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Implicit Euler Time-Discretization of a Class of Lagrangian Systems with Set-Valued Robust Controller

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A class of Lagrangian continuous dynamical systems with set-valued controller and subjected to a perturbation force has been thoroughly studied in [3]. In this paper, we study the time discretization of these set-valued systems with an implicit Euler scheme. Under some mild conditions, the well-posedness (existence and uniqueness of solutions) of the discrete-time scheme, as well as the convergence of the sequences of discrete positions and velocities in finite steps are assured. Furthermore, the approximate piecewise linear function generated by these discrete sequences is shown to converge to the solution of the continuous time differential inclusion with order $\frac{1}{2}$. Some numerical simulations on a two-degree of freedom example illustrate the theoretical developments.

Keywords: Lagrangian systems, set-valued systems, convergence in finite steps, implicit Euler time-discretization, set-valued analysis, convex analysis

1. Introduction

The dynamics of many systems in physics, mechanics [4], electrical circuits [15, 32], can be formulated by Lagrangian equations. The lagrangian function $\mathcal{L}(q, \dot{q})$ is defined by the difference between the kinetic energy $\mathcal{T}(q, \dot{q})$ and the potential energy $\mathcal{V}(q)$

$$\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{V}(q),$$

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where the kinetic energy is usually given by the quadratic form

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\langle M(q)\dot{q},\dot{q}\rangle.$$

The matrix $M(q) \in \mathbb{R}^{n\times n}$ is called the inertia matrix, which is symmetric and supposed to be positive definite and analytic with respect to q. With the generalized coordinates $q \in \mathbb{R}^n$, an external force $f \in \mathbb{R}^n$ and a perturbation $F(\cdot, q, \dot{q})$, the Lagrange equations have the form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + F(\cdot, q, \dot{q}) = f. \tag{1}$$

Using the Christoffel symbols of the first kind [14, 23, 33], we can rewrite (1) in the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(\cdot, q, \dot{q}) = f, \tag{2}$$

where $C(q,\dot{q})$ is called the centrifugal and Coriolis matrix, satisfying the relationship

$$\frac{d}{dt}\left(M(q(t))\right) = C(q(t),\dot{q}(t)) + C(q(t),\dot{q}(t))^T. \tag{3}$$

Indeed, the (j,k)-entry of the matrix $C(q,\dot{q})$ is given by

$$C_{jk}(q,\dot{q}) = \sum_{i=1}^{n} C_{ijk}(q)\dot{q}_i, \tag{4}$$

where C_{ijk} is defined by the Christoffel symbols of first kind

$$C_{ijk} = \frac{1}{2}\left(\frac{\partial M_{jk}}{\partial q_i} + \frac{\partial M_{ji}}{\partial q_k} - \frac{\partial M_{ik}}{\partial q_j}\right). \tag{5}$$

From (4), we can write the matrix $C(q,\dot{q})$ as $C(q,\dot{q}) = \sum_{i=1}^n \dot{q}_i C_i(q)$, where $C_i(q)$ has the entries $C_{ijk}(q)'s$ and satisfies the equality $C_i(q) + C_i^T(q) = \frac{\partial M}{\partial q_i}(q)$ for $i = 1, 2, \dots n$.
Therefore,

$$C(q,\dot{q}) + C(q,\dot{q})^T = \sum_{i=1}^{n} \dot{q}_i \frac{\partial M}{\partial q_i} = \frac{d}{dt}\left(M(q)\right). \tag{6}$$

Let us emphasize that the centrifugal and Coriolis matrix can be computed with only the knowledge of the inertia matrix, and the equation (3) holds for any differentiable function $q(\cdot)$.

In [3] the well-posedness and stability properties of set-valued Lagrangian systems like

$$\begin{aligned} &M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + \nabla\mathcal{V}(q(t)) + F(t,q(t),\dot{q}(t)) \\ \in\ &-\partial\Phi(\dot{q}(t)) \quad \text{a.e. } t \geq t_0, \end{aligned} \tag{7}$$

where $\Phi(\cdot)$ is a scalar convex function, has been analyzed. The terms $\nabla\mathcal{V}(q(t))$ and $\partial\Phi(\dot{q}(t))$ may represent a control input $u(q,\dot{q}) = -\nabla\mathcal{V}(q) - \partial\Phi(\dot{q})$ applied to stabilize the system at some fixed point. Thus $\mathcal{V}(q)$ may be seen as a desired (or controlled) potential energy (this is called in Control Theory the Energy shaping method [14]). This is a general framework for robust control, with a quite important particular case represented by sliding mode control [21, 30, 34, 35]. The example treated in section 6 illustrates this point. Time-discretization of set-valued, sliding mode controllers is crucial for practical implementations of such control techniques on computers, virtual prototyping, and chattering attenuation analysis [1, 2]. Experimental results reported in [20, 36] prove that implicit methods yield significant improvements in the closed-loop system behaviour (in particular the control input signal is much smoother and with smaller magnitude than with the usual explicit Euler implementation). Therefore, in this paper, we propose the following implicit time-discretized scheme of (7) and analyze its properties

$$\begin{cases} M(Q^k)\dfrac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1} \\ \qquad +\nabla\mathcal{V}(Q^k) + F(t_k, Q^k, \dot{Q}^k) \in -\partial\Phi(\dot{Q}^{k+1}) \\ Q^{k+1} = Q^k + h_k\dot{Q}^k, \end{cases} \tag{8}$$

where (h_k) is a sequence of positive steps. It is noteworthy that the set-valued term in (8) is discretized in an implicit way. The Euler implicit discretization of first order differential inclusions with maximal monotone operators has been considered in [18, 9, 8]. The scheme in (8) also belongs to the same family of numerical schemes as Moreau's catching-up algorithms. Moreau introduced the catching-up algorithm in [26] in order to prove existence results for first order convex sweeping processes, which are differential inclusions whose right-hand side is the normal cone to a time-varying or a moving closed and convex set. Then he extended both the sweeping process and the catching-up algorithm to Lagrangian systems subjected to unilateral constraints (second order sweeping process) in [27, 28, 22, 29]. The set-valued right-hand side of the Lagrangian sweeping process is, for given position q, a function of the velocity $\dot{q}$. The catching-up algorithm has been shown to converge and used for proving the existence of solutions of second order sweeping processes (which are measure differential inclusions) in [25, 24, 16, 17, 31].

Therefore the problem that is tackled in this article may be considered in-between these two groups of differential inclusions (first order with maximal monotone right-hand side on one hand, and Lagrangian sweeping process on the other hand), since we consider Lagrange dynamics with non-trivial mass matrix (see Remark 5.8), but time-invariant set-valued right-hand side with $\mathrm{dom}(\Phi) = \mathbb{R}^n$ (no unilateral constraint). It may also be seen as a (non-trivial) extension of the results in [22]. In [22] M. Jean and J. J. Moreau settled the dynamics of n-degree-of-freedom Lagrangian systems subject to unilateral constraints and Coulomb's friction, with a constant mass matrix. They proved, in a simple two dimensional case, that an implicit time-discretization as in (8) possesses some finite-step convergence capabilities towards the contact sticking mode: Theorem 4.6 below generalizes this result to the considered class of differential inclusions.

This article is organized as follows. In Section 2, we recall certain conventional notations and prerequisites which are used throughout the paper. Then some crucial properties and assumptions are proposed in Section 3. In Section 4, firstly the existence and uniqueness of (Q^k), $(\dot{Q}^k)$ satisfying the system (8) are proved. Then, under the assumptions on the boundedness, time independence of the perturbation force $F(\cdot,\cdot,\cdot)$, the boundedness of $\nabla\mathcal{V}(\cdot)$ and the domination of $\Phi(\cdot)$ over a quadratic form of norm function, the approximate velocity sequence $(\dot{Q}^k)$ is shown to vanish asymptotically and the approximate position (Q^k) converges to Q^∞, where Q^∞ satisfies the relation: $0 \in \partial\Phi(0) + F(Q^\infty, 0) + \nabla\mathcal{V}(Q^\infty)$, i.e., Q^∞ is an equilibrium point of (7) and (8). In addition, under the condition: $-F(Q^\infty, 0) - \nabla\mathcal{V}(Q^\infty) \in \mathrm{int}(\partial\Phi(0))$, the sequence (Q^k) is proved to converge in a finite number of steps. This kind of condition is often used in the literature to obtain the finite-time convergence property of the continuous and discrete systems [3, 5]. In Section 5, we use a sequence of piecewise linear functions to approximate the solutions of the continuous systems and it is proved that the convergence order is $\frac{1}{2}$. Section 6 is dedicated to numerical simulations on a nonlinear two-degree-of-freedom fully actuated robot manipulator. Conclusions end the paper in Section 7.

2. Notations and Mathematical Background

We first introduce some notations used in the paper. Denote by $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ the Euclidean scalar product and the corresponding norm in $\mathbb{R}^n$; by $\|\cdot\|_m$ the induced matrix norm defined as follows

$$\|M\|_m = \sup_{\|x\|=1, x\in\mathbb{R}^n} \|Mx\|, \tag{9}$$

where M is a square matrix of size n. The supremum norm, the L^∞-norm, the L^2-norm of a given function from $[0,T]$ to $\mathbb{R}^n$ are denoted by $\|\cdot\|_{\mathcal{C}(0,T;\mathbb{R}^n)}$, $\|\cdot\|_{L^\infty(0,T;\mathbb{R}^n)}$, $\|\cdot\|_{L^2(0,T;\mathbb{R}^n)}$ respectively. Denote by I the identity operator, I_n the identity matrix of size n; $\mathbb{B}_\varepsilon$ the closed ball of radius ε centered at 0 and $\mathbb{B}_\varepsilon(x)$ the closed ball of radius ε centered at x. The interior and boundary of a given set $S \subset \mathbb{R}^n$ are denoted by $\mathrm{int}(S)$ and $\mathrm{bd}(S)$ respectively. Suppose that $S \subset \mathbb{R}^n$ is a nonempty, closed and convex set. The normal cone to S at the point x in S is given by

$$N_S(x) = \{p \in \mathbb{R}^n : \langle p, y - x\rangle \leq 0 \text{ for all } y \in S\}.$$

The projection of a point x on the set S is the unique point belonging to S which is closest to x

$$\mathrm{proj}[S; x] = y \;\text{ such that } \|y - x\| = \min_{z\in S} \|z - x\|.$$

Let us recall from convex analysis that the subdifferential of the indicator function at $x \in S$ is the normal cone to S at x

$$\partial \Psi_S(x) = N_S(x). \tag{10}$$

In addition, let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, then we have the following relation

$$x^* \in \partial f(x) \quad \Leftrightarrow \quad x \in \partial f^*(x^*), \tag{11}$$

where $f^*(\cdot)$ is the conjugate function of $f(\cdot)$ defined by

$$f^*(x^*) = \sup_{x \in \mathbb{R}^n} \left\{ \langle x^*, x \rangle - f(x) \right\}.$$

Let $A(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $x \mapsto A(x) \subset \mathbb{R}^n$ be a set-valued map. The domain of $A(\cdot)$ is defined by $\mathrm{dom}(A) = \{x \in \mathbb{R}^n : A(x) \neq \emptyset\}$. The map $A(\cdot)$ is said *monotone* provided

$$\langle x_2 - x_1, y_2 - y_1 \rangle \geq 0 \quad \text{for all } x_1, x_2 \in \mathbb{R}^n \text{ and } y_1 \in A(x_1), y_2 \in A(x_2).$$

The monotone map $A(\cdot)$ is called *maximal monotone* if there is no other monotone set-valued map $B(\cdot)$ such that the graph $G(A) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in A(x)\}$ is contained strictly in the graph of $B(\cdot)$. Given a maximal monotone map $A(\cdot)$ and a positive real number λ, the *resolvent of* A (of index λ) defined by $J_\lambda^A := (I + \lambda A)^{-1}$ is a non-expansive single valued map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Denote by $m(A(x))$ the set of elements of $A(x)$ with minimal norm, i.e.,

$$m(A(x)) := \{y^* \in A(x) : \|y^*\| = \min_{y \in A(x)} \|y\|\}. \tag{12}$$

It is well-known that a positive semi-definite matrix $A \in \mathbb{R}^{n \times n}$ (i.e. $\langle Ax, x \rangle \geq 0$, for all $x \in \mathbb{R}^n$), is a maximal monotone operator.
The following result provides a sufficient criterion for verifying whether the sum of two maximal monotone operators is also maximal monotone. Let $A(\cdot)$ and $B(\cdot)$ be two maximal monotone operators on $\mathbb{R}^n$ with $\mathrm{dom}(A) \cap \mathrm{int}(\mathrm{dom}(B)) \neq \emptyset$, then $A(\cdot) + B(\cdot)$ is maximal monotone. Particularly, it is also true if $B : \mathbb{R}^n \to \mathbb{R}^n$ is single-valued, monotone and continuous, see [7].

3. Properties and Assumptions

Let us propose some important assumptions that are used in the analysis led in the sequel. Note that the Assumptions 3.1 and 3.2 may be considered as crucial properties of Lagrangian systems [11, 23, 33], for example the Assumption 3.2 $(\mathcal{C}_2)$, see (3) and the relevant comment in the introduction, is essential for the stability and the control of Lagrangian systems [14].

Assumption 3.1 (The inertia matrix).

($\mathcal{M}_1$) For all $q \in \mathbb{R}^n$, $M(q)$ is symmetric and there exist $k_1 > 0, k_2 > 0$ such that
$$k_1 I_n \leq M(q) \leq k_2 I_n, \tag{13}$$
i.e. for all $q, x \in \mathbb{R}^n$, we have: $k_1\|x\|^2 \leq \langle M(q)x, x\rangle \leq k_2\|x\|^2$.

($\mathcal{M}_2$) $M(\cdot) = (m_{ij}(\cdot))_{1\leq i,j\leq n} \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}^{n\times n})$, i.e., $m_{ij}(\cdot) \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$.

Assumption 3.2 (The centrifugal and Coriolis matrix).

($\mathcal{C}_1$) There exists $k_3 > 0$ such that for all $u, v \in \mathbb{R}^n$:
$$\|C(u, v)\|_m \leq k_3\|v\|.$$

($\mathcal{C}_2$) Given any differentiable function $u : \mathbb{R}_+ \to \mathbb{R}^n$, we have for all $t \geq t_0$:
$$\frac{d}{dt}(M(u(t))) = C(u(t), \dot{u}(t)) + C(u(t), \dot{u}(t))^T.$$

($\mathcal{C}_3$) The function $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $h(x_1, x_2, x_3) = C(x_1, x_2)x_3$ is locally Lipschitz.

Assumption 3.3.

($\mathcal{H}_F$) The function $F(t, x_1, x_2)$ from $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in t, uniformly locally Lipschitz in x_1, x_2 (i.e. the Lipschitz constant is independent of t).

($\mathcal{H}_\Phi$) $\Phi(\cdot)$ is a convex function on $\mathbb{R}^n$.

($\mathcal{H}_\mathcal{V}$) $\mathcal{V}(\cdot)$ is a differentiable function on $\mathbb{R}^n$ and $\nabla\mathcal{V}(\cdot)$ is locally Lipschitz.

The Assumption 3.1 ($\mathcal{M}_2$) implies the Lipschitzness on bounded sets of the inertia matrix $M(\cdot)$. Indeed, we have

Lemma 3.4. *Given $K > 0$, then there exists $k_4 > 0$ such that for all $q_1, q_2, u \in \mathbb{R}^n$ and $\|q_1\|, \|q_2\| \leq K$:*
$$\|M(q_1)u - M(q_2)u\| \leq k_4\|q_1 - q_2\|\|u\|. \tag{14}$$

Proof. Fix $u \in \mathbb{R}^n$. Let the function $G : \mathbb{R}^n \to \mathbb{R}^n$ be defined by
$$G(x) := M(x)u. \tag{15}$$

Then $G(\cdot)$ is a twice differentiable function using ($\mathcal{M}_2$). We have
$$G(x) = M(x)u = \begin{pmatrix} \sum_{j=1}^n m_{1j}(x)u_j \\ \dots \\ \sum_{j=1}^n m_{nj}(x)u_j \end{pmatrix} = \sum_{j=1}^n \begin{pmatrix} m_{1j}(x) \\ \dots \\ m_{nj}(x) \end{pmatrix} u_j, \tag{16}$$

and its Jacobian matrix

$$J_G(x) = \sum_{j=1}^{n} \begin{pmatrix} \nabla m_{1j}(x)^T \\ \dots \\ \nabla m_{nj}(x)^T \end{pmatrix} u_j. \tag{17}$$

Then using the Mean Value Theorem, the conclusion follows. Note that the constant k_4 depends only on K and the matrix $M(q)$. □

The following lemmas show that the inverse matrix $M^{-1}(\cdot)$ is also bounded and Lipschitz on bounded sets. Their proofs are analog to the proofs in Lemma 3.3 and Lemma 3.4 in [3].

Lemma 3.5. *For all $q \in \mathbb{R}^n$, $M^{-1}(q)$ is symmetric and there exist $k_5 > 0$, $k_6 > 0$ such that*

$$k_5 I_n \leq M^{-1}(q) \leq k_6 I_n. \tag{18}$$

Lemma 3.6. *Given $K > 0$, then there exists $k_7 > 0$ such that for all $q_1, q_2, u \in \mathbb{R}^n$ and $\|q_1\|, \|q_2\| \leq K$:*

$$\|M^{-1}(q_1)u - M^{-1}(q_2)u\| \leq k_7 \|q_1 - q_2\| \|u\|. \tag{19}$$

4. Well-posedness and Asymptotic Behavior Analysis

Let us begin with the existence and uniqueness results of the sequences (Q^k), $(\dot{Q}^k)$ defined in (8). Then under some mild conditions, the convergence of (Q^k), $(\dot{Q}^k)$ is obtained, even in finite steps. The following theorem assures the well-posedness of the scheme.

Theorem 4.1. *Let Assumptions 3.1 and 3.2 ($\mathcal{C}_1$) hold. Consider the implicit scheme in (8)*

$$\begin{cases} M(Q^k)\dfrac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1} + \nabla \mathcal{V}(Q^k) + F(t_k, Q^k, \dot{Q}^k) \in -\partial \Phi(\dot{Q}^{k+1}) \\ Q^{k+1} = Q^k + h_k \dot{Q}^k. \end{cases}$$

Fix some $h > 0$. Consider the sequences (Q^k), $(\dot{Q}^k)$ generated by the following algorithm:

1. *$k := 0, t_0 := 0$ and take the starting points $Q^0, \dot{Q}^0 \in \mathbb{R}^n$.*
2. *Given $(t_k, Q^k, \dot{Q}^k)$, we choose $h_k := \min\{h, \frac{k_1}{2k_3\|\dot{Q}^k\|}\}$, where k_1, k_3 are defined in the Assumptions 3.1, 3.2 and find $(Q^{k+1}, \dot{Q}^{k+1})$ satisfying (8). Let $t_{k+1} := t_k + h_k$.*
3. *Let $k := k + 1$ and go to step 2.*

Then at each step k, the approximated position Q^{k+1} and approximated velocity $\dot{Q}^{k+1}$ can be uniquely calculated from the knowledge of Q^k, $\dot{Q}^k, t_k$ and h_k.

Proof. Fix some positive integer k. Obviously, we can compute Q^{k+1} directly by the formula $Q^{k+1} = Q^k + h_k\dot{Q}^k$. From the Assumption 3.2 ($\mathcal{C}_1$) we have

$$\|C(Q^k,\dot{Q}^k)\|_m \le k_3\|\dot{Q}^k\|. \tag{20}$$

Since $h_k = \min\{h, \frac{k_1}{2k_3\|\dot{Q}^k\|}\} \le \frac{k_1}{2k_3\|\dot{Q}^k\|}$, it follows

$$h_k k_3\|\dot{Q}^k\| \le \frac{k_1}{2}. \tag{21}$$

Let $B^k := \frac{1}{h_k}M(Q^k) + C(Q^k,\dot{Q}^k) - \frac{k_1}{2h_k}I_n = \frac{1}{h_k}\{M(Q^k) - k_1 I_n\} + \{C(Q^k,\dot{Q}^k) + \frac{k_1}{2h_k}I_n\}$ and $\lambda_k := \frac{k_1}{2h_k} > 0$. Then $B^k(\cdot)$ is a linear single-valued mapping and for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\langle B^k x, x\rangle &= \frac{1}{h_k}\left\langle\left\{M(Q^k) - k_1 I_n\right\}x, x\right\rangle + \frac{1}{h_k}\left\langle\left\{h_k C(Q^k,\dot{Q}^k) + \frac{k_1}{2}I_n\right\}x, x\right\rangle \\ &\ge \frac{1}{h_k}\left(\frac{k_1}{2} - h_k k_3\|\dot{Q}^k\|\right)\|x\|^2 \ge 0, \end{aligned}\tag{22}$$

using (21) and Assumption 3.1. Hence $B^k(\cdot)$ is maximal monotone with $\mathrm{dom}(B^k) = \mathbb{R}^n$. Note that $\partial\Phi(\cdot)$ is maximal monotone since $\Phi(\cdot)$ is a convex function on $\mathbb{R}^n$, see [7]. The set-valued mapping $A^k := B^k + \partial\Phi$ is therefore also maximal monotone. After some simple computations, we obtain

$$M(Q^k)\frac{\dot{Q}^k}{h_k} - F(t_k, Q^k, \dot{Q}^k) - \nabla\mathcal{V}(Q^k) \in (A^k + \lambda_k I)(\dot{Q}^{k+1}),$$

which implies that

$$\dot{Q}^{k+1} = J_{\lambda_k}^{A^k}\left\{M(Q^k)\frac{\dot{Q}^k}{h_k} - F(t_k, Q^k, \dot{Q}^k) - \nabla\mathcal{V}(Q^k)\right\},$$

where $J_{\lambda_k}^{A_k} = (A^k + \lambda_k I)^{-1}$ is the resolvent of the maximal monotone mapping $A_k(\cdot)$ of index λ_k. In conclusion, we can compute $\dot{Q}^{k+1}$, Q^{k+1} in terms of Q^k, $\dot{Q}^k$ and h_k uniquely. $\square$

Remark 4.2. The choice of $h_k = \min\{h, \frac{k_1}{2k_3\|\dot{Q}^k\|}\} \le h$ is given in order to avoid large values of h_k when $\dot{Q}^k$ is very close to zero.

In the following part, we analyze the convergence of (Q^k), $(\dot{Q}^k)$ based on the existence of a Lyapunov's sequence which is non-increasing along the discrete trajectories. Let us begin with the following assumption:

Assumption 4.3. There exist $\alpha > \beta \geq 0$ and $\eta > 0$ such that

1) $\Phi(\cdot) \geq \eta\|\cdot\|^2 + \alpha\|\cdot\| + \Phi(0)$.
2) $\sup_{(t,x_1,x_2)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n} \|F(t,x_1,x_2)\| \leq \beta$.
3) $\|\nabla\mathcal{V}(x)\| \leq \frac{\alpha-\beta}{2}$ for all $x \in \mathbb{R}^n$.

Remark 4.4. If from the collected data, we can estimate an upper bound of the continuous perturbation force $F(\cdot)$ by some positive β, then we may choose $\Phi(\cdot) := \eta\|\cdot\|^2 + \alpha\|\cdot\|$ for some positive η, α with $\alpha > \beta$ and choose $\mathcal{V}(\cdot)$ by some constant functions, or linear functions, or $\mathcal{V}(\cdot) := \frac{\alpha-\beta}{2}\|\cdot\|_\lambda$ where $\|\cdot\|_\lambda$ is the Moreau-Yosida approximation of the Euclidean norm function of index λ for some $\lambda > 0$. Then Assumptions 3.3 and 4.3 are satisfied. Indeed, for each λ, it is known that $\frac{\alpha-\beta}{2}\|\cdot\|_\lambda$ is differentiable and $\|(\nabla\frac{\alpha-\beta}{2}\|\cdot\|_\lambda)\| \leq \frac{\alpha-\beta}{2}\|m(\partial\|x\|)\| \leq \frac{\alpha-\beta}{2}$.

Let us now investigate how property (3), which is instrumental for the analysis led in [3] and more generally for the stability analysis of controlled Lagrangian systems [14], may be transposed in the discrete time context. Given $(Q^k, \dot{Q}^k)$, we consider the following linear function $u : \mathbb{R}^+ \to \mathbb{R}^n$ defined by

$$u(t) = Q^k + \dot{Q}^k(t - t_0), \tag{23}$$

for some $t_0 > 0$. From Assumption 3.2 ($\mathcal{C}_2$), we obtain

$$\frac{d}{dt}M(u(t)) = C(u(t), \dot{u}(t)) + C(u(t), \dot{u}(t))^T = C(u(t), \dot{Q}^k) + C(u(t), \dot{Q}^k)^T, \tag{24}$$

for all $t \geq 0$. Take $t = t_0$ in (24), then

$$\frac{d}{dt}M(u(t_0)) = C(Q^k, \dot{Q}^k) + C(Q^k, \dot{Q}^k)^T. \tag{25}$$

On the other hand, by using Taylor's expansion and Assumption 3.1 ($\mathcal{M}_2$), we have

$$M(u(t_0 + h_k)) = M(u(t_0)) + h_k\frac{d}{dt}M(u(t_0)) + \mathcal{O}(h_k^2), \tag{26}$$

or equivalently,

$$\frac{M(Q^k + h_k\dot{Q}^k) - M(Q^k)}{h_k} = C(Q^k, \dot{Q}^k) + C(Q^k, \dot{Q}^k)^T + \mathcal{O}(h_k). \tag{27}$$

Let

$$\varepsilon_k = \varepsilon_k(h_k, Q^k, \dot{Q}^k) := \frac{M(Q^k + h_k\dot{Q}^k) - M(Q^k)}{h_k} - C(Q^k, \dot{Q}^k) - C(Q^k, \dot{Q}^k)^T.$$

We can define $h_k^* > 0$ as follows

$$h_k^* := \max\{h > 0 : \|\varepsilon_k\|_m \leq 2\eta \text{ for all } h_k \leq h\}, \tag{28}$$

where $\eta > 0$ is defined in Assumption 4.3. Note that h_k^* depends only on η, M, Q^k and $\dot{Q}^k$. Thus we have the following theorem.

Theorem 4.5. *Let Assumptions* 3.1, 3.2, 3.3, 4.3 *hold and consider the modified algorithm:*

Fix some $h > 0$. Consider the sequences (Q^k), $(\dot{Q}^k)$ generated by

1. $k := 0, t_0 = 0$ *and take the initial data* $Q^0, \dot{Q}^0 \in \mathbb{R}^n, h_{-1} = h$.
2. *Given* $(Q^k, \dot{Q}^k)$, *we choose* $h_k := \min\{h_{k-1}, h_k^*, \frac{k_1}{2k_3\|\dot{Q}^k\|}\}$ *where* h_k^* *is defined in* (28); k_1, k_3 *are defined in Assumptions* 3.1, 3.2. *Find* $(Q^{k+1}, \dot{Q}^{k+1})$ *satisfying* (8). *Let* $t_{k+1} := h_k + t_k$.
3. *Let* $k := k + 1$ *and go to step 2.*

Then the sequence (Q^k) *converges to a point denoted* Q^∞, $\lim_{k\to\infty} \dot{Q}^k = 0$ *and* $\lim_{k\to\infty} t_k = \infty$.

Proof. From Theorem 4.1, for each $k > 0$ we have that Q^{k+1} and $\dot{Q}^{k+1}$ can be uniquely computed in terms of Q^k, $\dot{Q}^k$ and h_k. So the sequences $(Q^k), (\dot{Q}^k)$ are well-defined. Furthermore, from (28) and the choice of h_k, we have

$$\begin{aligned} \varepsilon_k &= \frac{M(Q^k + h_k\dot{Q}^k) - M(Q^k)}{h_k} - C(Q^k, \dot{Q}^k) - C(Q^k, \dot{Q}^k)^T \\ &= \frac{M(Q^{k+1}) - M(Q^k)}{h_k} - C(Q^k, \dot{Q}^k) - C(Q^k, \dot{Q}^k)^T, \end{aligned} \tag{29}$$

and $\|\varepsilon_k\|_m \leq 2\eta$. Let us define the Lyapunov's sequence (E_k) by

$$E_k = \frac{1}{2}\langle M(Q^k)\dot{Q}^k, \dot{Q}^k\rangle,$$

for each positive integer k. We prove that (E_k) is non-increasing. Indeed, one has

$$E_{k+1} - E_k = \frac{1}{2}\langle M(Q^{k+1})\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle - \frac{1}{2}\langle M(Q^k)\dot{Q}^k, \dot{Q}^k\rangle.$$

Note that

$$\begin{aligned} &\langle M(Q^{k+1})\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle - \langle M(Q^k)\dot{Q}^k, \dot{Q}^k\rangle \\ &= \langle (M(Q^{k+1}) - M(Q^k))\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle + \langle M(Q^k)(\dot{Q}^{k+1} - \dot{Q}^k), \dot{Q}^{k+1}\rangle \\ &\qquad + \langle M(Q^k)\dot{Q}^k, \dot{Q}^{k+1} - \dot{Q}^k\rangle \\ &= \langle (M(Q^{k+1}) - M(Q^k))\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle + 2\langle M(Q^k)(\dot{Q}^{k+1} - \dot{Q}^k), \dot{Q}^{k+1}\rangle \\ &\qquad - \langle M(Q^k)(\dot{Q}^{k+1} - \dot{Q}^k), \dot{Q}^{k+1} - \dot{Q}^k\rangle \\ &\leq \langle (M(Q^{k+1}) - M(Q^k))\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle + 2\langle M(Q^k)(\dot{Q}^{k+1} - \dot{Q}^k), \dot{Q}^{k+1}\rangle \\ &\leq 2h_k\left(\left\langle M(Q^k)\frac{(\dot{Q}^{k+1} - \dot{Q}^k)}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1}, \dot{Q}^{k+1}\right\rangle + \frac{\|\varepsilon_k\|_m}{2}\|\dot{Q}^{k+1}\|^2\right) \\ &\leq 2h_k\left(\left\langle M(Q^k)\frac{(\dot{Q}^{k+1} - \dot{Q}^k)}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1}, \dot{Q}^{k+1}\right\rangle + \eta\|\dot{Q}^{k+1}\|^2\right), \end{aligned}$$

where we used Assumption 3.1 and (29). Let $r_{k+1} \in \partial\Phi(\dot{Q}^{k+1})$ be such that

$$M(Q^k)\frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1} + \nabla\mathcal{V}(Q^k) + F(t_k, Q^k, \dot{Q}^k) = -r_{k+1}. \quad (30)$$

Therefore

$$\begin{aligned}\frac{E_{k+1} - E_k}{h_k} &\leq \left\langle M(Q^k)\frac{(\dot{Q}^{k+1} - \dot{Q}^k)}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1}, \dot{Q}^{k+1}\right\rangle + \eta\|\dot{Q}^{k+1}\|^2 \\ &= -\langle\nabla\mathcal{V}(Q^k) + F(t_k, Q^k, \dot{Q}^k) + r_{k+1}, \dot{Q}^{k+1}\rangle + \eta\|\dot{Q}^{k+1}\|^2.\end{aligned}$$

From the definition of the subdifferential, we obtain that

$$\begin{aligned}\frac{E_{k+1} - E_k}{h_k} &\leq -\langle\nabla\mathcal{V}(Q^k) + F(t_k, Q^k, \dot{Q}^k), \dot{Q}^{k+1}\rangle - \Phi(\dot{Q}^{k+1}) + \Phi(0) + \eta\|\dot{Q}^{k+1}\|^2 \\ &\leq \frac{\alpha - \beta}{2}\|\dot{Q}^{k+1}\| + \beta\|\dot{Q}^{k+1}\| - \alpha\|\dot{Q}^{k+1}\| \leq -\frac{\alpha - \beta}{2}\|\dot{Q}^{k+1}\|, \end{aligned} \quad (31)$$

due to Assumption 4.3. Hence

$$\begin{aligned}E_{k+1} - E_k &\leq -h_k\frac{\alpha - \beta}{2}\|\dot{Q}^{k+1}\| \\ &\leq -h_{k+1}\frac{\alpha - \beta}{2}\|\dot{Q}^{k+1}\| = -\frac{\alpha - \beta}{2}\|Q^{k+2} - Q^{k+1}\|,\end{aligned} \quad (32)$$

since $h_{k+1} \leq h_k$, and thus

$$\|Q^{k+2} - Q^{k+1}\| \leq \frac{2}{\alpha - \beta}(E_k - E_{k+1}). \quad (33)$$

Then take the sum side by side of (33) from $k = 0$ to N for any positive integer N and let $N \to +\infty$. We obtain that $(\|Q^{k+1} - Q^k\|) \in l^1$ because (E_k) is bounded from below by zero. It means that (Q^k) is a Cauchy sequence and hence the convergence of (Q^k) follows due to the completeness of $\mathbb{R}^n$. Let $Q^\infty = \lim_{k\to\infty} Q^k$. In particular, (Q^k) is bounded. On the other hand, from (32), the sequence (E_k) is non-increasing. Therefore

$$\frac{1}{2}k_1\|\dot{Q}^k\|^2 \leq \frac{1}{2}\langle M(Q^k)\dot{Q}^k, \dot{Q}^k\rangle = E_k \leq E_0, \quad (34)$$

due to Assumption 3.1. This leads to the boundedness of $(\dot{Q}^k)$. It is obvious that (h_k) is bounded from above by h. The boundedness of $(Q^k), (\dot{Q}^k)$ implies the boundedness from below of (h_k^*) by some positive number (see (28) for the definition of h_k^*). Hence $h_k = \min\{h_{k-1}, h_k^*, \frac{k_1}{2k_3\|\dot{Q}^k\|}\}$ is also bounded from below by some positive number. The sequence (h_k) is non-increasing by construction, so there exists a $\bar{h} > 0$ such that $\lim_{k\to\infty} h_k = \bar{h}$. Hence for each $k > 0$, $h_k \geq \bar{h}$ and $t_{k+1} = t_k + h_k \geq t_k + \bar{h} \geq t_0 + (k+1)\bar{h}$ which implies the limit $\lim_{k\to\infty} t_k = +\infty$. From the formula $\dot{Q}^k = \frac{Q^{k+1} - Q^k}{h_k}$, one infers that $\lim_{k\to\infty} \dot{Q}^k = 0$. □

Theorem 4.6. *Suppose that the perturbation force is independent of time, i.e.*

$$F(t, x_1, x_2) \equiv F(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{35}$$

Let the assumptions of Theorem 4.5 hold. Then the sequence (Q^k) generated by the algorithm in Theorem 4.5 converges to an equilibrium of (8), i.e., $F(Q^\infty, 0) + \nabla\mathcal{V}(Q^\infty) \in -\partial\Phi(0)$. Furthermore, if $-F(Q^\infty, 0) - \nabla\mathcal{V}(Q^\infty) \in \mathrm{int}(\partial\Phi(0))$ then the sequence (Q^k) is finitely convergent, i.e., there exists $N \in \mathbb{Z}_+$ such that for all $k \geq N : Q^k = Q^\infty$.

Proof. For each $k = 1, 2...$, one knows that

$$M(Q^k)\frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1} + F(Q^k, \dot{Q}^k) + \nabla\mathcal{V}(Q^k) \in -\partial\Phi(\dot{Q}^{k+1}). \tag{36}$$

The Assumptions $3.1, 3.2$ $(\mathcal{C}_1)$ and Theorem 4.5 imply

$$\begin{aligned} &\left\| M(Q^k)\frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1} \right\| \\ &\leq k_2 \left\| \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} \right\| + k_3 \|\dot{Q}^k\| \|\dot{Q}^{k+1}\| \to 0 \end{aligned} \tag{37}$$

as $k \to +\infty$. Therefore

$$\begin{aligned} &M(Q^k)\frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1} + F(Q^k, \dot{Q}^k) + \nabla\mathcal{V}(Q^k) \\ &\to F(Q^\infty, 0) + \nabla\mathcal{V}(Q^\infty), \end{aligned} \tag{38}$$

as $k \to +\infty$ since $\nabla\mathcal{V}(\cdot), F(\cdot)$ are continuous. Using the closed graph property of the maximal monotone mapping $\partial\Phi(\cdot)$ (see for example [6]), we obtain: $F(Q^\infty, 0) + \nabla\mathcal{V}(Q^\infty) \in -\partial\Phi(0)$. It means that Q^∞ is an equilibrium of the scheme (8).

In the next part, we prove that the sequence (Q^k) converges in a finite number of steps under the condition $-F(Q^\infty, 0) - \nabla\mathcal{V}(Q^\infty) \in \mathrm{int}(\partial\Phi(0))$. Indeed, let $\mathcal{E}$ be the set $\{k \in \mathbb{N} : Q^{k+1} = Q^k\}$. Suppose that the sequence (Q^k) is not finitely convergent. Let Ω be the set of limit points of $(\frac{Q^{k+1} - Q^k}{\|Q^{k+1} - Q^k\|})_{k \in \mathbb{N} \setminus \mathcal{E}}$ and let $r \in \Omega$. Note that Ω is nonempty since $(\frac{Q^{k+1} - Q^k}{\|Q^{k+1} - Q^k\|})_{k \in \mathbb{N} \setminus \mathcal{E}}$ belongs to the unit sphere which is a compact subset of $\mathbb{R}^n$. Hence, there exists a subsequence of (Q^k), still denoted by (Q^k) such that $\lim_{k \to \infty} \frac{Q^{k+1} - Q^k}{\|Q^{k+1} - Q^k\|} = r$ with $\|r\| = 1$. Let us choose some $\zeta \in \partial\Phi(0)$ and take $r_{k+1} \in \partial\Phi(\dot{Q}^{k+1})$ defined in (30). Using the monotonicity of $\partial\Phi(\cdot)$, we obtain $\langle r_{k+1} - \zeta, \dot{Q}^{k+1} \rangle \geq 0$. Hence, from (8), one infers

$$\Big\langle -M(Q^k)\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k} - C(Q^k,\dot{Q}^k)\dot{Q}^{k+1} - F(Q^k,\dot{Q}^k) \\ - \nabla\mathcal{V}(Q^k) - \zeta, \frac{Q^{k+2}-Q^{k+1}}{\|Q^{k+2}-Q^{k+1}\|}\Big\rangle \geq 0.$$

By taking the limit when $k \to +\infty$, we have:$\langle -F(Q^\infty,0) - \nabla\mathcal{V}(Q^\infty) - \zeta, r\rangle \geq 0$ for all $\zeta \in \partial\Phi(0)$. Combining with $-F(Q^\infty,0) - \nabla\mathcal{V}(Q^\infty) \in \mathrm{int}(\partial\Phi(0))$, it is trivial to conclude that $r = 0$ since we can choose $\zeta \in \partial\Phi(0)$ such that $F(Q^\infty,0)+\nabla\mathcal{V}(Q^\infty)+\zeta = mr$ for some real number $m > 0$. It is a contradiction with the fact that $\|r\| = 1$. Hence, the sequence (Q^k) converges in finite steps. □

5. Convergence of piecewise linear approximations

We have proved that for each initial data $(Q^0, \dot{Q}^0)$ and for each positive h, one can generate the unique sequences $(Q^k), (\dot{Q}^k), (h_k), (t_k)$ by the algorithm in Theorem 4.5, satisfying

$$\begin{cases} M(Q^k)\dfrac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k} + C(Q^k,\dot{Q}^k)\dot{Q}^{k+1} + \nabla\mathcal{V}(Q^k) + F(t_k,Q^k,\dot{Q}^k) \in -\partial\Phi(\dot{Q}^{k+1}) \\ Q^{k+1} = Q^k + h_k\dot{Q}^k, \end{cases}$$

where $(Q^k), (\dot{Q}^k), (h_k)$ are bounded sequences (particularly (h_k) is bounded above from h) and $\lim_{k\to+\infty} Q^k = Q^\infty, \lim_{k\to+\infty}\dot{Q}^k = 0, \lim_{k\to+\infty} t_k = +\infty$. Given $T > 0$, let $q_h(\cdot)$ and $v_h(\cdot)$ be the linear interpolations of the Q^k's and $\dot{Q}^k$'s on $[0,T]$, respectively. It means that for each positive integer k and $t \in [t_k, t_{k+1})$

$$q_h(t) = Q^k + \frac{Q^{k+1}-Q^k}{t_{k+1}-t_k}(t-t_k) = Q^k + \dot{Q}^k(t-t_k), \tag{39}$$

$$v_h(t) = \dot{Q}^k + \frac{\dot{Q}^{k+1}-\dot{Q}^k}{t_{k+1}-t_k}(t-t_k) = \dot{Q}^k + \frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k}(t-t_k). \tag{40}$$

We define also the following step functions on $[0,T]$

$$q_{*h}(t) = Q^k \quad \text{on } [t_k, t_{k+1}), \tag{41}$$

$$v_h^*(t) = \dot{Q}^{k+1} \quad \text{on } [t_k, t_{k+1}), \tag{42}$$

$$v_{*h}(t) = \dot{Q}^k \quad \text{on } [t_k, t_{k+1}), \tag{43}$$

$$F_h(t) = F(t_k, Q^k, \dot{Q}^k) \quad \text{on } [t_k, t_{k+1}). \tag{44}$$

It is easy to verify that

$$\dot{q}_h(t) = \dot{Q}^k = v_{*h}(t) \quad \text{on } (t_k, t_{k+1}), \tag{45}$$

$$\dot{v}_h(t) = \frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k} \quad \text{on } (t_k, t_{k+1}), \tag{46}$$

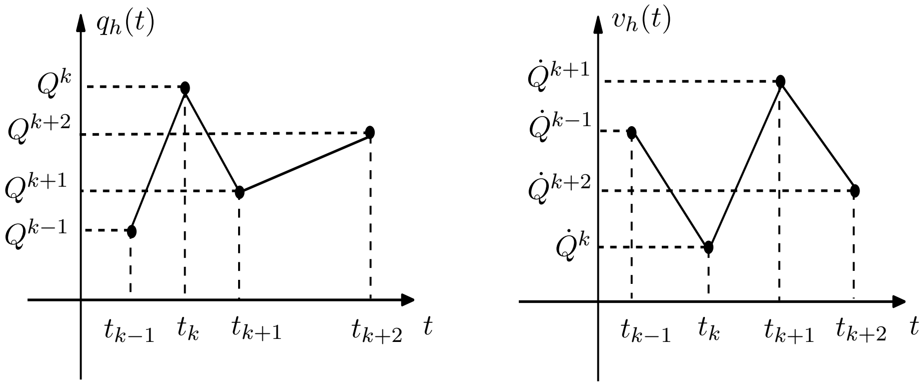

Figure 5.1: Piecewise linear approximation of position and velocity functions.

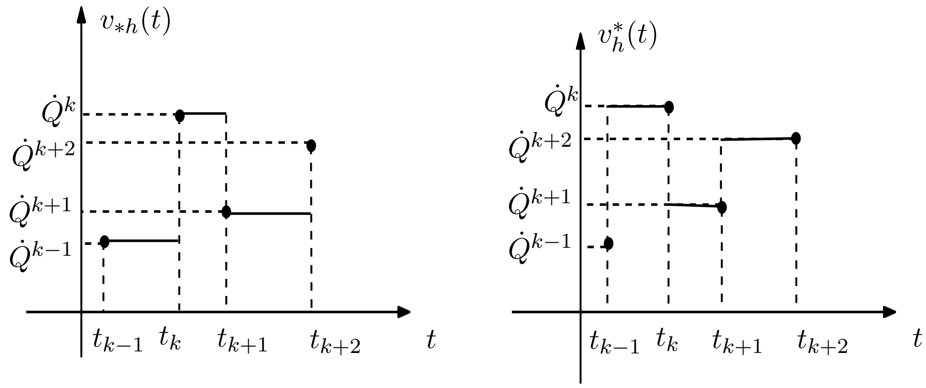

Figure 5.2: Step approximation of velocity functions from below (v_{*h}) and from above (v_h^*).

and

$$
\begin{aligned}
&M(q_{*h}(t))\dot{v_h}(t) + C(q_{*h}(t), \dot{q_h}(t))v_h^*(t) + \nabla\mathcal{V}(q_{*h}(t)) + F_h(t) \\
\in\ & -\partial\Phi(v_h^*(t)) \ \text{ a.e. on } [0,T].
\end{aligned} \tag{47}
$$

Let us denote

$$
C_1 := \sqrt{\frac{2E_0}{k_1}}, \tag{48}
$$

$$
C_2 := \|Q^0\| + C_1 T, \tag{49}
$$

where $E_0 = \frac{1}{2}\langle M(Q^0)\dot{Q}^0, \dot{Q}^0\rangle$. Note that C_1, C_2 are not dependent on h.

Lemma 5.1. *Let Assumptions* 3.1, 3.2, 3.3 *and* 4.3 *hold. There exists a constant M_1 which is not dependent on h such that*

$$
\begin{aligned}
&\|q_h\|_{\mathcal{C}(0,T;\mathbb{R}^n)} + \|q_{*h}\|_{L^\infty(0,T;\mathbb{R}^n)} + \|v_h\|_{\mathcal{C}(0,T;\mathbb{R}^n)} + \|\dot{q_h}\|_{L^\infty(0,T;\mathbb{R}^n)} \\
&\quad + \|v_h^*\|_{L^\infty(0,T;\mathbb{R}^n)} + \|\dot{v}_h\|_{L^\infty(0,T;\mathbb{R}^n)} \leq M_1.
\end{aligned} \tag{50}
$$

Proof. Indeed, from (34), it follows that $\|\dot{Q}^k\| \leq \sqrt{\frac{2E_0}{k_1}} = C_1$ for all $k = 1, 2...$ Note that $\dot{q_h}(t) = \dot{Q}^k$, for all $t \in (t_k, t_{k+1})$. Hence

$$\|v_h\|_{\mathcal{C}(0,T;\mathbb{R}^n)} + \|\dot{q_h}\|_{L^\infty(0,T;\mathbb{R}^n)} + \|v_h^*\|_{L^\infty(0,T;\mathbb{R}^n)} \leq 3C_1. \tag{51}$$

Furthermore, for each integer $k > 0$, $\|Q^k\| \leq \|Q^{k-1}\| + h_{k-1}\|\dot{Q}^{k-1}\| \leq \|Q^{k-1}\| + h_{k-1}C_1 \leq \ldots \leq \|Q^0\| + (h_{k-1} + \ldots + h_0)C_1 \leq \|Q^0\| + C_1 T = C_2$. Therefore

$$\|q_h\|_{\mathcal{C}(0,T;\mathbb{R}^n)} + \|q_{*h}\|_{L^\infty(0,T;\mathbb{R}^n)} \leq 2C_2. \tag{52}$$

We have proved that the sequences $(\|\dot{Q}^k\|), (\|Q^k\|)$ are bounded by C_1 and C_2, respectively. Using the boundedness of $F(\cdot)$, the boundedness on bounded sets of $\partial\Phi(\cdot)$ and Assumptions 3.1 and 3.2 ($\mathcal{C}_1$), from the differential inclusion (8), there exists a constant C_3, which is independent of h, such that for all positive integers k

$$\left\|\frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}\right\| \leq C_3. \tag{53}$$

It means that (see (46))

$$\|\dot{v}_h\|_{L^\infty(0,T;\mathbb{R}^n)} \leq C_3. \tag{54}$$

Thus we can find a constant M_1 which is not dependent on h such that (50) holds. □

Lemma 5.2. *Let Assumptions* 3.1*,* 3.2*,* 3.3 *and* 4.3 *hold. For all $h > 0$, there exists a constant M_2 which is not dependent on h such that*

$$\|q_h - q_{*h}\|_{L^2(0,T;\mathbb{R}^n)} + \|\dot{q_h} - v_h\|_{L^2(0,T;\mathbb{R}^n)} + \|v_h - v_h^*\|_{L^2(0,T;\mathbb{R}^n)} \leq M_2 h. \tag{55}$$

Proof. For any positive integer k and t in the interval (t_k, t_{k+1}), we have

$$q_h(t) - q_{*h}(t) = Q^k + \dot{Q}^k(t - t_k) - Q^k = \dot{Q}^k(t - t_k). \tag{56}$$

Therefore,

$$\begin{aligned}\int_{t_k}^{t_{k+1}} \|q_h(t) - q_{*h}(t)\|^2 dt &\leq C_1^2 \int_{t_k}^{t_{k+1}} (t - t_k)^2 dt \leq \frac{C_1^2}{3}(t_{k+1} - t_k)^3 \\ &\leq \frac{C_1^2 h^2}{3}(t_{k+1} - t_k).\end{aligned} \tag{57}$$

Hence

$$\int_0^T \|q_h(t) - q_{*h}(t)\|^2 dt \leq \frac{C_1^2 T h^2}{3}, \tag{58}$$

and thus $\|q_h - q_{*h}\|_{L^2(0,T;\mathbb{R}^n)} \leq \sqrt{\frac{T}{3}}C_1 h$. Similarly, one has on the interval (t_k, t_{k+1}) that

$$v_h(t) - \dot{q}_h(t) = \dot{Q}^k + \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}(t - t_k) - \dot{Q}^k = \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}(t - t_k), \tag{59}$$

$$\begin{aligned} v_h(t) - v_h^*(t) &= \dot{Q}^{k+1} + \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}(t - t_{k+1}) - \dot{Q}^{k+1} \\ &= \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}(t - t_{k+1}), \end{aligned} \tag{60}$$

and

$$\|v_h - \dot{q}_h\|_{L^2(0,T;\mathbb{R}^n)} \leq \sqrt{\frac{T}{3}}C_3 h, \tag{61}$$

$$\|v_h - v_h^*\|_{L^2(0,T;\mathbb{R}^n)} \leq \sqrt{\frac{T}{3}}C_3 h, \tag{62}$$

due to (53). Then the conclusion follows. □

Lemma 5.3. *Let Assumptions* 3.1, 3.2, 3.3 *and* 4.3 *hold. In addition, suppose that*

Assumption 5.4.

$$\begin{aligned} \forall R \geq 0, \quad \Gamma(R) = \sup \Bigg\{ \left\| \frac{\partial F}{\partial t}(\cdot, q, v) \right\|_{L^2(0,T;\mathbb{R}^n)} &: \\ \|q\|_{L^2(0,T;\mathbb{R}^n)}, \|v\|_{L^2(0,T;\mathbb{R}^n)} \leq R \Bigg\} &< +\infty. \end{aligned} \tag{63}$$

Then for all $h > 0$, there exists a constant M_3 which is not dependent on h such that

$$\|F(\cdot, q_{*h}, \dot{q}_h) - F_h\|_{L^2(0,T;\mathbb{R}^n)} \leq M_3 h. \tag{64}$$

Proof. Let us consider on the interval (t_k, t_{k+1})

$$\begin{aligned} F(t, q_{*h}(t), \dot{q}_h(t)) - F_h(t) &= F(t, Q^k, \dot{Q}^k) - F(t_k, Q^k, \dot{Q}^k) \\ &= \int_{t_k}^{t} \frac{\partial F}{\partial t}(s, Q^k, \dot{Q}^k) ds. \end{aligned} \tag{65}$$

Then the proof is similar to the proof of Lemma 2.4 in [9]. □

In the following theorem, we show that convergent subsequence of (q_h, v_h) can be extracted such that its limit function is a solution of the continuous system (7).

Theorem 5.5. *Let Assumptions* 3.1*,* 3.2*,* 3.3*,* 4.3 *and* 5.4 *hold. Then for any given initial conditions, we can find a subsequence of* (q_h, v_h) *(defined in* (39), (40)*) converging to a solution of the continuous system* (7).

Proof. Lemma 5.1 ensures that the sequences of continuous functions (q_h), (v_h) are uniformly bounded in $\mathcal{C}(0,T;\mathbb{R}^n)$ with the supremum norm and equicontinuous (since the sequences $(\dot{q_h})$, $(\dot{v_h})$ are uniformly bounded in $L^\infty(0,T;\mathbb{R}^n)$). Then, by using the Arzelà-Ascoli Theorem, there exist cluster points $q, v \in \mathcal{C}(0,T;\mathbb{R}^n)$ and subsequences of them, still denoted by (q_h), (v_h) such that

$$q_h \to q \ \text{ in } \mathcal{C}(0,T;\mathbb{R}^n), \tag{66}$$

$$v_h \to v \ \text{ in } \mathcal{C}(0,T;\mathbb{R}^n). \tag{67}$$

In particular $v_h \to v$ in $L^2(0,T;\mathbb{R}^n)$. On the other hand, Lemma 5.2 implies that $\|\dot{q_h} - v_h\|_{L^2(0,T;\mathbb{R}^n)} \to 0$. Therefore $\dot{q_h} \to v$ in $L^2(0,T;\mathbb{R}^n)$ due to the triangle inequality. We have $\dot{q_h} \to \dot{q}$ and $\dot{q_h} \to v$ in the sense of distribution on $(0,T)$. Identifying both limits, one infers that $\dot{q} = v$ in $L^2(0,T;\mathbb{R}^n)$. Since v is continuous on $(0,T)$, we deduce that the function $q(t) = q(0) + \int_0^t \dot{q}(s)ds = q(0) + \int_0^t v(s)ds$ is differentiable, or equivalently, $q \in \mathcal{C}^1(0,T;\mathbb{R}^n)$ (see for example, Theorem 8.2 in [13]). Lemma 5.1 also implies that the sequence $(\dot{v_h})$ is bounded in $L^\infty(0,T;\mathbb{R}^n)$. By using Banach-Alaoglu Theorem, we can find a function $w \in L^\infty(0,T;\mathbb{R}^n)$ and a subsequence of $(\dot{v_h})$, still denoted by $(\dot{v_h})$, such that

$$\dot{v_h} \to w \ \text{ for the topology } \sigma(L^\infty(0,T;\mathbb{R}^n), L^1(0,T;\mathbb{R}^n)). \tag{68}$$

One has $\dot{v_h} \to \dot{v} = \ddot{q}$ and $\dot{v_h} \to w$ in the sense of distribution on $(0,T)$. Identifying both limits, we obtain that $\ddot{q} \in L^\infty(0,T;\mathbb{R}^n)$, or equivalently, $q \in \mathcal{W}^{2,\infty}(0,T;\mathbb{R}^n)$. In the next part, we will prove that $q(\cdot)$ is a solution of (7).
Indeed, it is easy to verify that $q(\cdot)$ satisfies the initial condition. Note that $A(\cdot) := \partial\Phi(\cdot)$ is a maximal monotone operator from $\mathbb{R}^n$ into the subsets of $\mathbb{R}^n$. Let us define the new operator $\mathcal{A}(\cdot)$ from $L^2(0,T;\mathbb{R}^n)$ into the subsets $L^2(0,T;\mathbb{R}^n)$ by

$$f \in \mathcal{A}(u) \iff f(t) \in A(u(t)) \ \text{ a.e. } t \in [0,T]. \tag{69}$$

It is known that $\mathcal{A}(\cdot)$ is also maximal monotone, see [12]. From Lemma 5.2, $(\mathcal{H}_F)$ and (66), (67), (68) one obtains that

$$q_{*h} \to q \ \text{ strongly in } L^2(0,T;\mathbb{R}^n), \tag{70}$$

$$v_h^*, \dot{q_h} \to \dot{q} \ \text{ strongly in } L^2(0,T;\mathbb{R}^n), \tag{71}$$

$$F_h \to F(\cdot, q, \dot{q}) \ \text{ strongly in } L^2(0,T;\mathbb{R}^n), \tag{72}$$

$$\dot{v_h} \to \ddot{q} \ \text{ strongly in } L^2(0,T;\mathbb{R}^n). \tag{73}$$

Let us prove that

$$M(q_{*h})\dot{v_h} \to M(q)\ddot{q} \ \text{ weakly in } L^2(0,T;\mathbb{R}^n). \tag{74}$$

Given $\varphi \in L^2(0,T;\mathbb{R}^n)$, note that $M(q)\varphi$ also belongs to $L^2(0,T;\mathbb{R}^n)$. Then

$$\left|\int_0^T \langle M(q_{*h}(t))\dot{v_h}(t) - M(q(t))\ddot{q}(t), \varphi(t)\rangle dt\right|$$

$$\leq \left|\int_0^T \langle [M(q_{*h}(t)) - M(q(t))]\dot{v_h}(t), \varphi(t)\rangle dt\right| + \left|\int_0^T \langle M(q(t))(\dot{v_h}(t) - \ddot{q}(t)), \varphi(t)\rangle dt\right|$$

$$\leq k_4 \int_0^T \|q_{*h}(t) - q(t)\|\|\dot{v_h}(t)\|\|\varphi(t)\| dt + \left|\int_0^T \langle (\dot{v_h}(t) - \ddot{q}(t)), M(q(t))\varphi(t)\rangle dt\right| \to 0$$

for some constant $k_4 > 0$ by using Lemma 3.4, the boundedness of $q_{*h}, q, \dot{v_h}$, and since $\dot{v_h}$ converges weakly to $\ddot{q}$ while q_{*h} converges to q strongly in $L^2(0,T;\mathbb{R}^n)$. Therefore, one obtains

$$\begin{aligned} &M(q_{*h})\dot{v_h} + C(q_{*h}, \dot{q_h})v_h^* + \nabla\mathcal{V}(q_{*h}) + F_h \\ &\to M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(t,q,\dot{q}) \end{aligned}$$

weakly in $L^2(0,T;\mathbb{R}^n)$. From (47), (71) and the closed graph property of the maximal monotone operator $\mathcal{A}(\cdot)$, we infer that

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(t,q,\dot{q}) \in -\mathcal{A}(\dot{q}). \tag{75}$$

It means that $(q,\dot{q})$ is a solution of (7). □

In the following part, we show the convergence order of the scheme under the following assumption, which is about the local hypo-monotonicity of the operator $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto M^{-1}(x)\partial\Phi(y)$, for any $x \in \mathbb{R}^n$. Note that if this assumption holds, then it is easy to verify that (7) has at most one solution for given initial condition (see Theorem 5.1 in [3]). In particular, the assumption is satisfied for the cases in Proposition 5.1 and Proposition 5.2 in [3].

Assumption 5.6. Given $K > 0$, there exists a constant $c > 0$ such that for all $x_1, x_2, y_1, y_2 \in \mathbb{B}_K$ and $y_1^* \in \partial\Phi(y_1), y_2^* \in \partial\Phi(y_2)$

$$\langle M^{-1}(x_1)y_1^* - M^{-1}(x_2)y_2^*, y_1 - y_2\rangle \geq -c(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2). \tag{76}$$

Theorem 5.7. *Let Assumptions* 3.1*,* 3.2*,* 3.3*,* 4.3*,* 5.4 *and* 5.6 *hold. Then for any given initial conditions, the piecewise linear approximate function* (q_h, v_h) *converges to the unique solution of the continuous system* (7) *with order of convergence* 1/2.

Proof. For all $h, k > 0$ and accepting a usual abuse of notations, we get

$$-[M(q_{*h})\dot{v_h} + C(q_{*h}, \dot{q_h})v_h^* + \nabla\mathcal{V}(q_{*h}) + F_h] \in \partial\Phi(v_h^*), \tag{77}$$

$$-[M(q_{*k})\dot{v_k} + C(q_{*k}, \dot{q_k})v_k^* + \nabla\mathcal{V}(q_{*k}) + F_k] \in \partial\Phi(v_k^*). \tag{78}$$

By using Assumption 5.6, one infers

$$\begin{aligned}&\langle \dot{v}_h - \dot{v}_k + M^{-1}(q_{*h})[C(q_{*h}, \dot{q}_h)v_h^* + \nabla\mathcal{V}(q_{*h}) + F_h] \\ &\qquad - M^{-1}(q_{*k})[C(q_{*k}, \dot{q}_k)v_k^* + \nabla\mathcal{V}(q_{*k}) + F_k], v_h^* - v_k^*\rangle \\ &\le c(\|q_{*h} - q_{*k}\|^2 + \|v_h^* - v_k^*\|^2).\end{aligned} \tag{79}$$

Let us integrate both sides of (79) on $[0,t]$ $(t \in (0,T))$ and estimate all the terms. We begin with the first term by using Cauchy-Schwarz inequality

$$\begin{aligned}\int_0^t \langle \dot{v}_h - \dot{v}_k, v_h^* - v_k^*\rangle ds &= \frac{1}{2}\|v_h(t) - v_k(t)\|^2 + \int_0^t \langle \dot{v}_h - \dot{v}_k, v_h^* - v_h + v_k - v_k^*\rangle ds \\ &\ge \frac{1}{2}\|v_h(t) - v_k(t)\|^2 - (\|\dot{v}_h\|_{L^2(0,T;\mathbb{R}^n)} + \|\dot{v}_k\|_{L^2(0,T;\mathbb{R}^n)}) \\ &\qquad (\|v_h^* - v_h\|_{L^2(0,T;\mathbb{R}^n)} + \|v_k - v_k^*\|_{L^2(0,T;\mathbb{R}^n)}) \\ &\ge \frac{1}{2}\|v_h(t) - v_k(t)\|^2 - 2\sqrt{T}M_1M_2(h+k),\end{aligned} \tag{80}$$

where M_1, M_2 are defined in Lemma 5.1 and Lemma 5.2. Let us now deal with the remaining terms in (79). From the Assumptions 3.2 $(\mathcal{C}_1)$, $(\mathcal{C}_3)$, 3.3 $(\mathcal{H}_\mathcal{V})$, $(\mathcal{H}_F)$, Lemma 3.5, Lemma 3.6 and Lemma 5.1, it is not difficult to find some constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned}&\|M^{-1}(q_{*h})C(q_{*h}, \dot{q}_h)v_h^* - M^{-1}(q_{*k})C(q_{*k}, \dot{q}_k)v_k^*\| \\ &\le c_1(\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\| + \|v_h^* - v_k^*\|),\end{aligned} \tag{81}$$

$$\|M^{-1}(q_{*h})\nabla\mathcal{V}(q_{*h}) - M^{-1}(q_{*k})\nabla\mathcal{V}(q_{*k})\| \le c_2\|q_{*h} - q_{*k}\|, \tag{82}$$

$$\begin{aligned}&\|M^{-1}(q_{*h})F(\cdot, q_{*h}, \dot{q}_h) - M^{-1}(q_{*k})F(\cdot, q_{*k}, \dot{q}_k)\| \\ &\le c_3(\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\|).\end{aligned} \tag{83}$$

Indeed, let us prove the first inequality, then the two remaining inequalities are proved similarly. By using the triangle inequality, one infers

$$\begin{aligned}&\|M^{-1}(q_{*h})C(q_{*h}, \dot{q}_h)v_h^* - M^{-1}(q_{*k})C(q_{*k}, \dot{q}_k)v_k^*\| \\ &\le \|[M^{-1}(q_{*h}) - M^{-1}(q_{*k})]C(q_{*h}, \dot{q}_h)v_h^*\| \\ &\qquad + \|M^{-1}(q_{*k})[C(q_{*h}, \dot{q}_h)v_h^* - C(q_{*k}, \dot{q}_k)v_k^*]\| \\ &\le m_1\|q_{*h} - q_{*k}\| + m_2(\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\| + \|v_h^* - v_k^*\|) \\ &\le c_1(\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\| + \|v_h^* - v_k^*\|),\end{aligned} \tag{84}$$

for some constants $m_1, m_2 > 0$ and $c_1 := m_1 + m_2$ due to Assumption 3.2 $(\mathcal{C}_1)$, Lemma 3.5, Lemma 5.1, Lemma 3.6 and Assumptions $(\mathcal{C}_3)$.

From the inequality (83) and the triangle inequality, we have

$$\begin{aligned}
&\|M^{-1}(q_{*h})F_h - M^{-1}(q_{*k})F_k\| \\
&\leq \|M^{-1}(q_{*h})[F_h - F(\cdot, q_{*h}, \dot{q}_h)]\| \\
&\qquad + \|M^{-1}(q_{*h})F(\cdot, q_{*h}, \dot{q}_h) - M^{-1}(q_{*k})F(\cdot, q_{*k}, \dot{q}_k)\| \\
&\qquad + \|M^{-1}(q_{*k})[F_k - F(\cdot, q_{*k}, \dot{q}_k)]\| \\
&\leq k_6(\|F_h - F(\cdot, q_{*h}, \dot{q}_h)\| + \|F_k - F(\cdot, q_{*k}, \dot{q}_k)\|) \\
&\qquad + c_3(\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\|), \qquad (85)
\end{aligned}$$

where k_6 is defined in Lemma 3.5. By using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and Lemma 5.3, one gets

$$\begin{aligned}
&\int_0^t (\|F_h - F(\cdot, q_{*h}, \dot{q}_h)\| + \|F_k - F(\cdot, q_{*k}, \dot{q}_k)\|)\|v_h^* - v_k^*\| ds \\
&\leq \frac{1}{2}\int_0^T (\|F_h - F(\cdot, q_{*h}, \dot{q}_h)\|^2 + \|F_k - F(\cdot, q_{*k}, \dot{q}_k)\|^2) ds + \int_0^t \|v_h^* - v_k^*\|^2 ds \\
&\leq \frac{1}{2}M_3^2(h+k)^2 + \int_0^t \|v_h^* - v_k^*\|^2 ds, \qquad (86)
\end{aligned}$$

where M_3 is defined in Lemma 5.3. Let us recapitulate the inequalities in (80)–(86):

$$\int_0^t \langle \dot{v}_h - \dot{v}_k, v_h^* - v_k^* \rangle ds \geq \frac{1}{2}\|v_h(t) - v_k(t)\|^2 - 2\sqrt{T}M_1M_2(h+k); \qquad (87)$$

$$\begin{aligned}
&\int_0^t \langle M^{-1}(q_{*h})C(q_{*h}, \dot{q}_h)v_h^* - M^{-1}(q_{*k})C(q_{*k}, \dot{q}_k)v_k^*, v_h^* - v_k^* \rangle ds \\
&\geq -c_1\int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\| + \|v_h^* - v_k^*\|)\|v_h^* - v_k^*\| ds; \qquad (88)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t \langle M^{-1}(q_{*h})\nabla\mathcal{V}(q_{*h}) - M^{-1}(q_{*k})\nabla\mathcal{V}(q_{*k}), v_h^* - v_k^* \rangle ds \\
&\geq -c_2\int_0^t \|q_{*h} - q_{*k}\|\|v_h^* - v_k^*\| ds; \qquad (89)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t \langle M^{-1}(q_{*h})F_h - M^{-1}(q_{*k})F_k, v_h^* - v_k^* \rangle ds \\
&\geq -k_6\int_0^t (\|F_h - F(\cdot, q_{*h}, \dot{q}_h)\| + \|F_k - F(\cdot, q_{*k}, \dot{q}_k)\|)\|v_h^* - v_k^*\| ds \\
&\qquad - c_3\int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\|)\|v_h^* - v_k^*\| ds \qquad (90) \\
&\geq -\frac{1}{2}k_6M_3^2(h+k)^2 - k_6\int_0^t \|v_h^* - v_k^*\|^2 ds \\
&\qquad - c_3\int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\|)\|v_h^* - v_k^*\| ds.
\end{aligned}$$

Let $c_4 := c_1 + c_2 + c_3$. Taking the sum of (87)–(90) side by side and combining with (79), we obtain

$$
\begin{aligned}
&\frac{1}{2}\|v_h(t) - v_k(t)\|^2 - 2\sqrt{T}M_1M_2(h+k) - \frac{1}{2}k_6M_3^2(h+k)^2 \\
&\qquad - c_4\int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\| + \|v_h^* - v_k^*\|)\|v_h^* - v_k^*\|ds \\
&\qquad - k_6\int_0^t \|v_h^* - v_k^*\|^2 ds \\
&\leq \int_0^t \langle \dot{v}_h - \dot{v}_k + M^{-1}(q_{*h})[C(q_{*h}, \dot{q}_h)v_h^* + \nabla\mathcal{V}(q_{*h}) + F_h] \\
&\qquad - M^{-1}(q_{*k})[C(q_{*k}, \dot{q}_k)v_k^* + \nabla\mathcal{V}(q_{*k}) + F_k], v_h^* - v_k^*\rangle ds \\
&\leq c\int_0^t (\|q_{*h} - q_{*k}\|^2 + \|v_h^* - v_k^*\|^2)ds.
\end{aligned}
\tag{91}
$$

Thus, one infers

$$
\begin{aligned}
&\frac{1}{2}\|v_h(t) - v_k(t)\|^2 \\
&\leq 2\sqrt{T}M_1M_2(h+k) + \frac{1}{2}k_6M_3^2(h+k)^2 \\
&\qquad + c_4\int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q}_h - \dot{q}_k\| + \|v_h^* - v_k^*\|)\|v_h^* - v_k^*\|ds \\
&\qquad + \int_0^t (c\|q_{*h} - q_{*k}\|^2 + (c+k_6)\|v_h^* - v_k^*\|^2)ds \\
&\leq M_4(h+k) + \frac{c_4}{2}\int_0^t (\|q_{*h} - q_{*k}\|^2 + \|\dot{q}_h - \dot{q}_k\|^2)ds \\
&\qquad + 2c_4\int_0^t \|v_h^* - v_k^*\|^2 ds + \int_0^t (c\|q_{*h} - q_{*k}\|^2 + (c+k_6)\|v_h^* - v_k^*\|^2)ds \\
&\leq M_4(h+k) + c_5\int_0^t (\|q_{*h} - q_{*k}\|^2 + \|\dot{q}_h - \dot{q}_k\|^2 + \|v_h^* - v_k^*\|^2)ds
\end{aligned}
\tag{92}
$$

where $c_5 := c + 2c_4 + k_6$ and $M_4 := 2\sqrt{T}M_1M_2 + \frac{1}{2}k_6M_3^2$ (h and k are chosen small enough such that $h + k \leq 1$).

Lemma 5.2 implies

$$\begin{aligned}\int_0^t \|q_{*h}-q_{*k}\|^2 ds &\le \int_0^t (\|q_{*h}-q_h\|+\|q_h-q_k\|+\|q_k-q_{*k}\|)^2 ds\\ &\le 3\int_0^t (\|q_h-q_k\|^2+\|q_{*h}-q_h\|^2+\|q_k-q_{*k}\|^2)ds\\ &\le 3\int_0^t \|q_h-q_k\|^2 ds + 3\int_0^T (\|q_{*h}-q_h\|^2+\|q_k-q_{*k}\|^2)ds\\ &\le 3\int_0^t \|q_h-q_k\|^2 ds + 3M_2^2(h+k)^2. \end{aligned} \tag{93}$$

Similarly, we have

$$\int_0^t \|\dot{q}_h-\dot{q}_k\|^2 ds \le 3\int_0^t \|v_h-v_k\|^2 ds + 3M_2^2(h+k)^2 \tag{94}$$

$$\int_0^t \|v_h^*-v_k^*\|^2 ds \le 3\int_0^t \|v_h-v_k\|^2 ds + 3M_2^2(h+k)^2. \tag{95}$$

On the other hand

$$\frac{1}{2}\|q_h(t)-q_k(t)\|^2 = \int_0^t \langle \dot{q}_h-\dot{q}_k, q_h-q_k\rangle ds \le \frac{1}{2}\int_0^t (\|\dot{q}_h-\dot{q}_k\|^2+\|q_h-q_k\|^2)ds. \tag{96}$$

From (92)–(96), one can find some constants $M_5>0$ and $c_6>0$ (with $h+k\le 1$) such that

$$\begin{aligned}&\|q_h(t)-q_k(t)\|^2+\|v_h(t)-v_k(t)\|^2\\ \le\ & M_5(h+k)+c_6\int_0^t (\|q_h(s)-q_k(s)\|^2+\|v_h(s)-v_k(s)\|^2)ds.\end{aligned}$$

By using the Gronwall's inequality with the function $\Gamma(\cdot) := \|q_h(\cdot)-q_k(\cdot)\|^2 + \|v_h(\cdot)-v_k(\cdot)\|^2$, one deduces

$$\Gamma(t) = \|q_h(t)-q_k(t)\|^2+\|v_h(t)-v_k(t)\|^2 \le M_6(h+k), \tag{97}$$

where $M_6 = M_5 e^{c_6 T}$. Let $k\to 0$, we obtain

$$\|q_h(t)-q(t)\|^2+\|v_h(t)-v(t)\|^2 \le M_6 h. \tag{98}$$

It means that the piecewise linear approximate function converges to the unique solution of the continuous system (7) with order 1/2. □

Remark 5.8. If the matrix $M(q)$ is constant, by using a transformation of variables, we can reduce our system to a first order differential inclusion where the right-hand side is the subdifferential of a convex function. Then the convergence order is 1, see [8].

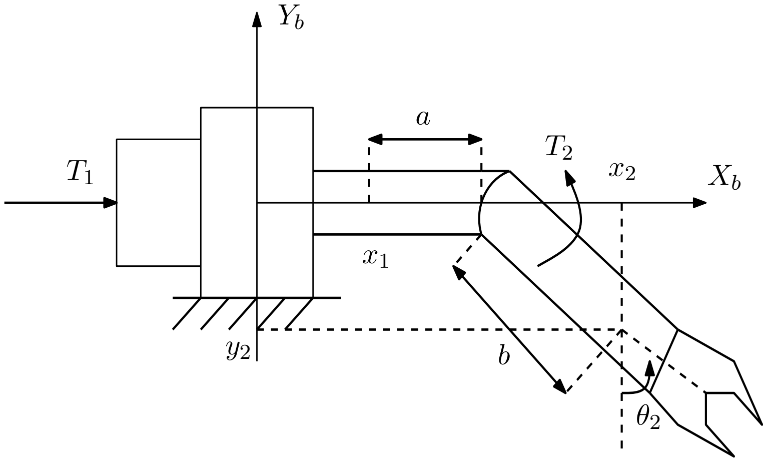

Figure 6.1: A robot manipulator.

6. Numerical Simulations

Let us consider the following dynamical model of a robot manipulator (see [10]) without perturbation ($F \equiv 0$), which is only for illustrative purpose:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) \in -\partial\Phi(\dot{q}), \tag{99}$$

where the state $q = \begin{pmatrix} x_1 \\ \theta_2 \end{pmatrix}$ and

$$M(q) = \begin{pmatrix} m_1 + m_2 & m_2 b\cos\theta_2 \\ m_2 b\cos\theta_2 & I + m_2 b^2 \end{pmatrix}, \qquad C(q,\dot{q}) = \begin{pmatrix} 0 & -m_2 b\dot{\theta}_2 \sin\theta_2 \\ 0 & 0 \end{pmatrix},$$

$$\nabla\mathcal{V}(q) = \begin{pmatrix} 0 \\ bm_2 g\sin\theta_2 \end{pmatrix}, \qquad \partial\Phi(\dot{q}) = -\begin{pmatrix} T_1 \\ T_2 \end{pmatrix},$$

for some positive constants m_1, m_2, b, I and g. If we consider the forces T_1, T_2 with the following form

$$T_1 = -(2\eta_1\dot{x}_1 + \alpha_1 \mathrm{Sign}(\dot{x}_1)), \qquad T_2 = -(2\eta_2\dot{\theta}_2 + \alpha_2 \mathrm{Sign}(\dot{\theta}_2)),$$

for some positive constants $\eta_1, \eta_2, \alpha_1, \alpha_2$, then for $y = (y_1\ y_2)^T \in \mathbb{R}^2$

$$\Phi(y) = \eta_1 y_1^2 + \alpha_1|y_1| + \eta_2 y_2^2 + \alpha_2|y_2| \geq \min\{\eta_1,\eta_2\}\|y\|^2 + \min\{\alpha_1,\alpha_2\}\|y\|.$$

The control forces $T_1(\cdot)$ and $T_2(\cdot)$ correspond to velocity plus sliding-mode feedback controller, a control technique that is widely used in practice [34]. It is not difficult to show that Assumptions 3.1, 3.2, 3.3, 4.3 and 5.4 are satisfied with suitable parameters, but not Assumption 5.6. This assumption, which is also a sufficient condition to obtain the uniqueness, holds for a narrow class of mechanical systems. In fact, the property of solution uniqueness of nonsmooth problems is not attained generally. However, it is important and interesting to

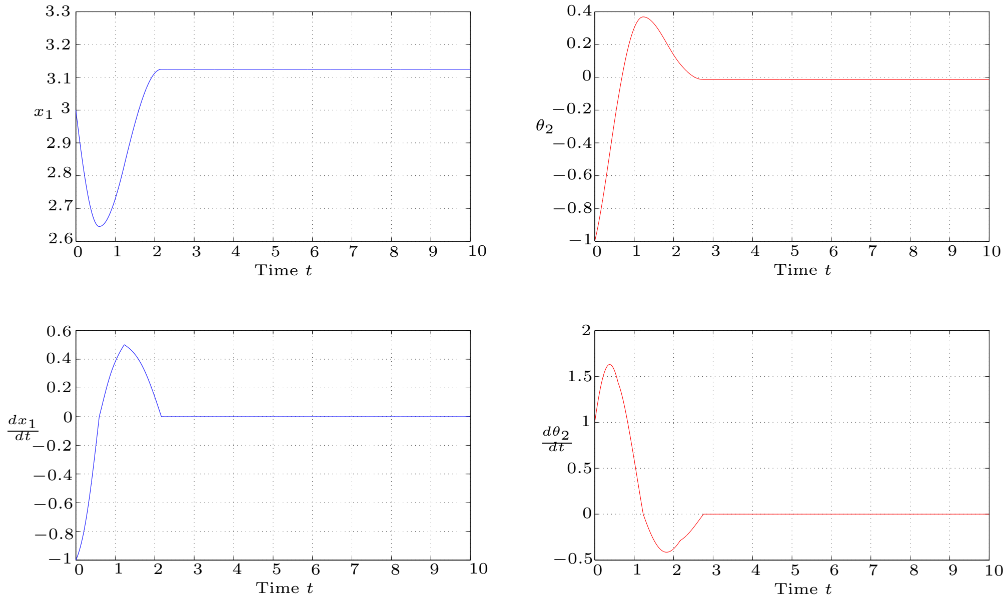

Figure 6.2: Simulation for the position and velocity of (99) with $(x_1^0, \theta_2^0) = (3, -1)$ and $(\dot{x}_1^0, \dot{\theta}_2^0) = (-1, 1)$.

simulate at least one solution of the continuous problem. For the sake of simplicity, we consider $\Phi(y) = \frac{1}{2}\|y\|^2 + |y_1| + |y_2|$, which implies $\partial\Phi(y) = y + \text{Sign}(y)$ where

$$\text{Sign}(y) = \left(\begin{array}{c} \text{Sign}(y_1) \\ \text{Sign}(y_2) \end{array} \right), \quad \text{and for } m \in \mathbb{R},\ \text{Sign}(m) = \begin{cases} -1 & \text{if } m < 0, \\ [-1, +1] & \text{if } m = 0, \\ +1 & \text{if } m > 0. \end{cases} \tag{100}$$

From the first line of (8), we have

$$M(Q^k)\frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1} + \nabla\mathcal{V}(Q^k) \in -\dot{Q}^{k+1} - \text{Sign}(\dot{Q}^{k+1}). \tag{101}$$

Let us denote

$$P := \frac{1}{h_k}M(Q^k) + C(Q^k, \dot{Q}^k) + I_2 \quad \text{and} \quad x := \nabla\mathcal{V}(Q^k) - \frac{1}{h_k}M(Q^k)\dot{Q}^k. \tag{102}$$

This leads to find a solution y for given x of the following inclusion

$$Py + x \in -\text{Sign}(y) = -\partial\phi(y), \tag{103}$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a convex function defined by

$$\phi(y) = |y_1| + |y_2|. \tag{104}$$

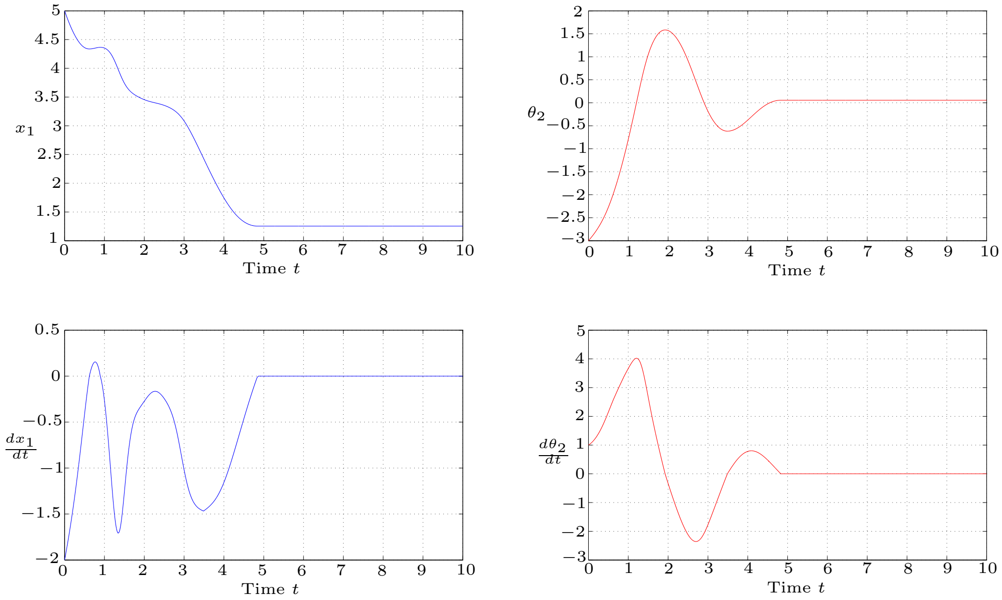

Figure 6.3: Simulation for the position and velocity of (99) with $(x_1^0, \theta_2^0) = (5, -3)$ and $(\dot{x}_1^0, \dot{\theta}_2^0) = (-2, 1)$.

Remark 6.1. (i) Note that at each step, the matrix P is positive definite (in fact $P \geq I_2$, see the proof of Theorem 4.1) but non-symmetric since the Coriolis matrix C is non-symmetric. The existence and uniqueness of y in (103) are assured by a well-known result for general functions $\phi(\cdot)$ (proper, convex and lower semicontinuous), see for example [7]. Furthermore, the operator $(P+\partial\phi)^{-1}$ is single-valued and Lipschitz continuous.
(ii) There exists various efficient iterative algorithms in the literature for solving the variational inclusion (103) even for a general convex function $\phi : \mathbb{R}^n \to \mathbb{R}$. However, exploiting the specific structure of $\phi(\cdot)$ in (104), we show that (103) is equivalent to a nonlinear equation for which the semismooth Newton method can be applied.

Let's denote the symmetric and skew-symmetric parts of P by

$$P_s := \frac{P + P^T}{2}; \qquad P_a := \frac{P - P^T}{2}. \tag{105}$$

Let $K := [-1, 1]^2$ be the square containing the origin in $\mathbb{R}^2$ and $L := \{z \in \mathbb{R}^2 : P_s^{1/2} z \in K\} = P_s^{-1/2} K$. The following result transforms the inclusion (103) to an implicit equation.

Lemma 6.2. *For a given $x \in \mathbb{R}^2$, the inclusion* (103) *is equivalent to the equation*

$$[I + P_s^{-1} P_a]y + P_s^{-1} x = P_s^{-1/2} \text{proj}[L; P_s^{-1/2} P_a y + P_s^{-1/2} x]. \tag{106}$$

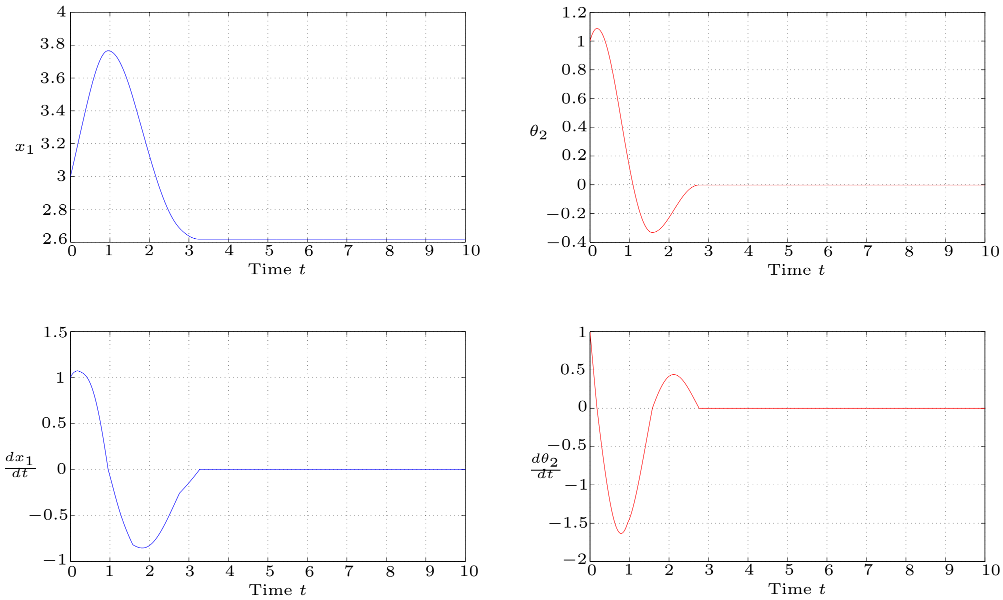

Figure 6.4: The system under perturbation with $(x_1^0, \theta_2^0) = (3, 1)$ and $(\dot{x}_1^0, \dot{\theta}_2^0) = (1, 1)$.

Proof. Note that the inclusion

$$-Py - x \in \partial\phi(y) \tag{107}$$

is equivalent to

$$y \in \partial\phi^*(-Py - x) = \partial\Psi_K(-Py - x) = N_K(-Py - x) = -N_K(Py + x), \tag{108}$$

by using (10), (11) and the symmetry of the set K. Let $x' := P_a y + x$. From (108), we obtain

$$y \in -N_K(P_s y + x').$$

Let the vector z be such that $P_s^{1/2} z = P_s y + x'$, which implies $y = P_s^{-1/2} z - P_s^{-1} x'$. We have

$$P_s^{-1/2} z - P_s^{-1} x' \in -N_K(P_s^{1/2} z) \quad \Leftrightarrow \quad z - P_s^{-1/2} x' \in -P_s^{1/2} N_K(P_s^{1/2} z).$$

Equivalently,

$$z - P_s^{-1/2} x' \in -N_L(z) \;\text{ with } L = \{z \in \mathbb{R}^2 : P_s^{1/2} z \in K\} = P_s^{-1/2} K.$$

Using the definition of the projection, we have

$$\begin{aligned}
& z - P_s^{-1/2} x' \in -N_L(z) \\
\Leftrightarrow\; & z = \mathrm{proj}[L; P_s^{-1/2} x'] \\
\Leftrightarrow\; & y = P_s^{-1/2} \mathrm{proj}[L; P_s^{-1/2} x'] - P_s^{-1} x' \\
\Leftrightarrow\; & [I + P_s^{-1} P_a] y + P_s^{-1} x = P_s^{-1/2} \;\mathrm{proj}[L;\; P_s^{-1/2} P_a y + P_s^{-1/2} x].
\end{aligned}$$

This completes the proof. $\square$

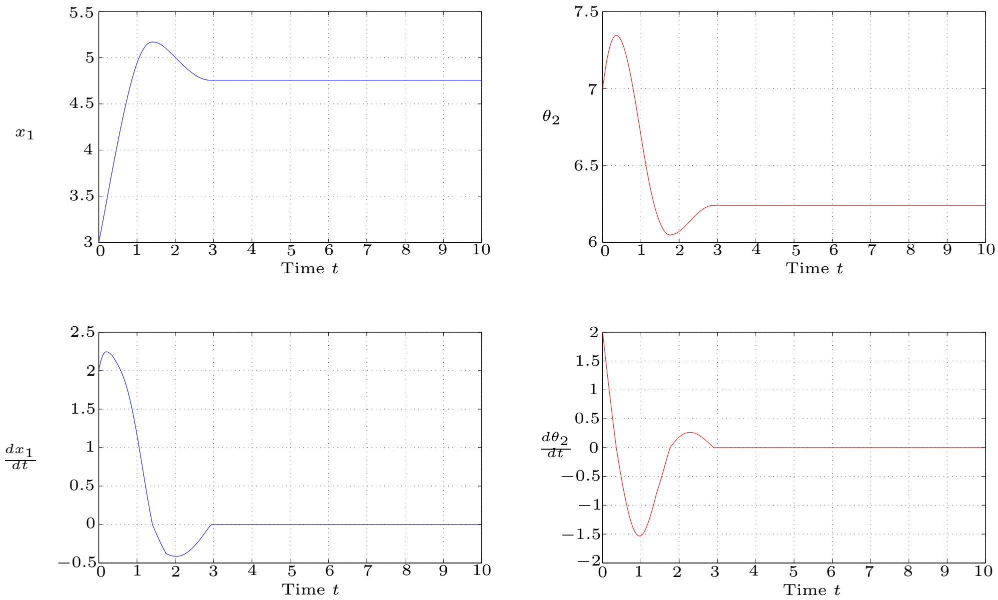

Figure 6.5: The system under perturbation with $(x_1^0, \theta_2^0) = (3, 7)$ and $(\dot{x}_1^0, \dot{\theta}_2^0) = (2, 2)$.

Remark 6.3. For each fixed $x \in \mathbb{R}^2$, we define the following function $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$, by

$$y \mapsto \mathcal{F}(y) = [I + P_s^{-1} P_a] y + P_s^{-1} x - P_s^{-1/2} \text{proj}[L; P_s^{-1/2} P_a y + P_s^{-1/2} x].$$

The equality (106) can be rewritten as a nonlinear equation $\mathcal{F}(y) = 0$. Since $\mathcal{F}$ is semismooth (see [19] for the definition), the Semismooth Newton Method may be used to solve this equation [19].

Now, let us simulate the system (99) with different initial conditions (Figures 6.2 and 6.3). We also consider the case with nonzero perturbation, with $F(t, x_1) = (\sin(t)\ \sin(x_1))^T$, with the same analysis (Figures 6.4 and 6.5). The finite-time convergence and smooth stabilization at zero are clearly illustrated on all the figures. Another important point is that implicit methods allow to obtain the exact (at the machine precision) stabilization at the equilibrium, which is not the case with explicit Euler methods. This type of behaviour for implicit methods is also studied for other classes of dynamical systems in [1, 2]. Furthermore, the example allows us to recover numerically the finite-time convergence property studied in [3].

7. Conclusions

This paper contains the analysis of the time-discretization of the differential inclusions studied in [3] with an implicit Euler-like scheme. These differential inclusions stem from the robust, finite-time global stabilization of fully actuated Lagrangian systems. Applications are mainly in the discrete-time robust sliding mode control of mechanical systems. Experimental results reported eleswhere have proved that such implicit methods allow to significantly improve the controllers performance when a digital implementation is made. The convergence of the approximated positions and velocities is shown, as well as the order of the method. The proof is based on the use of a Lyapunov function, and of fundamental properties of Lagrangian systems. The convergence to the equilibrium point, when it exists, is proved to occur in a finite number of steps. Numerical simulations illustrate the theoretical analysis. Future research will concern the problem of trajectory tracking, using set-valued extensions of passivity-based controllers [14].

References

[1] V. Acary, B. Brogliato: Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems, Syst. Control Lett. 59 (2010) 284–293.

[2] V. Acary, B. Brogliato, Y. Orlov: Chattering-free digital sliding-mode control with state observer and disturbance rejection, IEEE Trans. Autom. Control 57(5) (2012) 1087–1101.

[3] S. Adly, B. Brogliato, B. K. Le: Well-posedness, robustness and stability analysis of a set-valued controller for Lagrangian systems, SIAM J. Control Optim. 51(2) (2013) 1592–1614.

[4] V. I. Arnold: Mathematical Methods of Classical Mechanics, Springer, New York (1989).

[5] B. Baji, A. Cabot: An inertial proximal algorithm with dry friction: finite convergence results, Set-Valued Anal. 14(1) (2006) 1–23.

[6] D. P. Bertsekas, A. Nedic, A. E. Ozdaglaz: Convex Analysis and Optimization, Athena Scientific, Belmont (2003).

[7] V. Barbu: Optimal Control of Variational Inequalities, Pitman, London (1984).

[8] J. Bastien: Convergence order of implicit Euler numerical scheme for maximal monotone differential inclusions, Z. Angew. Math. Phys. 64 (2013) 955–966.

[9] J. Bastien, M. Schatzman: Numerical precision for differential inclusions with uniqueness, M2AN, Math. Model. Numer. Anal. 36 (2002) 427–460.

[10] G. Bastin: Modélisation et Analyse des Systèmes Dynamiques, Lectures Notes, Louvain School of Engineering, Louvain-la-Neuve (2013).

[11] G. Besançon: Global output feedback tracking control for a class of lagrangian systems, Automatica 36 (2000) 1915–1921.

[12] H. Brezis: Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, Math. Studies 5, North-Holland, Amsterdam (1973).

[13] H. Brezis: Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York (2011).

[14] B. Brogliato, R. Lozano, B. Maschke, O. Egeland: Dissipative Systems Analysis and Control, 2nd Ed., Springer, London (2007).

[15] J. Clemente-Gallardo, J. M. A. Scherpen: Relating Lagrangian and Hamiltonian formalisms of LC circuits, IEEE Trans. Circuits Syst. I 50(10) (2003) 1359–1363.

[16] R. Dzonou, M. Monteiro-Marques: A sweeping process approach to inelastic contact problems with general inertia operators, Eur. J. Mech., A, Solids 26(3) (2007) 474–490.

[17] R. Dzonou, M. Monteiro-Marques, L. Paoli: A convergence result for a vibro-impact problem with a general inertia operator, Nonlinear Dyn. 58(1-2) (2009) 361–384.

[18] C. M. Elliot: On the convergence of a one-step method for the numerical solution of an ordinary differential inclusion, IMA J. Numer. Anal. 5(1) (1985) 3–21.

[19] F. Facchinei, J. S. Pang: Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer, New York (2003).

[20] O. Huber, V. Acary, B. Brogliato, F. Plestan: Discrete-time twisting controller without numerical chattering: analysis and experimental results with an implicit method, Proc. IEEE Decision and Control (CDC) (Los Angeles, 2014) (2014) 4373–4378.

[21] IEEE Trans. Ind. Electron., Special Issues, O. Kaynak, A. Bartoszewicz, V. I. Utkin (Guest eds.), 55(11) (2008); 56(9) (2009).

[22] M. Jean, J. J. Moreau: Dynamics in the presence of unilateral contacts and dry friction: a numerical approach, in: Unilateral Problems in Structural Analysis II (Prescudin, 1985), G. Del Piero, F. Maceri (eds.), CISM Courses and Lectures 304, Springer, Wien (1987) 151–196.

[23] M. Mabrouk, S. Ammar, J. C. Vivalda: Transformation synthesis for Euler-Lagrange systems, Nonlinear Dyn. Syst. Theory 7(2) (2007) 197–216.

[24] M. Mabrouk: A unified variational model for the dynamics of perfect unilateral constraints, Eur. J. Mech., A, Solids 17(5) (1998) 819–842.

[25] M. D. P. Monteiro Marques: Differential Inclusions in Nonsmooth Mechanical Problems: Shocks and Dry Friction, Birkhäuser, Basel (1993).

[26] J. J. Moreau: Evolution problem associated with a moving convex set in a Hilbert space, J. Differ. Equations 26(3) (1977) 347–374.

[27] J. J. Moreau: Liaisons unilatérales sans frottement et chocs inélastiques, C. R. Acad. Sci., Paris, Sér. II 296(19) (1983) 1473–1476.

[28] J. J. Moreau: Standard inelastic shocks and the dynamics of unilateral constraints, in: Unilateral Problems in Structural Analysis (Ravello, 1983), G. Del Piero, F. Maceri (eds.), CISM Courses and Lectures 288, Springer, Wien (1985) 173–221.

[29] J. J. Moreau: Unilateral contact and dry friction in finite freedom dynamics, in: Nonsmooth Mechanics and Applications, J. J. Moreau, P. D. Panagiotopoulos (eds.), CISM Courses and Lectures 302, Springer, Wien (1988) 1–82.

[30] Y. Orlov: Discontinuous Systems-Lyapunov Analysis and Robust Synthesis under Uncertainty Conditions, Communications and Control Engineering Ser., Springer, Berlin (2009).

[31] L. Paoli: A proximal-like method for a class of second order measure-differential inclusions describing vibro-impact problems, J. Differ. Equations 250(1) (2001) 476–514.

[32] J. M. A. Scherpen, D. Jeltsema, J. B. Ben Klaassens: Lagrangian modeling of switching electrical networks, Syst. Control Lett. 48(5) (2003) 365–374.

[33] M. W. Spong, S. Hutchinson, M. Vidyasagar: Robot Modeling and Control, Wiley, New York (2005).

[34] V. I. Utkin: Sliding Modes in Control Optimization, Springer, Berlin (1992).

[35] V. I. Utkin, J. Guldner, J. Shi: Sliding Mode Control in Electro-Mechanical Systems, 2nd Ed., Automation and Control Engineering Ser., CRC Press, Boca Raton (2009).

[36] B. Wang, B. Brogliato, V. Acary, A. Boubakir, F. Plestan: Experimental comparisons between implicit and explicit implementations of discrete-time sliding mode controllers: towards input and output chattering suppression, IEEE Trans. Control Syst. Technol. 23(5) (2015) 2071–2075.