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Optimal Control of a Cahn-Hilliard-Navier-Stokes Model with State Constraints

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We investigate in this article the Pontryagin's maximum principle for a class of control problems associated with a coupled Cahn-Hilliard-Navier-Stokes model in a two dimensional bounded domain. The model consists of the Navier-Stokes equations for the velocity v, coupled with a Cahn-Hilliard model for the order (phase) parameter ϕ. The optimal problems involve a state constraint similar to that considered in [21]. We derive the Pontryagin's maximum principle for the control problems assuming that a solution exists. Let us note that the coupling between the Navier-Stokes and the Cahn-Hilliard systems makes the analysis of the control problem more involved.

Keywords: Two-phase flow model, maximum principle, state-constrained

1991 Mathematics Subject Classification: 93C05, 93B50, 93C35

1. Introduction

This article deals with some optimal control problems (with a state constraint) associated with a two phase flow model. The purpose is to derive a Pontryagin's maximum principle for these control problems assuming that a solution exists. The model consists of the Navier-Stokes equation coupled with a phase-field system.

It is well accepted that the incompressible Navier-Stokes equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [12]. For instance, this approach is used in [4] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with the phase-field system, [9, 12, 11, 13]. In the isothermal compressible case, the existence of

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a global weak solution is proved in [10]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity v and the order parameter ϕ. This system can be written as a NS equation coupled with a convective Allen-Cahn equation, [12]. The associated initial and boundary value problem was studied in [12] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor. They also established the existence of an exponential attractor. This entails that the global attractor has a finite fractal dimension, which is estimated in [12] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated although some important issues remain unresolved, [20]. In the case of binary fluids, the analysis is even more complicate and the mathematical studied is still at it infancy as noted in [12].

As noted in [11], the mathematical analysis of binary fluid flows is far from being well understood. For instance, the spinodal decomposition under shear consists of a two-stage evolution of a homogeneous initial mixture: a phase separation stage in which some macroscopic patterns appear, then a shear stage in which these patters organize themselves into parallel layers (see, e.g. [19] for experimental snapshots). This model has to take into account the chemical interactions between the two phases at the interface, achieved using a Cahn-Hilliard approach, as well as the hydrodynamic properties of the mixture (e.g., in the shear case), for which a Navier-Stokes equations with surface tension terms acting at the interface are needed. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but non-zero thickness, a well-known model is the so-called "Model H" (cf. [15, 14]). This is a system of equations where an incompressible Navier-Stokes equation for the (mean) velocity v is coupled with a convective Cahn-Hilliard equation for the order parameter ϕ, which represents the relative concentration of one of the fluids.

The necessary conditions for optimal control problems governed by fluid mechanic models such as the NS systems have been studied by several authors (see for instance [23, 24, 22, 21, 16, 17, 18, 3]). In [21], the authors studied a Pontryagin's maximum principle for optimal control problems (with a state constraint) governed by the 3D NS equations. In order to overcome the problem associated with the state constraint, the authors first defined a new penalty functional depending on a small parameter ϵ with which they approximated the original problem with a family of optimal control problems P_ϵ without state constraint. A Pontryagin's maximum principle is derived for the approximate problem P_ϵ and the limit as ϵ goes to 0 yields an optimality condition for the original control problem with a state constraint.

In this article, we investigate the Pontryagin's maximum principle for some optimal control problems associated with the Cahn-Hilliard-Navier-Stokes model. Assuming that a solution exits, we derive the Pontryagin's maximum principle following some idea from [21]. Let us point out that the coupling between the Cahn-Hilliard equation and the Navier-Stokes systems introduces in the model a coupling mapping R_0, which enjoys less regularity than the nonlinear term of the associated Navier-Stokes equations. This adds to the level of difficulty of the analysis of optimal control problems studied in this article.

The article is divided as follows. In the next section we present Cahn-Hilliard-Navier-Stokes system and its mathematical setting. We recall from [7, 5, 6, 12, 11] some existence and uniqueness results for the model and we derive some a priori estimates on the solution. We also present some estimates for the linearized system. The third section presents the control framework and the the main results is given in the fourth and fifth sections.

2. The CH-NS model and its mathematical setting

2.1. Governing equations

In this article, we consider a model of homogenuous incompressible two-phase flow. More precisely, we assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\Re^2$. Then, we consider the system

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu_1 \Delta v + (v \cdot \nabla) v + \nabla p - \mathcal{K} \mu \nabla \phi = \tilde{D} Q, \\ \operatorname{div} v = 0, \\ \dfrac{\partial \phi}{\partial t} + v \cdot \nabla \phi - \nu_3 \Delta \mu = 0, \\ \mu = -\nu_2 \Delta \phi + \alpha f(\phi), \end{cases} \tag{1}$$

in $\mathcal{M} \times (0, +\infty)$.

In (1), the unknown functions are the velocity $v = (v_1, v_2)$ of the fluid, its pressure p and the order (phase) parameter ϕ. The quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_{\mathcal{M}} \Big(\frac{\nu_2}{2} |\nabla \phi|^2 + \alpha F(\phi) \Big)\, ds, \tag{2}$$

where, e.g., $F(r) = \int_0^r f(\zeta) d\zeta$. Here, the constants $\nu_1 > 0,\ \nu_3 > 0$ and $\mathcal{K} > 0$ correspond to the kinematic viscosity of the fluid, the mobility constant and the capillarity (stress) coefficient respectively. Here $\nu_2,\ \alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, ν_2 is related with the thickness of the interface separating the two fluids.

In (1), $\tilde{D}$ is a linear operator from a real Hilbert space $\mathcal{U}$ to $(L^2(\mathcal{M}))^2$. The norm in $\mathcal{U}$ will be denoted $\| \cdot \|_{\mathcal{U}}$ and the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{U}}$.

A typical example of potential F is that of logarithmic type. However, this potential is often replaced by a polynomial approximation of the type $F(r) = \gamma_1 r^4 - \gamma_2 r^2$, γ_1, γ_2 being positive constants. As noted in [11], $(1)_1$ can be replaced by

$$\frac{\partial v}{\partial t} - \nu_1 \Delta v + (v \cdot \nabla) v + \nabla \tilde{p} = -\mathcal{K} \operatorname{div}(\nabla \phi \oplus \nabla \phi) + \tilde{D} Q, \tag{3}$$

where $\tilde{p} = p - \mathcal{K}(\frac{\nu_2}{2}|\nabla\phi|^2 + \alpha F(\phi))$, since $\mathcal{K}\mu\nabla\phi = \nabla(\mathcal{K}(\frac{\nu_2}{2}|\nabla\phi|^2 + \alpha F(\phi))) - \mathcal{K}\operatorname{div}(\nabla\phi\oplus\nabla\phi)$. The stress tensor $\nabla\phi\oplus\nabla\phi$ is considered the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

Regarding the boundary conditions for these models, we assume that the boundary conditions for ϕ are the natural no-flux condition

$$\partial_\eta \phi = \partial_\eta \Delta \phi = 0, \quad \text{on } \partial\mathcal{M} \times (0, \infty), \tag{4}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and η is the outward normal to $\partial\mathcal{M}$. These conditions ensure the mass conservation. Note that (4) implies that

$$\partial_\eta \mu = 0, \quad \text{on } \partial\mathcal{M} \times (0, \infty). \tag{5}$$

From (5), we deduce the conservation of the following quantity

$$\langle \phi(t) \rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \phi(x, t) dx, \tag{6}$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of $\mathcal{M}$. More precisely, we have

$$\langle \phi(t) \rangle = \langle \phi(0) \rangle, \quad \forall t \geq 0. \tag{7}$$

Concerning the boundary condition for v, we assume the Dirichlet (no-slip) boundary condition

$$v = 0, \quad \text{on } \partial\mathcal{M} \times (0, \infty). \tag{8}$$

Therefore we assume that there is no relative motion at the fluid-solid interface.

The initial condition is given by

$$(v, \phi)(0) = (v_0, \phi_0), \quad \text{in } \mathcal{M}. \tag{9}$$

2.2. Mathematical setting

We first recall from [11] a weak formulation of (1), (4), (8)–(9). Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial\mathcal{M}$ (e.g., of class $\mathcal{C}^3$). We also assume that $f \in \mathcal{C}^2(\Re)$ satisfies

$$\begin{cases} \lim\limits_{|r| \to +\infty} f'(r) > 0, \\ |f''(r)| \leq c_f(1 + |r|^{m-1}), \quad \forall r \in \Re, \end{cases} \tag{10}$$

where c_f is some positive constant and $m \in [1, +\infty)$ is fixed. It follows from (10) that

$$|f'(r)| \leq c_f(1 + |r|^m), \quad |f(r)| \leq c_f(1 + |r|^{m+1}), \quad \forall r \in \Re. \tag{11}$$

Note that the derivative of the typical double-well potential f satisfies (10) with $m = 2$. Let us now recall from [11] the functional set up of the model (1), (4), (8), (9).

If X is a real Hilbert space with inner product $(\cdot, \cdot)_X$, we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. We set

$$\mathcal{V}_1 = \{u \in \mathcal{C}_c^\infty(\mathcal{M}) : \operatorname{div} u = 0 \text{ in } \mathcal{M}\}.$$

We denote by H_1 and V_1 the closure of $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^2$ and $(H_0^1(\mathcal{M}))^2$ respectively. The scalar product in H_1 is denoted by $(\cdot, \cdot)_{L^2}$ and the associated norm by $|\cdot|_{L^2}$. Moreover, the space V_1 is endowed with the scalar product

$$((u, v)) = \sum_{i=1}^{2} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \qquad \|u\| = ((u, u))^{1/2}.$$

We now define the operator A_0 by

$$A_0 u = -\mathcal{P}_1 \Delta u, \quad \forall u \in D(A_0) = H^2(\mathcal{M}) \cap V_1,$$

where $\mathcal{P}_1$ is the Leray-Helmotz projector in $L^2(\mathcal{M})$ onto H_1. Then, A_0 is a self-adjoint positive unbounded operator in H_1 which is associated with the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and $|A_0 \cdot|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the H^2-norm.

We introduce the linear nonnegative unbounded operator on $L^2(\mathcal{M})$

$$A_N \phi = -\Delta \phi, \quad \forall \phi \in D(A_N) = \{\phi \in H^2(\mathcal{M}), \ \partial_\eta \phi = 0, \text{ on } \partial\mathcal{M}\}, \tag{12}$$

and we endow $D(A_N)$ with the norm $|A_N \cdot|_{L^2} + |\langle \cdot \rangle|_{L^2}$, which is equivalent to the H^2-norm. We also define the linear positive unbounded operator on the Hilbert space $L_0^2(\mathcal{M})$ of the L^2- functions with null mean

$$B_N \phi = -\Delta \phi, \ \forall \phi \in D(B_N) = D(A_N) \cap L_0^2(\mathcal{M}). \tag{13}$$

Note that B_N^{-1} is a compact linear operator on $L_0^2(\mathcal{M})$. More generally, we can define B_N^s for any $s \in \Re$, noting that $|B_N^{s/2} \cdot|_{L^2}, \ s > 0$, is an equivalent norm to the canonical H^s- norm on $D(B_N^{s/2}) \subset H^s(\mathcal{M}) \cap L_0^2(\mathcal{M})$. Also note that $A_N = B_N$ on $D(B_N)$. If ϕ is such that $\phi - \langle\phi\rangle \in D(B_N^{s/2})$, we have that $|B_N^{s/2}(\phi - \langle\phi\rangle)|_{L^2} + |\langle\phi\rangle|_{L^2}$ is equivalent to the H^s-norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*$, whenever $s < 0$.

We introduce the bilinear operators B_0, B_1 (and their associated trilinear forms b_0, b_1) as well as the coupling mapping R_0, which are defined from $D(A_0) \times D(A_0)$ into H_1, $D(A_0) \times D(A_N)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M}) \times (D(A_N) \cap H^3(\mathcal{M}))$ into H_1, respectively. More precisely, we set

$$
\begin{aligned}
\langle B_0(u,v),w\rangle &= \int_{\mathcal{M}}[(u\cdot\nabla)v]\cdot w dx = b_0(u,v,w),\ \forall u,v,w\in D(A_0),\\
\langle B_1(u,\phi),\rho\rangle &= \int_{\mathcal{M}}[(u\cdot\nabla)\phi]\rho dx = b_1(u,\phi,\rho),\\
&\qquad \forall u\in D(A_0),\ \phi,\rho\in D(A_N),\\
\langle R_0(\mu,\phi),w\rangle &= \int_{\mathcal{M}}\mu[\nabla\phi\cdot w]dx = b_1(w,\phi,\mu),\\
&\qquad \forall w\in D(A_0),\ \phi\in D(A_N)\cap H^3(\mathcal{M}),\ \mu\in L^2(\mathcal{M}).
\end{aligned}
\tag{14}
$$

Note that

$$
R_0(\mu,\phi) = \mathcal{P}\mu\nabla\phi.
$$

We recall that B_0, B_1 and R_0 satisfy the following estimates

$$
\begin{aligned}
|B_0(u,v)|_{V_1^*} &\le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|v\|,\quad \forall u,v\in V_1,\\
|B_0(u,v)|_{L^2} &\le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|v\|^{1/2}|A_0v|_{L^2}^{1/2},\quad \forall u\in V_1,\ v\in D(A_0),
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
|B_1(u,\phi)|_{V_2^*} &\le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|\phi\|,\quad \forall u\in V_1,\ \phi\in V_2,\\
|B_1(u,\phi)|_{L^2} &\le c|u|_{L^2}^{1/2}\|u\|^{1/2}\|\phi\|^{1/2}|A_N\phi|_{L^2}^{1/2},\quad \forall u\in V_1,\ \phi\in D(A_N),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
|R_0(A_N\phi,\rho)|_{V_1^*} &\le c|A_N\phi|_{L^2}^{1/2}|\phi|_{H^3}^{1/2}\|\rho\|,\quad \forall\phi\in D(A_N),\ \rho\in V_2,\\
|R_0(A_N\phi,\rho)|_{L^2} &\le c\|\rho\|^{1/2}|A_N\rho|_{L^2}^{1/2}|A_N\phi|_{L^2}^{1/2}|\phi|_{H^3}^{1/2},\\
&\qquad \forall\phi\in D(A_N),\ \rho\in D(A_N^{3/2}).
\end{aligned}
\tag{17}
$$

We recall that (due to the mass conservation) we have

$$
\langle\phi(t)\rangle = \langle\phi(0)\rangle = M_0,\quad \forall t>0. \tag{18}
$$

Thus, up to a shift of the order parameter field, we can always assume that the mean of ϕ is zero a the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

$$
\langle\phi(t)\rangle = \langle\phi(0)\rangle = 0,\quad \forall t>0. \tag{19}
$$

We set

$$
\mathbb{Y} = H_1\times D(B_N^{1/2}). \tag{20}
$$

The space $\mathbb{Y}$ is a complete metric space with respect to the norm

$$
|(v,\phi)|_{\mathbb{Y}}^2 = \mathcal{K}^{-1}|v|_{L^2}^2 + \nu_2|\nabla\phi|_{L^2}^2. \tag{21}
$$

We define the Hilbert space $\mathbb{V}$ by

$$\mathbb{V} = V_1 \times D(B_N), \tag{22}$$

endowed with the scalar products whose associated norm is

$$\|(v,\phi)\|_{\mathbb{V}}^2 = \|v\|^2 + |B_N\phi|_{L^2}^2. \tag{23}$$

Finally we set

$$H_2 = D(B_N^0), \qquad V_2 = D(B_N^{1/2}), \qquad H = H_1 \times H_2, \qquad V = V_1 \times V_2. \tag{24}$$

Throughout this article, we shall denote by c_i, K_i, K several positive constants that depend on the data (v_0,ϕ_0) and Q. We will also denote by c a generic positive constant that depends on the domain $\mathcal{M}$. To simplify the notations, we set (without loss of generality) $\nu_1 = \nu_2 = \nu_3 = \alpha = \mathcal{K} = 1$.

Using the notations above, we rewrite (1), (4), (8)–(6) as

$$\begin{cases} \dfrac{dv}{dt} + A_0 v + B_0(v,v) - R_0(A_N\phi,\phi) = DQ, \ \text{ in } V_1^*, \text{ a.e. in } (0,T), \\ \mu = A_N\phi + \alpha f(\phi), \ \text{ a.e. in } \mathcal{M}\times(0,T), \\ \dfrac{d\phi}{dt} + A_N\mu + B_1(v,\phi) = 0, \ \text{ in } H^{-1}, \text{ a.e. in } (0,T), \\ (v,\phi)(0) = (v_0,\phi_0), \end{cases} \tag{25}$$

where hereafter $D = \mathcal{P}_1\tilde{D}$.

Remark 2.1. In the weak formulation (25), the term $\mu\nabla\phi$ is replaced by $A_N\nabla\phi$. This is justified since $f'(\phi)\nabla\phi$ is the gradient $F(\phi)$ and can be incorporated into the pressure gradient, see [11] for details. For the sake of convenience, as in [11] we will replace μ in $(25)_3$ by $\bar{\mu} = \mu - \langle\mu\rangle$, that is $\bar{\mu} = A_N\phi + f(\phi) - \langle f(\phi)\rangle$, a.e. in $\mathcal{M}\times(0,+\infty)$. Obviously we have $\langle\bar{\mu}(t)\rangle = 0 \ \forall t > 0$.

Definition 2.2. Suppose that $(v_0,\phi_0) \in \mathbb{Y}$, $Q \in L^2(0,T;\mathcal{U})$ and $T > 0$. A pair (v,ϕ) is called a weak solution to (25) and (9) on $[0,T]$ if it satisfies (25) and (9) in a weak sense on $[0,T]$ and

$$\begin{gathered} (v,\phi) \in \mathcal{C}([0,T];\mathbb{Y}) \cap L^2([0,T];\mathbb{V}), \\ \frac{dv}{dt} \in L^2([0,T];V_1^*), \qquad \frac{d\phi}{dt} \in L^2([0,T];V_2^*). \end{gathered} \tag{26}$$

If $(u_0,\phi_0) \in \mathbb{V}$, a weak solution (v,ϕ) is called a strong solution on the time interval $[0,T]$ if in addition to (26), it satisfies

$$(v,\phi) \in \mathcal{C}([0,T];\mathbb{V}) \cap L^2(0,T;D(A_0)\times(D(B_N)\cap H^3(\mathcal{M}))). \tag{27}$$

The weak formulation of (25) was proposed and studied in [7, 5, 6, 12, 11] (see also [2, 1, 8]), where the existence and uniqueness results for weak and strong solutions were proved.

2.3. Some a priori estimates

In this part, we first derive some a priori estimates on the solution to (25), (9).

Note that if (v,ϕ) is a smooth solution to (1), by taking the scalar product in H_1 of $(1)_1$ with v, then taking the scalar product in $L^2(\mathcal{M})$ of $(1)_3$ with μ, we derive that

$$\frac{d}{dt}\left[\frac{1}{2}|v|^2_{L^2}+\mathcal{F}(\phi)\right]-(v,DQ)_{L^2}+\|v\|^2+|\nabla\mu|^2_{L^2}=0. \tag{28}$$

Proposition 2.3. *For $(v_0,\phi_0)\in\mathbb{Y}$ and $f\in\mathcal{C}^1(\Re)$. The system (25), (9) has a unique weak solution $(v,\phi)(t)$. Moreover, the following estimate holds:*

$$\begin{aligned}
&|(v,\phi)(t)|^2_{\mathbb{Y}}+\int_0^t\left(\|v(s)\|^2+|\mu(s)|^2_{H^1}\right)ds\\
&\leq Q_0(|(v,\phi)(0)|^2_{\mathbb{Y}})+c\int_0^t|DQ|^2_{V^*}ds,\quad \forall t\geq 0,\\
&\int_0^t(|A_N\phi(s)|^2_{L^2}+|\phi(s)|^2_{H^3})ds\\
&\leq Q_0(|(v,\phi)(0)|^2_{\mathbb{Y}})+c\int_0^t|DQ|^2_{V^*}ds,\ \forall t\geq 0,\\
&\int_0^t\left|\frac{d}{dt}(v,\phi)\right|^2_{V^*}ds\leq Q_0(|(v,\phi)(0)|^2_{\mathbb{Y}})+c\int_0^t|DQ|^2_{V^*}ds,\ \forall t\geq 0,
\end{aligned} \tag{29}$$

where hereafter, Q_0 will denote a monotone non-decreasing function independent of the time t and the initial conditions.

Proof. The existence and uniqueness of weak solutions is proved in [7, 5, 6]. To derive (29), we proceed as in [11], (see Proposition 3.1 an Lemma 3.3 in [11]). To derive (29), we proceed follows. We take the scalar product in $L^2(\mathcal{M})$ of $(25)_3$ with 2ϕ. Adding the resulting equation to (28) gives

$$\frac{dE}{dt}-2(v,DQ)_{L^2}+2\|v\|^2+2|\nabla\mu|^2_{L^2}=0, \tag{30}$$

where

$$E(t)=|(v,\phi)(t)|^2_{\mathbb{Y}}+2(F(\phi(t)),1)_{L^2}+C_e, \tag{31}$$

and $C_e=2C_F|\mathcal{M}|>0$, $|\mathcal{M}|>0$ being the Lebesgue measure of $\mathcal{M}$, and $C_F>0$ is a constant large enough to ensure that $E(t)$ is nonnegative.

Note that F is bounded from below by a constant. With this choice of C_e, we can find $C_f>0$ such that

$$|(v(t),\phi(t))|^2_{\mathbb{Y}}\leq E(t)\leq Q_0(|(v(t),\phi(t))|^2_{\mathbb{Y}}). \tag{32}$$

It follows from (30) that

$$\frac{dE}{dt} + \|v(t)\|^2 + 2|\nabla\mu(t)|^2_{L^2} \le c|DQ|^2_{V^*}, \tag{33}$$

which gives

$$E(t) + \int_0^t (\|v(s)\|^2 + 2|\nabla\mu(s)|^2_{L^2})ds \le E(0) + c\int_0^t |DQ|^2_{V^*}ds, \tag{34}$$

and $(29)_1$ follows from (34) and (32).

Using $(25)_1$, we can check (using well known regularity result) that

$$\int_0^t (|A_N\phi(s)|^2_{L^2} + |\phi(s)|^2_{H^3})ds \le Q_0(|(v,\phi)(0)|^2_{\mathbb{Y}}) + c\int_0^t |DQ|^2_{V^*}ds, \tag{35}$$

and $(29)_2$ follows. Finally, it is clear that $(29)_3$ follows from $(29)_1, (29)_2$ and (15)–(17). □

Proposition 2.4. *Let $Q \in \mathcal{U}$, and $(v_0, \phi_0) \in \mathbb{V}$, the system (25), (9) has a unique strong solution $(v,\phi)(t)$. Moreover the following estimate holds:*

$$\begin{aligned}&\|(v,\phi)(t)\|^2_{\mathbb{V}} + \int_0^T (|A_0v(s)|^2_{L^2} + |B_N^2\phi(s)|^2_{L^2} + |B_N\bar{\mu}(s)|^2_{L^2})ds\\ &\le c_1\|(v,\phi)(0)\|^2_{\mathbb{V}} + c_1\int_0^T |DQ(s)|^2_{L^2}ds.\end{aligned} \tag{36}$$

$$\int_0^T \left|\frac{d}{dt}(v,\phi)\right|^2_{L^2} \le c_1\|(v,\phi)(0)\|^2_{\mathbb{V}} + c_1\int_0^T |DQ(s)|^2_{L^2}ds, \tag{37}$$

$$|B_N^{3/2}\phi(t)|^2_{L^2} \le c_1\|(v,\phi)(0)\|^2_{\mathbb{V}} + c_1\int_0^T |DQ(s)|^2_{L^2}ds. \tag{38}$$

Proof. The existence and uniqueness of strong solution is proved as in [7, 5, 6]. We only need to prove (36). We introduce the function $\bar{\mu}(t) = \mu(t) - \langle\mu(t)\rangle$, and we observe that $\langle\bar{\mu}(t)\rangle = 0$.

Taking the inner product in H_1 of $(25)_1$ with $2A_0v$, the inner product in $L^2(\mathcal{M})$ of $(25)_2$ and $(25)_3$ with $2B_N^2\bar{\mu}+2\xi B_N^3\phi$ ($\xi > 0$ small enough) and $2B_N^2\phi$ respectively. Adding the resulting equalities gives

$$\begin{aligned}&\frac{d\mathcal{Y}}{dt} + 2|A_0v|^2_{L^2} + 2\xi|B_N^2\phi|^2_{L^2} + 2|B_N\bar{\mu}|^2_{L^2}\\ &= 2(R_0(A_N\phi,\phi), A_0v)_{L^2} - 2(A_N(f(\phi) - \langle f(\phi)\rangle), B_N\bar{\mu})_{L^2}\\ &\quad - 2(B_1(v,\phi), B_N^2\phi)_{L^2} - 2(B_0(v,v), A_0v)_{L^2} + (DQ, A_0v)_{L^2}\\ &\quad + 2\xi(B_N\bar{\mu}, B_N^2\phi)_{L^2} + 2\xi(B_N(f(\phi) - \langle f(\phi)\rangle), B_N^2\phi)_{L^2},\end{aligned} \tag{39}$$

where

$$\mathcal{Y}(t) = \|v(t)\|^2 + |B_N\phi(t)|_{L^2}^2.$$

Note that

$$\begin{aligned} |2(R_0(A_N\phi,\phi),A_0v)_{L^2}| &\leq 2|B_N\phi|_{L^2}|\nabla\phi|_{L^\infty}|A_0v|_{L^2} \\ &\leq \frac{1}{8}|A_0v|_{L^2}^2 + \frac{\xi}{8}|B_N^2\phi|_{L^2}^2 + c|B_N\phi|_{L^2}^4|\nabla\phi|_{L^2}^2, \end{aligned} \tag{40}$$

$$|2(B_0(v,v),A_0v)_{L^2}| \leq \frac{1}{8}|A_0v|_{L^2}^2 + c|v|_{L^2}^2\|v\|^4, \tag{41}$$

$$|2(B_1(v,\phi),B_N^2\phi)_{L^2}| \leq \frac{\xi}{8}|B_N^2\phi|_{L^2}^2 + c(|v|_{L^2}^2\|v\|^2 + |\nabla\phi|_{L^2}^2|B_N\phi|_{L^2}^2), \tag{42}$$

$$|2\xi(B_N\bar{\mu},B_N^2\phi)_{L^2}| \leq \frac{\xi}{4}|B_N^2\phi|_{L^2}^2 + 4\xi|B_N\bar{\mu}|_{L^2}^2, \tag{43}$$

$$\begin{aligned} &|2(A_N(f(\phi)-\langle f(\phi)\rangle),B_N\bar{\mu})_{L^2}| \\ &\leq \xi|B_N\bar{\mu}|_{L^2}^2 + c|B_N(f(\phi)-\langle f(\phi)\rangle)|_{L^2}^2 \\ &\leq \xi|B_N\bar{\mu}|_{L^2}^2 + Q_0(|\phi|_{H^1})(1+|B_N\phi|_{L^2}^2)|B_N\phi|_{L^2}^2, \end{aligned} \tag{44}$$

$$\begin{aligned} &|2\xi(B_N(f(\phi)-\langle f(\phi)\rangle),B_N^2\phi)_{L^2}| \\ &\leq \frac{\xi}{4}|B_N^2\phi|_{L^2}^2 + Q_0(|\phi|_{H^1})(1+|B_N\phi|_{L^2}^2)|B_N\phi|_{L^2}^2. \end{aligned} \tag{45}$$

$$\frac{d\mathcal{Y}}{dt} + |A_0v|_{L^2}^2 + \frac{\xi}{4}|B_N^2\phi|_{L^2}^2 + (2-5\xi)|B_N\bar{\mu}|_{L^2}^2 \leq \mathcal{G}(t)\mathcal{Y}(t) + c|DQ|_{L^2}^2, \tag{46}$$

where

$$\mathcal{G}(t) = c(|\nabla\phi|_{L^2}^2|B_N\phi|_{L^2}^2 + |v|_{L^2}^2 + |\nabla\phi|_{L^2}^2) + Q_0(|\phi|_{H^1})(1+|B_N\phi|_{L^2}^2). \tag{47}$$

Note that from (29), we have

$$\int_0^t \mathcal{Y}(s)ds \leq Q_1\left(T, |(v,\phi)(0)|_{\mathbb{Y}}^2, \int_0^t |DQ|_{L^2}^2 ds\right), \tag{48}$$

$$\begin{aligned} &\int_0^t \mathcal{G}(s)ds \\ &\leq c\int_0^t \left(|\nabla\phi|_{L^2}^2|B_N\phi|_{L^2}^2 + |v|_{L^2}^2 + |\nabla\phi|_{L^2}^2 + Q_0(|\phi|_{H^1})(1+|B_N\phi|_{L^2}^2)\right) ds \\ &\leq Q_1\left(T, |(v,\phi)(0)|_{\mathbb{Y}}^2, \int_0^t |DQ|_{L^2}^2 ds\right). \end{aligned} \tag{49}$$

Choosing $\xi = \frac{1}{5}$, it follows from (46)–(49) that

$$\begin{aligned} &\mathcal{Y}(t) + \int_0^t (|A_0v|_{L^2}^2 + |B_N^2\phi|_{L^2}^2 + |B_N\bar{\mu}|_{L^2}^2)ds \\ &\leq c_1\|(v,\phi)(0)\|_{\mathbb{V}}^2 + c_1\int_0^t |DQ|_{L^2}^2 ds, \quad \forall t\in[0,T], \end{aligned} \tag{50}$$

and (36) follows. We can easily check that (37) follows from (36) and (15)–(17). To derive (38), we set $w = \dfrac{dv}{dt}$, $\psi = \dfrac{d\phi}{dt}$. Then, differentiating $(25)_3$ with respect to time gives,

$$\frac{d\psi}{dt} + A_N^2\psi + B_1(w,\phi) + B_1(v,\psi) + A_N f'(\phi)\psi = 0. \tag{51}$$

We multiply (51) by $A_N\psi$ to derive that

$$\begin{aligned} &\frac{d}{dt}\|\psi\|^2 + |B_N^{3/2}\psi|_{L^2}^2 \\ &\leq |b_1(w,\phi,A_N\psi)| + |b_1(v,\psi,A_N\psi)| + |(A_N f'(\phi)\psi, A_N\psi)_{L^2}|. \end{aligned} \tag{52}$$

Note that

$$|b_1(w,\phi,A_N\psi)| \leq \frac{1}{4}|B_N^{3/2}\psi|_{L^2}^2 + c|w|_{L^2}^2\|\phi\||A_N\phi|_{L^2}, \tag{53}$$

$$|b_1(v,\psi,A_N\psi)| \leq \frac{1}{4}|B_N^{3/2}\psi|_{L^2}^2 + c|v|_{L^2}^2|A_0 v|_{L^2}^2\|\psi\|^2, \tag{54}$$

$$|(A_N f'(\phi)\psi, A_N\psi)_{L^2}| \leq \frac{1}{4}|B_N^{3/2}\psi|_{L^2}^2 + Q_0(\|\phi\|)\|\psi\|^2. \tag{55}$$

It follows from (52)–(55) and (36)–(37) that

$$\|\psi(t)\|^2 + \int_0^t |B_N^{3/2}\psi(s)|^2 ds \leq c_1\|(v,\phi)(0)\|_{\mathbb{V}}^2 + c_1\int_0^t |DQ|_{L^2}^2 ds. \tag{56}$$

Finally, (38) follows from (56) and $(25)_3$. Note that

$$\begin{aligned} |B_N^{3/2}\phi|_{L^2}^2 &\leq c\left|\frac{d\phi}{dt}\right|_{L^2}^2 + c|B_1(v,\phi)|_{L^2}^2 + c|A_N^{1/2}f(\phi)|_{L^2}^2 \\ &\leq c\left|\frac{d\phi}{dt}\right|_{L^2}^2 + c|v|_{L^2}\|v\|\|\phi\||\mathcal{A}_N\phi|_{L^2} + Q_0(\|\phi\|). \end{aligned} \tag{57}$$

□

2.4. Linearized system

Let (v,ϕ) be a strong solution to (25) given by Proposition 2.2. We consider the linearized system

$$\begin{cases} \dfrac{dw}{dt} + A_0 w + B_0(v,w) + B_0(w,v) = R_0(A_N\phi,\psi) + R_0(A_N\psi,\phi) + g_1, \\ \mu = A_N\psi + f'(\phi)\psi, \\ \dfrac{d\psi}{dt} + A_N\mu + B_1(v,\psi) + B_1(w,\phi) = g_2, \\ (w,\psi)(0) = (w_0,\psi_0), \end{cases} \tag{58}$$

or equivalently

$$\begin{cases} \dfrac{dw}{dt} + A_0 w + B_0(v,w) + B_0(w,v) = R_0(A_N\phi,\psi) + R_0(A_N\psi,\phi) + g_1, \\ \bar{\mu} = A_N\psi + f^{'}(\phi)\psi - \langle f^{'}(\phi)\rangle, \\ \dfrac{d\psi}{dt} + A_N\bar{\mu} + B_1(v,\psi) + B_1(w,\phi) = g_2, \\ (w,\psi)(0) = (w_0,\psi_0), \end{cases} \tag{59}$$

Proposition 2.5. *For $(w_0,\psi_0) \in \mathbb{Y}$ and $g = (g_1,g_2) \in L^2(0,T;H)$, the system (59) has a unique solution $(w,\psi) \in L^2(0,T;\mathbb{V}) \cap C(0,T;\mathbb{Y})$. Moreover, the following estimate holds true:*

$$\begin{aligned} &|(w,\psi)(t)|^2_{\mathbb{Y}} + \int_0^T (\|(w,\psi)(s)\|^2_{\mathbb{V}} + |\nabla\bar{\mu}(s)|^2_{L^2})ds \\ &\leq c_1|(w_0,\psi_0)|^2_{\mathbb{Y}} + c_1\int_0^T |g(s)|^2_{L^2}ds. \end{aligned} \tag{60}$$

Furthermore if $(w_0,\psi_0) \in \mathbb{V}$, then the solution (w,ψ) to (59) satisfies the regularity

$$(w,\phi) \in L^2(0,T;D(A_0)\times D(B_N^2)) \cap C(0,T;\mathbb{V}).$$

and the following estimates hold:

$$\begin{aligned} &\|(w,\psi)(t)\|^2_{\mathbb{V}} + \int_0^T (|A_0w(s)|^2_{L^2} + |B_N^2\psi(s)|^2_{L^2} + |B_N\bar{\mu}(s)|^2)ds \\ &\leq c_1\|(w_0,\psi_0)\|^2_{\mathbb{V}} + c_1\int_0^T |g(s)|^2_{L^2}ds, \\ &\int_0^T \left|\frac{d}{dt}(w,\psi)\right|^2_{L^2} dt \leq c_1\|(w_0,\psi_0)\|^2_{\mathbb{V}} + c_1\int_0^T |g(s)|^2_{L^2}ds. \end{aligned} \tag{61}$$

Proof. Since $(v,\phi) \in L^\infty(0,T;\mathbb{V}) \cap L^2(0,T;D(A_0)\times D(B_N^2))$, using the known regularity results for parabolic equations such as the linearized Navier-Stokes system (see e.g., [20]), we can rigorously prove the existence and uniqueness of solution to (59) or (58). To derive (60), we multiplying $(59)_1$ by w. Then we take the duality of $(59)_2$ and $(59)_3$ with $A_N\bar{\mu} - \xi A_N\psi$ and $A_N\psi$ respectively, where $\xi > 0$ is small enough and will be selected later. We derive that

$$\begin{aligned} &\frac{d}{dt}(|w|^2_{L^2} + \|\psi\|^2) + \|w\|^2 + \xi|A_N\bar{\psi}|^2_{L^2} + |\nabla\bar{\mu}|^2_{L^2} \\ = \; &-b_0(w,v,w) + (R_0(A_N\phi,\psi),w)_{L^2} + (R_0(A_N\psi,\phi),w)_{L^2} \\ &-b_1(w,\phi,A_N\psi) - b_1(v,\psi,\mathcal{A}_N\psi) + \xi(\bar{\mu},A_N\psi)_{L^2} \\ &-\xi(f^{'}(\phi)\psi,A_N\psi)_{L^2} + (f^{'}(\phi)\psi,A_N\bar{\mu})_{L^2} + (g_1,w)_{L^2} + (g_2,\mathcal{A}_N\psi)_{L^2}. \end{aligned} \tag{62}$$

Note that

$$|b_0(w,v,w)| \leq \frac{1}{8}\|w\|^2 + c\|v\|^2|w|_{L^2}^2, \tag{63}$$

$$\begin{aligned}|(R_0(A_N\psi,\phi),w)_{L^2}| &= |b_1(w,\phi,A_N\psi)| \\ &\leq \frac{1}{8}(\|w\|^2+\xi|A_N\psi|_{L^2}^2) + c|w|_{L^2}^2\|\phi\|^2|A_N\phi|_{L^2}^2,\end{aligned} \tag{64}$$

$$\begin{aligned}|(R_0(A_N\phi,\psi),w)_{L^2}| &= |b_1(w,\psi,A_N\phi)| \\ &\leq \frac{1}{8}(\|w\|^2+\xi|A_N\psi|_{L^2}^2) + c(|w|_{L^2}^2+|\nabla\psi|_{L^2}^2)\|\phi\|^2|\phi|_{H^2}^2,\end{aligned} \tag{65}$$

$$\xi|(f'(\phi)\psi, A_N\psi)_{L^2}| \leq \frac{\xi}{8}|A_N\psi|_{L^2}^2 + Q_0(|\phi|_{H^1})\|\psi\|^2, \tag{66}$$

$$\begin{aligned}\xi|(f'(\phi)\psi, A_N\bar{\mu})_{L^2}| &\leq \xi|A_N^{1/2}f'(\phi)\psi|_{L^2}|A_N^{1/2}\bar{\mu}|_{L^2} \\ &\leq \frac{1}{8}|\nabla\bar{\mu}|_{L^2}^2 + Q_0(|\phi|_{H^1})\|\psi\|^2,\end{aligned} \tag{67}$$

$$|b_1(v,\psi,A_N\psi)| \leq \frac{\xi}{8}|A_N\psi|_{L^2}^2 + c|v|_{L^2}^2\|v\|^2\|\psi\|^2, \tag{68}$$

$$\xi|(\bar{\mu},A_N\psi)_{L^2}| \leq \frac{\xi}{8}|A_N\psi|_{L^2}^2 + c\xi|\nabla\bar{\mu}|_{L^2}^2. \tag{69}$$

Let

$$\mathcal{Y}_1(t) = |w(t)|_{L^2}^2 + |\nabla\psi|_{L^2}^2.$$

It follows from (62)–(69) that

$$\frac{d\mathcal{Y}_1}{dt} + c\|w\|^2 + (1-\xi c)|\nabla\bar{\mu}|_{L^2}^2 + c\xi|A_N\psi|_{L^2}^2 \leq \mathcal{Y}_1(t)\mathcal{G}(t) + c|g_1|_{L^2}^2 + c|g_1|_{L^2}^2, \tag{70}$$

where

$$\mathcal{G}(t) = c(\|v\|^2 + \|\phi\|^2|A_N\phi|_{L^2}^2 + |v|_{L^2}^2\|v\|^2) + Q_0(|\phi|_{H^1}). \tag{71}$$

and finally (60) follows from the Gronwall lemma. Note that $\mathcal{G}(t) \in L^1(0,T)$.

To prove $(61)_1$, we take the inner product of $(59)_1$ with A_0w in $L^2(\mathcal{M})$. Then we take the inner product in $L^2(\mathcal{M})$ of $(59)_2$ and $(59)_3$ with $B_N^2\bar{\mu} - \xi B_N^2\psi$ and $B_N^2\psi$ respectively, where $\xi > 0$ will be selected later. We derive that

$$\frac{d\mathcal{Y}}{dt} + |A_0w|_{L^2}^2 + \xi|B_N^2\psi|_{L^2}^2 + |B_N\bar{\mu}|_{L^2}^2 = \Lambda_3(t), \tag{72}$$

where

$$\begin{aligned}\Lambda_3(t) = &-b_0(w,v,A_0w) - b_0(v,w,A_0w) + (R_0(B_N\phi,\psi),A_0w)_{L^2} \\ &+ (R_0(B_N\psi,\phi),A_0w)_{L^2} - b_1(w,\phi,B_N^2\psi) - b_1(v,\psi,B_N^2\psi) \\ &- (f'(\phi)\psi, B_N^2\bar{\mu})_{L^2} + \xi(B_N\psi,B_N\bar{\mu})_{L^2} - \xi(f'(\phi)\psi,B_N^2\psi)_{L^2} \\ &+ (g_1,A_0w)_{L^2} + (g_2,B_N^2\psi)_{L^2}.\end{aligned} \tag{73}$$

We note that

$$|b_0(v,w,A_0w)| \le \frac{1}{8}|A_0w|_{L^2}^2 + c|v|_{L^2}^2\|v\|^2\|w\|^2, \tag{74}$$

$$|b_0(w,v,A_0w)| \le \frac{1}{8}|A_0w|_{L^2}^2 + c\|v\|^2|A_0v|_{L^2}^2\|w\|^2, \tag{75}$$

$$\begin{aligned}
&|(R_0(A_N\phi,\psi),A_0w)_{L^2}| + |(R_0(B_N\psi,\phi),A_0w)_{L^2}| \\
&\le \frac{1}{8}(|A_0w|_{L^2}^2 + \xi|B_N^2\psi|_{L^2}^2) + c\|\psi\|^2|B_N\phi|_{L^2}^2|\phi|_{H^2}^2 + c|B_N\psi||\phi|_{W^{1,\infty}}|A_0w|_{L^2} \qquad (76)\\
&\le \frac{1}{8}(|A_0w|_{L^2}^2 + \xi|B_N^2\psi|_{L^2}^2) + c\|\psi\|^2|B_N\phi|_{L^2}^2|\phi|_{H^2}^2 + c|B_N\psi|_{L^2}^2|\phi|_{H^1}^2|\phi|_{H^3}^2
\end{aligned}$$

$$\begin{aligned}
|b_1(w,\phi,B_N^2\psi)| &\le |(B_1(w,\phi),B_N^2\psi)_{L^2}| \\
&\le \frac{1}{8}(|A_0w|_{L^2}^2 + \xi|B_N^2\psi|_{L^2}^2) + c\|w\|^2|\phi|_{H^2}^2\|\phi\|^2 \qquad (77)\\
&\quad + c\|w\|^2|\phi|_{H^1}^2 + c\|w\|^2|\phi|_{H^1}|\phi|_{H^3},
\end{aligned}$$

$$\begin{aligned}
|(f'(\phi)\psi,B_N^2\bar{\mu})_{L^2}| &\le c|(A_Nf'(\phi)\psi,B_N\bar{\mu})_{L^2}| \\
&\le \frac{\xi}{8}|B_N\bar{\mu}|_{L^2}^2 + Q_0(|\phi|_{H^1})|B_N\psi|_{L^2}^2.
\end{aligned} \tag{78}$$

$$\xi|(B_N\psi,B_N\bar{\mu})_{L^2}| \le \frac{\xi}{8}|B_N\bar{\mu}|_{L^2}^2 + c|B_N\psi|_{L^2}^2. \tag{79}$$

It follows from (72)–(79) that (with $0<\xi<1$ small enough)

$$\frac{d\mathcal{Y}}{dt} + |A_0w|_{L^2}^2 + \tfrac{\xi}{2}|B_N^2\psi|_{L^2}^2 + (1-\xi)|B_N\bar{\mu}|_{L^2}^2 \le \mathcal{G}(t)\mathcal{Y}(t) + c|g|_{L^2}^2, \tag{80}$$

where

$$\mathcal{G}(t) = c(\|v\|^2|v|_{L^2}^2 + \|v\|^2|A_0v|_{L^2}^2 + |\phi|_{H^1}^2|\phi|_{H^2}^2 + |\phi|_{H^1}|\phi|_{H^3} + |\phi|_{H^2}^2 + 1) + Q_0(|\phi|_{H^1}), \tag{81}$$

and $(61)_1$ follows from the Gronwall lemma. Note that $\mathcal{G}(t) \in L^1(0,T)$.

It is easy to check that $(61)_2$ follows from $(61)_1$ and the properties of the operators B_0, B_1 and R_0 given in (15)–(17). □

3. Control setting

For both physical and mathematical considerations, hereafter we take the state space to be the space $\mathcal{Y}$ defined below. Physically, one needs to have some smoothness on the solutions. Moreover, the coupling between the Navier-Stokes and the Cahn-Hilliard equations make more involved the analysis of the adjoint system associated to the control problem (see Appendix).

Hereafter, we assume the following regularity for the initial condition: $u_0 = (v_0, \theta_0) \in \mathbb{V}$.

Then the control problem we will study in this article is given by:

$$(P): \ \inf L(v,\phi,Q) \tag{82}$$

over all $(v,\phi,Q) \in \mathcal{Y} \times L^2(0,T;\mathcal{U})$ satisfying (25) and

$$\mathcal{F}(v,\phi) \in W, \tag{83}$$

where

$$\begin{aligned}
&X \text{ is a Banach space with dual } X^* \text{ strictly convex,}\\
&W \subset X \text{ is convex and closed subset,}\\
&\mathcal{F} : L^2(0,T;V) \to X \text{ is continuously Frechet differentiable,}\\
&\mathcal{Y} = W^{1,2}(0,T;H) \cap L^2(0,T;D(A_0) \times D(B_N^2)),
\end{aligned} \tag{84}$$

and $W^{1,2}(0,T;H)$ is the space of all absolutely continuous functionals $(v,\phi) : [0,T] \to H$ such that $\left(\frac{dv}{dt}, \frac{d\phi}{dt}\right) \in L^2(0,T;H)$. Note that $\mathcal{Y} \subset C(0,T;\mathbb{V})$.

We will also define the space $\mathcal{W}(0,T)$ by

$$\mathcal{W}(0,T) = \left\{(v,\phi) \in L^2(0,T;V), \ \frac{d}{dt}(v,\phi) \in L^2(0,T;V^*)\right\}. \tag{85}$$

Let us recall that $\mathcal{W}(0,T) \subset C(0,T;H)$.

The cost functional L is given by

$$L(v,\phi,Q) = \int_0^T (g^0(t,v(t),\phi(t)) + h(Q(t)))dt, \tag{86}$$

where

$$\begin{gathered}
g^0 : [0,T] \times V \to \Re^+ \text{ is measurable in the first variable,}\\
g^0(t,0,0) \in L^\infty(0,T),\\
\text{and for every } k > 0, \text{ there exists an}\\
C_k > 0 \text{ independent of } t \text{ such that}\\
|g^0(t,v_1,\phi_1) - g^0(t,v_2,\phi_2)| \le C_k \|(v_1,\phi_1) - (v_2,\phi_2)\|\\
\forall t \in [0,T], \ \|(v_1,\phi_1)\| + \|(v_1,\phi_1)\| \le k,
\end{gathered} \tag{87}$$

$$\begin{gathered}
h : \mathcal{U} \to \bar{\Re} \equiv (-\infty,+\infty] \text{ is convex and lower semicontinuous,}\\
\text{there exist constants } c_1 > 0, \ c_2 \in \Re\\
\text{such that } h(Q) \ge c_1 \|Q\|_{\mathcal{U}}^2 - c_2 \ \forall Q \in \mathcal{U}.
\end{gathered} \tag{88}$$

We say that $(v,\phi,Q) \in \mathcal{Y} \times L^2(0,T;\mathcal{U})$ is admissible for problem (P) if $(v,\phi) \in \mathcal{Y}$ is a solution to (25) which satisfies $\mathcal{F}(v,\phi) \in W$.

The purpose of this article is to obtain the Pontryagin's maximum principle for Problem (P) assuming that the solution exists.

Lemma 3.1. *Let $(v_{0n}, \phi_{0n}, Q_n), (\tilde{v}_0, \tilde{\phi}_0, \tilde{Q}) \in \mathbb{V} \times \mathcal{U}_{ad}$ and $(v_n, \phi_n), (\tilde{v}, \tilde{\phi})$ be the solution to (25) corresponding to (v_{0n}, ϕ_{0n}, Q_n) and $(\tilde{v}_0, \tilde{\phi}_0, \tilde{Q})$ respectively. Suppose that $(v_{0n}, \phi_{0n}, Q_n) \to (\tilde{v}_0, \tilde{\phi}_0, \tilde{Q})$ strongly in $\mathbb{V} \times L^2(0,T;\mathcal{U})$. Then there exists a subsequence of (v_n, ϕ_n) (still denoted (v_n, ϕ_n) such that $(v_n, \phi_n) \to (\tilde{v}, \tilde{\phi})$ strongly in $\mathcal{Y}$.*

Proof. Let us recall that (v_n, ϕ_n, Q_n) satisfies

$$\begin{cases} \dfrac{dv_n}{dt} + A_0 v_n + B_0(v_n, v_n) = R_0(A_N\phi_n, \phi_n) + DQ_n, \\ \dfrac{d\phi_n}{dt} + A_N\mu_n + B_1(v_n, \phi_n) = 0, \\ \mu_n = A_N\phi_n + f(\phi_n), \quad (v_n, \phi_n)(0) = (v_{0n}, \phi_{0n}). \end{cases} \tag{89}$$

Since (v_{0n}, ϕ_{0n}, Q_n) is bounded in $\mathbb{V} \times L^2(0,T;\mathcal{U})$, it follows from Proposition 2.4 that

$$\begin{gathered} \|(v_n, \phi_n)\|_{\mathbb{V}}^2 + \int_0^T (|A_0 v_n|_{L^2}^2 + |B_N^2\phi_n|_{L^2}^2 + |B_N\bar{\mu}_n|_{L^2}^2) ds \le C, \\ \int_0^T \left| \frac{d}{dt}(v_n, \phi_n) \right|_{L^2}^2 ds \le C, \end{gathered} \tag{90}$$

where $C > 0$ is independent of n.

If we set $(w, \psi) = (v_n, \phi_n) - (v_m, \phi_m)$. Then (w, ψ) satisfies

$$\begin{cases} \dfrac{dw}{dt} + A_0 w + B_0(v_m, w) + B_0(w, v_n) \\ \qquad = R_0(A_N\phi_m, \psi) + R_0(A_N\psi, \phi_n) + D(Q_n - Q_m), \\ \dfrac{d\psi}{dt} + A_N^2\psi + A_N(f(\phi_n) - f(\phi_m)) + B_1(v_m, \psi) + B_1(w, \phi_n) = 0, \\ (w, \psi)(0) = (v_{0n}, \phi_{0n}) - (v_{0m}, \phi_{0m}). \end{cases} \tag{91}$$

We can prove as in Proposition 2.3 that

$$\begin{aligned} &\|(v_n, \phi_n)(t) - (v_m, \phi_m)(t)\|_{\mathbb{V}}^2 \\ &\qquad + \int_0^T (|A_0(v_n - v_m)(s)|_{L^2}^2 + |B_N^2(\phi_n - \phi_m)(s)|_{L^2}^2) ds \\ &\le c_1 \|(v_{0n}, \phi_{0n}) - (v_{0m}, \phi_{0m})\|_{\mathbb{V}}^2 + c_1 \int_0^T \|(Q_n - Q_m)(s)\|_{\mathcal{U}}^2 ds, \end{aligned} \tag{92}$$

and

$$\begin{aligned} &\left| \frac{d}{dt}(v_n, \phi_n) - \frac{d}{dt}(v_m, \phi_m) \right|_{L^2(0,T;H)}^2 \\ &\le c_1 \|(v_{0n}, \phi_{0n}) - (v_{0m}, \phi_{0m})\|_{\mathbb{V}}^2 + c_1 \int_0^T \|Q_n - Q_m\|_{\mathcal{U}}^2 ds. \end{aligned} \tag{93}$$

This shows that the sequence (v_n, ϕ_n) is a Cauchy sequence in $\mathcal{Y}$, which proves that

$$(v_n, \phi_n) \to (\tilde{v}, \tilde{\phi}) \text{ strongly in } \mathcal{Y}. \tag{94}$$

We can pass in the limit in (89) to obtain that $(\tilde{v}, \tilde{\phi}) \in \mathcal{Y}$ is the solution to (25) corresponding to $(\tilde{v}_0, \tilde{\phi}_0, \tilde{Q})$. □

Let (v_*, ϕ_*, Q_*) be optimal for Problem (P). Hereafter, we denote by $(\mathcal{F}'(v_*, \phi_*))^*$ the adjoint operator of $\mathcal{F}'(v_*, \phi_*)$ and by D^* the adjoint operator of D. To derive a necessary condition for (v_*, ϕ_*, Q_*), we first define the following approximate functions h_ϵ and g_ϵ.

Let $\{e_i\}$ be an orthonormal basis of $\mathbb{V}$ and $m = [\epsilon^{-1}]$, the integer part of ϵ^{-1}. We denote by X_m the finite dimensional space generated by $\{e_1, e_2, \cdots, e_m\}$ and by $P_m : \mathbb{V} \to X_m$ the projection operator from $\mathbb{Y}$ to X_m. Let $\wedge_m : \Re^m \to X_m$ be the operator defined by:

$$\wedge_m(\tau) = \sum_{i=1}^{m} \tau_i e_i, \quad \text{for } \tau = (\tau_1, \tau_2, \cdots, \tau_m) \tag{95}$$

and ρ_m be a mollifier in $\Re^m$. We define $g_\epsilon(t, v, \phi)$ by

$$g_\epsilon(t, v, \phi) = \int_{\Re^m} g^0(t, P_m(v, \phi) - \epsilon \wedge_m \tau)\rho_m(\tau) d\tau, \quad \forall (v, \phi) \in \mathbb{V}. \tag{96}$$

Let $h_\epsilon : \mathcal{U} \to \Re$ defined by

$$h_\epsilon(Q) = \inf_{Q_1 \in \mathcal{U}} \left\{ \frac{1}{2\epsilon} \|Q - Q_1\|^2_{\mathcal{U}} + h(Q) \right\}. \tag{97}$$

For $\epsilon > 0$, we define $L_\epsilon : \mathcal{Y} \times L^2(0, T; \mathcal{U}) \to \Re$ by:

$$\begin{aligned} &L_\epsilon(v, \phi, Q) \\ = &\int_0^T (g_\epsilon(t, v, \phi) + h_\epsilon(Q)) dt + \frac{1}{2} \int_0^T \|Q - Q_*\|^2_{\mathcal{U}} dt \\ &+ \frac{1}{2\epsilon} \int_0^T \left| \frac{dv}{dt} + A_0 v + B_0(v, v) - R_0(A_N \phi, \phi) - DQ \right|^2_{L^2} dt \\ &+ \frac{1}{2\epsilon} \int_0^T \left| \frac{d\phi}{dt} + A_N^2 \phi + B_1(v, \phi) + A_N f(\phi) \right|^2_{L^2} dt \\ &+ \frac{1}{2\epsilon} (\epsilon + d_W \mathcal{F}(v, \phi))^2. \end{aligned} \tag{98}$$

Now we introduce the approximate problem

$$(P_\epsilon) : \ \inf L_\epsilon(v, \phi, Q) \tag{99}$$

over all $(v, \phi, Q) \in \mathcal{Y}_0 \times L^2(0, T; \mathcal{U})$, where

$$\mathcal{Y}_0 = \{(v, \phi) \in \mathcal{Y}, \ (v, \phi)(0) = (v_0, \phi_0)\}. \tag{100}$$

Lemma 3.2. *Problem (P_ϵ) has at least one solution.*

Proof. Let $d = \inf(L_\epsilon)$ and $(v_n, \phi_n, Q_n) \in \mathcal{Y}_0 \times L^2(0,T;\mathcal{U})$ such that

$$d \le L_\epsilon(v_n, \phi_n, Q_n) \le d + \frac{1}{n}. \tag{101}$$

From (98), we have

$$\int_0^T \|Q_n\|_{\mathcal{U}}^2 dt \le C. \tag{102}$$

Moreover, there exists $f_n = (f_n^1, f_n^2) \in L^2(0,T;H)$ such that

$$\int_0^T |f_n|_{L^2}^2 dt \le C \tag{103}$$

and

$$\begin{cases} \dfrac{dv_n}{dt} + A_0 v_n + B_0(v_n, v_n) = DQ_n + R_0(A_N\phi_n, \phi_n) + f_n^1, \\ \mu_n = A_n\phi_n + f(\phi_n), \\ \dfrac{d\phi_n}{dt} + A_N\mu_n + B_1(v_n, \phi_n) = f_n^2, \quad (v_n, \phi_n)(0) = (v_0, \phi_0). \end{cases} \tag{104}$$

Since f_n is bounded in $L^2(0,T;H)$, it follows from Proposition 2.4 that

$$\begin{aligned} &\|(v_n, \phi_n)(t)\|_{\mathbb{V}}^2 + \int_0^t (|A_0 v_n|_{L^2}^2 + |B_N^2\phi_n|_{L^2}^2 + |B_N\bar{\mu}_n|_{L^2}^2) ds \le C, \\ &\int_0^T \left|\left(\frac{dv_n}{dt}, \frac{d\phi_n}{dt}\right)\right|_{L^2}^2 dt \le C. \end{aligned} \tag{105}$$

Therefore, we conclude that there exists a subsequence (v_n, ϕ_n, Q_n) still denoted (v_n, ϕ_n, Q_n) such that

$$\begin{aligned} (v_n, \phi_n) &\to (\tilde{v}, \tilde{\phi}) \text{ strongly in } C(0,T;\mathbb{Y}) \cap L^2(0,T;\mathbb{V}), \\ (v_n, \phi_n) &\to (\tilde{v}, \tilde{\phi}) \text{ weakly in } L^2(0,T;D(A_0) \times D(B_N^2)), \\ \left(\frac{dv_n}{dt}, \frac{d\phi_n}{dt}\right) &\to \left(\frac{d\tilde{v}}{dt}, \frac{d\tilde{\phi}}{dt}\right) \text{ weakly in } L^2(0,T;H), \\ Q_n &\to \tilde{Q} \text{ weakly in } L^2(0,T;\mathcal{U}). \end{aligned} \tag{106}$$

Let us prove that

$$\begin{aligned} &R_0(A_N\phi_n, \phi_n) \to R_0(A_N\tilde{\phi}, \tilde{\phi}) \\ &\text{and } \quad B_0(v_n, v_n) \to B_0(\tilde{v}, \tilde{v}) \text{ strongly in } L^2(0,T;H_1), \\ &B_1(v_n, \phi_n) \to B_1(\tilde{v}, \tilde{\phi}) \\ &\text{and } \quad f(\phi_n) \to f(\tilde{\phi}) \text{ strongly in } L^2(0,T;H_2). \end{aligned} \tag{107}$$

For $w \in H_1$, we have

$$
\begin{aligned}
&|\langle R_0(A_N\phi_n, \phi_n) - R_0(A_N\tilde{\phi}, \tilde{\phi}), w\rangle| \\
&\le |b_1(w, \phi_n - \tilde{\phi}, A_N\tilde{\phi})| + |b_1(w, \phi_n, A_N(\phi_n - \tilde{\phi}))| \\
&\le c|w|\|\phi_n - \tilde{\phi}\|^{1/2}|A_N(\phi_n - \tilde{\phi})|_{L^2}^{1/2}|A_N\tilde{\phi}|_{L^2}^{1/2}|A_N^{3/2}\tilde{\phi}|_{L^2}^{1/2} \\
&\qquad + c|w|\|\phi_n\|^{1/2}|A_N\phi_n|_{L^2}^{1/2}|A_N(\phi_n - \tilde{\phi})|_{L^2}^{1/2}|A_N^{3/2}(\phi_n - \tilde{\phi})|_{L^2}^{1/2},
\end{aligned}
\tag{108}
$$

which gives $\left(\text{since } \int_0^T |A_N^{3/2}(\phi_n - \tilde{\phi})|_{L^2}^2 dt \le C, \int_0^T |A_N^{3/2}\tilde{\phi}|_{L^2}^2 dt \le C\right)$

$$
\begin{aligned}
&\int_0^T |R_0(A_N\phi_n, \phi_n) - R_0(A_N\tilde{\phi}, \tilde{\phi})|_{L^2}^2 dt \\
&\le C\int_0^T \|\phi_n - \tilde{\phi}\||A_N(\phi_n - \tilde{\phi})|_{L^2}|A_N\tilde{\phi}|_{L^2}|A_N^{3/2}\tilde{\phi}|_{L^2} dt \\
&\qquad + C\int_0^T \|\phi_n\||A_N\phi_n|_{L^2}|A_N(\phi_n - \tilde{\phi})|_{L^2}|A_N^{3/2}(\phi_n - \tilde{\phi})|_{L^2} dt \\
&\le C\int_0^T |A_N(\phi_n - \tilde{\phi})|_{L^2}|A_N^{3/2}\tilde{\phi}|_{L^2} dt \\
&\qquad + C\int_0^T |A_N(\phi_n - \tilde{\phi})|_{L^2}|A_N^{3/2}(\phi_n - \tilde{\phi})|_{L^2} dt \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
\tag{109}
$$

We can also check that

$$
\begin{aligned}
&|B_0(v_n, v_n) - B_0(\tilde{v}, \tilde{v})|_{L^2} \\
&\le c|v_n - \tilde{v}|_{L^2}^{1/2}\|v_n - \tilde{v}\|^{1/2}\|v_n\|^{1/2}|A_0v_n|_{L^2}^{1/2} \\
&\qquad + c|\tilde{v}|_{L^2}^{1/2}\|\tilde{v}\|^{1/2}|A_0(v_n - \tilde{v})|_{L^2}^{1/2}\|v_n - \tilde{v}\|^{1/2},
\end{aligned}
\tag{110}
$$

which gives (since $\|v_n - \tilde{v}\| \le C,\ \|\tilde{v}\| \le C$)

$$
\int_0^T |B_0(v_n, v_n) - B_0(\tilde{v}, \tilde{v})|_{L^2}^2 dt \to 0 \quad \text{as } n \to \infty. \tag{111}
$$

This proves that $B_0(v_n, v_n) \to B_0(\tilde{v}, \tilde{v})$ and $R_0(A_N\phi_n\phi_n) \to R_0(A_N\tilde{\phi}, \tilde{\phi})$ strongly in $L^2(0, T; H_1)$. Similarly, we can also prove $(107)_2$.

Now using (106) and (107), we derive that

$$
\begin{aligned}
&\liminf_{n\to\infty} \int_0^T \left|\frac{dv_n}{dt} + \ A_0v_n + B_0(v_n, v_n) - R_0(A_N\phi_n, \phi_n) - DQ_n\right|_{L^2}^2 dt \\
&\ge \int_0^T \left|\frac{dv}{dt} + A_0v + B_0(v, v) - R_0(A_N\phi, \phi) - DQ\right|_{L^2}^2 dt,
\end{aligned}
\tag{112}
$$

$$\begin{aligned}&\liminf_{n\to\infty}\int_0^T\left|\frac{d\phi_n}{dt}+\ A_N^2\phi_n+B_1(v_n,\phi_n)+A_Nf(\phi_n)\right|_{L^2}^2 dt\\ \geq\ &\int_0^T\left|\frac{d\phi}{dt}+A_N^2\phi+B_1(v,\phi)+A_Nf(\phi)\right|_{L^2}^2 dt,\end{aligned}\tag{113}$$

From (87), (96) and (106), we also have

$$\begin{aligned}&\int_0^T|g_\epsilon(t,v_n,\phi_n)-g_\epsilon(t,\tilde{v},\tilde{\phi})|dt\\ &\leq C\int_0^T\|(v_n,\phi_n)-(\tilde{v},\tilde{\phi})\|dt\to 0\ \text{ as } n\to\infty,\end{aligned}\tag{114}$$

where C is independent of n.

Since h_ϵ is convex and continuous, we derive from (106) that

$$\liminf_{n\to\infty}\int_0^T h_\epsilon(v_n,\phi_n)dt\geq\int_0^T h_\epsilon(\tilde{v},\tilde{\phi})dt.\tag{115}$$

From (106) and (84), we also have

$$\lim_{n\to\infty}\mathcal{F}(v_n,\phi_n)=\mathcal{F}(\tilde{v},\tilde{\phi})\tag{116}$$

and therefore

$$\lim_{n\to\infty}\frac{1}{2\epsilon}(\epsilon+d_W\mathcal{F}(v_n,\phi_n))^2=\frac{1}{2\epsilon}(\epsilon+d_W\mathcal{F}(v,\phi))^2.\tag{117}$$

Since $(v_n,\phi_n)(0)=(v_0,\phi_0)$ and $(v_n,\phi_n)(0)\to(\tilde{v},\tilde{\phi})(0)$ strongly in $\mathbb{Y}$, we have $(\tilde{v},\tilde{\phi})(0)=(v_0,\phi_0)$. This shows that $(\tilde{v},\tilde{\phi})\in\mathcal{Y}_0$ and finally it follows from (101) and (112)–(117) that $(\tilde{v},\tilde{\phi},\tilde{Q})$ is optimal for problem (P_ϵ).

Lemma 3.3. *Let $(v_\epsilon,\phi_\epsilon,Q_\epsilon)$ be optimal for problem (P_ϵ). Then there exists a generalized subsequence $(v_\epsilon,\phi_\epsilon,Q_\epsilon)$ such that $(v_\epsilon,\phi_\epsilon)\to(v_*,\phi_*)$ strongly in $\mathcal{Y}$ and $Q_\epsilon\to Q_*$ strongly in $L^2(0,T;\mathcal{U})$ as $\epsilon\to 0$.*

Proof. Since $(v_\epsilon,\phi_\epsilon,Q_\epsilon)$ is solution to (P_ϵ), from (98) we have

$$L_\epsilon(v_\epsilon,\phi_\epsilon,Q_\epsilon)\leq L_\epsilon(v_*,\phi_*,Q_*)=\int_0^T(g_\epsilon(t,v_*,\phi_*)+h_\epsilon(Q_*))dt+\frac{\epsilon}{2}.\tag{118}$$

It follows from (96) that (see [21])

$$|g_\epsilon(t,v_*,\phi_*)-g^0(t,v_*,\phi_*)|\leq C(\|(v_*,\phi_*)-P_m(v_*,\phi_*)\|+\epsilon),\tag{119}$$

where C is independent of ϵ.

From (118), (119), we deduce as in [21] that

$$\limsup_{\epsilon \to 0} L(v_\epsilon, \phi_\epsilon, Q_\epsilon) \le L(v_*, \phi_*, Q_*). \tag{120}$$

From (98), we have

$$\begin{gathered}
\int_0^T \left| \frac{dv_\epsilon}{dt} + A_0 v_\epsilon + B_0(v_\epsilon, v_\epsilon) - R_0(A_N \phi_\epsilon, \phi_\epsilon) - DQ_\epsilon \right|_{L^2}^2 dt \le C\epsilon, \\
\int_0^T \left| \frac{d\phi_\epsilon}{dt} + A_N^2 \phi_\epsilon + B_1(v_\epsilon, \phi_\epsilon) + A_N f(\phi_\epsilon) \right|_{L^2}^2 dt \le C\epsilon, \\
(\epsilon + d_W \mathcal{F}(v_\epsilon, \phi_\epsilon))^2 \le C\epsilon.
\end{gathered} \tag{121}$$

Using (98), there exists $f_\epsilon = (f_\epsilon^1, f_\epsilon^2) \in L^2(0,T;H)$ such that

$$\begin{cases}
\dfrac{dv_\epsilon}{dt} + A_0 v_\epsilon + B_0(v_\epsilon, v_\epsilon) - R_0(A_N \phi_\epsilon, \phi_\epsilon) - DQ_\epsilon = f_\epsilon^1, \\
\dfrac{d\phi_\epsilon}{dt} + A_N^2 \phi_\epsilon + B_1(v_\epsilon, \phi_\epsilon) + A_N f(\phi_\epsilon) = f_\epsilon^2, \\
(v_\epsilon, \phi_\epsilon)(0) = (v_0, \phi_0),
\end{cases} \tag{122}$$

and

$$f_\epsilon = (f_\epsilon^1, f_\epsilon^2) \to 0 \text{ strongly in } L^2(0,T;H). \tag{123}$$

Using the same argument as in the proof of Lemma 3.2, we conclude that there exists a subsequence $(v_\epsilon, \phi_\epsilon, Q_\epsilon)$ of $(v_\epsilon, \phi_\epsilon, Q_\epsilon)$ and $(\tilde{v}, \tilde{\phi}, \tilde{Q}) \in \mathcal{Y} \times L^2(0,T;\mathcal{U})$ such that as $\epsilon \to 0$, we have

$$\begin{gathered}
(v_\epsilon, \phi_\epsilon) \to (\tilde{v}, \tilde{\phi}) \ \text{ strongly in } C(0,T;\mathbb{Y}) \cap L^2(0,T;\mathbb{V}), \\
(v_\epsilon, \phi_\epsilon) \to (\tilde{v}, \tilde{\phi}) \ \text{ weakly in } L^2(0,T;D(A_0) \times D(B_N^2)), \\
\frac{d}{dt}(v_\epsilon, \phi_\epsilon) \to \frac{d}{dt}(\tilde{v}, \tilde{\phi}) \ \text{ weakly in } L^2(0,T;H), \\
Q_\epsilon \to \tilde{Q} \ \text{ weakly in } L^2(0,T;\mathcal{U}), \\
B_0(v_\epsilon, v_\epsilon) \to B_0(\tilde{v}, \tilde{v}) \\
\text{and} \quad R_0(A_N \phi_\epsilon, \phi_\epsilon) \to R_0(A_N \tilde{\phi}, \tilde{\phi}) \text{ strongly in } L^2(0,T,H_1), \\
B_1(v_\epsilon, \phi_\epsilon) \to B_1(\tilde{v}, \tilde{\phi}) \\
\text{and} \quad f(\phi_\epsilon) \to f(\tilde{\phi}) \text{ strongly in } L^2(0,T,H_2).
\end{gathered} \tag{124}$$

Taking the limit in (122), we derive that

$$\begin{cases}
\dfrac{d\tilde{v}}{dt} + A_0 \tilde{v} + B_0(\tilde{v}, \tilde{v}) = R_0(A_N \tilde{\phi}, \tilde{\phi}) + D\tilde{Q}, \\
\dfrac{d\tilde{\phi}}{dt} + A_N^2 \tilde{\phi} + B_1(\tilde{v}, \tilde{\phi}) + A_N f(\tilde{\phi}) = 0, \\
(\tilde{v}, \tilde{\phi})(0) = (v_0, \phi_0).
\end{cases} \tag{125}$$

More precisely, at the limit we derive that $(\tilde{v}, \tilde{\phi})$ is a weak solution to (125). Since $\tilde{Q} \in L^2(0, T; \mathcal{U})$, it follows from Proposition 2.4 that $(\tilde{v}, \tilde{\phi}) \in \mathcal{Y}$ is the strong solution to (125).

From (84) and (124), we deduce that

$$\mathcal{F}(v_\epsilon, \phi_\epsilon) \to \mathcal{F}(\tilde{v}, \tilde{\phi}) \;\text{ strongly in } X \text{ as } \epsilon \to 0. \tag{126}$$

By (121), we have

$$\lim_{\epsilon \to 0} d_W \mathcal{F}(v_\epsilon, \phi_\epsilon) = 0, \tag{127}$$

which proves that

$$\mathcal{F}(\tilde{v}, \tilde{\phi}) \in W \tag{128}$$

since W is closed.

From (125) and (128), we obtain that

$$L(v_*, \phi_*, Q_*) \leq L(\tilde{v}, \tilde{\phi}, \tilde{Q}) \tag{129}$$

since (v_*, ϕ_*, Q_*) is optimal for problem (P).

From (87), (124) and (96), we also have

$$\begin{aligned}
|g_\epsilon(t, v_\epsilon, \phi_\epsilon) - g_\epsilon(t, \tilde{v}, \tilde{\phi})| &\leq C\|(v_\epsilon, \phi_\epsilon) - (\tilde{v} - \tilde{\phi})\|, \\
\lim_{\epsilon \to 0} g_\epsilon(t, \tilde{v}(t), \tilde{\phi}(t)) &= g^0(t, \tilde{v}(t), \tilde{\phi}) \;\; \forall t \in [0, T], \\
|g_\epsilon(t, \tilde{v}(t), \tilde{\phi}(t)) - g^0(t, \tilde{v}(t), \tilde{\phi}(t))| &\leq C(\|(\tilde{v}, \tilde{\phi}) - P_m(\tilde{v}, \tilde{\phi})\| + \epsilon),
\end{aligned} \tag{130}$$

where C is independent of ϵ.

Using (130) and the Lebesgue dominated theorem, we derive that

$$\begin{aligned}
&\int_0^T |g_\epsilon(t, v_\epsilon, \phi_\epsilon) - g^0(t, \tilde{v}, \tilde{\phi})| dt \\
\leq &\int_0^T (|g_\epsilon(t, v_\epsilon, \phi_\epsilon) - g_\epsilon(t, \tilde{v}, \tilde{\phi})| + |g_\epsilon(t, \tilde{v}, \tilde{\phi}) - g^0(t, \tilde{v}, \tilde{\phi})|) dt \\
\leq &\int_0^T (C\|(v_\epsilon, \phi_\epsilon)(t) - (\tilde{v}, \tilde{\phi})(t)\| + |g_\epsilon(t, \tilde{v}, \tilde{\phi}) - g^0(t, \tilde{v}, \tilde{\phi})|) dt \\
\to &\; 0 \;\text{ as } \epsilon \to 0.
\end{aligned} \tag{131}$$

Using the same argument as in [21], we also have

$$\liminf_{\epsilon \to 0} \int_0^T (h_\epsilon(Q_\epsilon) + \frac{1}{2}\|Q_\epsilon - Q_*\|_{\mathcal{U}}^2) dt \geq \int_0^T (h(\tilde{Q}) + \frac{1}{2}\|\tilde{Q} - Q_*\|_{\mathcal{U}}^2) dt. \tag{132}$$

It follows from (129), (131) and (132) that

$$\liminf_{\epsilon \to 0} L_\epsilon(v_\epsilon, \phi_\epsilon, Q_\epsilon) \geq L(\tilde{v}, \tilde{\phi}, \tilde{Q}) \geq L(v_*, \phi_*, Q_*). \tag{133}$$

Therefore using (29) and (133), we conclude that $(\tilde{v}, \tilde{\phi}) = (v_*, \phi_*)$, $\tilde{Q} = Q_*$ and

$$Q_\epsilon \to Q_* \ \text{ strongly in } L^2(0, T, \mathcal{U}). \tag{134}$$

Let us prove that

$$(v_\epsilon, \phi_\epsilon) \to (v_*, \phi_*) \ \text{ strongly in } \mathcal{Y}. \tag{135}$$

Let $(w, \psi) = (v_\epsilon, \phi_\epsilon) - (v_*, \phi_*)$. Then (w, ψ) satisfies

$$\begin{cases} \dfrac{dw}{dt} + A_0 w + B_0(v_*, w) + B(w, v_\epsilon) \\ \qquad = R_0(A_N \phi_*, \psi) + R_0(A_N \psi, \phi_\epsilon) + D(Q_\epsilon - Q_*) + f_\epsilon^1, \\ \dfrac{d\psi}{dt} + A_N^2 \psi + B_1(v_*, \psi) + B_1(w, \phi_\epsilon) + A_N(f(\phi_\epsilon) - f(\phi_*)) = f_\epsilon^2, \\ (w, \psi)(0) = 0. \end{cases} \tag{136}$$

Following the proof of Proposition 2.5, we derive the following estimates on w.

$$\begin{aligned} &\|(w, \psi)(t)\|_{\mathbb{V}}^2 + \int_0^t (|A_0 w|_{L^2}^2 + |B_N^2 \psi|_{L^2}^2) ds \\ &\le c_1 \int_0^T \left(\|Q_\epsilon - Q_*\|_{\mathcal{U}}^2 + |f_\epsilon|_{L^2}^2 \right) ds, \int_0^T \left| \frac{d}{dt}(w, \psi) \right|_{L^2}^2 dt \\ &\le c_1 \int_0^T \left(\|Q_\epsilon - Q_*\|_{\mathcal{U}}^2 ds + |f_\epsilon|_{L^2}^2 \right) ds. \end{aligned} \tag{137}$$

It follows from (123), (134) and (137) that $(v_\epsilon, \phi_\epsilon) \to (v_*, \phi_*)$ strongly in $\mathcal{Y}$.

We make the following assumption: for some $r > 0$,

$$\mathcal{F}'(v_*, \phi_*) R_r - W \ \text{ has finite codimensionality in } X, \tag{138}$$

where

$$\begin{aligned} R_r = \Bigg\{ &(z, \varphi) \in \mathcal{Y} : \ \frac{dz}{dt} + A_0 z + B_0(v_*, z) + B_0(z, v_*) \\ &= R_0(A_N \phi_*, \varphi) + R_0(A_N \varphi, \phi_*) + DQ, \\ &\frac{d\varphi}{dt} + A_N^2 \varphi + B_1(v_*, \varphi) + B_1(z, \phi_*) + A_N f'(\phi_*) \varphi = 0, \\ &(z, \varphi)(0) = 0, \ \text{for some } Q \in M(0, r) \Bigg\}, \\ M(0, r) = \Bigg\{ &Q \in L^2(0, T; \mathcal{U}) : \ \int_0^T \|Q\|_{\mathcal{U}}^2 dt \le r^2 \Bigg\}. \end{aligned} \tag{139}$$

□

4. Main result

Before we state the main result of this paper, we will first introduce some notations.

We define the linear operator $A : V \to V^*$ by

$$\langle Au_1, u_2\rangle = \langle A_0 v_1, v_2\rangle + \langle A_N^2 \phi_1, \phi_2\rangle, \tag{140}$$

for $u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2) \in V$, where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between V and V^* or between V_i and V_i^*, $i = 1, 2$.

We also define the following trilinear functional and associated operator

$$\langle B(u_1, u_2), u_3\rangle = b(u_1, u_2, u_3) = b_0(v_1, v_2, v_3) + b_1(v_1, \phi_2, \phi_3), \tag{141}$$

for $u_1 = (v_1, \phi_1)$, $u_2 = (v_2, \phi_2)$, $u_3 = (v_3, \phi_3)$ in V.

We introduce the bilinear operator $R : \mathbb{V} \times \mathbb{V} \to \mathbb{V}^*$ by

$$R(u_1, u_2) = (R_0(A_N \phi_1, \phi_2), 0), \tag{142}$$

for $u_1 = (v_1, \phi_1)$, $u_2 = (v_2, \phi_2) \in \mathbb{V}$.

Finally we set

$$E(u_1) = (0, f(\phi_1)), \qquad (E^{'}(u_1))u_2 = (0, A_N f^{'}(\phi_1)\phi_2). \tag{143}$$

for $u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2) \in H$.

To simplify the notations, we will also set

$$B(u_1) = B(u_1, u_1),\ \ R(u_1) = R(u_1, u_1). \tag{144}$$

Then we can check that

$$\begin{aligned} B^{'}(u_1)u_2 &= B(u_1, u_2) + B(u_2, u_1), \\ R^{'}(u_1)u_2 &= R(u_1, u_2) + R(u_2, u_1), \quad \forall u_1, u_2 \in \mathbb{V}. \end{aligned} \tag{145}$$

With the above notations, if we set $u = (v, \phi)$, $u_0 = (v_0, \phi_0)$ and $DQ \equiv (DQ, 0)$, then we can rewrite (25) as:

$$\frac{du}{dt} + Au + B(u, u) + Eu = R(u, u) + DQ, \quad u(0) = u_0. \tag{146}$$

Let $u = (v, \phi)$ be the strong solution to (146) given by Proposition 2.4. If we set $\omega = (w, \psi)$, $\omega_0 = (w_0, \psi_0)$ and $g = (g_1, g_2)$, then the linearized system (59) can be rewritten as:

$$\frac{d\omega}{dt} + A\omega + B(u, \omega) + B(\omega, u) + E^{'}(u)\omega = R(u, \omega) + R(\omega, u) + g,\ \omega(0) = \omega_0. \tag{147}$$

We will also consider the following adjoint system for which a result on the existence and uniqueness of solutions is given in Proposition A.1 in the Appendix.

$$-\frac{d\omega}{dt} + A\omega + (B^{'}(u))^{*}\omega + (E^{'}(u))^{*}\omega = (R^{'}(u))^{*}\omega + g, \ \omega(T) = \omega_0. \tag{148}$$

Let us now state and prove the main result of this article.

Theorem 4.1. *Suppose that* (87), (88), (84) *and* (138) *hold, where* (v_*, ϕ_*, Q_*) *is optimal for problem (P) and we set* $u_* = (v_*, \phi_*)$. *Then there exists* $(\lambda_0, p, \zeta_0) \in \Re \times \mathcal{W}(0,T) \times X^*$ *with* $(\lambda_0, \zeta_0) \neq 0$ *such that*

$$\begin{cases} -\frac{dp}{dt} + Ap + (B^{'}(u_*))^{*}p + (E^{'}(u_*))^{*}p + (\mathcal{F}^{'}(v_*, \phi_*))^{*}\zeta_0 \\ -(R^{'}(u_*))^{*}p \in -\lambda_0 \partial g(t, v_*, \phi_*) \quad a.e. \ in \ (0,T), \\ p(T) = 0, \end{cases} \tag{149}$$

and

$$\begin{aligned} &D^{*}p(t) \in \lambda_0 \partial h(Q_*(t)) \quad a.e. \ in \ (0,T), \\ &\left\langle \zeta_0, \Theta - \mathcal{F}^{'}(v_*, \phi_*) \right\rangle_{X^*, X} \leq 0 \quad \forall \Theta \in W. \end{aligned} \tag{150}$$

Moreover if $\mathcal{F}^{'}(v_*, \phi^*)$ *is injective, then* $(\lambda_0, p) \neq 0$.

Proof. Hereafter we set $u_\epsilon = (v_\epsilon, \phi_\epsilon)$, $u_\epsilon^\rho = (v_\epsilon^\rho, \phi_\epsilon^\rho)$. Let

$$Z = \{(v, \phi) \in \mathcal{Y} : \ (v, \phi)(0) = 0\}. \tag{151}$$

For $\Theta \in Z$ and $Q \in L^2(0,T;\mathcal{U})$, we set $u_\epsilon^\rho = u_\epsilon + \rho\Theta$, $Q_\epsilon^\rho = Q_\epsilon + \rho Q$, where (u_ϵ, Q_ϵ) is optimal for problem (P_ϵ). Then

$$\begin{aligned} &(u_\epsilon^\rho, Q_\epsilon^\rho) \equiv (v_\epsilon^\rho, \phi_\epsilon^\rho, Q_\epsilon^\rho) \\ &\to (u_\epsilon, Q_\epsilon) \equiv (v_\epsilon, \phi_\epsilon, Q_\epsilon) \ \text{ strongly in } \mathcal{Y} \times L^2(0,T;\mathcal{U}) \text{ as } \rho \to 0. \end{aligned} \tag{152}$$

We notice that

$$\begin{aligned} \frac{B(u_\epsilon^\rho) - B(u_\epsilon)}{\rho} &= B^{'}(u_\epsilon)\Theta + \rho B(u_\epsilon), \\ \frac{R(u_\epsilon^\rho) - R(u_\epsilon)}{\rho} &= R^{'}(u_\epsilon)\Theta + \rho R(u_\epsilon). \end{aligned} \tag{153}$$

Therefore

$$\begin{aligned} &\frac{B(u_\epsilon^\rho) - B(u_\epsilon)}{\rho} \to B^{'}(u_\epsilon)\Theta, \\ &\frac{R(u_\epsilon^\rho) - R(u_\epsilon)}{\rho} \to R^{'}(u_\epsilon)\Theta \ \text{ strongly in } L^2(0,T;H) \text{ as } \rho \to 0 \end{aligned} \tag{154}$$

since $B(u_\epsilon), R(u_\epsilon) \in L^2(0,T;H)$.

From (154), we also have

$$\begin{aligned}
&\frac{1}{2\epsilon\rho}\int_0^T \left\{ \left| \frac{du_\epsilon^\rho}{dt} + Au_\epsilon^\rho + Eu_\epsilon^\rho + B(u_\epsilon^\rho, u_\epsilon^\rho) - R(u_\epsilon^\rho, u_\epsilon^\rho) - DQ_\epsilon^\rho \right|_{L^2}^2 \right. \\
&\qquad \left. - \left| \frac{du_\epsilon}{dt} + Au_\epsilon + Eu_\epsilon + B(u_\epsilon, u_\epsilon) - R(u_\epsilon, u_\epsilon) - DQ_\epsilon \right|_{L^2}^2 \right\} dt \\
&\to \int_0^T \left\langle q_\epsilon, \frac{d\Theta}{dt} + A\Theta + E'(u_\epsilon)\Theta + B'(u_\epsilon)\Theta - R'(u_\epsilon)\Theta - DQ \right\rangle ds \\
&\qquad \text{as } \rho \to 0,
\end{aligned} \tag{155}$$

where q_ϵ is given by

$$q_\epsilon = \frac{1}{\epsilon}\left[\frac{du_\epsilon}{dt} + Au_\epsilon + Eu_\epsilon + B(u_\epsilon, u_\epsilon) - R(u_\epsilon, u_\epsilon) - DQ_\epsilon \right]. \tag{156}$$

As in [21], we can check that

$$\lim_{\rho\to 0}\frac{1}{\rho}\int_0^T (g_\epsilon(t, v_\epsilon^\rho, \phi_\epsilon^\rho) - g_\epsilon(t, v_\epsilon, \phi_\epsilon))dt = \int_0^T \langle \nabla g_\epsilon(t, v_\epsilon, \phi_\epsilon), \Theta\rangle\, dt, \tag{157}$$

$$\begin{aligned}
&\frac{1}{\rho}\int_0^T \left\{ (h_\epsilon(Q_\epsilon^\rho) - h_\epsilon(Q_\epsilon)) + \frac{1}{2}(\|Q_\epsilon^\rho - Q_*\|_{\mathcal{U}}^2 - \|Q_\epsilon - Q_*\|_{\mathcal{U}}^2) \right\} dt \\
&\to \int_0^T \langle \nabla h_\epsilon(Q_\epsilon) + Q_\epsilon - Q_*, Q\rangle_{\mathcal{U}}\, dt \quad \text{as } \rho \to 0,
\end{aligned} \tag{158}$$

$$\begin{aligned}
&\left\{ (\epsilon + d_W\mathcal{F}(v_\epsilon^\rho, \phi_\epsilon^\rho))^2 - (\epsilon + d_W\mathcal{F}(v_\epsilon, \phi_\epsilon))^2 \right\} \\
&\to \frac{\epsilon + d_W\mathcal{F}(v_\epsilon, \phi_\epsilon)}{\epsilon}\left\langle \zeta_\epsilon, \mathcal{F}'(v_\epsilon, \phi_\epsilon)\Theta \right\rangle_{X^*,X} \quad \text{as } \rho \to 0,
\end{aligned} \tag{159}$$

where $\nabla g_\epsilon(t, v_\epsilon, \phi_\epsilon)$ denotes the gradient of g_ϵ with respect to the variable $(v_\epsilon, \phi_\epsilon)$ at $(v_\epsilon, \phi_\epsilon)$ and $\nabla h_\epsilon(Q_\epsilon)$ denotes the gradient of h_ϵ at Q_ϵ, while $\zeta_\epsilon \in \partial d_W(\mathcal{F}(v_\epsilon, \phi_\epsilon))$, ∂d_W being the sub-differential of d_W. Note that

$$\|\zeta_\epsilon\|_{X^*} = \begin{cases} 1 & \text{if } \mathcal{F}(v_\epsilon, \phi_\epsilon) \notin W, \\ 0 & \text{if } \mathcal{F}(v_\epsilon, \phi_\epsilon) \in W, \end{cases} \tag{160}$$

since W is convex and closed and X^* is strictly convex.

Let

$$\lambda_\epsilon = \frac{\epsilon}{\epsilon + d_W\mathcal{F}(v_\epsilon, \phi_\epsilon)}, \qquad p_\epsilon = \lambda_\epsilon q_\epsilon \in L^2(0, T; H). \tag{161}$$

Since

$$\frac{L_\epsilon(v_\epsilon^\rho, \phi_\epsilon^\rho, Q_\epsilon^\rho) - L_\epsilon(v_\epsilon, \phi_\epsilon, Q_\epsilon)}{\rho} \geq 0 \quad \forall \rho > 0, \tag{162}$$

it follows from (98), (155)–(162) that

$$
\begin{aligned}
0 \le \lambda_\epsilon \int_0^T & [\langle \nabla g_\epsilon(t, v_\epsilon, \phi_\epsilon), \Theta\rangle + \langle \nabla h_\epsilon(Q_\epsilon), Q\rangle_{\mathcal{U}} + \langle Q_\epsilon - Q_*, Q\rangle_{\mathcal{U}}]\, dt \\
& + \int_0^T \left\langle p_\epsilon, \frac{d\Theta}{dt} + A\Theta + E^{'}(u_\epsilon)\Theta + B^{'}(u_\epsilon)\Theta - R^{'}(u_\epsilon)\Theta - DQ \right\rangle dt \\
& + \int_0^T \langle (\mathcal{F}^{'}(v_\epsilon, \phi_\epsilon))^* \zeta_\epsilon, \Theta\rangle dt \quad \forall (\Theta, Q) \in Z \times L^2(0, T; \mathcal{U}).
\end{aligned} \tag{163}
$$

Taking $\Theta = 0$ in (163) gives

$$
D^* p_\epsilon = \lambda_\epsilon \nabla h_\epsilon(Q_\epsilon) + \lambda_\epsilon (Q_\epsilon - Q_*) \quad \text{a.e. in } (0, T). \tag{164}
$$

Taking $Q = 0$ in (163) yields

$$
\begin{aligned}
0 = & \int_0^T \langle \lambda_\epsilon \nabla g_\epsilon(t, v_\epsilon, \phi_\epsilon) + (\mathcal{F}^{'}(v_\epsilon, \phi_\epsilon))^* \zeta_\epsilon, \Theta\rangle dt \\
& \quad + \int_0^T \left\langle p_\epsilon, \frac{d\Theta}{dt} + A\Theta + E^{'}(u_\epsilon)\Theta + B^{'}(u_\epsilon)\Theta - R^{'}(u_\epsilon)\Theta \right\rangle dt \\
& \qquad \forall \Theta = (z, \varphi) \in Z,
\end{aligned} \tag{165}
$$

which can be viewed as necessary conditions for $(v_\epsilon, \phi_\epsilon, Q_\epsilon)$.

We will now pass to the limit in (164)–(165) as $\epsilon \to 0$ to derive necessary conditions for (v_*, ϕ_*, Q_*).

Let us set

$$
\alpha_\epsilon = \lambda_\epsilon \nabla g_\epsilon(t, v_\epsilon, \phi_\epsilon) + (\mathcal{F}^{'}(v_\epsilon, \phi_\epsilon))^* \zeta_\epsilon \in L^2(0, T; V^*). \tag{166}
$$

Then (α_ϵ) is bounded in $L^2(0, T; V^*)$. Let $\tilde{p}_\epsilon \in \mathcal{W}(0, T)$ be the unique solution to (see Proposition A.1 in the Appendix)

$$
-\frac{d\tilde{p}_\epsilon}{dt} + A\tilde{p}_\epsilon + (E^{'}(u_\epsilon))^* \tilde{p}_\epsilon + (B^{'}(u_\epsilon))^* \tilde{p}_\epsilon - (R^{'}(u_\epsilon))^* \tilde{p}_\epsilon = -\alpha_\epsilon, \quad \tilde{p}_\epsilon(T) = 0. \tag{167}
$$

Multiplying (167) by Θ and integrating over $(0, T)$ gives

$$
\int_0^T \left\langle \tilde{p}_\epsilon, \frac{d\Theta}{dt} + A\Theta + E^{'}(u_\epsilon)\Theta + B^{'}(u_\epsilon)\Theta - R^{'}(u_\epsilon)\Theta \right\rangle dt = -\int_0^T \langle \alpha_\epsilon, \Theta\rangle dt, \tag{168}
$$

which implies that (see (165))

$$
\int_0^T \left\langle p_\epsilon - \tilde{p}_\epsilon, \frac{d\Theta}{dt} + A\Theta + E^{'}(u_\epsilon)\Theta + B^{'}(u_\epsilon)\Theta - R^{'}(u_\epsilon)\Theta \right\rangle dt = 0 \quad \forall \Theta \in Z. \tag{169}
$$

Since for every $g \in L^2(0,T;H)$, there exists $\Theta = (z,\varphi) \in Z$ such that (see Proposition 2.5)

$$\frac{d\Theta}{dt} + A\Theta + E^{'}(u_\epsilon)\Theta + B^{'}(u_\epsilon)\Theta - R^{'}(u_\epsilon)\Theta = g \text{ a.e. in } (0,T), \tag{170}$$

it follows from (169) that $p_\epsilon(t) = \tilde{p}_\epsilon(t)$ a.e. in $(0,T)$. Therefore $p_\epsilon = (p^1_\epsilon, p^2_\epsilon) \in \mathcal{W}(0,T)$ and satisfies (see Proposition A.1)

$$|(p^1_\epsilon, p^2_\epsilon)(t)|^2_{\mathbb{Y}} + \int_0^T (\|(p^1_\epsilon, p^2_\epsilon)(s)\|^2_{\mathbb{V}} ds \le c_1 \int_0^T |\alpha_\epsilon|^2_{V^*} dt \le C. \tag{171}$$

We conclude that there exists $p = (p^1, p^2) \in \mathcal{W}(0,T)$ and a subsequence (p_ϵ) of (p_ϵ) such that

$$\begin{gathered}
(p^1_\epsilon, p^2_\epsilon) \to (p^1, p^2) \text{ strongly in } L^2(0,T;H) \text{ as } \epsilon \to 0,\\
(p^1_\epsilon, p^2_\epsilon) \to (p^1, p^2) \text{ weakly in } L^2(0,T;V) \text{ as } \epsilon \to 0,\\
\left(\frac{dp^1_\epsilon}{dt}, \frac{dp^2_\epsilon}{dt}\right) \to \left(\frac{dp^1}{dt}, \frac{dp^2}{dt}\right) \text{ weakly in } L^2(0,T;V^*) \text{ as } \epsilon \to 0,\\
(p^1_\epsilon, p^2_\epsilon)(0) \to (p^1, p^2)(0) \text{ weakly in } H \text{ as } \epsilon \to 0.
\end{gathered} \tag{172}$$

From (160) and (161), we also have

$$1 \le \lambda_\epsilon + \|\zeta_\epsilon\|_{X'} \le 2 \quad \forall \epsilon > 0. \tag{173}$$

Therefore, there exists a generalized sequence $(\lambda_\epsilon, \zeta_\epsilon)$ of $(\lambda_\epsilon, \zeta_\epsilon)$ such that

$$\lambda_\epsilon \to \lambda_0, \ \zeta_\epsilon \to \zeta_0 \text{ weakly- star in } X^* \text{ as } \epsilon \to 0. \tag{174}$$

Using Lemma 3.3 and (172), we can pass to the limit in (164) and derive $(150)_1$. Now let us pass to the limit in (165). First we note that

$$\nabla g_\epsilon(t, v_\epsilon, \phi_\epsilon) \to \beta(t) \text{ weakly in } L^2(0,T;V^*) \text{ as } \epsilon \to 0, \tag{175}$$

where

$$\beta(t) \in \partial g(t, v_*(t), \phi_*(t)) \text{ a.e. in } (0,T). \tag{176}$$

From (84) and Lemma 3.3, we derive that

$$(\mathcal{F}^{'}(v_\epsilon, \phi_\epsilon))^* \zeta_\epsilon \to (\mathcal{F}^{'}(v_*, \phi_*))^* \zeta_0 \text{ weakly in } L^2(0,T;V^*) \text{ as } \epsilon \to 0. \tag{177}$$

We also have (see the proof in the Appendix)

$$\begin{gathered}
(B^{'}(u_\epsilon))^* p_\epsilon \to (B^{'}(u_*))^* p \text{ weakly star in } L^2(0,T;V^*) \text{ as } \epsilon \to 0,\\
(R^{'}(u_\epsilon))^* p_\epsilon \to (R^{'}(u_*))^* p \text{ weakly star in } L^2(0,T;V^*) \text{ as } \epsilon \to 0.
\end{gathered} \tag{178}$$

From (172), (174) and (175)–(178), we can pass to the limit in (167) to derive that $p \in \mathcal{Y}$ and satisfies (149).

Since $\zeta_\epsilon \in \partial d_W(\mathcal{F}(v_\epsilon, \phi_\epsilon))$, we also have

$$\langle \zeta_\epsilon, \Theta - \mathcal{F}(v_\epsilon, \phi_\epsilon)\rangle_{X^*,X} \leq 0, \quad \forall \Theta = (w, \psi) \in W, \tag{179}$$

which gives

$$\langle \zeta_\epsilon, \Theta - \mathcal{F}(v_*, \phi_*)\rangle_{X^*,X} \leq \langle \zeta_\epsilon, \mathcal{F}(v_\epsilon, \phi_\epsilon) - \mathcal{F}(v_*, \phi_*)\rangle_{X^*,X} \quad \forall \Theta \in W. \tag{180}$$

From (174) and Lemma 3.3, we can pass to the limit in (180) as $\epsilon \to 0$ and derive that $(150)_2$ holds true.

Next we prove that $(\lambda_0, \zeta_0) \neq 0$. We assume that $\lambda_0 = 0$. From (173), (174), there exist $\epsilon_1 > 0$ and $\delta > 0$ such that

$$2 \geq \|\zeta_\epsilon\|_{X^*} \geq \delta > 0 \quad \forall \epsilon < \epsilon_1. \tag{181}$$

From (163) and (180), we obtain that

$$\begin{aligned} & - \eta_\epsilon(\vartheta, Q) \\ \leq & \left\langle \zeta_\epsilon, \mathcal{F}'(v_*, \phi_*)\vartheta - \Theta + \mathcal{F}(v_*, \phi_*) \right\rangle_{X^*,X} \\ & \quad + \int_0^T \left\langle p_\epsilon, \frac{d\vartheta}{dt} + A\vartheta + E'(u_*)\vartheta + B'(u_*)\vartheta - R'(u_*)\vartheta - DQ \right\rangle dt \\ & \quad \forall (\vartheta, Q) \in Z \times L^2(0, T; \mathcal{U}), \Theta \in W, \end{aligned} \tag{182}$$

where

$$\begin{aligned} & \eta_\epsilon(\vartheta, Q) \\ = & \lambda_\epsilon \left\{ \int_0^T [\langle \nabla g_\epsilon(t, v_\epsilon, \phi_\epsilon), \vartheta\rangle + \langle \nabla h_\epsilon(Q_\epsilon), Q\rangle_{\mathcal{U}}]\, dt + \int_0^T \langle Q_\epsilon - Q_*, Q\rangle_{\mathcal{U}} dt \right\} \\ & + \int_0^T \left\langle p_\epsilon, (B'(u_\epsilon) - B'(u_*))\vartheta \right\rangle dt - \int_0^T \left\langle p_\epsilon, (R'(u_\epsilon) - R'(u_*))\vartheta \right\rangle dt \\ & + \left\langle \zeta_\epsilon, (\mathcal{F}'(v_\epsilon, \phi_\epsilon) - \mathcal{F}'(v_*, \phi_*))\vartheta + \mathcal{F}(v_\epsilon, \phi_\epsilon) - \mathcal{F}(v_*, \phi_*) \right\rangle_{X^*,X}. \end{aligned} \tag{183}$$

For every $\epsilon > 0$ and $Q \in M(0, r)$, let $\vartheta_\epsilon = (w_\epsilon, \psi_\epsilon) \equiv (w_\epsilon, \psi_\epsilon)(Q)$ be the solution to

$$\frac{d\vartheta_\epsilon}{dt} + A\vartheta_\epsilon + E'(u_\epsilon)\vartheta_\epsilon + B'(u_\epsilon)\vartheta_\epsilon = R'(u_\epsilon)\vartheta_\epsilon + DQ, \quad \vartheta_\epsilon(0) = 0, \tag{184}$$

where r is given by (138). Then from Proposition 2.5, $\vartheta_\epsilon \equiv (\xi_\epsilon, \varphi_\epsilon) \in Z$ and

$$\|(\xi_\epsilon, \varphi_\epsilon)\|_{\mathbb{V}}^2 + \int_0^T \left| \frac{d\vartheta_\epsilon}{dt} \right|_{L^2}^2 dt + \int_0^T (|A_0 \xi_\epsilon|_{L^2}^2 + |B_N^2 \varphi_\epsilon|_{L^2}^2) dt \leq C \quad \forall Q \in M(0, r), \tag{185}$$

where C is independent of ϵ and Q.

We can also check that (see (135), (124) and (185))

$$\begin{aligned} &\int_0^T |\langle (B^{'}(u_\epsilon) - B^{'}(u_*))\vartheta_\epsilon, p_\epsilon\rangle| dt \to 0 \quad \text{as } \epsilon \to 0, \\ &\int_0^T |\langle (R^{'}(u_\epsilon) - R^{'}(u_*))\vartheta_\epsilon, p_\epsilon\rangle| dt \to 0 \quad \text{as } \epsilon \to 0. \end{aligned} \tag{186}$$

From (183)–(186) and Lemma 3.3, we conclude that

$$\eta_\epsilon(\zeta_\epsilon(Q), Q) \to 0 \quad \text{as } \epsilon \to 0 \text{ uniformly in } Q \in M(0,r). \tag{187}$$

It follows from (139) and (182) that

$$\begin{gathered} \langle \zeta_\epsilon, \mathcal{F}^{'}(v_*, \phi_*)\vartheta - \Theta + \mathcal{F}(v_*, \phi_*)\rangle_{X^*,X} \geq -\eta_\epsilon \\ \forall \vartheta = (z, \varphi) \in R_r, \Theta = (w, \psi) \in W, \end{gathered} \tag{188}$$

where

$$\eta_\epsilon \to 0 \quad \text{as } \epsilon \to 0. \tag{189}$$

Using (138) we know that $\mathcal{F}^{'}(v_*, \phi_*)R_r - W$ and $\mathcal{F}^{'}(v_*, \phi_*)R_r - W - \{\mathcal{F}(v_*, \phi_*)\}$ have finite co-dimensionality in X. From (181), (187)–(188), we conclude as in [21] that $(\lambda_0, \zeta_0) \neq 0$.

Moreover, if $\mathcal{F}^{'}(v_*, \phi_*)$ is injective and $(\lambda_0, p) = 0$, then from (149) we have $(\mathcal{F}^{'}(v_*, \phi_*))^*\zeta_0 = 0$, which yields $\zeta_0 = 0$ and $(\zeta_0, \lambda_0) = 0$. This is a contradiction since $(\zeta_0, \lambda_0) \neq 0$. Therefore $(\lambda_0, p) \neq 0$ if $\mathcal{F}^{'}(v_*, \phi_*)$ is injective. □

A. Appendix

Let $u = (v, \phi)$ be the strong solution to (146) given by Proposition 2.4.

We now consider the following adjoint system:

$$-\frac{d\omega}{dt} + A\omega + (B^{'}(u))^*\omega + (E^{'}(u))^*\omega = (R^{'}(u))^*\omega + g, \quad \omega(T) = \omega_0, \tag{190}$$

where the unknown is $\omega = (w, \psi)$ and $\omega_0 = (w_0, \psi_0)$ is given.

Proposition A.1. *For $\omega_0 = (w_0, \psi_0) \in \mathbb{Y}$ and $g \in L^2(0,T;V^*)$, the system (190) has a unique solution $\omega = (w, \psi) \in L^2(0,T;\mathbb{V}) \cap C(0,T;\mathbb{Y})$. Moreover, the following estimate holds true:*

$$|(w,\psi)(t)|^2_{\mathbb{Y}} + \int_0^T \|(w,\psi)\|^2_{\mathbb{V}} ds \leq c_1|(w_0,\psi_0)|^2_{\mathbb{Y}} + c_1\int_0^T |g|^2_{V^*} ds. \tag{191}$$

Furthermore, if $(w_0,\psi_0)\in\mathbb{V}$ *and* $g\in L^2(0,T;H)$*, then the solution* $(w,\psi)\in C(0,T;\mathbb{V})\cap L^2(0,T;D(A_0)\times D(B_N^2))$ *and the following estimates hold*

$$\begin{aligned}
&\|(w,\psi)(t)\|_{\mathbb{V}}^2\int_0^T(|A_0w(s)|_{L^2}^2+|B_N^2\psi(s)|_{L^2}^2)ds\\
&\le c_1\|(w_0,\psi_0)\|_{\mathbb{V}}^2+c_1\int_0^T|g(s)|_{L^2}^2ds, \\
&\int_0^T\left|\frac{d}{dt}(w,\psi)\right|_{L^2}^2ds\le c_1\|(w_0,\psi_0)\|_{\mathbb{V}}^2+c_1\int_0^T|g(s)|_{L^2}^2ds.
\end{aligned}\tag{192}$$

Proof. We recall that $u=(v,\phi)$, $\omega=(w,\psi)$. Since $(v,\phi)\in L^\infty(0,T;\mathbb{V})\cap L^2(0,T;D(A_0)\times D(B_N^2))$, using the known regularity results for parabolic equations such as the linearized Navier-Stokes system (see e.g., [20]), we can rigorously prove the existence and uniqueness of solution to (190). To derive (192), we first note that for $\Theta=(\zeta,\varphi)$ regular enough, we have

$$\begin{aligned}
&\langle(B'(u))^*\omega,\Theta\rangle\\
&=\langle\omega,B'(u)\Theta\rangle=\langle w,B_0(v,\zeta)+B_0(\zeta,v)\rangle+\langle\psi,B_1(v,\varphi)+B_1(\zeta,\phi)\rangle\\
&=b_0(v,\zeta,w)+b_0(\zeta,v,w)+b_1(v,\varphi,\psi)+b_1(\zeta,\phi,\psi).
\end{aligned}\tag{193}$$

We also have

$$\begin{aligned}
\langle(R'(u))^*\omega,\Theta\rangle&=\langle\omega,R'(u)\Theta\rangle=\langle w,R_0(A_N\phi,\varphi)+R_0(A_N\varphi,\phi)\rangle\\
&=b_1(w,\varphi,A_N\phi)+b_1(w,\phi,A_N\varphi).
\end{aligned}\tag{194}$$

Multiplying (190) with $\Theta=(w,\psi)$ in H and using (193)–(194) give

$$\begin{aligned}
&\frac{d}{dt}(|w|_{L^2}^2+|\psi|_{L^2}^2)+2\|w\|^2+2|B_N\psi|_{L^2}^2\\
&\qquad+2b_0(w,v,w)+2b_1(v,\psi,\psi)+2b_1(w,\phi,\psi)+2(A_Nf'(\phi)\psi,\psi)_{L^2}\\
&=2b_1(w,\psi,A_N\phi)+2b_1(w,\phi,A_N\psi)+2\langle g_1,w\rangle+2\langle g_2,\psi\rangle.
\end{aligned}\tag{195}$$

It follows that

$$\begin{aligned}
&|(w,\psi)(t)|_{L^2}^2+\int_0^T(\|w(s)\|^2+|B_N\psi(s)|_{L^2}^2)ds\\
&\le c_1|(w_0,\psi_0)|_{L^2}^2+c_1\int_0^T|g|_{V^*}^2.
\end{aligned}\tag{196}$$

Multiplying (190) with $\Theta=(w,A_N\psi)$ in H and using (193)–(194) give

$$\begin{aligned}
&\frac{d}{dt}(|w|_{L^2}^2+\|\psi\|^2)+2\|w\|^2+2|B_N^{3/2}\psi|_{L^2}^2+2b_0(w,v,w)\\
&\qquad+2b_1(v,A_N\psi,\psi)+2b_1(w,\phi,\psi)+2(A_Nf'(\phi)\psi,A_N\psi)_{L^2}\\
&=2b_1(w,A_N\psi,A_N\phi)+2b_1(w,\phi,A_N^2\psi)+2\langle g_1,w\rangle+2\langle g_2,A_N\psi\rangle.
\end{aligned}\tag{197}$$

We note that

$$|b_0(w,v,w)| \leq \frac{1}{8}\|w\|^2 + c|w|_{L^2}^2\|v\|^2, \tag{198}$$

$$\begin{aligned} |b_1(v,A_N\psi,\psi)| &= |b_1(v,\psi,A_N\psi)| \\ &\leq \frac{1}{8}|B_N^{3/2}\psi|_{L^2}^2 + c|v|_{L^2}^2\|v\|^2\|\psi\|^2, \end{aligned} \tag{199}$$

$$\begin{aligned} |b_1(w,\phi,\psi)| &\leq c|w|_{L^2}^{1/2}\|w\|^{1/2}\|\phi\||\psi|_{L^2}^{1/2}\|\phi\|^{1/2} \\ &\leq \frac{1}{8}(|B_N^{3/2}\psi|_{L^2}^2 + \|w\|^2) + c|w|_{L^2}^2\|\phi\|^4, \end{aligned} \tag{200}$$

$$\begin{aligned} |b_1(w,A_N\psi,\phi)| &= |b_1(w,\phi,A_N\psi)| \\ &\leq c|w|_{L^2}^{1/2}\|w\|^{1/2}\|\phi\|^{1/2}|B_N\phi|_{L^2}^{1/2}|A_N\psi|_{L^2} \\ &\leq \frac{1}{8}(\|w\|^2 + |B_N^{3/2}\psi|_{L^2}^2) + c|w|_{L^2}^2|\phi|_{L^2}^2|B_N\phi|_{L^2}^2. \end{aligned} \tag{201}$$

$$|(A_Nf'(\phi)\psi, A_N\psi)_{L^2}| \leq c\frac{1}{8}|B_N^{3/2}\psi|_{L^2}^2 + Q_0(\|\phi\|)\|\psi\|^2. \tag{202}$$

$$\begin{aligned} &|b_1(w,\phi,A_N^2\phi)| \\ &\leq |A_N^{1/2}B_1(w,\phi)|_{L^2}|A_N^{3/2}\phi|_{L^2} \\ &\leq c(\|w\|\|\nabla\phi|_\infty + |w|^{1/2}\|w\|^{1/2}|A_N\phi|_{L^4})|A_N^{3/2}\phi|_{L^2} \\ &\leq \frac{1}{8}|A_N^{3/2}\phi|_{L^2}^2 + c\|w\|^2\|\phi\||A_N^{3/2}\phi|_{L^2} + c\|w\|^2|A_N\phi|_{L^2}|A_N^{3/2}\phi|_{L^2}. \end{aligned} \tag{203}$$

It follows from (197)–(203) that

$$\begin{aligned} &\frac{d}{dt}(|w|_{L^2}^2 + \|\psi\|^2) + c(\|w\|^2 + |B_N^{3/2}\psi|_{L^2}^2) \\ &\leq \mathcal{G}_1(t)(|w|_{L^2}^2 + \|\psi\|^2) + c|g|_{V^*}^2 + c\|w\|^2\|\phi\||A_N^{3/2}\phi|_{L^2}, \end{aligned} \tag{204}$$

where

$$\mathcal{G}_1(t) = c\|v\|^2 + c|v|_{L^2}^2\|v\|^2 + \|\phi\|^2|B_N\phi|_{L^2}^2 + Q_0(\|\phi\|)), \tag{205}$$

which gives

$$\begin{aligned} &|w(t)|_{L^2}^2 + \|\psi(t)\|^2 + \int_0^T (\|w\|^2 + |B_N^{3/2}\psi|_{L^2}^2)ds \\ &\leq c_1(|w_0|_{L^2}^2 + \|\psi_0\|^2) + c_1\int_0^T (|g(s)|_{V^*}^2 + \|w\|^2\|\phi\||A_N^{3/2}\phi|_{L^2})ds, \end{aligned} \tag{206}$$

and

$$|(w,\psi)(t)|_{\mathbb{Y}}^2 + \int_0^T \|(w,\psi)(s)\|_{\mathbb{V}}^2 ds \leq c_1|(w_0,\psi_0)|_{\mathbb{Y}}^2 + c_1\int_0^T |g(s)|_{V^*}^2 ds, \tag{207}$$

and (191) follows. Note that $\mathcal{G}_1(t) \in L^1(0,T)$.

To prove $(192)_1$, we multiply (190) by $(A_0 w, A_N\psi)$ and we use (193)–(194) to derive that

$$\begin{aligned}
&\frac{d}{dt}(\|w\|^2+\|\psi\|^2)+2|A_0w|_{L^2}^2+2|B_N^{3/2}\psi|_{L^2}^2+2b_0(v,A_0w,w)\\
&\qquad +2b_0(A_0w,v,w)+2b_1(v,A_N\psi,\psi)\\
&\qquad +2b_1(A_0w,\phi,\psi)+2(A_Nf^{'}(\phi)\psi,A_N\psi)_{L^2}\\
&=2b_1(w,A_N\psi,A_N\phi)+2b_1(w,\phi,A_N^2\psi)+2(g_1,A_0w)_{L^2}+2(g_2,A_N\psi)_{L^2}
\end{aligned}\tag{208}$$

We note that

$$\begin{aligned}
|b_0(v,A_0w,w)| &\le c|v|_{L^2}^{1/2}\|v\|^{1/2}\|w\|^{1/2}|A_0w|_{L^2}^{3/2}\\
&\le \frac{1}{8}|A_0w|_{L^2}^2+c|v|_{L^2}^2\|v\|^2\|w\|^2,
\end{aligned}\tag{209}$$

$$\begin{aligned}
|b_0(A_0w,v,w)| &\le c|A_0w|_{L^2}\|v\|^{1/2}|A_0v|_{L^2}^{1/2}\|w\|^{1/2}|w|_{L^2}^{1/2}\\
&\le \frac{1}{8}|A_0w|_{L^2}^2+c\|v\||A_0v|_{L^2}\|w\|^2,
\end{aligned}\tag{210}$$

$$\begin{aligned}
|b_1(A_0w,\phi,\psi)| &\le c|A_0w|_{L^2}\|\phi\|^{1/2}|A_N\phi|_{L^2}^{1/2}\|\psi\|^{1/2}|\psi|_{L^2}^{1/2}\\
&\le \frac{1}{8}(|A_0w|_{L^2}^2+|B_N^{3/2}\psi|_{L^2}^2)+c\|\phi\|^2|B_N\phi|_{L^2}^2\|\psi\|^2,
\end{aligned}\tag{211}$$

$$\begin{aligned}
ll|b_1(v,A_N\psi,\psi)| = |b_1(v,\psi,A_N\psi)| &\le c|v|_{L^2}^{1/2}\|v\|^{1/2}|B_N^{3/2}\psi|_{L^2}^{3/2}\|\psi\|^{1/2}\\
&\le \frac{1}{8}|B_N^{3/2}\psi|_{L^2}^2+c|v|_{L^2}^2\|v\|^2\|\psi\|^2,
\end{aligned}\tag{212}$$

$$\begin{aligned}
|b_1(w,A_N\psi,A_N\phi)| &\le c|w|_{L^2}^{1/2}\|w\|^{1/2}|B_N\phi|_{L^2}^{1/2}|B_N^{3/2}\phi|_{L^2}^{1/2}|B_N^{3/2}\psi|_{L^2}\\
&\le \frac{1}{8}|B_N^{3/2}\psi|_{L^2}^2+c\|w\|^2|B_N\phi|_{L^2}|B_N^{3/2}\phi|_{L^2},
\end{aligned}\tag{213}$$

$$\begin{aligned}
&|b_1(w,\phi,A_N^2\psi)|\\
&\le |\langle A_N^{1/2}B_1(w,\phi),A_N^{3/2}\psi\rangle|\\
&\le c\|w\|\|\phi\|^{1/2}|B_N^{3/2}\phi|_{L^2}^{1/2}|B_N^{3/2}\psi|_{L^2}\\
&\qquad +c|w|_{L^2}^{1/2}\|w\|^{1/2}|B_N\phi|_{L^2}^{1/2}|B_N^{3/2}\phi|_{L^2}^{1/2}|B_N^{3/2}\psi|_{L^2}\\
&\le \frac{1}{8}|B_N^{3/2}\psi|_{L^2}^2+c\|w\|^2\|\phi\||B_N^{3/2}\phi|_{L^2}+c\|w\|^2|B_N\phi|_{L^2}|B_N^{3/2}\phi|_{L^2},
\end{aligned}\tag{214}$$

$$|\langle f^{'}(\phi)\psi,A_N\psi\rangle| \le \tfrac{1}{8}|B_N^{3/2}\psi|_{L^2}^2+Q_0(\|\phi\|)\|\psi\|^2 \tag{215}$$

It follows from (208)–(215) that

$$\begin{aligned}
&\frac{d}{dt}(\|w(t)\|^2+\|\psi(t)\|^2)+|A_0w|_{L^2}^2+|B_N^{3/2}\psi|_{L^2}^2\\
&\le \mathcal{G}_2(t)(\|w\|^2+\|\psi(t)\|^2)+c|g|_{L^2}^2,
\end{aligned}\tag{216}$$

where

$$\begin{aligned}\mathcal{G}_2(t) = c(&|v|_{L^2}^2\|v\|^2 + \|v\||A_0 v|_{L^2} + \|\phi\|^2|B_N\phi|_{L^2}^2 \\ &+ |B_N\phi|_{L^2}|B_N^{3/2}\phi|_{L^2} + \|\phi\||B_N^{3/2}\phi|_{L^2}) + Q_0(\|\phi\|),\end{aligned} \tag{217}$$

which gives

$$\begin{aligned}&\|w(t)\|^2 + \|\psi(t)\|^2 + \int_0^T (|A_0 w|_{L^2}^2 + |B_N^{3/2}\psi|_{L^2}^2)ds \\ &\le c_1(\|w_0\|^2 + \|\psi_0\|^2) + c_1\int_0^T |g(s)|_{L^2}^2 ds,\end{aligned} \tag{218}$$

and $(192)_1$ follows. Note that $\mathcal{G}_2(t) \in L^1(0,T)$.

We multiply (190) by $(A_0 w, A_N^2\psi)$ and we use (193)–(194) to derive that

$$\begin{aligned}&\frac{d}{dt}(\|w\|^2 + |B_N\psi|_{L^2}^2) + 2|A_0 w|_{L^2}^2 + 2|B_N^2\psi|_{L^2}^2 \\ &\quad + 2b_0(v, A_0 w, w) + 2b_0(A_0 w, v, w) + 2b_1(v, A_N^2\psi, \psi) \\ &\quad + 2b_1(A_0 w, \phi, \psi) + 2(A_N f'(\phi)\psi, A_N^2\psi)_{L^2} \\ &= 2b_1(w, A_N^2\psi, A_N\phi) + 2b_1(w, \phi, A_N^3\psi) + 2(g_1, A_0 w)_{L^2} + 2(g_2, A_N\psi)_{L^2}.\end{aligned} \tag{219}$$

We note that

$$\begin{aligned}|b_1(v, A_N^2\psi, \psi)| &\le |b_1(v, \psi, A_N^2\psi)| \le c|v|_{L^2}^{1/2}\|v\|^{1/2}\|\psi\|^{1/2}|A_N\psi|_{L^2}^{1/2}|A_N^2\psi|_{L^2} \\ &\le \frac{1}{8}|A_N^2\psi|_{L^2}^2 + c|v|_{L^2}\|v\||A_N\psi|_{L^2}^2,\end{aligned} \tag{220}$$

$$\begin{aligned}|b_1(w, A_N^2\psi, A_N\phi)| &= |b_1(w, A_N\phi, A_N^2\psi)| \\ &\le c|w|_{L^2}^{1/2}|A_0 w|_{L^2}^{1/2}|A_N^{3/2}\phi|_{L^2}|A_N^2\psi|_{L^2} \\ &\le \frac{1}{8}(|A_0 w|_{L^2}^2 + |A_N^2\psi|_{L^2}^2) + c|w|_{L^2}^2|A_N^{3/2}\phi|_{L^2}^4,\end{aligned} \tag{221}$$

$$|2\langle A_N f'(\phi)\psi, A_N^2\psi\rangle| \le \frac{1}{8}|A_N^2\psi|_{L^2}^2 + Q_0(|B_N\phi|_{L^2})|B_N\psi|_{L^2}^2, \tag{222}$$

$$\begin{aligned}&|b_1(w, \phi, A_N^3\psi)| \\ &= |A_N B_1(w, \phi), A_N^2\psi)| \\ &\le c|A_0 w|_{L^2}\|\phi\|^{1/2}|A_N^{3/2}\phi|_{L^2}^{1/2}|A_N^2\psi|_{L^2} + c|w|_{L^2}^{1/2}|A_0 w|_{L^2}^{1/2}|A_N^{3/2}\phi|_{L^2}|A_N^2\psi|_{L^2} \\ &\le \frac{1}{8}|A_N^2\psi|_{L^2}^2 + c|A_0 w|_{L^2}\|\phi\||A_N^{3/2}\phi|_{L^2} + c|w|_{L^2}^2|A_N^{3/2}\phi|_{L^2}^4.\end{aligned} \tag{223}$$

It follows from (219)–(223) that

$$\begin{aligned}&\frac{d}{dt}(\|w(t)\|^2 + |B_N\psi|_{L^2}^2) + |A_0 w|_{L^2}^2 + |B_N^2\psi|_{L^2}^2 \\ &\le \mathcal{G}_3(t)(\|w\|^2 + |B_N\psi|_{L^2}^2) + \mathcal{H}_1(t),\end{aligned} \tag{224}$$

where

$$\begin{aligned}
G_3(t) &= \mathcal{G}_2(t) + c|A_N^{3/2}\phi|_{L^2}^4 Q_0(|B_N\phi|_{L^2}), \\
\mathcal{H}_1(t) &= c|g|_{L^2}^2 + c|A_0 w|_{L^2}\|\phi\||A_N^{3/2}\phi|_{L^2},
\end{aligned} \tag{225}$$

and $(192)_1$ follows. Note that $\mathcal{G}_3(t), \mathcal{H}(t) \in L^1(0,T)$. As in the proof of Proposition 2.5, we can easily prove that $(192)_2$ follows from $(192)_1$.

A.1. Proof of (178)

For $\Theta = (w, \psi) \in L^2(0,T;V)$, we have

$$\begin{aligned}
&|\langle B^{'}(u_\epsilon)^* p_\epsilon - B^{'}(u_*)^* p, \Theta\rangle| \\
&= |b_0(v_\epsilon, w, p_\epsilon^1) + b_0(w, v_\epsilon, p_\epsilon^1) - b_0(v_*, w, p^1) - b_0(w, v_*, p^1) \\
&\qquad + b_1(v_\epsilon, \psi, p_\epsilon^2) + b_0(w, \phi_\epsilon, p_\epsilon^2) - b_1(v_*, \psi, p^2) - b_1(w, \phi_*, p^2) \\
&\leq |b_0(v_\epsilon - v_*, w, p_\epsilon^1)| + |b_0(v_*, w, p_\epsilon^1 - p^1)| + |b_0(w, v_\epsilon - v_*, p_\epsilon^1)| \\
&\qquad + |b_0(w, v_*, p_\epsilon^1 - p^1)| + |b_1(v_\epsilon - v_*, \psi, p_\epsilon^2)| + |b_1(v_*, \psi, p_\epsilon^2 - p^2)| \\
&\qquad + |b_1(w, \phi_\epsilon - \phi_*, p_\epsilon^2)| + |b_1(w, \phi_*, p_\epsilon^2 - p^2)| \\
&\equiv M_1 + M_2 + \cdots + M_8.
\end{aligned} \tag{226}$$

Note that

$$M_1 = |b_0(v_\epsilon - v_*, w, p_\epsilon^1)| \leq c|v_\epsilon - v_*|_{L^2}^{1/2}\|v_\epsilon - v_*\|^{1/2}\|w\||p_\epsilon^1|_{L^2}^{1/2}\|p_\epsilon^1\|^{1/2}, \tag{227}$$

$$M_2 = |b_0(v_*, w, p_\epsilon^1 - p^1)| \leq c|v_*|_{L^2}^{1/2}\|v_*\|^{1/2}\|w\||p_\epsilon^1 - p^1|_{L^2}^{1/2}\|p_\epsilon^1 - p^1\|^{1/2}, \tag{228}$$

$$M_3 = |b_0(w, v_\epsilon - v_*, p_\epsilon^1)| \leq c|w|_{L^2}^{1/2}\|w\|^{1/2}\|v_\epsilon - v_*\||p_\epsilon^1|_{L^2}^{1/2}\|p_\epsilon^1\|^{1/2}, \tag{229}$$

$$M_4 = |b_0(w, v_*, p_\epsilon^1 - p^1)| \leq c|w|_{L^2}^{1/2}\|w\|^{1/2}\|v_*\||p_\epsilon^1 - p^1|_{L^2}^{1/2}\|p_\epsilon^1 - p^1\|^{1/2}, \tag{230}$$

$$M_5 = |b_1(v_\epsilon - v_*, \psi, p_\epsilon^2)| \leq c|v_\epsilon - v_*|_{L^2}^{1/2}\|v_\epsilon - v_*\|^{1/2}\|\psi\||p_\epsilon^2|_{L^2}^{1/2}\|p_\epsilon^2\|^{1/2}, \tag{231}$$

$$M_6 = |b_1(v_*, \psi, p_\epsilon^2 - p^2)| \leq c|v_*|_{L^2}^{1/2}\|v_*\|^{1/2}\|\psi\||p_\epsilon^2 - p^2|_{L^2}^{1/2}\|p_\epsilon^2 - p^2\|^{1/2}, \tag{232}$$

$$M_7 = |b_1(w, \phi_\epsilon - \phi_*, p_\epsilon^2)| \leq c|w|_{L^2}^{1/2}\|w\|^{1/2}\|\phi_\epsilon - \phi_*\||p_\epsilon^2|_{L^2}^{1/2}\|p_\epsilon^2\|^{1/2}, \tag{233}$$

$$M_8 = |b_1(w, \phi_*, p_\epsilon^2 - p^2)| \leq c|w|_{L^2}^{1/2}\|w\|^{1/2}\|\phi_*\||p_\epsilon^2 - p^2|_{L^2}^{1/2}\|p_\epsilon^2 - p^2\|^{1/2}, \tag{234}$$

and it follows from (226)–(234) that (see (135) and (172))

$$\int_0^T |\langle B^{'}(u_\epsilon)^* p_\epsilon - B^{'}(u_*)^* p, \Theta\rangle| dt \to 0 \quad \text{as } \epsilon \to 0, \tag{235}$$

which proves $(178)_1$.

To prove $(178)_2$, we note that

$$\begin{aligned}
&|\langle R^{'}(u_\epsilon)^* p_\epsilon - R^{'}(u_*)^* p, \Theta\rangle| \\
&\le |b_1(p^1_\epsilon - p^1, \psi, A_N\phi_\epsilon)| + |b_1(p^1, \psi, A_N\phi_\epsilon - A_N\phi_*)| \\
&\qquad + |b_1(p^1_\epsilon - p^1, \phi_\epsilon, A_N\psi)| + |b_1(p^1, \phi_\epsilon - \phi_*, A_N\psi)| \\
&\equiv N_1 + N_2 + N_3 + N_4
\end{aligned} \tag{236}$$

We note that

$$\begin{aligned}
N_1 &= |b_1(p^1_\epsilon - p^1, \psi, A_N\phi_\epsilon)| \\
&\le c|p^1_\epsilon - p^1|^{1/2}_{L^2}\|p^1_\epsilon - p^1\|^{1/2}\|\psi\||A_N\phi_\epsilon|^{1/2}_{L^2}\|B_N^{3/2}\phi_\epsilon\|^{1/2},
\end{aligned} \tag{237}$$

$$\begin{aligned}
N_2 &= |b_1(p^1, \psi, A_N\phi_\epsilon - A_N\phi_*)| \\
&\le c|p^1|^{1/2}_{L^2}\|p^1\|^{1/2}\|\psi\||A_N(\phi_\epsilon - \phi_*)|^{1/2}_{L^2}\|B_N^{3/2}(\phi_\epsilon - \phi_*)\|^{1/2},
\end{aligned} \tag{238}$$

$$\begin{aligned}
N_3 &= |b_1(p^1_\epsilon - p^1, \phi_\epsilon, A_N\psi)| \\
&\le c|\langle A_N^{1/2}B_1(p^1_\epsilon - p^1, \phi_\epsilon), A_N^{1/2}\psi)\rangle| \\
&\le c\|p^1_\epsilon - p^1\|^{1/2}|A_N(p^1_\epsilon - p^1)|^{1/2}_{L^2}\|\phi_\epsilon\|^{1/2}|A_N\phi_\epsilon|^{1/2}_{L^2}\|\psi\| \\
&\qquad + c|p^1_\epsilon - p^1|^{1/2}_{L^2}\|p^1_\epsilon - p^1\|^{1/2}|A_N\phi_\epsilon|^{1/2}_{L^2}|B_N^{3/2}\phi_\epsilon|^{1/2}_{L^2}, \|\psi\|,
\end{aligned} \tag{239}$$

$$\begin{aligned}
N_4 &= |b_1(p^1, \phi_\epsilon - \phi_*, A_N\psi)| \le c|\langle A_N^{1/2}B_1(p^1, \phi_\epsilon - \phi_*), A_N^{1/2}\psi)\rangle| \\
&\le c\|p^1\|\|\phi_\epsilon - \phi_*\|^{1/2}|A_N^{3/2}(\phi_\epsilon - \phi_*)|^{1/2}_{L^2}\|\psi\| \\
&\qquad + c|p^1|^{1/2}_{L^2}\|p^1\|^{1/2}|A_N(\phi_\epsilon - \phi_*)|^{1/2}_{L^2}|B_N^{3/2}(\phi_\epsilon - \phi_*)|^{1/2}_{L^2}\|\psi\|,
\end{aligned} \tag{240}$$

and it follows from (236)–(240) that (see (135) and (172))

$$\int_0^T |\langle R^{'}(u_\epsilon)^* p_\epsilon - R^{'}(u_*)^* p, \Theta\rangle| dt \to 0 \quad \text{as } \epsilon \to 0, \tag{241}$$

which proves $(178)_2$. □

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References

[1] H. Abels: On a diffuse interface model for a two-phase flow of compressible viscous fluids, Indiana Univ. Math. J. 57 (2008) 659–698.

[2] H. Abels: On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities, Arch. Ration. Mech. Anal. 194 (2009) 463–506.

[3] F. Abergel, R. Temam: On some control problems in fluid mechanics, Theor. Comput. Fluid Dyn. 1 (1990) 303–325.

[4] T. Blesgen: A generalization of the Navier-Stokes equation to two-phase flow, J. Phys. D: Appl. Phys. 32 (1999) 1119–1123.

[5] F. Boyer: Mathematical study of multi-phase flow under shear through order parameter formulation, Asymptotic Anal. 20 (1999) 175–212.

[6] F. Boyer: Nonhomogeneous cahn-hilliard fluids, Ann. Inst. H. Poincaré, Anal. Non Linéaire 18 (2001) 225–259.

[7] F. Boyer: A theoretical and numerical model for the study of incompressible mixture flows, Comput. Fluids 31 (2002) 41–68.

[8] C. Cao, C. G. Gal: Global solutions for the 2d ns-ch model for a two-phase flow of viscous, incompressible fluids with mixed partial viscosity and mobility, Nonlinearity 25(11) (2012) 3211–3234.

[9] G. Caginalp: An analysis of a phase field model of a free boundary, Arch. Rational Mech. Anal. 92(3) (1986) 205–245.

[10] E. Feireisl, H. Petzeltová, E. Rocca, G. Schimperna: Analysis of a phase-field model for two-phase compressible fluids, Math. Models Methods Appl. Sci. 20(7) (2010) 1129–1160.

[11] C. G. Gal, M. Grasselli: Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27(1) (2010) 401–436.

[12] C. G. Gal, M. Grasselli: Longtime behavior for a model of homogeneous incompressible two-phase flows, Discrete Contin. Dyn. Syst. 28(1) (2010) 1–39.

[13] C. G. Gal, M. Grasselli: Trajectory attractors for binary fluid mixtures in 3D, Chin. Ann. Math., Ser. B 31(5) (2010) 655–678.

[14] M. E. Gurtin, D. Polignone, J. Viñals: Two-phase binary fluids and immiscible fluids described by an order parameter, Math. Models Methods Appl. Sci. 6 (1996) 8–15.

[15] P. C. Hohenberg, B. I. Halperin: Theory of dynamic critical phenomena, Rev. Mod. Phys. 49 (1977) 435–479.

[16] S. Li: Optimal controls of Boussinesq equations with state constraints, Nonlinear Anal. 60 (2005) 1485–1508.

[17] X. Li, J. Yong: Optimal Control Theory for Infinite Dimensional Systems, Birkhäuser, Boston (1995).

[18] J. L. Lions: Optimal Control of Systems Governed by Partial Differential Equations, Springer, New York (1970).

[19] A. Onuki: Phase transition of fluids in shear flow, J. Phys.: Condens. Matter 9 (1997) 6119–6157.

[20] R. Temam: Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd Ed., Appl. Math. Sci. 68, Springer, New York (1997).

[21] G. Wang: Optimal controls of 3-dimensional Navier-Stokes equations with state constraints, SIAM J. Control Optim. 41(2) (2002) 583–606.

[22] G. Wang: Pontryagin maximum principle of optimal control governed by fluid dynamic systems with two point boundary state constraint, Nonlinear Anal. 51 (2002) 509–536.

[23] G. Wang: Pontryagin's maximum principle for optimal control of the stationary Navier-Stokes equations, Nonlinear Anal. 52 (2003) 1853–1866.

[24] G. Wang, L. Wang: Maximum principle of state-constrained optimal control governed by fluid dynamic systems, Nonlinear Anal. 52 (2003) 1911–1931.