minimum in (5) is achieved by a function u satisfying

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0 & \text{on } \mathbb{R}^N \setminus E \\ u = 1 & \text{on } E \\ \lim_{|x| \to +\infty} u(x) = 0. \end{cases} \tag{6}$$

Such a function u is usually called *capacitary function* of the set E. It is worth noticing that 2-Riesz capacity and the 2-capacity coincide on compact sets. We refer to [20, Section 11.15] for a discussion on this topic.

In a series of works (see for instance [7, 11, 18] and the monograph [17]) it has been proved that the solutions of (6) are level set convex provided E is a convex body, that is, a compact convex set with non-empty interior. Moreover, in [2] (and in [10] in a more general setting) it has been proved that the 2-capacity satisfies a suitable version of the Brunn-Minkowski inequality: given two convex bodies K_0 and K_1 in $\mathbb{R}^N$, for any $\lambda \in [0,1]$ it holds

$$\mathcal{C}_2(\lambda K_1 + (1-\lambda) K_0)^{\frac{1}{N-2}} \geq \lambda\, \mathcal{C}_2(K_1)^{\frac{1}{N-2}} + (1-\lambda)\, \mathcal{C}_2(K_0)^{\frac{1}{N-2}}.$$

We refer to [21, 15] for comprehensive surveys on the Brunn-Minkowski inequality.

The main purpose of this paper is to show analogous results in the fractional setting $\alpha = 1$, that is, $s = 1/2$ in Problem (1). More precisely, we shall prove the following result.

Theorem 1.1. *Let $K \subset \mathbb{R}^N$ be a convex body and let u be the solution of Problem (1) with $s = 1/2$ and $E = K$. Then*

(*i*) *u is continuous and level set convex, that is, for every $c \in \mathbb{R}$ the level set $\{u > c\}$ is convex;*

(*ii*) *the 1-Riesz capacity $\mathrm{Cap}_1(K)$ satisfies the following Brunn-Minkowski inequality: for any couple of convex bodies K_0 and K_1 and for any $\lambda \in [0, 1]$ we have*

$$\mathrm{Cap}_1(\lambda K_1 + (1-\lambda)K_0)^{\frac{1}{N-1}} \geq \lambda \mathrm{Cap}_1(K_1)^{\frac{1}{N-1}} + (1-\lambda)\mathrm{Cap}_1(K_0)^{\frac{1}{N-1}}. \quad (7)$$

The strategy of the proof of Theorem 1.1 is the following. First, for a continuous and bounded function $u : \mathbb{R}^N \to \mathbb{R}$, we consider the (unique) bounded solution (see [14]) of the problem

$$\begin{cases} -\Delta_{(x,t)} U = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ U(\cdot, 0) = u(\cdot) & \text{in } \mathbb{R}^N. \end{cases} \quad (8)$$

It holds true the following classical result (see for instance [9]).

Lemma 1.2. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be smooth and bounded, and let $U : \mathbb{R}^N \times [0, +\infty)$ be the solution of Problem (8). There holds*

$$\lim_{t \to 0^+} \partial_t U(x, t) = (-\Delta)^{\frac{1}{2}} u(x) \quad \textit{for any } x \in \mathbb{R}^N. \quad (9)$$

Thanks to Lemma 1.2 we are able to show that the (unique) solution of Problem (1) with $s = 1/2$ is the trace of the capacitary function U in $\mathbb{R}^{N+1}$ of $K \subset \mathbb{R}^{N+1}$. This allows us to exploit results which hold for capacitary functions of a convex set contained in [10, 11]. Eventually we show that the results obtained in the $(N + 1)$-dimensional setting for U hold true as well for u.

We point out that our results do not extend straightfowardly to solutions of (1) with a general s. Indeed, for $s \neq 1/2$ the extension function U in Lemma 1.2 satisfies a more complicated equation (see [9]), to which the results in [10, 11] do not directly apply.

Eventually, in Section 3 we provide an application of Theorem 1.1 and we state some open problems.

2. Proof of the main result

This section is devoted to the proof of Theorem 1.1. We start by an approximation result, which extends most of the results in [11, 10], originally aimed to convex bodies, to general convex sets with positive capacity.

Lemma 2.1. *Let K be a compact convex set with positive 2-capacity and let, for $\varepsilon > 0$, $K_\varepsilon = \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) \leq \varepsilon\}$. Letting u_ε and u be the capacitary functions of K_ε and K respectively, we have that u_ε converges uniformly to u as $\varepsilon \to 0$. Moreover we have that the level sets $\{u_\varepsilon > s\}$ converge to $\{u > s\}$ for any $1 > s > 0$, with respect to the Hausdorff distance.*

Proof. Let us prove that $u_\varepsilon \to u$ uniformly as $\varepsilon \to 0$. Since $u_\varepsilon - u$ is a harmonic function on $\mathbb{R}^N \setminus K_\varepsilon$, we have that

$$\sup_{\mathbb{R}^N \setminus K_\varepsilon} |u_\varepsilon - u| \leq \sup_{\partial K_\varepsilon} |u_\varepsilon - u| \leq 1 - \min_{\partial K_\varepsilon} u. \tag{10}$$

Since ∂K_ε Hausdorff converge to ∂K, we get that the right-hand side of (10) converges to 0 as $\varepsilon \to 0$. To prove that the level sets $\{u_\varepsilon > s\}$ converge to $\{u > s\}$ for any $1 > s > 0$, with respect to the Hausdorff distance, we begin by showing that the following equality

$$\overline{\{v > s\}} = \{v \geq s\} \tag{11}$$

holds true for $v = u$ or $v = u_\varepsilon$, $\varepsilon > 0$. Indeed, let $x \in \overline{\{v > s\}}$, with $v = u$ or $v = u_\varepsilon$. Then there exists $x_k \to x$ such that $v(x_k) > s$. Thus

$$v(x) = \lim_{k \to \infty} v(x_k) \geq s,$$

and so $x \in \{v \geq s\}$.

Suppose now that $v(x) \geq s$ but $x \notin \overline{\{v > s\}}$. Notice that in this case $v(x) = s$. Suppose moreover that there exists an open neighborhood A of x such that $A \cap \{v > s\} = \emptyset$, so that $v \leq s$ on A. In this case we get that x is a local maximum in A, and v is harmonic in a neighborhood of A. This leads to a contradiction thanks to the maximum principle for hamonic functions. To conclude the proof of (11), we just notice that if such an A does not exist, then x is an adjacency point for $\{v > s\}$ so that it belongs to $\overline{\{v > s\}}$.

Suppose by contradiction that there exist $c > 0$ and a sequence $x_\varepsilon \in \{u_\varepsilon > s\}$ such that $\operatorname{dist}(x_n, \{u > s\}) \geq c > 0$. Recalling that $K_{\varepsilon_0} \subset K_{\varepsilon_1}$ if $\varepsilon_0 < \varepsilon_1$, and applying again the maximum principle, it is easy to show that $\{u_\varepsilon \geq s\}$ is a family of uniformly bounded compact sets. Thus, there exists an $x \in \mathbb{R}^N$ such that x_ε converges to x, up to extracting a (not relabeled) subsequence. By uniform convergence we have that

$$|u_\varepsilon(x_\varepsilon) - u(x)| \leq |u_\varepsilon(x_\varepsilon) - u(x_\varepsilon)| + |u(x_\varepsilon) - u(x)| \to 0$$

as $\varepsilon \to 0$. Thus, for any $\delta > 0$ there exists ε such that

$$u(x) \geq u_\varepsilon(x_\varepsilon) - \delta \geq s - \delta,$$

whence $u(x) \geq s$. But thanks to (11) we have that

$$0 < c \leq \mathrm{dist}(x, \{u > s\}) = \mathrm{dist}(x, \overline{\{u > s\}}) = \mathrm{dist}(x, \{u \geq s\}) = 0.$$

This is a contradiction and thus the proof is concluded. $\square$

Remark 2.2. Notice that a compact convex set has positive 2-capacity if and only if its $\mathcal{H}^{N-1}$-measure is non-zero (see for instance [13]). In particular if K is a convex body of $\mathbb{R}^N$, then, although its $(N+1)$-Lebesgue measure is 0, K has positive capacity in $\mathbb{R}^{N+1}$.

To prove Theorem 1.1, we wish to apply Lemma 1.2 to the function $u = u_K$. Since u_K is not a smooth function (being non-regular on the boundary of K) for our purposes we need a weaker version of Lemma 1.2.

Lemma 2.3. *Let $u : \mathbb{R}^N \to [0, +\infty)$ be a continuous, bounded function and let U be the extension of u in the sense of Lemma 1.2. Suppose that u is of class C^1 in a neighbourhood of $x \in \mathbb{R}^N$. Then the partial derivative $\partial_t U$ of U with respect to the last coordinate is well defined at the point $(x, 0)$, and it holds*

$$\partial_t U(x, 0) = (-\Delta)^{1/2} u(x).$$

Proof. For $\varepsilon > 0$ let $u_\varepsilon = u * \rho_\varepsilon$ where ρ_ε is a mollifying smooth kernel and U_ε is the extension of u_ε in the sense of Lemma 1.2. Then, since u_ε is a regular bounded function, by Lemma 1.2 we have $\partial_t U_\varepsilon(x, 0) = (-\Delta)^{1/2} u_\varepsilon(x)$ for every $x \in \mathbb{R}^N$. If u is C^1-regular in a neighbourhood A of x, then so is U on $A \times [0, \infty)$ and it holds $\partial_t U_\varepsilon(x, 0) \to \partial_t U(x, 0)$ as $\varepsilon \to 0$. Hence, in order to conclude, we only need to check that $(-\Delta)^{1/2} u_\varepsilon(x)$ converges to $(-\Delta)^{1/2} u(x)$ as $\varepsilon \to 0$. To do this, it is sufficient to show that it holds $(-\Delta)^{1/2}(u * \rho_\varepsilon)(x) = ((-\Delta)^{1/2} u) * \rho_\varepsilon(x)$. This latter fact is true since, recalling that u is bounded, we have

$$\begin{aligned}\mathcal{F}^{-1}\left(\mathcal{F}((-\Delta)^{1/2}(u * \rho_\varepsilon))\right)(x) &= \mathcal{F}^{-1}\left(|\xi|^{1/2} \mathcal{F}(u) \mathcal{F}(\rho_\varepsilon)\right)(x) \\ &= (-\Delta)^{1/2} u * \rho_\varepsilon(x).\end{aligned}$$

$\square$

We recall the well known fact that the capacitary function of a compact set $K \subset \mathbb{R}^N$ of positive capacity is continuous on $\mathbb{R}^N$. We offer a simple proof of this fact for the reader's convenience.

Lemma 2.4. *Let $K \subset \mathbb{R}^N$ be a compact set of strictly positive capacity and let u be its capacitary function. Then u is continuous on $\mathbb{R}^N$.*

Proof. Since u is harmonic on $\mathbb{R}^N \setminus K$ and it is constantly equal to 1 on K, we only have to show that if $x \in \partial K$ then $u(y) \to 1$ as $y \to x$. To do this, we recall that u is a lower semicontinuous function, which follows, for instance, from the fact that u is the convolution of a positive kernel and a non-negative Radon measure (see for instance [19, p. 59]). Hence, since K is closed, we get

$$1 = u(x) \leq \liminf_{y \to x} u(y) \leq \limsup_{y \to x} u(y) \leq 1,$$

where the last inequality is due to the fact that, thanks to the maximum principle, $0 \leq u \leq 1$. □

Proof of Theorem 1.1. Let us prove claim (i). We begin by showing that u is a continuous function. Indeed, let U be the capacitary function of K in $\mathbb{R}^{N+1}$ (in this setting, K is contained in the hyperspace $\{x_{N+1} = 0\}$), that is, let U be the solution of

$$\begin{cases} -\Delta_{(x,t)} U = 0 & \text{in } \mathbb{R}^{N+1} \\ U(x,0) = 1 & x \in K \\ \lim_{|(x,t)| \to +\infty} U(x,t) = 0. \end{cases} \tag{12}$$

Then, since K is symmetric with respect to the hyperplane $\{x_{N+1} = 0\} = \mathbb{R}^N$, also U is symmetric with respect to the same hyperplane, as can be shown by applying a suitable version of the Pólya-Szegö inequality for the Steiner symmetrization (see for instance [3, 5]). Let $v(x) = U(x,0)$. Notice that v is a continuous function, since U is continuous, being the capacitary function of a compact set of positive capacity (and thanks to Lemma 2.4).

Let us prove that v is the solution of (1). It is clear that $\lim_{|x| \to \infty} v(x) = 0$ and that $v(x) = 1$ if $x \in K$. Moreover we have that v is bounded and regular on $\mathbb{R}^N \setminus K$ (being so U) thus we can apply Lemma 2.3 to get that for every $x \in \mathbb{R}^N \setminus K$ we have $(-\Delta)^{s/2} v(x) = \partial_t U(x,0) = 0$, and thus v solves (1). By uniqueness it then follows that $u = v$ is a continuous function.

We now prove that u is level set convex. Notice first that, for any $c \in \mathbb{R}$ we have

$$\{u \geq c\} = \{(x,t) : U(x,t) \geq c\} \cap \{t = 0\}.$$

In particular, the claim is proved if we show that U is level set convex.

We recall from [10] that the capacitary function of a convex body is always level set convex. Let now $K_\varepsilon = \{x : \operatorname{dist}(x,K) \leq \varepsilon\}$ and let u_ε be the capacitary function of K_ε. From Lemma 2.1 we know that, for any $s \in (0,1)$ the level set $\{U > s\}$ is the Hausdorff limit of the level sets $\{u_\varepsilon > s\}$, which are convex by the result in [10]. It follows that U is level set convex, and this concludes the proof of (i).

To prove (ii) we start by noticing that the 1-Riesz capacity is a $(1-N)$-homogeneous functional, hence inequality (7) can be equivalently stated (see for instance [2]) by requiring that, for any couple of convex sets K_0 and K_1 and for any $\lambda \in [0,1]$, the inequality

$$\operatorname{Cap}_1(\lambda K_1 + (1-\lambda) K_0) \geq \min\{\operatorname{Cap}_1(K_0), \operatorname{Cap}_1(K_1)\} \tag{13}$$

holds true.

We divide the proof of (13) into two steps.

Step 1. We characterize the 1-Riesz capacity of a convex set K as the behaviour at infinity of the solution of the following PDE

$$\begin{cases} (-\Delta)^{1/2} u_K = 0 & \text{in } \mathbb{R}^N \setminus K \\ u_K = 1 & \text{in } K \\ \lim_{|x|\to\infty} |x|^{N-1} u_K(x) = \mathrm{Cap}_1(K). \end{cases}$$

We recall that, if μ_K is the optimal measure for the minimum problem in (2), then the function

$$u(x) = \int_{\mathbb{R}^N} \frac{d\mu_K(y)}{|x-y|^{N-1}}$$

is harmonic on $\mathbb{R}^N \setminus K$ and is constantly equal to $I_1(K)$ on K (see for instance [16]). Moreover the optimal measure μ_K is supported on K, so that $|x|^{N-1}u(x) \to \mu_K(K) = 1$ as $|x| \to \infty$. The claim follows by letting $u_K = u/I_1(K)$.

Step 2. Let $K_\lambda = \lambda K_1 + (1-\lambda)K_0$ and $u_\lambda = u_{K_\lambda}$. We want to prove that

$$u_\lambda(x) \geq \min\{u_0(x), u_1(x)\}$$

for any $x \in \mathbb{R}^N$. To this aim we define the auxiliary function (first introduced in [2])

$$\widetilde{u}_\lambda(x) = \sup\left\{\min\{u_0(x_0), u_1(x_1)\} : x_0, x_1 \in \mathbb{R}^N,\ x = \lambda x_1 + (1-\lambda)x_0\right\},$$

and we notice that the claim follows if we show that $u_\lambda \geq \widetilde{u}_\lambda$. An equivalent formulation of this statement is to require that for any $s > 0$ we have

$$\{\widetilde{u}_\lambda > s\} \subseteq \{v_\lambda > s\}. \tag{14}$$

A direct consequence of the definition of $\widetilde{u}_\lambda$ is that

$$\{\widetilde{u}_\lambda > s\} = \lambda\{u_1 > s\} + (1-\lambda)\{u_0 > s\}.$$

For all $\lambda \in [0,1]$, we let U_λ be the harmonic extension of u_λ on $\mathbb{R}^N \times [0,\infty)$, which solves

$$\begin{cases} -\Delta_{(x,t)} U_\lambda = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ U_\lambda(x,0) = u_\lambda(x) & \text{in } \mathbb{R}^N \times \{0\} \\ \lim_{|(x,t)|\to\infty} U_\lambda(x,t) = 0. \end{cases} \tag{15}$$

Notice that U_λ is the capacitary function of K_λ in $\mathbb{R}^{N+1}$, restricted to $\mathbb{R}^N \times [0,+\infty)$. Letting $H = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} : t = 0\}$, for any $\lambda \in [0,1]$ and $s \in \mathbb{R}$ we have

$$\{U_\lambda > s\} \cap H = \{u_\lambda > s\}.$$

Letting also

$$\begin{aligned} &\widetilde{U}_\lambda(x,t) \\ &= \sup\{\min\{U_0(x_0,t_0), U_1(x_1,t_1)\} : (x,t) = \lambda(x_1,t_1) + (1-\lambda)(x_0,t_0)\}, \end{aligned} \tag{16}$$

as above we have that

$$\{\widetilde{U}_\lambda > s\} = \lambda\{U_1 > s\} + (1-\lambda)\{U_0 > s\}.$$

By applying again Lemma 2.1 to the sequences $K_0^\varepsilon = K_0 + B(\varepsilon)$ and $K_1^\varepsilon = K_1 + B(\varepsilon)$, we get that the corresponding capacitary functions, denoted respectively as U_0^ε and U_1^ε, converge uniformly to U_0 and U_1 in $\mathbb{R}^N$, and that $\widetilde{U}_\lambda^\varepsilon$, defined as in (16), converges uniformly to $\widetilde{U}_\lambda$ on $\mathbb{R}^N \times [0,+\infty)$.

Since $\widetilde{U}_\lambda^\varepsilon(x,t) \le U_\lambda^\varepsilon(x,t)$ for any $(x,t) \in \mathbb{R}^N \times [0,+\infty)$, as shown in [10, pages 474–476], we have that $\widetilde{U}_\lambda(x,t) \le U_\lambda(x,t)$. As a consequence, we get

$$\begin{aligned} \{u_\lambda > s\} &= \{U_\lambda > s\} \cap H \supseteq \{\widetilde{U}_\lambda > s\} \cap H \\ &= [\lambda\{U_1 > s\} + (1-\lambda)\{U_0 > s\}] \cap H \\ &\supseteq \lambda\{U_1 > s\} \cap H + (1-\lambda)\{U_0 > s\} \cap H \\ &= \lambda\{u_1 > s\} + (1-\lambda)\{u_0 > s\} \end{aligned}$$

for any $s > 0$, which is the claim of *Step 2*.

We conclude by observing that inequality (13) follows immediately, by putting together *Step 1* and *Step 2*. This concludes the proof of (*ii*), and of the theorem. □

Remark 2.5. The fact that the solution of (1) is a continuous function can be proved without using the extension problem thanks to the formulation (3) which entails that u is a superharmonic function and the Evans Theorem (see for instance [6, Theorem 1]). We used a less direct approach to show at once the fact that u can be seen as the trace of the capacitary function U of K in $\mathbb{R}^{N+1}$.

Remark 2.6. The equality case in the Brunn-Minkowski inequality (7) is not easy to address by means of our techniques. The problem is not immediate even in the case of the 2-capacity. In that case it has been studied in [8, 10].

3. Applications and open problems

In this section we state a corollary of Theorem 1.1. To do this we introduce some tools which arise in the study of convex bodies. The *support function* of a convex body $K \subset \mathbb{R}^N$ is defined on the unit sphere centred at the origin $\partial B(1)$ as

$$h_K(\nu) = \sup_{x \in \partial K} \langle x, \nu \rangle.$$

The *mean width* of a convex body K is

$$M(K) = \frac{2}{\mathcal{H}^{N-1}(\partial B(1))} \int_{\partial B(1)} h_K(\nu)\, d\mathcal{H}^{N-1}(\nu).$$

We refer to [21] for a complete reference on the subject. We observe that, if $N = 2$, then $M(K)$ coincides up to a constant with the perimeter $P(K)$ of K (see [4]).

We denote by $\mathcal{K}_N$ the set of convex bodies of $\mathbb{R}^N$ and we set

$$\mathcal{K}_{N,c} = \{K \in \mathcal{K}_N,\ M(K) = c\}.$$

The following result has been proved in [4, 1].

Theorem 3.1. *Let $F : \mathcal{K}_N \to [0,\infty)$ be a q-homogeneous functional which satisfies the Brunn-Minkowski inequality, that is, such that $F(K+L)^{1/q} \geq F(K)^{1/q} + F(L)^{1/q}$ for any $K, L \in \mathcal{K}_N$. Then the ball is the unique solution of the problem*

$$\min_{K\in\mathcal{K}_N} \frac{M(K)}{F^{1/q}(K)}\,. \tag{17}$$

An immediate consequence of Theorem 3.1, Theorem 1.1 and Definition (4) is the following result.

Corollary 3.2. *The minimum of I_1 on the set $\mathcal{K}_{N,c}$ is achieved by the ball of mean width c. In particular, if $N = 2$, the ball of radius r solves the isoperimetric type problem*

$$\min_{K\in\mathcal{K}_2, P(K)=2\pi r} I_1(K). \tag{18}$$

Motivated by Theorem 1.1 and Corollary 3.2 we conclude the paper with the following conjecture:

Conjecture 3.3. *For any $N \geq 2$ and $\alpha \in (0,N)$, the α-Riesz capacity $\mathrm{Cap}_\alpha(K)$ satisfies the following Brunn-Minkowski inequality:*
for any couple of convex bodies K_0 and K_1 and for any $\lambda \in [0,1]$ we have

$$\mathrm{Cap}_\alpha(\lambda K_1 + (1-\lambda)K_0)^{\frac{1}{N-\alpha}} \geq \lambda \mathrm{Cap}_\alpha(K_1)^{\frac{1}{N-\alpha}} + (1-\lambda)\mathrm{Cap}_\alpha(K_0)^{\frac{1}{N-\alpha}}. \tag{19}$$

In particular, for any $\alpha \in (0,2)$ the ball of radius r is the unique solution of the isoperimetric type problem

$$\min_{K\in\mathcal{K}_2, P(K)=2\pi r} I_\alpha(K). \tag{20}$$

Acknowledgements. The authors wish to thank G. Buttazzo and D. Bucur for useful discussions on the subject of this paper. The authors were partially supported by the Project ANR-12-BS01-0014-01 *Geometrya*, and by the GNAMPA Project 2014 *Analisi puntuale ed asintotica di energie di tipo non locale collegate a modelli della fisica*.

References

[1] C. Bianchini, P. Salani: Concavity properties for elliptic free boundary problems, Nonlinear Anal. 71(10) (2009) 4461–4470; 72(7–8) (2010) 3551.

[2] C. Borell: Capacitary inequalities of the Brunn-Minkowski type, Math. Ann. 263 (1984) 179–184.

[3] F. Brock: Weighted Dirichlet-type inequalities for Steiner symmetrization, Calc. Var. Partial Differ. Equ. 8 (1999) 15–25.

[4] D. Bucur, I. Fragalà, J. Lamboley: Optimal convex shapes for concave functionals, ESAIM, Control Optim. Calc. Var. 18 (2012) 693–711.

[5] A. Burchard: Steiner symmetrization is continuous in $W^{1,p}$, Geom. Funct. Anal. 7 (1997) 823–860.

[6] L. A. Caffarelli: The obstacle problem revisited, J. Fourier Anal. Appl. 4(4) (1998) 383–402.

[7] L. A. Caffarelli, J. Spruck: Convexity of solutions to some classical variational problems, Commun. Partial Differ. Equations 7 (1982) 1337–1379.

[8] L. A. Caffarelli, D. Jerison, E. H. Lieb: On the case of equality in the Brunn-Minkowski inequality for capacity, Adv. Math. 117 (1996) 193–207.

[9] L. A. Caffarelli, L. Silvestre: An extension problem related to the fractional Laplacian, Commun. Partial Differ. Equations 32 (2007) 1245–1260.

[10] A. Colesanti, P. Salani: The Brunn-Minkowski inequality for p-capacity of convex bodies, Math. Ann. 327 (2003) 459–479.

[11] A. Colesanti, P. Salani: Quasi-concave envelope of a function and convexity of level sets of solutions to elliptic equations, Math. Nachr. 258 (2003) 3–15.

[12] E. Di Nezza, G. Palatucci, E. Valdinoci: Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521–573.

[13] L. C. Evans, R. F. Gariepy: Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton (1992).

[14] G. Folland: Lectures on Partial Differential Equations, Princeton Univ. Press, Princeton (1976).

[15] R. Gardner: The Brunn-Minkowski inequality, Bull. Amer. Math. Soc. 353 (2002) 355–405.

[16] M. Goldman, M. Novaga, B. Ruffini: Existence and stability for a non-local isoperimetric model of charged liquid drops, Arch. Ration. Mech. Anal. 217(1) (2015) 1–36.

[17] B. Kawohl: Rearrangements and Convexity of Level Sets in PDE, Lecture Notes in Mathematics 1150, Springer, Berlin (1985).

[18] N. Korevaar: Convexity of level sets for solutions to elliptic ring problems, Commun. Partial Differ. Equations 15 (1990) 541–556.

[19] N. S. Landkof: Foundations of Modern Potential Theory, Springer, Berlin (1972).

[20] E. H. Lieb, M. Loss: Analysis, Graduate Studies in Mathematics 14, AMS, Providence (2001).

[21] R. Schneider: Convex Bodies: The Brunn-Minkowski Theory, Cambridge Univ. Press, Cambridge (1993).