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Strictly Convex Space: Strong Orthogonality and Conjugate Diameters

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In a normed linear space X an element x is said to be orthogonal to another element y in the sense of Birkhoff-James, written as $x \perp_B y$, iff $\|x\| \leq \|x + \lambda y\|$ for all scalars λ. We prove that a normed linear space X is strictly convex iff for any two elements x, y of the unit sphere S_X, $x \perp_B y$ implies $\|x + \lambda y\| > 1 \ \forall \ \lambda \neq 0$. We apply this result to find a necessary and sufficient condition for a Hamel basis to be a strongly orthonormal Hamel basis in the sense of Birkhoff-James in a finite dimensional real strictly convex space X. Applying the result we give an estimation for lower bounds of $\|tx + (1-t)y\|$, $t \in [0, 1]$ and $\|y + \lambda x\|$, $\forall \ \lambda$ for all elements $x, y \in S_X$ with $x \perp_B y$. We find a necessary and sufficient condition for the existence of conjugate diameters through the points $e_1, e_2 \in S_X$ in a real strictly convex space of dimension 2. The concept of generalized conjugate diameters is then developed for a real strictly convex smooth space of finite dimension.

Keywords: Orthogonality, strict convexity, extreme point, conjugate diameters

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1. Introduction

Suppose $(X, \|.\|)$ is a normed linear space, real or complex. X is said to be strictly convex iff every element of the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ is an extreme point of the unit ball $B_X = \{x \in X : \|x\| \leq 1\}$. There are many equivalent characterizations of the strict convexity of a normed space, some of them given in [11, 16] are

(i) If $x, y \in S_X$ are linearly independent then we have $\|x + y\| < 2$,
(ii) Every non-zero continuous linear functional attains a maximum on at most

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one point of the unit sphere,

(iii) If $\|x+y\| = \|x\| + \|y\|,\ x \neq 0$ then $y = cx$ for some $c \geq 0$.

A normed linear space X is said to be uniformly convex iff given $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in S_X$ and $\|x - y\| \geq \epsilon$ then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

The number $\delta(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| \ : \ x, y \in S_X,\ \|x - y\| \geq \epsilon\}$ is called the modulus of convexity of X. A space X is strictly convex iff $\delta(2) = 1$.

The concept of strictly convex and uniformly convex spaces have been extremely useful in the studies of the geometry of Banach Spaces. One may go through [2, 5–8, 10, 11, 14, 16–18] for more information related to strictly convex spaces and [4, 6, 7, 11] for uniformly convex spaces.

In a normed linear space X an element x is said to be orthogonal to another element y in the sense of Birkhoff-James [3, 10, 11], written as $x \perp_B y$, iff

$$\|x\| \leq \|x + \lambda y\| \ \text{ for all scalars } \lambda.$$

Recently in [13] we introduced the notions of strongly orthogonal set and strongly orthonormal Hamel basis in the sense of Birkhoff-James. For the sake of clarity we mention them briefly here.

Strongly orthogonal in the sense of Birkhoff-James: In a normed linear space X an element x is said to be strongly orthogonal to another element y in the sense of Birkhoff-James, written as $x \perp_{SB} y$, iff

$$\|x\| < \|x + \lambda y\| \ \text{ for all scalars } \lambda \neq 0.$$

If $x \perp_{SB} y$ then $x \perp_B y$ but the converse is not true. In $(\mathbb{R}^2, \|.\|_\infty)$ the element $(1, 0)$ is orthogonal to $(0, 1)$ in the sense of Birkhoff-James but not strongly orthogonal.

Strongly Orthogonal Set in the sense of Birkhoff-James: A finite set of elements $\{x_1, x_2, \ldots, x_n\}$ in a normed linear space X is said to be a strongly orthogonal set in the sense of Birkhoff-James iff for each $i \in \{1, 2, \ldots, n\}$,

$$\|x_i\| < \left\| x_i + \sum_{j=1, j\neq i}^{n} \lambda_j x_j \right\|,$$

whenever not all λ_j's are 0.

In addition if $\|x_i\| = 1$, for each i, then the set is called strongly orthonormal set in the sense of Birkhoff-James.

Strongly orthonormal Hamel basis in the sense of Birkhoff-James: A finite set of elements $\{x_1, x_2, \ldots, x_n\}$ in a normed linear space X is said to be a strongly orthonormal Hamel basis in the sense of Birkhoff-James iff the set is a Hamel basis of X and is a strongly orthonormal set in the sense of Birkhoff-James. As for example the set $\{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots (0, 0, \ldots, 0, 1)\}$

is a strongly orthonormal Hamel basis in the sense of Birkhoff-James in $(\mathbb{R}^n, \|.\|_p)$ for $(1 \leq p < \infty)$ but not in $(\mathbb{R}^n, \|.\|_\infty)$.

We show that a normed linear space X is strictly convex iff for $x, y \in S_X$ we have $x \perp_B y$ implies $x \perp_{SB} y$. Using this result we find a necessary and sufficient condition for a Hamel basis to be a strongly orthonormal Hamel basis in the sense of Birkhoff-James. We also show that in a normed linear space X if $x, y \in S_X$ and $x \perp_B y$ then $\|y + \lambda x\| \geq \frac{1}{2}$, for all λ. If the space is strictly convex then we show that $\|y + \lambda x\| > \frac{1}{2}$ for all λ. That the condition is only necessary but not sufficient is illustrated with an example. The result does not get better in uniformly convex spaces too. Following Theorem 1 of James [9] we conclude that a real normed linear space X (of dimension > 2) is an inner product space iff for $x, y \in S_X$ we have $x \perp_B y$ implies $y \perp_{SB} x$. We also find a necessary and sufficient condition for two diameters of S_X to be conjugate diameters in a real strictly convex space of dimension 2. Thus given any two diameters in a strictly convex space we can say whether they are conjugate or not. Finally we show that if X is a real strictly convex smooth space of dimension 2 then S_X is a Radon curve iff at each point of S_X there exists a strongly orthonormal Hamel basis in the sense of Birkhoff-James containing that point. The concept of generalized conjugate diameters is then developed for a real strictly convex smooth space of finite dimension higher than 2.

2. Main Results

We first obtain the characterization of strictly convex spaces.

Theorem 2.1. *Suppose X is a real normed linear space. If for $x, y \in X - \{0\}$, $x \perp_B y$ implies $x \perp_{SB} y$ then X is strictly convex.*

Proof. If possible let the unit sphere S_X contains a straight line segment i.e., there exists $\|u\| = \|v\| = 1$ with $\|tu + (1-t)v\| = 1 \ \forall t \in [0, 1]$.

Let $x = \frac{1}{2}u + \frac{1}{2}v$, $y = v - u$. Consider $x + \lambda y$.

If $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$ then $\|x + \lambda y\| = \|tu + (1-t)v\| = 1$ where $t = \frac{1}{2} - \lambda$.

If $\lambda < -\frac{1}{2}$ then $\frac{1}{2} - \lambda > 1$ and so we can write $\frac{1}{2} - \lambda = t\alpha$ for some $t \in (0, 1)$ and $\alpha > 1$. In this case

$$\begin{aligned}
\|x + \lambda y\| &= \|t\alpha\ u + (1 - t\alpha)\ v\| \\
&= \|t\alpha\ u + (\alpha - t\alpha)\ v + (1-\alpha)v\| \\
&\geq\ |\|t\alpha\ u + (\alpha - t\alpha)\ v\| - \|(\alpha - 1)v\|| \\
&=\ ||\alpha| - |\alpha - 1|| \\
&= 1
\end{aligned}$$

If $\lambda > \frac{1}{2}$ then $\frac{1}{2} + \lambda > 1$ and so we can write $\frac{1}{2} + \lambda = t\alpha$ for some $t \in (0, 1)$ and

$\alpha > 1$. In this case

$$\begin{aligned} \|x + \lambda y\| &= \|(1 - t\alpha)\ u + t\alpha\ v\| \\ &= \|(\alpha - t\alpha)\ u + t\alpha\ v + (1 - \alpha)u\| \\ &\geq\ |\|(\alpha - t\alpha)\ u + t\alpha\ v\| - \|(\alpha - 1)u\|| \\ &=\ ||\alpha| - |\alpha - 1|| \\ &= 1 \end{aligned}$$

Thus $\|x + \lambda y\| \geq \|x\|\ \forall \lambda$ but $\|x + \lambda_0 y\| = \|x\|$, for $\lambda_0 \in [-1/2, 1/2]$. This is a contradiction to our hypothesis and so X is strictly convex.

Alternatively the result can be proved using the convexity of the unit ball B_X and noting that for each λ, $x + \lambda y = \alpha u + \beta v$ where $\alpha + \beta = 1$.

Conversely we prove the following

Theorem 2.2. *Suppose X is a strictly convex space and $x, y \in X - \{0\}$ with $x \perp_B y$, then $x \perp_{SB} y$.*

Proof. Without loss of generality we assume that there exists $x, y \in X$, $\|x\| = 1$ such that $\|x + \lambda y\| \geq 1\ \forall\ \lambda$ but $\|x + \lambda_0 y\| = 1$ for some $\lambda_0 \neq 0$.

Let $0 < t < 1$, then

$$\begin{aligned} & 1 = t\|x\| + (1 - t)\|x + \lambda_0 y\| \geq \|tx + (1 - t)(x + \lambda_0 y)\| \\ \Rightarrow\quad & 1 \geq \|tx + (1 - t)(x + \lambda_0 y)\| = \|x + (1 - t)\lambda_0 y\| \geq 1 \end{aligned}$$

Thus $\|tx + (1 - t)(x + \lambda_0 y)\| = 1$ – which contradicts the fact that X is strictly convex.

Corollary 2.3. *If a normed linear space X is strictly convex and $x_1, x_2, \ldots, x_n \in S_X$ $(n \geq 2)$ are linearly independent then we have $\|x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n\| \geq 1$ for all $\lambda_2, \lambda_3, \ldots, \lambda_n$ implies $\|x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n\| > 1$ for all $\lambda_2, \lambda_3, \ldots, \lambda_n$ with $(\lambda_2, \lambda_3, \ldots, \lambda_n) \neq (0, 0, \ldots, 0)$.*

Proof. Follows easily from the last theorem.

Thus we have the following characterization of a real strictly convex space which follows easily from the last two theorems.

Theorem 2.4. *A real normed linear space X is strictly convex iff*

$$x, y \in S_X \text{ and } x \perp_B y \Rightarrow x \perp_{SB} y.$$

James [9] proved that a real normed linear space X of dimension > 2 is an inner product space iff Birkhoff-James orthogonality is symmetric. The following characterization of a real inner product space now follows easily from Theorem 2.4 mentioned above and Theorem 1 of James[9].

Theorem 2.5. *A real normed linear space X of dimension > 2 is an inner product space iff*

$$x, y \in S_X \text{ and } x \perp_B y \Rightarrow x \perp_{SB} y \text{ and } y \perp_{SB} x.$$

We next try to calculate a lower bound for $\|y+\lambda x\| \ \forall \lambda$ if $x, y \in S_X$ and $x \perp_B y$ in a normed linear space. We first prove the following two theorems:

Theorem 2.6. *Suppose X is a normed linear space and $x, y \in S_X$ with $\|x + \lambda y\| \geq 1 \ \forall \lambda$. Then there exists a $\delta \geq \frac{1}{3}$ such that $\|tx + (1-t)y\| \geq \delta \ \forall t \in [0, 1]$. In addition $\delta > \frac{1}{3}$ if X is strictly convex.*

Proof. Consider $f : [0, 1] \longrightarrow [0, 1]$ defined as $f(t) = \|tx + (1-t)y\|$. Then f is a continuous function on a compact set and so attains its minimum value. As $f(0) = 1$ by the property of continuity we can find $\delta_0 \in (0, 1)$ such that $f(t) > \frac{1}{2} \ \forall \ t \in (0, \delta_0)$.
Again for all t with $t \geq \delta_0$ we get $f(t) = \|tx + (1-t)y\| \geq \delta_0$. Thus we get

$$\|tx + (1-t)y\| \geq \min\left\{\delta_0, \frac{1}{2}\right\} = \delta \quad \forall t \in [0, 1].$$

Clearly for all $t \in [0, 1]$, $\|tx + (1-t)y\| \geq |t|$.
Also for all $t \in [0, 1]$, $\|tx + (1-t)y\| \geq ||t| - |(1-t)|| = |2t - 1|$. Thus $\|tx + (1-t)y\| \geq \frac{1}{3} \ \forall t \in [0, 1]$.
If X is strictly convex then by Theorem 2.2, $\|tx + (1-t)y\| > t$ for all $t \in [0, 1)$, also for all $t \in [0, 1]$, $\|tx + (1-t)y\| \geq |2t - 1|$ and so $\delta > \frac{1}{3}$.

Remark 2.7. Geometrically this shows that if an element x of the unit sphere is orthogonal to an element y of the unit sphere then the line segment joining x and y lies outside the open ball of radius $\frac{1}{3}$.

Remark 2.8. If X is finite dimensional then there exists $\delta > 0$ such that $\|tx + (1-t)y\| \geq \delta \ \forall t \in [0, 1]$ and for all $x, y \in S_X$ with $x \perp_B y$ (consider the continuous function $f : M \times [0, 1] \longrightarrow \mathbb{R}$ defined on the compact set $M \times [0, 1]$ as $f(x, y, t) = \|tx + (1-t)y\|$ where $M = \{(x, y) \in S_X \times S_X : x \perp_B y\}$).

Theorem 2.9. *Suppose $(X, \|.\|)$ is a normed linear space and $x, y \in S_X$ such that $\|x + \lambda y\| \geq 1 \ \forall \lambda$. Then there exists $\delta \geq \frac{1}{2}$ such that $\|y + \lambda x\| \geq \delta \ \forall \lambda$. In addition $\delta > \frac{1}{2}$ if X is a strictly convex normed linear space.*

Proof. Clearly for all λ, $\|y + \lambda x\| \geq |\lambda|$.
Also for all λ, $\|y + \lambda x\| \geq |1 - |\lambda||$.
Thus for all λ, $\|y + \lambda x\| \geq \frac{1}{2}$.
If X is strictly convex then using Theorem 2.1, $\|y + \lambda x\| > |\lambda|$ for all λ and $\|y + \lambda x\| \geq |1 - |\lambda||$ so that $\delta > \frac{1}{2}$.

Remark 2.10. Geometrically this shows that if an element x of the unit sphere is orthogonal to an element y of the unit sphere then the line through y parallel to x lies outside the open ball of radius $\frac{1}{2}$.

Remark 2.11. If X is finite dimensional then one can find a $\delta > 0$ such that $\|y + \lambda x\| \geq \delta \ \forall \lambda$ and for all $x, y \in S_X$ with $x \perp_B y$ (consider the continuous function $f : M \times S_K \longrightarrow \mathbb{R}$ defined on the compact set $M \times S_K$ as $f(x, y, \lambda) = \|y + \lambda x\|$ where $M = \{(x, y) \in S_X \times S_X : x \perp_B y\}$).

We next give an example to show that the bounds can not be improved on in an arbitrary normed linear space.

Example 2.12. Consider $(\mathbb{R}^2, \|.\|_\infty)$ with $x = (1, 1)$ and $y = (-1, 0)$, then $\|x + \lambda y\| \geq \|x\| \ \forall \ \lambda$ but $\|tx + (1 - t)y\| = \frac{1}{3}$ for $t = \frac{1}{3}$. Also $\|y + \lambda x\| = \frac{1}{2}$ for $\lambda = \frac{1}{2}$. This shows that in an arbitrary normed linear space one can not improve on the bounds.

That the condition of $\delta > \frac{1}{2}$ is necessary but not sufficient for strict convexity follows from the example given below.

Example 2.13. Consider the normed linear space $X = \mathbb{R}^2$ with $S_X = \{(x, y) : -1 \leq x \leq 1, \ y = \pm 1\} \cup \{(x, y) : 1 \leq x \leq \sqrt{2}, y = \pm\sqrt{2 - x^2}\} \cup \{(x, y) : -\sqrt{2} \leq x \leq -1, y = \pm\sqrt{2 - x^2}\}$. Then it is easy to see that X is not strictly convex and with a little bit of calculation we can show that if $x, y \in S_X$ such that $\|x + \lambda y\| \geq \|x\| \ \forall \lambda$ then there exists $\delta > \frac{1}{2}$ such that $\|y + \lambda x\| > \delta \ \forall \lambda$.

We can sum up our results in the following two theorems which give a better idea of how the estimation of the bounds discussed above behave in different types of normed linear spaces.

Theorem 2.14. *Suppose X is a normed linear space.*

- *If $x, y \in S_X$ and $x \perp_B y$ then $\|tx + (1 - t)y\| \geq \frac{1}{3} \ \ \forall t \in [0, 1]$.*
- *If X is strictly convex then $x, y \in S_X$ and $x \perp_B y$ implies $\|tx + (1 - t)y\| > \frac{1}{3} \ \forall t \in [0, 1]$.*
- *If X is an inner product space then $x, y \in S_X$ and $x \perp_B y$ implies $\|tx + (1 - t)y\| \geq \frac{1}{\sqrt{2}} \ \forall t \in [0, 1]$.*

Theorem 2.15. *Suppose X is a normed linear space.*

- *If $x, y \in S_X$ and $x \perp_B y$ then $\|y + \lambda x\| \geq \frac{1}{2} \ \forall \ \lambda$.*
- *If X is strictly convex then $x, y \in S_X$ and $x \perp_B y$ implies $\|y + \lambda x\| > \frac{1}{2} \ \forall \ \lambda$.*
- *If X is an inner product space then $x, y \in S_X$ and $x \perp_B y$ implies $\|y + \lambda x\| > 1 \ (\lambda \neq 0)$.*

Remark 2.16. Considering the uniformly convex spaces $(\mathbb{R}^2, \|.\|_p)$ $(p > 1)$ it may be remarked that a better estimation of the bounds discussed above may not be possible in an arbitrary uniformly convex space.

3. Strongly orthonormal Hamel basis in the sense of Birkhoff-James

In this section we find a necessary and sufficient condition for a Hamel basis to be a strongly orthonormal Hamel basis in the sense of Birkhoff-James in a finite dimensional real strictly convex space. We begin with the following lemma.

Lemma 3.1. *Let X be a finite dimensional real normed linear space and $\{e_1, e_2, \ldots, e_m\}$ be a Hamel basis of X with $\|e_i\| = 1, \forall i = 1, 2, \ldots, m$.*
For $i = 1, 2, \ldots, m$ let

$$S_i = \{\alpha_{iz} \ : \ z = \alpha_{1z}e_1 + \alpha_{2z}e_2 + \ldots + \alpha_{mz}e_m \in S_X\}.$$

Then S_i attains its maximum value for each $i = 1, 2, \ldots, m$.

Proof. We first show that each S_i is closed and bounded.
Let $\{\alpha_{in}\}_{n=1}^{\infty}$ be a sequence in S_i which converges to some real number β where $z_n = \alpha_{1n}e_1 + \alpha_{2n}e_2 + \ldots + \alpha_{mn}e_m$ and $\|z_n\| = 1$ for all n.
Since S_X is compact so $\{z_n\}$ has a convergent subsequence z_{n_k} converging to some z_0 in S_X. Let $z_0 = \gamma_1 e_1 + \gamma_2 e_2 + \ldots + \gamma_m e_m$. Then

$$\begin{aligned}\|z_{n_k} - z_0\| = & \ |(\alpha_{1n_k} - \gamma_1)e_1 + (\alpha_{2n_k} - \gamma_2)e_2 + \ldots + (\alpha_{mn_k} - \gamma_m)e_m| \\ & > c\,[|(\alpha_{1n_k} - \gamma_1)| + |(\alpha_{2n_k} - \gamma_2)| + \ldots + |(\alpha_{mn_k} - \gamma_m)|]\end{aligned}$$

for some $c > 0$ as any two norms on a finite dimensional space are equivalent.
As $\{z_{n_k}\}$ converges to z_0 so $\{\alpha_{in_k}\}$ converges to γ_i as $k \longrightarrow \infty$.
This shows that $\beta = \gamma_i$. Thus S_i is closed for each i.
Similarly using the fact that any two norms on a finite dimensional space are equivalent it is easy to show that each S_i is bounded. So each S_i is compact and hence attains its maximum value. This completes the proof of the lemma.

In the next lemma we show that if X is strictly convex then there exists a unique element of S_X for which the coefficient of e_i attains the maximum value of S_i.

Lemma 3.2. *Let X be a finite dimensional real strictly convex space. Let $\alpha_{i_0 z_0} = \max S_{i_0}$ which is attained for $z_0 = \alpha_{1z_0}e_1 + \alpha_{2z_0}e_2 + \ldots + \alpha_{i_0 z_0}e_{i_0} + \ldots + \alpha_{mz_0}e_m \in S_X$. If $y = \beta_1 e_1 + \beta_2 e_2 + \ldots + \beta_m e_m \in S_X$ with $\beta_{i_0} = \alpha_{i_0 z_0}$ then $y = z_0$.*

Proof. For any $t \in [0, 1]$ we have $\|ty + (1-t)z_0\| \leq 1$.
Also for any $t \in [0, 1]$,

$$\begin{aligned} ty + (1-t)z_0 &= t(\beta_1 e_1 + \beta_2 e_2 + \ldots + \beta_m e_m) \\ &\qquad + (1-t)(\alpha_{1z_0}e_1 + \alpha_{2z_0}e_2 + \ldots + \alpha_{i_0 z_0}e_{i_0} + \ldots + \alpha_{mz_0}e_m) \\ &= \gamma_1 e_1 + \gamma_2 e_2 + \ldots + \gamma_{i_0}e_{i_0} + \ldots + \gamma_m e_m \end{aligned}$$

where $\gamma_1 = t\beta_1 + (1-t)\alpha_{1z_0}$, $\gamma_2 = t\beta_2 + (1-t)\alpha_{2z_0}, \ldots, \gamma_{i_0} = \alpha_{i_0 z_0}, \ldots, \gamma_m = t\beta_m + (1-t)\alpha_{mz_0}$.

If $\|ty + (1-t)z_0\| < 1$ then $\frac{ty+(1-t)z_0}{\|ty+(1-t)z_0\|} \in S_X$ and

$$\frac{ty + (1-t)z_0}{\|ty + (1-t)z_0\|} = \frac{1}{\|ty + (1-t)z_0\|}(\gamma_1 e_1 + \gamma_2 e_2 + \ldots + \gamma_{i_0} e_{i_0} + \ldots + \gamma_m e_m).$$

As $\frac{\gamma_{i_0}}{\|ty+(1-t)z_0\|} > \gamma_{i_0} = \alpha_{i_0 z_0} = \max S_{i_0}$ so we get a contradiction.
Hence $\|ty + (1-t)z_0\| = 1$ for all $t \in [0, 1]$.
Since X is a strictly convex space so we must have $y = z_0$.

We finally prove the theorem which gives the necessary and sufficient condition for a Hamel basis to be a strongly orthonormal Hamel basis in the sense of Birkhoff-James.

Theorem 3.3. *Let X be a finite dimensional real strictly convex space and $\{e_1, e_2, \ldots, e_m\}$ be a Hamel basis of X with $\|e_i\| = 1, \forall i = 1, 2, \ldots, m$. Then a necessary and sufficient condition for $\{e_1, e_2, \ldots, e_m\}$ to be a strongly orthonormal Hamel basis in the sense of Birkhoff-James is* $\max S_i = 1 \ \ \forall\ i = 1, 2, \ldots, m$.

Proof. Let $\max S_i = 1$ for all $i = 1, 2, \ldots, m$.
Now

$$\left\| e_i + \sum_{j=1, j\neq i}^{m} \lambda_j e_j \right\| \geq 1 \ \ \forall\ \lambda_j \text{ with } 1 \leq j \leq m,\ j \neq i.$$

for otherwise the coefficient of e_i in the unit element $\frac{e_i+\sum_{j=1,j\neq i}^{m}\lambda_j e_j}{\|e_i+\sum_{j=1,j\neq i}^{m}\lambda_j e_j\|}$ will be greater than 1 which is not possible as $\max S_i = 1$. Since the normed space X is strictly convex so by applying Corollary 2.3, we obtain that

$$\left\| e_i + \sum_{j=1, j\neq i}^{m} \lambda_j e_j \right\| > 1 \ \ \forall\ \lambda_j \text{ with } 1 \leq j \leq m,\ j \neq i, \text{ whenever not all } \lambda_j' s \text{ are } 0.$$

This proves that $\{e_1, e_2, \ldots, e_m\}$ is a strongly orthonormal Hamel basis in the sense of Birkhoff-James of X.
Conversely let $\{e_1, e_2, \ldots, e_m\}$ be a strongly orthonormal Hamel basis in the sense of Birkhoff-James.
For $z = \alpha_{1z} e_1 + \alpha_{2z} e_2 + \ldots + \alpha_{mz} e_m \in S_X$, we have
$1 = \|z\| = \|\alpha_{1z} e_1 + \alpha_{2z} e_2 + \ldots + \alpha_{mz} e_m\| \geq |\alpha_{iz}| \ \forall\ i = 1, 2, \ldots, m$.
Thus $\max S_i \leq 1$. On the other hand, $\max S_i \geq 1$ since $\|e_i\| = 1$. This proves that $\max S_i = 1 \ \forall\ i = 1, 2, \ldots, m$.

Remark 3.4. In $(\mathbb{R}^2, \|.\|_p), (p > 1)$ the set $\{(1, 0), (0, 1)\}$ is a strongly orthonormal Hamel basis in the sense of Birkhoff-James since the maximum possible value of the x-coordinate of any point on the unit sphere of $(\mathbb{R}^2, \|.\|_p)$ is attained at the point (1,0), the same holds for the point (0,1).

4. Conjugate diameters and strongly orthonormal Hamel basis in the sense of Birkhoff-James

If the dimension of the real strictly convex space X is 2 then the concept of strongly orthonormal Hamel basis in the sense of Birkhoff-James is connected with the concept of conjugate diameters. We first give the definition of conjugate diameters in a conic. Two diameters of a conic are said to be conjugate iff each of them bisects the chords of the conic that are parallel to the other. If X is a real normed linear space with $\dim X = 2$ then two diameters of the unit sphere S_X passing through x and y are said to be conjugate iff $x \perp_B y$ and $y \perp_B x$. It follows from Auerbach [1] that S_X has at least one pair of conjugate diameters. We now state the following theorem the proof of which is clear from definition of conjugate diameters and Theorem 2.3.

Theorem 4.1. *Let X be a real strictly convex space of dimension 2. Then S_X has a pair of conjugate diameters through e_1, e_2 iff $\{e_1, e_2\}$ is a strongly orthonormal Hamel basis in the sense of Birkhoff-James.*

We now give an example of a real normed linear space of dimension 2 which has no strongly orthonormal Hamel basis in the sense of Birkhoff-James but has a pair of conjugate diameters to show that strict convexity in the last theorem is necessary.

Example 4.2. Consider $(\mathbb{R}^2, \|.\|)$ where the unit sphere is given by $S_{\mathbb{R}^2} = \pm A \cup \pm B \cup \pm C$ where
$A = \{(x, y) : \ x^4 + y^4 = 1, \ 0 \leq x \leq \frac{1}{3}, \ y \geq 0\}$,
$B = \{(x, y) : \ (-x)^3 + y^3 = 1, \ -\frac{1}{4} \leq x \leq 0, \ y \geq 0\}$,
$C = \{t(\frac{1}{3}, y_1) + (1-t)(\frac{1}{4}, y_2) : \ 0 \leq t \leq 1, (\frac{1}{3})^4 + (y_1)^4 = 1, (\frac{1}{4})^3 + (-y_2)^3 = 1\}$.
It is easy to check that $(\mathbb{R}^2, \|.\|)$ has no strongly orthonormal Hamel basis in the sense of Birkhoff-James but has a pair of conjugate diameters.

We next explore the relation between the concept of strongly orthonormal Hamel basis in the sense of Birkhoff-James and the notion of Radon curve [12, 15], the definition of which is given below.

Radon Curve. A curve is a Radon curve iff it is the unit sphere of a two dimensional real normed linear space (also known as Minkowski plane) in which the notion of orthogonality (also called normality) in the sense of Birkhoff-James is symmetric. We now give the following theorem.

Theorem 4.3. *Suppose X is a real strictly convex smooth space of dimension 2. Then S_X is a Radon curve iff at each point of S_X there exists a strongly orthonormal Hamel basis in the sense of Birkhoff-James containing that point.*

Proof. The proof follows from the fact that in a smooth space the supporting line to S_X at any point on S_X is unique.

So using the concept of strongly orthonormal Hamel basis in the sense of Birkhoff-

James one can generalize the notion of conjugate diameters and Radon curve in a real strictly convex smooth space even if the dimension of the space is greater than 2.

Generalized conjugate diameters. Suppose X is a real normed linear space of dimension n. Then a set of n diameters of S_X are said to be generalized conjugate diameters iff any two of them are conjugate diameters. As for example in $(\mathbb{R}^n, \|.\|_p)$ $(1 < p < \infty)$ diameters passing through $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0)$, $\ldots\ldots, (0, 0, \ldots, 1)$ are generalized conjugate diameters since $\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots\ldots, (0, 0, \ldots, 1)\}$ is a strongly orthonormal Hamel basis in the sense of Birkhoff-James in $(\mathbb{R}^n, \|.\|_p)$.

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