Journal of Convex Analysis
Volume 23 (2016), No. 1, 5–22

Jean Jacques Moreau.
A Selected Review of his Mathematical Works

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The aim of this review is to analyze the impact of a personal list of selected mathematical contributions of Jean Jacques Moreau. I start with Legendre's transform, Mandelbrojt's transform and Fenchel's conjugate to come to the concept of Moreau and Rockafellar of the conjugate as a function defined on the whole topological dual space taking on values in the extended real line. Then I discuss a large number of Moreau's fundamental ideas in modern convex analysis.

Mechanical systems, on which unilateral constraints (that is, one-sided constraints) are imposed, as they appeared in his articles [2, 10, 12] (to cite just a few of them), motivated Jean Jacques Moreau to be heavily involved in the analysis of variational properties of nondifferentiable convex functions as well as duality aspects. Before carrying out the discussion of some fundamental contributions of J. J. Moreau in Convex Analysis we take a look at two earlier papers of Szolem Mandelbrojt and Werner Fenchel, respectively. Considering a function f of one real variable and using the notation $\overline{\text{borne}}$ to mean the supremum, Szolem Mandelbrojt stated in his 1939 article [50] the following theorem[1] (the range space therein is not explicitly stated):

Theorem 1. *Toute fonction $f(x)$ de la forme*

$$f(x) = \overline{\text{borne}}_{-\infty<t<+\infty}[xt - \varphi(t)], \tag{1}$$

où $\varphi(t)$ est une fonction quelconque, définie sur la droite entière ($-\infty < t < +\infty$), est une fonction convexe. Réciproquement: quelle que soit la fonction convexe $f(x)$, il existe une fonction $\varphi(t)$ telle que la relation (1) *ait lieu; on peut, en particulier, poser*

$$\varphi(t) = \overline{\text{borne}}_{-\infty<x<+\infty}[xt - f(x)]. \tag{2}$$

Before beginning with the proof, Mandelbrojt added: "À chaque fonction convexe $f(x)$ correspond une autre fonction convexe, $\varphi(t)$ qu'on peut appeler la

[1]At the end of this 1939 paper [50] Mandelbrojt wrote: "We believe that this result is new, but the actual events do not allow us to be sure".

ISSN 0944-6532 / \$ 2.50 © Heldermann Verlag

fonction convexe *associée* à $f(x)$ et qui est liée à celle-ci par la double relation (1) et (2)".

Here are the English translations.

Theorem (Mandelbrojt). *Every function $f(x)$ of the form*

$$f(x) = \overline{\text{borne}}_{-\infty<t<+\infty}[xt - \varphi(t)],$$

where $\varphi(t)$ is any function, defined on the whole real line ($-\infty < t < +\infty$), is a convex function. Conversely: For any convex function $f(x)$, there exists a function $\varphi(t)$ such that the relation (1) *holds; one can, in particular, choose*

$$\varphi(t) = \overline{\text{borne}}_{-\infty<x<+\infty}[xt - f(x)].$$

"With any convex function $f(x)$ is associated another convex function, $\varphi(t)$, which can be called the convex function *associated* to $f(x)$ and which is related to the latter by both relations (1) and (2)".

With $\varphi(t)$ as above, to prove for all t, the inequality

$$\overline{\text{borne}}_{-\infty<t<+\infty}[xt - \varphi(t)] \geq f(x),$$

Mandelbrojt used the right side derivative $f'_+(x)$ to write for every x

$$\overline{\text{borne}}_{-\infty<t<+\infty}\left\{xt - \overline{\text{borne}}_{-\infty<\tau<+\infty}[t\tau - f(\tau)]\right\} \geq xf'_+(x) - \overline{\text{borne}}_{-\infty<\tau<+\infty}[\tau f'_+(x) - f(\tau)].$$

This right side derivative is also used to write in a second step, for all τ

$$\tau f'_+(x) - f(\tau) \leq xf'_+(x) - f(x), \text{ hence } \overline{\text{borne}}_{-\infty<\tau<+\infty}[\tau f'_+(x) - f(\tau)] = xf'_+(x) - f(x).$$

Then, the conclusion in [50] is

$$\overline{\text{borne}}_{-\infty<t<+\infty}\left\{xt - \overline{\text{borne}}_{-\infty<\tau<+\infty}[t\tau - f(\tau)]\right\} \geq xf'_+(x) - xf'_+(x) + f(x) = f(x).$$

It is clear through the arguments reproduced above that Mandelbrojt implicitly assumed that the convex function f is finite on the whole real line. However, the function φ is finite-valued on the interval of all reals t such that the function $\tau \mapsto \tau t - f(\tau)$ is bounded from above, and this interval may fail to be the whole real line. Consequently, there is no direct duality between f and φ.

Taking this fact into account Werner Fenchel dealt in the 1949 paper [43] with convex functions defined on convex subsets of $\mathbb{R}^n$. As in [43] denoting $\Sigma x\xi = \Sigma_{i=1}^n x_i\xi_i$ for $x, \xi \in \mathbb{R}^n$, and saying interior point (resp. boundary point) set of a convex set G of $\mathbb{R}^n$ instead of relative interior (resp. relative boundary) point set of G, the formulation word for word of Fenchel's result is the following.

Theorem 2 (Fenchel). *Let G be a convex point set in $\mathbb{R}^n$ and $f(x)$ a function defined in G convex and semicontinuous from below and such that $\lim_{x\to x^*} f(x) = \infty$ for each boundary point x^* of G which does not belong to G. Then there exists one and only one point set Γ in $\mathbb{R}^n$ and one and only one function $\varphi(\xi)$ defined on Γ with exactly the same properties as G and $f(x)$ such that*

$$\Sigma x\xi \leq f(x) + \varphi(\xi), \tag{3}$$

where to every interior point x of G there corresponds at least one point ξ of Γ for which equality holds.

In the same way G, $f(x)$ correspond to Γ, $\varphi(\xi)$.

In the proof in [43] the set Γ is defined as the set of $\xi \in \mathbb{R}^n$ such that $x \mapsto \Sigma x\xi - f(x)$ is bounded from above in G, and φ in turn is defined by

$$\varphi(\xi) = \sup_{x\in G} (\Sigma x\xi - f(x)) \quad \text{for all } \xi \in \Gamma.$$

Then, one of the main tools in the development of the proof is the (finite-dimensional) property that at each relative interior point of G there is a support hyperplane of the hypersurface determined by f on G.

The pair $[\Gamma, \varphi]$ is called in [43] the conjugate of $[G, f]$. Fenchel's theorem establishes a complete duality in the sense that $[G, f]$ is also the conjugate of $[\Gamma, \varphi]$. However, the statement is not so amenable. On the one hand, the property $\lim_{x\to x^*} f(x) = \infty$ at every relative boundary point x^* of G which does not belong to G is required in the involved class of concerned lower semicontinuous convex functions. So, for functions constructed from finitely/infinitely many others, working both with the latter and with the set where the new function is defined is of course not amenable.

In the early Sixties Jean Jacques Moreau and Ralph Tyrrell Rockafellar realized independently that the right setting which one must work with is the one of functions taking on values in the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Because of the need for Mechanics of Continua (see [14, Section 6]) Moreau's main concern was with infinite-dimensional spaces. Considering two real vector spaces X and Y with a separating bilinear functional $\langle \cdot, \cdot \rangle$ and endowing X and Y with the locally convex topologies $w(X, Y)$ and $w(Y, X)$, respectively, the set $\Gamma_0(X)$ involved by Moreau can be obtained as the set of all lower $w(X, Y)$-semicontinuous convex functions from X into $\overline{\mathbb{R}}$ which are proper, that is, $f(X) \subset \mathbb{R} \cup \{+\infty\}$ and $f \not\equiv +\infty$. The set $\Gamma(X)$ can then be defined as

$$\Gamma(X) = \Gamma_0(X) \cup \{-\omega_X, +\omega_X\},$$

where ω_X denotes the function identically equal to $+\infty$ on X. A function $f : X \to \overline{\mathbb{R}}$ is then declared in this context to be convex whenever the classical inequality holds true with the extended addition $\dot{+}$ and the real scalar in $]0, 1[$.

Following Mandelbrojt and Fenchel, the *polar* or *conjugate* function of any extended real-valued function $h : X \to \overline{\mathbb{R}}$, is defined by Moreau [3] as the extended real-valued function $h^* : Y :\to \overline{\mathbb{R}}$ with

$$h^*(y) := \sup_{x \in X} (\langle x, y \rangle - h(x)) \quad \text{for all } y \in Y.$$

This point of view of extended real-valued functions offers the flexibility to deal with addition, infimum or supremum of finitely/infinitely many functions, as well as with composition.

One of the deep results in [3] (see also [15]) can then be formulated as follows.

Theorem 3 (Moreau [3]). *Any function f in $\Gamma_0(X)$ coincides with the pointwise supremum of a family of $w(X,Y)$-continuous affine functions majorized by f, that is, there exist two families $(y_i)_{i\in I}$ in Y and $(\beta_i)_{i\in I}$ in $\mathbb{R}$ such that*

$$f(x) = \sup_{i\in I} (\langle x, y_i \rangle - \beta_i).$$

This yields to the following duality.

Theorem 4 (Moreau [3]). *Any function $f \in \Gamma(X)$ coincides with its bipolar or biconjugate f^{**}.*

Further, for any $f \in \Gamma_0(X)$ one has $f^* \in \Gamma_0(Y)$.

The latter result obviously encompasses Theorem 2 of Fenchel. Connections are made in [43, 55, 58] with the transform by Adrien-Marie Legendre in [49], known as Legendre transform. This was born from Legendre's investigations on the (PDE) partial differential equation (whose earlier studies of a particular case by Gaspard Monge motivated Legendre, as written in [49, page 309])

$$A(p,q)\frac{\partial^2 u}{\partial x^2}(x,y) + B(p,q)\frac{\partial^2 u}{\partial x \partial y}(x,y) + C(p,q)\frac{\partial^2 u}{\partial y^2}(x,y) = 0,$$

where (as in [49]) $p := \frac{\partial u}{\partial x}(x,y)$ and $q := \frac{\partial u}{\partial y}(x,y)$, and the coefficients A, B, C depend merely on p, q (see [49, page 314]). The first step in [49, Section II] was to transform the above equation in a suitable way into another one with p, q as variables. Assuming ∇u is a diffeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$, Legendre's idea is to write $\nabla u\,(x(p,q), y(p,q)) = (p,q)$, or equivalently $(x(p,q), y(p,q)) = (\nabla u)^{-1}(p,q)$. Then x, y, u can be considered as functions of (p,q). Noting that the regularity of u ensures that $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$, calculation gives $\frac{\partial x}{\partial q}(p,q) = \frac{\partial y}{\partial p}(p,q)$, so $x(p,q)\,dp + y(p,q)\,dq$ is an exact differential form on $\mathbb{R}^2$. Then let $\mathbb{R}^2 \ni (p,q) \mapsto \omega(p,q) \in \mathbb{R}$ be such that $d\omega = x\,dp + y\,dq$. One sees that

$$x(p,q) = \frac{\partial \omega}{\partial p}(p,q), \qquad y(p,q) = \frac{\partial \omega}{\partial q}(p,q), \qquad d(px + yq - \omega) = du,$$

so a suitable choice of the constants allows one to obtain the function ω such that $p\,x(p,q) + q\,y(p,q) - \omega(p,q) = u(p,q)$ for all $(p,q) \in \mathbb{R}^2$; I followed Legendre's notation in denoting by $\omega(\cdot)$ the function determined in this way. Altogether this yields the dual pair of equalities

$$p = \frac{\partial u}{\partial x}(x,y), \qquad q = \frac{\partial u}{\partial y}(x,y), \qquad u(x,y) + \omega(p,q) = px + qy,$$

and

$$x = \frac{\partial \omega}{\partial p}(p,q), \qquad y = \frac{\partial \omega}{\partial q}(p,q), \qquad u(x,y) + \omega(p,q) = px + qy.$$

Further, ω is of class C^2 and after calculation Legendre obtained the new (PDE)

$$A(p,q)\frac{\partial^2 \omega}{\partial q^2}(p,q) - B(p,q)\frac{\partial^2 \omega}{\partial p \partial q}(p,q) + C(p,q)\frac{\partial^2 \omega}{\partial p^2}(p,q) = 0,$$

and he noticed that the new (PDE) is of the same type but much easier since it does not contain first order partial derivatives. In fact, before the development reproduced above Legendre utilized the method to study the famous equation with $A(p,q) = 1 + q^2$, $B(p,q) = 2pq$ and $C(p,q) = 1 + p^2$ (related to Monge's works). In [49] Legendre also applied his transform to various classes of (PDE).

It is worth pointing out another viewpoint from the book of Richard Courant and David Hilbert [42] concerning Legendre's transform. In the first step of the process, the idea was to establish a correspondence between a smooth surface (regular in a certain sense) and the family of its tangent planes. Of course, the main point is to recover the surface knowing the family of tangent planes. Let us follow the presentation in [42, Chapter I, Section 6]. Consider the surface (S) which is the graph of a function $x \mapsto u(x)$, where $x := (x_1, x_2) \in \mathbb{R}^2$; suppose that u is smooth on $\mathbb{R}^2$ (say C^1 at this stage). This surface can be seen also as the envelope of its tangent planes. Denote by $(\overline{x}, \overline{z}) \in \mathbb{R}^2 \times \mathbb{R}$ the generic point of a nonvertical plane whose equation is

$$(\mathrm{P}_{\xi,\omega}) \qquad \overline{z} - \xi_1 \overline{x}_1 - \xi_2 \overline{x}_2 + \omega = 0, \quad \text{or equivalently} \quad \overline{z} - \langle \xi, \overline{x} \rangle + \omega = 0;$$

the pair $(\xi, \omega) \in \mathbb{R}^2 \times \mathbb{R}$ is called in [42] the *plane coordinates* of $(\mathrm{P}_{\xi,\omega})$.

Given $x \in \mathbb{R}^2$ one knows that, with $(\overline{x}, \overline{z}) \in \mathbb{R}^2 \times \mathbb{R}$ as generic point, the equation of the plane tangent to (S) at the point $(x, u(x)) \in S$ has as equation

$$\overline{z} - u(x) - \langle \overline{x} - x, \nabla u(x) \rangle = 0,$$

so its plane coordinate (ξ, ω) satisfies

$$\xi = \nabla u(x) \quad \text{and} \quad \omega = \langle x, \nabla u(x) \rangle - u(x).$$

The surface (S) is determined also by the family of its tangent planes, that is, the family $\big((\mathrm{P}_{\xi,\omega(\xi)})\big)_\xi$, where $\omega(\xi)$ needs to be such that $(\mathrm{P}_{\xi,\omega(\xi)})$ is tangent to

(S). Given $\xi \in \mathbb{R}^2$, if x is the element in $\mathbb{R}^2$ such that $(\mathrm{P}_{\xi,\omega(\xi)})$ is tangent to (S) at $(x, u(x))$, then calculating x in terms of ξ (when it is possible) by means of the equality $\xi = \nabla u(x)$ furnishes the value in terms of ξ of

$$\omega(\xi) = \langle \xi, x \rangle - u(x)$$

as a function of ξ. This is then feasible with $\xi \mapsto \omega(\xi)$ as a C^1 mapping provided that ∇u is a diffeomorphism from $\mathbb{R}^2$ onto itself. So, under this additional assumption on ∇u, with the surface (S) is associated the surface (Σ) which is the graph of $\xi \mapsto \omega(\xi)$, where $(\xi, \omega(\xi))_{\xi \in \mathbb{R}^2}$ are as above the plane coordinates of planes tangent to (S).

Writing $\omega(\xi) = \langle \xi, x(\xi) \rangle - u\,(x(\xi)) = \xi_1 x_1(\xi) + \xi_2 x_2(\xi) - u\,(x_1(\xi), x_2(\xi))$ it results also that

$$\begin{aligned}\frac{\partial \omega}{\partial \xi_1}(\xi) &= x_1(\xi) + \xi_1 \frac{\partial x_1}{\partial \xi_1}(\xi) + \xi_2 \frac{\partial x_2}{\partial \xi_1}(\xi) \\ &\qquad - \frac{\partial u}{\partial x_1}\,(x(\xi))\,\frac{\partial x_1}{\partial \xi_1}(\xi) - \frac{\partial u}{\partial x_2}\,(x(\xi))\,\frac{\partial x_2}{\partial \xi_1}(\xi) = x_1(\xi),\end{aligned}$$

and, in the same way, $\frac{\partial \omega}{\partial \xi_2}(\xi) = x_2(\xi)$. This leads to the equality $u(x) = \langle x, \xi \rangle - \omega(\xi)$. Consequently, one sees that $\nabla \omega$ exists and is a diffeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^2$ and that (S) is in turn the surface associated with (Σ) through the same above process, in the sense that (S) is the surface described in $\mathbb{R}^2 \times \mathbb{R}$ by the *plane coordinates* of planes tangent to (Σ). This duality is confirmed also by the following dual equalities obtained above:

$$\omega(\xi) + u(x) = \langle \xi, x \rangle, \qquad \xi = \nabla u(x), \;\; x = \nabla \omega(\xi).$$

Of course, the above analysis is still valid with $\mathbb{R}^n \times \mathbb{R}$ in place of $\mathbb{R}^2 \times \mathbb{R}$.

When the function u is in addition convex, it is clearly seen in the two above presentations that ω is the conjugate function of u as it was pointed out by Fenchel [43, 44].

Moreau [15, Chapter 14, 1rst ed.; Section 14.4, 2nd ed.] linked the conjugate transform also to the famous 1912 inequality of William Henry Young[2] (see (1) in [63]).

The setting of extended real-valued convex functions also allowed Moreau to reformulate in [7] (see also [16]) the *inf-convolution* (also called *infimal convolution*) $f \nabla g$ of two extended real-valued functions $f, g : X \to \overline{\mathbb{R}}$ in the form

$$(f \nabla g)(x) := \inf_{y+z=x} \big(f(y) \dot{+} g(z) \big) \quad \text{ for all } x \in X;$$

[2]With Young's notation in [63] the inequality can be stated in the form: If U is an increasing function of $u \geq 0$ with $U(0) = 0$, then with $v \geq 0$,

$$uv \leq \int_0^u U(t)\, dt + \int_0^v U^{-1}(t)\, dt.$$

It was established in [63] in view of applications to Fourier series.

the operation is even considered in the more general situation when X is a semigroup. Topological properties of lower/upper semicontinuity as well as condition for exactness are provided in Moreau [3]. Recall that the exactness property holds at x_0 if there are $y_0, z_0 \in X$ with $y_0 + z_0 = x_0$ such that $(f \nabla g)(x_0) = f(y_0) + g(z_0)$.

A large part of three papers in this Volume are devoted to inf-convolutions of functions.

Of course, with n functions $f_1, \cdots, f_n$ their inf-convolution is expressed as

$$(f_1 \nabla \cdots \nabla f_n))(x) := \inf_{x_1 + \cdots + x_n = x} (f_1(x_1) + \cdots + f_n(x_n)) .$$

Infinite families may also be involved. In this respect, it is worth pointing out that an integral extension of infimal convolution in the form

$$\inf \left\{ \int_T \varphi\,(t, u(t))\; d\mu(t) : u(\cdot) \in L^1_X(T, \mu),\; \int_T u(t)\, d\mu(t) = x \right\}$$

has been studied under suitable assumptions by Michel Valadier [60] with a normal convex integrand $\varphi : T \times X \to \mathbb{R} \cup \{+\infty\}$, where $(T, \mathcal{T}, \mu)$ is a measure space. Previously to [60], defining $I_\varphi : L^p_X(T, \mu) \to \mathbb{R} \cup \{+\infty\}$ by

$$I_\varphi(x) := \int_T \varphi\,(t, x(t))\; d\mu(t) \quad \text{for all } x(\cdot) \in L^p_X(T, \mu),$$

R. T. Rockafellar initiated the study of convex integral functionals and established (under appropriate natural hypotheses) the deep and nice duality theorem saying that the polar or conjugate of I_φ coincides with I_{φ^*}, where

$$\varphi^*(t, y(t)) := \varphi(t, \cdot)^*(y(t)) \quad \text{for } y(\cdot) \in L^q_{X^*}(T, \mu),$$

and $p \in [1, +\infty[$, X is a separable reflexive Banach space and X^* is the topological dual of X. Various other deep results in this line have been provided by Charles Castaing [39, 40] (see also [41]), Alexander D. Ioffe and Vladimir L. Levin [46], A. D. Ioffe and Vladimir M. Tikhomirov [47], Michel Valadier [61], to cite just a few authors. With the convex integral functionals, in a long series of papers (see [58] for references), R. Tyrrell Rockafellar and Roger J.-B. Wets provided a complete study of stochastic optimization problems with recourse.

Concerning the behaviour of the polar or conjugate function f^* under some continuity property of the function f, Moreau [11] showed the following strong equivalence.

Theorem 5 (Moreau [11]). *Let $\mathcal{T}$ be a vector topology on X (not necessarily compatible with the duality between X and Y). A function $f \in \Gamma_0(X)$ is bounded from above on a $\mathcal{T}$-neighbourhood of a point $a \in X$ if and only if the set $\{f^*(\cdot) - \langle a, \cdot\rangle \le \rho\}$ is $\mathcal{T}$-equicontinuous for every $\rho \in \mathbb{R}$; in fact, it is enough that the equicontinuity property holds for some real $\rho > \inf_X (f^* - \langle a, \cdot\rangle)$.*

As a consequence one finds in Moreau [11] this elegant characterisation of $w(X,Y)$-continuity.

Corollary 6 (Moreau [11]). *A function $f \in \Gamma_0(X)$ is $w(X,Y)$-continuous at a point $a \in X$ if and only if, for every $\rho \in \mathbb{R}$, the set $\{f^*(\cdot) - \langle a, \cdot\rangle\}$ is bounded and included in some finite dimensional subspace of Y.*

In such a case, there is a vector subspace E of X with finite codimension such that

$$f(x) = f(x') \quad \textit{for all } x, x' \in X \textit{ with } x - x' \in E.$$

J. J. Moreau and R. T. Rockafellar introduced (again independently), for an extended real-valued function $h : X \to \mathbb{R}$ and a point $a \in X$ where h is finite, what should be called its *Moreau-Rockafellar subdifferential* at the point a, as the set

$$\partial h(a) := \{\, y \in Y : \langle x - a, y\rangle + h(a) \le h(y),\ \forall x \in X\};$$

by convention $\partial h(a) = \emptyset$ if $h(a)$ is infinite.

Among the important properties are the following evident ones: The point a is a minimizer of h on X if and only if $0 \in \partial h(a)$; the subdifferential $\partial h(a)$ is related to the polar or conjugate of f via the equality

$$\partial h(a) = \{\, y \in Y : h(a) + h^*(y) = \langle a, y\rangle\}.$$

Moreau [9] provided the first infinite-dimensional subdifferential additivity in the following form:

Theorem 7 (Moreau [9]). *Let $f, g \in \Gamma_0(X)$ be two functions. If there is some point $x_0 \in X$ such that both functions are finite at x_0 and g is continuous at x_0 with respect to the Mackey topology $\tau(X,Y)$ of X, then*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x) \quad \textit{for all } x \in X.$$

The proof in Moreau [9] is based on the exactness property of the inf-convolution $f^* \nabla g^*$. Given a Hausdorff topological vector space X, through the use of the Hahn-Banach separation theorem of the epigraph of some suitable function related to g and with the continuity of g at x_0 where f, g are finite (as above), Rockafellar [54] extended Theorem 7 in removing the lower semicontinuity assumption of f, g.

For applications of subdifferentials of convex functions to Optimisation Algorithms and to other fields, we refer to the book of Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal [45] and the references therein. Hedy Attouch also employed (see, e.g., [34]) subdifferentials of convex functions to show, for a sequence $(f_n)_n$ in $\Gamma_0(X)$, that the Peano-Painlevé convergence of the graphs of ∂f_n in $X \times X^*$ plus a constant condition characterises the *Mosco convergence* of the sequence $(f_n)_n$ provided that X is a reflexive Banach space (and Y is the topological dual X^* of X). Further results in this line are contained in the book [37] of Gerald Beer. The impact of the concept of sudifferentials (normal cones) of convex functions (sets) in Economic and Game theories is testified, for example, in Jean-Pierre Aubin's book [35]. Many other viewpoints in the Banach space setting can be found in [36, 38, 41, 52, 59, 64].

Very early J. J. Moreau noticed that, when X is a Hilbert space H, he could take advantage of the possibility of adding together elements of a pair of dual problems. Given a function $f \in \Gamma_0(H)$, we saw above that

$$y \in \partial f(x) \iff x \in \partial f^*(y) \iff f(x) + f^*(y) = \langle x, y \rangle, \tag{4}$$

where $\langle x, y \rangle$ stands here for the inner product in the Hilbert space H. Such points x, y are declared by Moreau as a *conjugate pair* with respect to the dual functions f, f^*. What can be said if we (take advantage of the Hilbert setting to) add the dual points to obtain $z = x + y$?

Let $\|\cdot\|$ be the norm associated with the inner product. With the new element $z = x + y$ at hand, we can continue the process of equivalences in (4) as follows:

$$\begin{aligned} y \in \partial f(x) &\iff z - x \in \partial f(x) \\ &\iff 0 \in \partial f(x) + (x - z) \\ &\iff x \text{ is a minimizer of } f(\cdot) + \frac{1}{2}\|z - \cdot\|^2. \end{aligned}$$

Of course, the latter function invokes the *inf-convolution* operation to which Moreau devoted a large number of his papers.

Then, to develop the analysis, Moreau first observed the exactness in a unique point of the inf-convolution $f \nabla (\frac{1}{2}\|\cdot\|^2)$, that is, for any $z \in H$ there is a unique point $x =: \mathrm{prox}_f z$ such that

$$\left(f \nabla \frac{1}{2}\|\cdot\|^2 \right)(z) = \inf_{u \in H} \left(f(u) + \frac{1}{2}\|z - u\|^2 \right) = f(x) + \frac{1}{2}\|z - x\|^2.$$

In [4, 5, 6, 13] Moreau called the point $\mathrm{prox}_f z$ the *proximal point* of z relative to f and called $\mathrm{prox}_f : H \to H$ the *prox mapping* of f. The radiant and powerful duality result established by Moreau in [6, 13] is the following.

Theorem 8 (Moreau [6, 13]). *Let $f \in \Gamma_0(H)$ and x, y, z in the Hilbert space H. The following assertions (a) and (b) are equivalent:*

(*a*) $z = x + y$ *and* $f(x) + f^*(y) = \langle x, y \rangle$;
(*b*) $x = \mathrm{prox}_f z$ *and* $y = \mathrm{prox}_{f^*} z$.

Through this theorem Moreau showed in a subtle way in [13, Proposition 12.b] with a few lines that the graph of the subdifferential of any function $f \in \Gamma_0(H)$ is maximal monotone in the Hilbert space $H \times H$. Various other basic useful features and properties are achieved in [6, 13]. In order to analyse some among them, let us call, following R. T. Rockafellar and R. J.-B. Wets, *Moreau λ-envelope* of f (where $\lambda > 0$) the function $e_\lambda f$ defined by

$$(e_\lambda f)(z) := \inf_{u \in H} \left(f(u) + \frac{1}{2\lambda} \|u - z\|^2 \right) \quad \text{for all } u \in H;$$

it is just the same as above endowing H with the new inner product $\langle \lambda^{-1/2} \cdot, \lambda^{-1/2} \cdot \rangle$. This function with $\lambda = 1$ is called the primitive[3] of the prox-mapping prox_{f^*} in Moreau [13, Section 7.a] according to the property that $\mathrm{prox}_{f^*}(z)$ is the Fréchet gradient of $e_1 f$ at z, established for the first time in [13].

Two questions then arise. How can functions be recognized which are envelopes of convex functions (or primitives of prox mappings), and how can prox mappings be characterized? Anwers to both questions were provided by Moreau [6, 13].

Moreau's answer to the first question is a piece of elegance:

Theorem 9 (Moreau [6, 13]). *With the Hilbert space H let $F \in \Gamma_0(H)$. The following assertions are equivalent:*

(a) *The function F is the Moreau envelope with index $\lambda = 1$ of some function in $\Gamma_0(H)$ (or equivalently the primitive of some function in $\Gamma_0(H)$);*

(b) *There exists a convex function $G : H \to \mathbb{R} \cup \{+\infty\}$ such that $F + G = \frac{1}{2}\| \cdot \|^2$;*

(c) *There exists a convex function $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ such that $F^* = \Phi + \frac{1}{2}\| \cdot \|^2$.*

The next theorem furnishes the answer to the second question.

Theorem 10 (Moreau [13]). *Let $P : H \to H$ be a mapping defined on the whole Hilbert space H. Then P is a prox mapping (or equivalently, there is some $f \in \Gamma_0(H)$ with $P = \mathrm{prox}_f$) if and only if both properties (i) and (ii) hold:*

(i) *the mapping P is non-expansive, that is,*

$$\|P(z) - P(z')\| \le \|z - z'\| \quad \textit{for all } z \in H;$$

(ii) *there exists a convex function $\varphi : H \to \mathbb{R}$ such that*

$$P(z) \in \partial \varphi(z) \quad \textit{for all } z, z' \in H.$$

Six papers in this Volume use either Moreau's envelope or prox mapping in their development.

Through that concept of envelopes, in the early 1970s, Moreau established in [18] his first result for "Sweeping Processes". Moreau called Sweeping Process

[3]I use as in [58] the terminology of Moreau envelope because of so many other properties that it enjoys even when f is not convex; see, e.g., [58, 48].

the differential inclusion

$$\begin{cases} du \in -N\left(C(t); u(t)\right) \\ u(T_0) = a \in C(T_0), \end{cases} \tag{5}$$

where $C(t)$ are closed sets of the Hilbert space H for $t \in I := [T_0, T]$, $N(C(t); \cdot)$ is the normal cone of $C(t)$ and $a \in C(T_0)$ is the initial state. Moreau's mechanical motivation was the mathematical treatment of elastoplastic systems; details can be found in Moreau [17, 21, 30] and in the book [51] by Manuel D. P. Monteiro Marques. When the sets $C(t)$ move in an absolutely continuous way with respect to Hausdorff distance, a solution $u(\cdot)$ is an absolutely continuous mapping from $[T_0, T]$ into H such that $u(T_0) = a$,

$$\frac{du}{dt}(t) \in -N\left(C(t); u(t)\right) \quad \text{a.e. } t \in [T_0, T] \text{ and } u(t) \in C(t) \ \forall t \in [T_0, T].$$

Considering with $\lambda > 0$ the solution u_λ of the differential equation

$$\frac{du_\lambda}{dt}(t) = -\nabla\left(e_\lambda \psi_{C(t)}\right)\left(u_\lambda(t)\right), \quad u_\lambda(0) = a,$$

and passing to the limit as $\lambda \downarrow 0$, Moreau proved in [18] that (5) has a unique solution provided that the closed sets $C(t)$ are convex and move in an absolutely continuous way. The function $\psi_{C(t)}$ denotes the *indicator function*[4] defined (as introduced at the same time by Moreau and Rockafellar) by $\psi_{C(t)}(x) = 0$ if $x \in C(t)$ and $\psi_{C(t)}(x) = +\infty$ if $x \in H \setminus C(t)$. Other Moreau's approaches can be found in [19]. The sets $C(t)$ involved in the modeling of mechanical problems which led Moreau to such dynamical processes, incorporate jumps very often. In view of such dynamical systems, Moreau developed first in [20, 22] the analysis of multimappings with bounded/finite retractions (see also [29]). Differential measures associated with vector valued mappings with bounded variations are studied in Moreau [23, 24, 27]. Assume that the closed sets $C(t)$ move with a bounded/finite retraction; recall that the retraction of $C(\cdot)$ on a subinterval J of I is defined (see [22, 25]) as

$$\operatorname{ret}(C; J) := \sup\left(\sum_i \operatorname{exc}\left(C(s_i), C(s_{i+1})\right)\right),$$

where the supremum is taken over all finite nondecreasing sequences $(s_i)_i$ in J, and $\operatorname{exc}(S, S')$ denotes the excess of a set $S \subset H$ over S' given by $\operatorname{exc}(S, S') := \sup_{x \in S} d(x, S')$. This retraction coincides with the usual variation $\operatorname{var}(\gamma; J)$ whenever $C(t)$ is a singleton $\{\gamma(t)\}$. Let $P : I_0, I_1, \cdots$, be a finite sequence of successive intervals included in I with $t_0 := T_0 \in I_0$, and, for $i > 0$, call t_i the

[4]I follow here Moreau's notation $\psi_{C(t)}$.

origin of I_i (not necessarily in I_i). With such a sequence P Moreau associated in [25] the step multimapping C_P defined by

$$C_P(I_i) := \begin{cases} C(t_i) & \text{if } t_i \in I_i \\ C({t_i}^+) & \text{if } t_i \notin I_i, \end{cases}$$

where $C(t^+)$ denotes the Peano-Painlevé limit of $C(s)$ as $s \downarrow t$ (such a limit exists according to the bounded retraction of $C(\cdot)$). The value $C_P(I_i)$ has to be understood as the value that C_P takes on the interval I_i, that is, $C_P(t) = C_P(I_i)$ for all $t \in I_i$. To each sequence P is also associated the step mapping u_P taking on each interval I_i the constant value u_i with

$$u_0 := a \quad \text{and} \quad u_{i+1} := \text{proj}(u_i, C_{i+1}),$$

where $\text{proj}(u_i, C_{i+1})$ denotes the metric projection of u_i on the closed convex set $C_{i+1} := C_P(I_{i+1})$. Clearly $(u_P)_P$ is a net with respect to the usual preorder of sequences P.

Theorem 11. *Assume that the closed sets $C(t)$ are convex and move with a bounded retraction. Then the net $(u_P(\cdot))_P$ converges uniformly on the interval $I = [T_0, T]$ to a mapping $u(\cdot)$. Further, $u(T_0) = a$, $u(t) \in C(t)$ for all $t \in I$ and $u(\cdot)$ is of bounded variation; in fact*

$$\text{var}(u; s, t) \leq \text{ret}(C; s, t) \quad \textit{for all } s < t \textit{ in } I,$$

where $\text{var}(u; s, t)$ *denotes the variation of the mapping* $u(\cdot)$ *on* $[s, t]$ *and* $\text{ret}(C; s, t)$ *the retraction of the multimapping* $C(\cdot)$ *on* $[s, t]$.

Arguing the interpretation of the latter limit, Moreau declared it the *weak solution* of the sweeping process (5). Various properties are investigated: limit from the right, restriction and piecing, estimation of $\|u(t) - u'(t)\|$ where u' is the weak solution corresponding to closed convex sets $C'(t)$ with bounded retraction, change of time variable, etc.

To any mapping $u : I \to H$ with bounded variation one can associate (as is well-known) its differential vector measure du satisfying $u(t^+) - u(s^-) = \int_{[s,t]} du$; in such a general case the vector measure does not furnish any information on the value of $u(\cdot)$ at discontinuity points. However, if $u(\cdot)$ is in addition right continuous, then

$$u(t) = u(s) + \int_{]s,t]} du \quad \text{for all } s < t.$$

Moreau then declared a *right continuous* mapping $u : I \to H$ with *bounded variation* to be a *strong solution* of (5) whenever $u(T_0) = a$, $u(t) \in C(t)$ for all $t \in I$, and there exists a positive Radon measure μ on I with respect to which the differential measure du is absolutely continuous and such that

$$\frac{du}{d\mu}(t) \in -N\left(C(t); u(t)\right) \quad \mu - a.e.\ t \in I.$$

Theorem 12 (Moreau [25]). *Assume that the closed convex sets $C(t)$ of the Hilbert space H move with a bounded retraction and that the retraction function* $\mathrm{ret}(C; T_0, \cdot)$ *is right continuous. Then the sweeping process* (5) *has a unique strong solution and the strong and weak solutions coincide.*

Connections with aforementioned absolutely continuous solutions are also established in [25] and the so-called *Moreau's catching-up algorithm* is shown to be very efficient.

Moreau's next step was the treatment of multibody multicontact dynamical systems. Let us suppose that the configuration at time $t \in I$ of a body collection is parametrized by means of coordinates $x(t) := (x_1(t), \cdots, x_N(t)) \in H := \mathbb{R}^N$. Taking constraints of *non-interpenetrability* into account, suppose also that $x(t)$ is required to satisfy, for every $t \in I$, the inequality constraints

$$g_j(t, x(t)) \leq 0, \quad \text{for all } j = 1, \cdots, k,$$

where $g_j : I \times H \to \mathbb{R}$ are C^2 given functions. The sets $C(t)$

$$C(t) := \{z \in H : g_j(t, z) \leq 0, \ \forall j = 1, \cdots, k\}$$

are considered as normally regular in the sense that the normal cone (see [58]) $N(C(t); z)$ at $z \in C(t)$ coincides with the convex cone generated by the gradients $\nabla_z g_j(t, z)$ with $j \in \{i \in \{1, \cdots, k\} : g_i(t, z) = 0\}$ (with the convention $N(C(t); z) = H$ if the latter index set is empty). Let the force field acting to the system be given by

$$I \times H \ni (t, z) \mapsto F(t, z) \in H.$$

Moreau showed in [32, 33, 26] that the process is characterized by a second order differential inclusion in the form

$$\begin{cases} F(t, x(t), u(t))\, dt - A(t, x(t))\, du \in N(C(t); x(t)) \\ x(t) = a + \int_{T_0}^{t} u(\tau)\, d\tau \\ x(T_0) = a \in C(T_0), \end{cases} \tag{6}$$

where $A(t, z)$ is a positive definite symmetric square $N \times N$ matrix related to the problem, and the mapping $x(\cdot)$ is assumed to be absolutely continuous on $I = [T_0, T]$ with $u(\cdot)$ as derivative. According to Moreau [26, Section 4] a solution $x(\cdot)$ of (6) is an absolutely continuous mapping on I for which there exists a (right continuous) with bounded variation mapping $u : I \to H$ such that

(i) $x(t) = a + \int_{T_0}^{t} u(\tau)\, d\tau$ for all $t \in I$,

(ii) there is a non-negative Radon measure μ on I with respect to which both Lebesgue measure λ and the Radon differential measure du are absolutely continuous, and such that the following inclusion holds:

$$F(t, x(t), u(t)) \frac{d\lambda}{d\mu}(t) - A(t, x(t)) \frac{du}{d\mu}(t) \in N(C(t); x(t)) \quad \mu - \text{a.e. } t \in I.$$

When the mapping $u(\cdot)$ is also absolutely continuous, the measure μ can be taken to be the usual Lebesgue measure, so the latter inclusion is reduced to

$$F\left(t, x(t), u(t)\right) - A\left(t, x(t)\right) \frac{du}{dt}(t) \in N\left(C(t); x(t)\right) \quad \lambda - \text{a.e. } t \in I;$$

in such a case $x(\cdot)$ is a usual solution.

Involving, for each pair (t, z) the index set

$$J(t, z) := \{j \in \{1, \cdots, k\} : g_j(t, z) \geq 0\}$$

and the polyhedral closed convex set

$$W(t, z) := \left\{ v \in H : \frac{\partial g_j}{\partial t}(t, z) + \langle v, \nabla_z g_j(t, z) \rangle \leq 0, \ \forall j \in J(t, z) \right\}$$

(hence $W(t, z) = H$ if $J(t, z) = \emptyset$), Moreau [32, 33] considered, with the initial value $x(T_0) = a \in C(T_0)$, the differential inclusion

$$\begin{cases} F\left(t, x(t), u(t)\right) \, dt - A\left(t, x(t)\right) \, du \in N\left(W(t, x(t)); x(t)\right) \\ x(t) = a + \int_{T_0}^{t} u(\tau) \, d\tau \, ; \end{cases} \tag{7}$$

such a dynamical system is nowadays called a *state-dependent second order sweeping process*. Fixing $a \in C(T_0)$, he proved in [28, 31, 33] that $x(\cdot)$ is a usual solution of (6) with initial condition $x(T_0) = a$ if and only if it is a usual solution of (7) with initial condition $x(T_0) = a$. Of course, such a property can be applied in a suitable way on any interval over which the derivative $u(\cdot)$ of the generalized solution is absolutely continuous.

Through (7) Moreau [32, 33] developed what he called *CD algorithms* (Contact Dynamics algorithms) in various situations.

For more on Moreau's works in the context of *second order sweeping process* I refer to the paper by Laetitia Paoli in the present Volume which also contains six articles with various new results concerning the sweeping process.

As best I could I have tried through this brief account to relate certain faces of Moreau's fascinating mathematical works. He also knew an outstanding trajectory in mechanics, namely in the mechanics of fluid flows and non-regular mechanics. For other points of his scientific career, I invite the reader to consult the tribute paid in [62] by M. Valadier[5] to Jean Jacques Moreau.

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