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A Survey on Galois Stratifications and Measures of Viability Risk*

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Dedicated to the memory of Jean Jacques Moreau.

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The hyperspace of an underlying space is the family of its subsets and hypermaps are maps from one hyperspace to another one. Set-valued maps generate some of them, either "pointwise", as image and focus hypermaps and their inverses, or indirectly, as "viability kernels" of subsets under a differential inclusion, and several other viability concepts (capture basins, invariant kernels and absorption basins, and many combinations of them). When the underlying space is itself the product of two spaces, the hyperspace is the space of graphs of set-valued maps, or binary relations. When the arrival space is the space of real numbers, the hyperspace of the product contains the graphs, epigraphs and hypographs of numerical functions, so that hypermaps actually operate on functions and provide adequate "solutions" of first-order partial differential equations, the graph of which are capture basins, for instance. When the underlying space is the product of more than two spaces, it contains the graphs of "relations": they pop-up whenever it is impossible to partition a group of variables in only two classes, the inputs and the outputs.
The framework of hyperspaces and hypermaps is useful for studying many problems at this level. Some general results are surveyed in the first section. The main examples concern the powers of hypermaps from one hyperspace into itself. Under adequate assumptions, each hypermap leads to a stratification of the underlying space, associating with each set a denumerable partition telling "how deep" an element belongs to this set or "how far" it does not belong to it. This is in this sense that we speak of "Galois measure of viability risk". Indeed, the concept of "Galois measure" allows not only to measure the localization of a given element in a subset of this partition, but to locate any other subsets in the stratification. The simplest example is the hypermap mapping any subset to its interior: the partition of each states it generates is made of its interior, its topological boundary and its exterior. Other hypermaps induce more sophisticated stratifications, such as the ones generated by viability concepts, the

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elements of which are the iterations of "viability algorithms". The analysis of many examples can be done at the level of hyperspaces and hypermaps or hyperrelations: we briefly mention the atypical logics and mathematical morphology. Sociology (studying societies of individual agents sharing a same subset of cultural codes) and, in particular, economics, when the economic codes are made of commodities on one hand, prices and units of numéraire measuring them, and the societies are the agents who consumer or producer, buyer or seller, debtor or creditor. This opens avenues for further research in the sciences of life.

Keywords: Hypermap, focus, mutational analysis, Galois stratification, Galois quotation, Galois transform, Galois measure, atypical logic, mathematical morphology, society, cultures, economic classes, dynamic economic behavior

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A tribute to Jean Jacques Moreau and Terry Rockafellar Although this volume is dedicated to Jean Jacques Moreau, Terry Rockafellar and him are twinned mathematicians, founding together and independently the foundations of convex analysis, with different motivations (see *Fonctionnelles convexes*, [44], and *Convex analysis*, [48]). The first author is old enough to have had the opportunity to meet them both in the middle of the 1960's. Even though he was deeply involved in the numerical functional analysis of boundary value problems of partial differential equations, he was stroke by the beauty of convex analysis he discovered then, although far away from his preoccupations. He enjoyed their intellectual elegance and culture besides their mathematical achievements and still remembers long walks during which many issues were disputed. This was the hidden part of an unfathomable scientific friendly influence on his conceptual ideas. He did not anticipate at this time that he will be happy to use them. For the events of May 1968 triggered the creation of the Université de Paris-Dauphine which offered him the opportunity to experiment his dream to switch mathematical motivations from physical sciences to the sciences of life. The first example was economics, which, at the time, relied on equilibria and optima. Then, convex analysis became a main topic of investigation, although there was no much left after their pioneering contributions. Sweeping processes went in pair with differential inclusions and viability questions. This is a personal and grateful tribute to them, as well as to Jacques-Louis Lions, Laurent Schwartz and so many other mentors too numerous to be cited here. Their deep although confidential influence is a debt impossible to refund.

Contents

1. Introduction

1.1. Galois Measure

The hyperspace $\mathcal{P}(X)$ is the family of all subsets of an underlying space X and an hypermap is a single-valued map $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$. We are lead to solve *hyperequations*

$$\forall\ M \in \mathcal{P}(Y),\ \text{find } K \in \mathcal{P}(X) \text{ such that } \Omega(K) = M$$

which can be split in into:

1. the *lower hyperinclusion*

$$\forall\ M \in \mathcal{P}(Y),\ \text{find } K \in \mathcal{P}(X) \text{ such that } \Omega(K) \subset M.$$

2. the *upper hyperinclusion*

$$\forall\ M \in \mathcal{P}(Y),\ \text{find } K \in \mathcal{P}(X) \text{ such that } \Omega(K) \supset M.$$

The Boolean operations on the hyperspaces, independent of the structures of the underlying spaces, thus play an important role and justifies to use the lattice structure and its properties. For instance, the Galois transform associates with M the minimal selection $K^\flat =: \Omega^\flat(M)$ of the upper hyperinclusion (whenever it exists under adequate assumptions). Hence the *Galois* "transform" and *Stone* functor should be revisited. (see Section 2.6, p. 195).

However, other questions of set-valued analysis and viability theory could be revisited in the framework of hyperspace for laying the foundation of a forthcoming "hyperspace analysis" which should take over set-valued analysis. Surveying part of what already exists is the purpose of this study.

When the hypermap is generated by a set-valued map $F : X \rightsquigarrow Y$ in the sense that $\Omega(K) = F(K)$, we are naturally led for instance to select the maximal pairs of subsets $(K, M) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ such that $K \times M \subset \text{Graph}(F)$: they constitute the graph of the "focus" hypermap $\widehat{F}$ defined by $\widehat{F}(K) := \bigcap_{x \in K} F(x)$.

1.2. Time Arrowlets

One of the question presented in this study boils down to define the *Galois quotation* $\alpha_\Omega : \mathcal{P}(X) \times X \mapsto \mathbb{Z}$ associating with any set $K \subset X$ and any $x \in X$ an integer $\alpha_\Omega(K, x) \in \mathbb{Z}$ for *measuring* by an integer "how deep" x belongs to K for positive values or "how far" it is from K for negative values:

1. for each fixed subset K, the function $x \mapsto \alpha_\Omega(K, x)$ plays the role of a *membership function* (such as the ones defining *fuzzy set*) constructed from the datum of the hypermap Ω;
2. for each fixed x, the function $K \mapsto \alpha_\Omega(K, x)$ plays the role of a Dirac measure of K. We associate the lower and upper Galois measures $K \mapsto \alpha^\flat_\Omega(K, M) := \inf_{x \in M} \alpha_\Omega(K, x)$ and $K \mapsto \alpha^\sharp_\Omega(K, M) := \sup_{x \in M} \alpha_\Omega(K, x)$ of subsets K that we shall connect later with the solutions of the hyperinclusions mentioned at the beginning of this introduction:

This was motivated for defining a tychastic measure of viability risk (see *Tychastic Viability Measure of Risk Eradication. A Viabilist Portfolio Performance and Insurance Approach*, [16]) as *"insurance durations"*, a universal measure of "value[1]", instead of one of the statistical concepts of the populous jungle of *statistical risk measures* (we refer to the elegant contributions [49], as well as to the tutorial [47] and [50], in which they use convex analysis and the Legendre-Fenchel Transform as an umbrella to cover many of these value-at-risk measures by adding to the expectation of a random variable all kinds of deviations which are variants of the variance, as well as [55]).

As a motivating example, we describe the Galois quotation derived from the basic concepts of viability theory, when the hypermap Ω is associated with the right hand side F of a differential inclusion[2] $x'(t) \in F(x(t))$: the Galois quotation associates with each element x the "insurance duration" describing the number of seconds such that *all evolutions* starting from x are viable in K. They play the role of "time arrowlets"[3].

The best is to give an example provided by viability theory. The following figures symbolize the concepts, with the danger to be too simple[4] and conceal more complex issues (investigated in Section 4, p. 211).

A tychastic evolutionary system associates with any initial state a subset of evolutions. Following *Charles Peirce*, they form a "tychastic reservoir" in which the goddess *Tyche* chooses any evolution *with no statistic regularity.* The tychastic viability problem requires that *all evolutions* triggered by an initial positions are viable (or remain) in a given subset K for some durations: n units of times (time arrowlets) or forever (time arrow).

The first stratum is the set of initial states from which all successors are *viable* in K at least during one unit of time (tychastic viability kernel of duration 1). Its complement is the subset of initial states from which one successor evolution reaches (captures) K immediately. Their intersection delineate is a kind of "thick boundary", called a (internal) "marche" of zero duration (see Definition 3.1, p. 205), symbolized in Figure 1.1.

One can iterate this construction and define successively the strata of durations n and, inside, the marches of duration n of elements which at least one evolution leaves K for $n+1$ units of duration. Applying the same strategy to the

[1]We refer to *La valeur n'existe pas. À moins que [...]*, [7], [15], and *Traffic Networks as Information Systems. A viability Approach*, [17], for viability approaches to time.

[2]Differential inclusions are closely related to sweeping processes introduced by Jean Jacques Moreau in [43]. See also *Differential Inclusions in Nonsmooth Mechanical Problems - Shocks and Dry Friction*, [42], for instance, and).

[3]In French, *fléchettes du temps*, as a diminutive of *flèche du temps.* We could have suggested "time needles", as a tribute to *Buffon* who used them for computing π by throwing them on a parquet, and not as measuring durations as we do.

[4]We avoided nonsmooth sets just for the purpose of clarity, but this study is valid for any set.

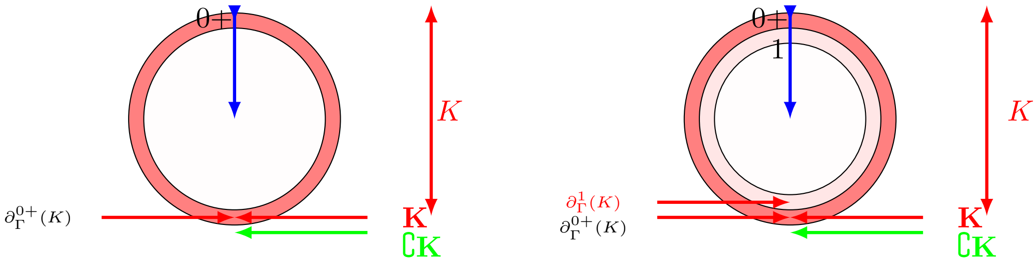


Figure 1.1: Internal Marches of Duration 0 and 1. The subset K is symbolized by a ball. The first marche is displayed in the figure at left. We carve in the first stratum a second one, the tychastic viability kernel of duration 2 of elements from which all evolutions are viable in the first stratum during one duration, and thus, viable in the environment during 2 units of duration. Its complement in the first stratum is the subset of initial states from which all evolutions are viable during one unit of duration, but such that at least one evolution leaves K in 2 units of duration. This is the marche of order 1.

complement of K, the strata are regarded as restoration since they measure the durations needed for restoring viability. The associated marches are numbered from $-\infty$ to $0-$. Taking the union of the stratifications of K and of its complement, we obtain a partition by marches associated with both the underlying discrete system and the subset K.

This Galois stratification process can be associated with any "intensive hypermap" and can be used in numerous fields. For instance, in finance, the insurance duration n assigned to the elements of the marche of duration n defines the *tychastic insurance duration of the viability risk* used *Tychastic Viability Measure of Risk Eradication. A Viabilist Portfolio Performance and Insurance Approach*, [16]). This insurance duration n is an example of Galois quotation that we shall define in the general case (see Section 4, p. 211).

We will evoke also stratifications appearing in atypical logics (see Section 2.4, p. 193), definition of societies by the cultural codes they share and economic classes of agents sharing the same dynamical economic behavior (see Section 2.8, p. 200), and mathematical morphology (see Section 5, p. 220, and Figure 5, p. 222).

1.3. Organization of this Study

Section 2, p. 188, is devoted to an overview on hypermaps, and, among them, the ones generated by set-valued maps, in particular, their focus and inverse focus hypermaps (Subsection 2.3.2, p. 192). These hypermaps are used in many areas, such atypical logics (Subsection 2.4, p. 193), and one of its motivation, the definitions of societies as set of individual sharing the same cultural codes (Subsection 2.8, p. 200) and mathematical morphology (Section 5, p. 220). For

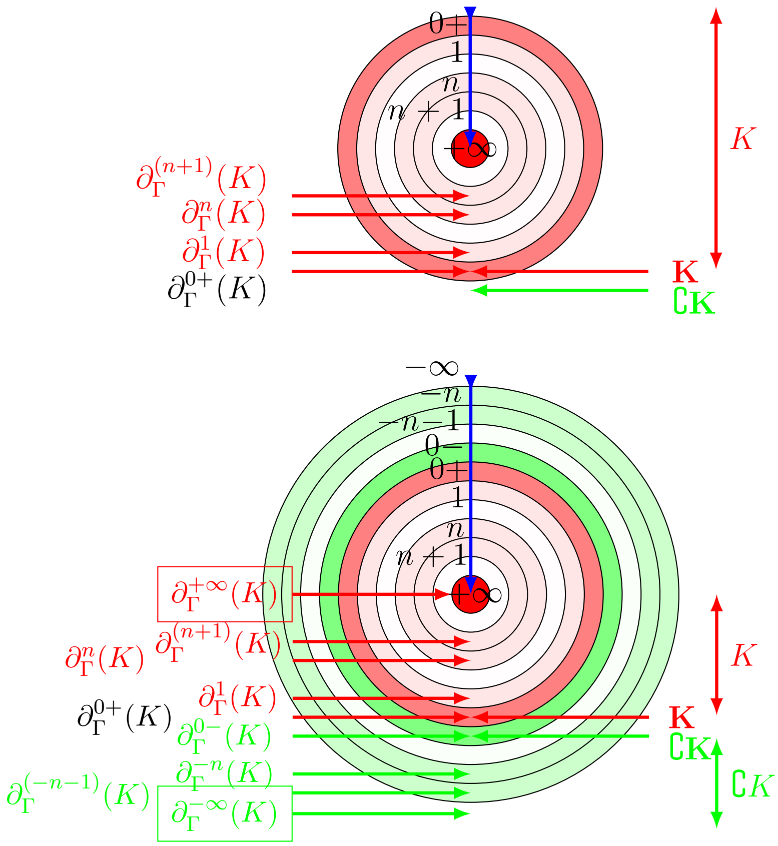


Figure 1.2: Time Arrowlets. This figure symbolizes the "Galois stratification" of the space by marches are numbered above the abscissa. The arrow from $-\infty$ to $+\infty$ represents the (discrete) "time arrow", and the "time arrowlets" symbolize the time needed to passe from one marche to the next one.

defining and studying them, we need to define relations (Subsection 2.5, p. 194): they extend the graphical approach regarding set-valued maps as binary relations, subsets of the product of two spaces, by subsets of any product, which also define relations. We next recall the concept of Galois transforms of hypermaps in the context of solving upper and lower hyperinclusions (Section 2.6, p. 195). We end this section by a too short overview of the legacy by hyperspaces of some structures of their underlying spaces: vector space structure, topological structure and, above all, mutational structure. The latter allows us to define a differential calculus on hyperspaces of vector spaces, devoid of a linear structure. Governing the evolutions of subsets as differential inclusions govern the evolution of vectors is therefore available.

Section 3, p. 203, studies the structuration of subsets associated with intensive hypermaps, playing a different role than the Galois transform, using instead the reflection transforms. Such a structuration allows us to construct a partition of the underlying space. Consequently, it provides an alternative to fuzzy sets (Subsection 3.1, p. 203), and generates kind of measures of subset allowing them to be located precisely in the stratification (Subsection 3.5, p. 209).

Section 4, p. 211, uses Galois stratification associated with "viability hypermaps", which are not associated with set-valued maps. The contingent stratification by viability kernels and absorption basins, the tychastic stratification by invariance kernels and capture basins, which were described by arrowlets described in the introduction. They are actually the viability algorithms for computing kernels and basins under differential inclusions and are connected with the concept of Minkowki-Bouligand dimensions and the construction of fractals as viability kernels.

2. Hypermaps

2.1. Hyperspaces

We introduce the following notations:

Definition 2.1 (Demarcation). Let K and L be subsets of X. We denote by

1. $K \cup_{\emptyset} L$ the union of two *disjoint* subsets;
2. by $K \setminus L := K \cap \complement L$ the *abrasion* (or *set difference*) of K by L;
3. $K \mho L$ the *demarcation* between K and L, the subset

$$K \mho L := \complement K \cap \complement L = \complement(K \cup L) = \complement L \setminus K = \complement K \setminus L \qquad (1)$$

of elements that belong neither to K, nor to L.

The etymology of "demarcation" is Spanish (*demarcar*), such as the demarcation dividing the New World between Spain and Portugal in 1493, whereas "marche" comes from Frankish *marka*, which designates a border or a boundary, such as the border districts between England and Scotland or England and Wales. *Markon* means to walk ("marcher", in French). Even though "markets" took place on "marches", it comes from Latin *mercor*, meaning "trade". *Terminology*, separating and naming the words of a sentence, is derived from the Roman deity *Terminus* (without legs nor arms), who remained immobile for guaranteeing boundaries delimitating fields. From spatial, the "term" became a metaphor of the boundary of a temporal window, before they became deadlines, since. In Chinese, these notions are associated with the sinogram 边, *bian1*, meaning "separation".

We denote by $\mathcal{P}(X)$ the *hyperspace* (also called *power space* 2^X) of an underlying space X made of all subsets of X. Subsets of hyperspaces are called *families* (of subsets). There exist numerous types of families, among which

1. *partitions* of a subset K, families of subsets mutually disjoint $S_i \subset K$, $i \in I$ covering: $K := \bigcup_{i \in I} S_i$;
2. *trees*, families of subsets $S_i \subset K$ such that, $\forall\, i \neq j$, $S_i \cap S_j = \emptyset$, or $S_i \subset S_j$ or $S_j \subset S_i$,

in addition to families of open subsets, of nonempty compact subsets for defining *Maslov* measures, σ-algebras defining *Kolmogorov* measures, *Choquet* capacitable subsets, etc.

2.2. Hypermaps and their Reflection

We next introduce hypermaps:

Definition 2.2 (Hypermaps). Introducing two spaces X and Y and their corresponding hyperspaces $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, we shall investigate "hypermaps" $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$. A hypermap Ω is

1. *increasing* (resp. *decreasing*) if $K \subset L$ implies $\Omega(K) \subset \Omega(L)$ (resp. $\Omega(K) \supset \Omega(L)$);
2. *a dilation* if for every $\mathcal{J} \subset \mathcal{P}(X)$, $\Omega\left(\bigcup_{K\in\mathcal{J}} K\right) = \bigcup_{K\in\mathcal{J}} \Omega(K)$;
3. *an erosion* if for every $\mathcal{J} \subset \mathcal{P}(X)$, $\Omega\left(\bigcap_{K\in\mathcal{J}} K\right) = \bigcap_{K\in\mathcal{J}} \Omega(K)$.

We observe that *erosions are increasing* and *dilations are increasing.*

We introduce also the reflection transform of a hypermap:

Definition 2.3 (Reflections of Hypermaps). Let $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$. We denote by $\Omega^\ddagger : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$ the *reflection* of $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$, the hypermap defined by

$$\Omega^\ddagger(K) := \complement\Omega(\complement K).$$

The *reflection transform* mapping hypermaps Ω to their reflection $\Omega^\ddagger$ is a bijection.

We observe that

1. Any hypermap is increasing if and only if its reflection is increasing;
2. An hypermap is an erosion if and only if its reflection is a dilation;
3. The reflection of the product $\Delta\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Z)$ of the hypermaps $\Delta : \mathcal{P}(Y) \mapsto \mathcal{P}(Z)$ and Ω is the product of their reflections:
$$(\Delta\Omega)^\ddagger = \Delta^\ddagger\Omega^\ddagger.$$
4. We define the union $\bigcup_{i\in\mathbb{I}} \Omega_I$ and the intersection $\bigcup_{i\in\mathbb{I}} \Omega_I$ of family of hypermaps $\Omega^i : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$ by
$$\begin{cases} \forall\, K \subset X, \ \left(\bigcup_{i\in\mathbb{I}} \Omega_i\right)(K) := \bigcup_{i\in\mathbb{I}} \Omega_i(K) \\ \forall\, K \subset X, \ \left(\bigcap_{i\in\mathbb{I}} \Omega_i\right)(K) := \bigcap_{i\in\mathbb{I}} \Omega_i(K). \end{cases} \tag{2}$$
Then
$$\left(\bigcup_{i\in\mathbb{I}} \Omega_i\right)^\ddagger = \bigcap_{i\in\mathbb{I}} \Omega_i^\ddagger \quad \text{and} \quad \left(\bigcap_{i\in\mathbb{I}} \Omega_i\right)^\ddagger := \bigcup_{i\in\mathbb{I}} \Omega_i^\ddagger. \tag{3}$$

When $Y = X$, we single out other useful properties:

Definition 2.4 (Intensivity and Idempotence). The n-th power of an hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ is defined recursively by $\Omega^n(K) := \Omega(\Omega^{n-1}(K))$. We shall say that Ω is

1. *intensive* (resp. *extensive*) if $\Omega(K) \subset K$ (resp. $\Omega(K) \supset K$);
2. *idempotent* if $\Omega^2 = \Omega$.

If the hypermaps Ω_i are dilations, then

$$\left(\bigcup_{i\in\mathbb{I}} \Omega_i\right)^n (K) := \bigcup_{(i_1,\ldots i_n)\in\mathbb{I}^n} \Omega_{i_1}(\Omega_{i_2}(\ldots(\Omega_{i_n}(K)))). \tag{4}$$

It is easy to check

Lemma 2.5 (Partitions associated with intensions). *The reflection of an intensive hypermap is extensive and the reflection of an idempotent hypermap is idempotent.*
Any intensive hypermap Ω associates with any subset $K \subset X$ the partition

$$X = \Omega(K) \cup_\emptyset [K \cap \Omega^\ddagger(\complement K)] \cup_\emptyset [\complement K \cap \Omega^\ddagger(K)] \cup_\emptyset \Omega(\complement K). \tag{5}$$

Example (Partition associated with the abrasion). We observe that the reflection $\Lambda_L^\ddagger$ of the abrasive hypermap Λ_L by L defined by $\Lambda(K) := K \setminus L$ is the union: $\Lambda_L^\ddagger(K) := K \cup L$. Hence $\Lambda_L(\complement K) = K \mho L$, $K \cap \Lambda_L^\ddagger(K) = K \cap L$ and $\complement K \cap \Lambda_L^\ddagger(K) = L \setminus K$, so that we recover the well known partition

$$X = (K \setminus L) \cup_\emptyset [K \cap L] \cup_\emptyset [(L \setminus K)] \cup_\emptyset (K \mho L) \tag{6}$$

defining the symmetric difference $(K \setminus L) \cup (L \setminus K)$. □

Remark (Comparison between hypermaps). We say that $\Omega_1 \subset \Omega_2$ if, for all subsets $K \subset X$, $\Omega_1(K) \subset \Omega_2(K)$. Their reflections satisfy $\Omega_2^\ddagger(K) \subset \Omega_1^\ddagger(K)$. If the hypermaps Ω_1 and Ω_2 are intensive, we observe that for any $K \subset X$,

$$X = \Omega_1(K) \cup_\emptyset (\Omega_2^\ddagger(K) \setminus \Omega_1(K)) \cup_\emptyset \complement\Omega_2^\ddagger(K). \tag{7}$$ □

Remark (Construction of intensive hypermaps). If a hypermap $\Omega : \mathcal{P}(X) \rightsquigarrow \mathcal{P}(X)$ is not intensive (respectively, not extensive), we associate with it its

1. *intension* $\widehat{\Omega}$ defined by $\widehat{\Omega}(K) := K \cap \Omega(K)$;
2. *extension* $\widehat{\Omega}$ defined by $\widetilde{\Omega}(K) := K \cup \Omega(K)$.

The reflection of an intension is the extension of its intension

$$\widehat{\Omega}^\ddagger = \widetilde{\Omega^\ddagger}. \tag{8}$$

Hence, the results about intensive and extensive hypermaps extend naturally to their intension and extension thanks to these formulas. □

Example (Topological partition). If X and Y are topological spaces, the topological opening $K \mapsto \overset{\circ}{K} := \mathrm{Int}(K)$ is an intensive idempotent increasing hypermap. Its reflection is the topological closing $\overline{K}$, an extensive idempotent increasing hypermap. Note that, for any hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$, the reflection of the hypermap $\overset{\circ}{\Omega}(K) := \Omega(\overset{\circ}{K})$ is the hypermap $\overline{\Omega^{\ddagger}}(K) := \Omega^{\ddagger}(\overline{K})$. $\square$

2.3. Hypermaps Associated with Set-Valued Maps

Any hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$ generates a set-valued map $F_{\Omega} : X \rightsquigarrow Y$ defined by

$$\forall\, x \in X, \quad F_{\Omega}(x) := \Omega(\{x\}). \tag{9}$$

It is the *restriction of* Ω to X identified to a subset of $\mathcal{P}(X)$ by the injection $x \in X \mapsto \{x\} \in \mathcal{P}(X)$.

However, there are several ways to associate hypermaps with set-valued maps and their inverses.

2.3.1. Inverses and Cores

The canonical way to associate with a set-valued map $F : X \rightsquigarrow Y$ an hypermap (again denoted by $\Omega_F = F$) is the hypermap defined by

$$\forall\, K \subset X, \quad F(K) := \bigcup_{x \in K} F(x). \tag{10}$$

We recall that there are at least two manners to define the inverse image by a set-valued map F of a subset L:

Definition 2.6 (Inverse Image and Core). Let us consider a *set-valued* map $F : X \rightsquigarrow Y$.

1. The (canonical) *inverse image* $F^{-1} : \mathcal{P}(Y) \mapsto \mathcal{P}(X)$ of F^{-1} of F is the hypermap defined by:

$$\forall\, L \in \mathcal{P}(Y), \quad F^{-1}(L) := \bigcup_{y \in L} F^{-1}(y) = \{x \mid L \cap F(x) \neq \emptyset\} \in \mathcal{P}(Y) \tag{11}$$

the intension of which is defined by $F^{-1}(L) \cap L$;

2. The *core* (or *(inverse) core*) $F^{\ominus 1} : \mathcal{P}(Y) \mapsto \mathcal{P}(X)$ of F is defined by

$$\forall\, L \in \mathcal{P}(Y), \quad F^{\ominus 1}(L) := \{x \mid F(x) \subset L\}$$

the intension of which is defined by $F^{\ominus 1}(L) \cap L$.

We observe that the reflection of the hypermap F^{-1} (inverse of F) is the hypermap $F^{\ominus 1}$ (core of F):

$$\forall\, L \subset Y,\;\; F^{\ominus 1}(L) = \left(F^{-1}\right)^{\ddagger}(L)\; (= \complement(F^{-1}(\complement L))) \tag{12}$$

and the hypermap core satisfy

$$\forall\, K \subset X,\;\; K \subset F^{\ominus 1}F(K) \text{ and } \forall\, L \subset Y,\;\; FF^{\ominus 1}(L) \subset L. \tag{13}$$

2.3.2. Focus and Inverse Focus of a Set-Valued Map

We associate with a set-valued map F another hypermap associating with K its focus:

Definition 2.7 (Focus and Inverse Focus of a Set-Valued Map). Let $F : X \rightsquigarrow Y$ be a set-valued map. It generates its *focus hypermap* $\widehat{F} : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$ of F defined by

$$\forall\, K \subset X,\;\; \widehat{F}(K) := \bigcap_{x\in K} F(x) = \{y \in Y \text{ such that } K \subset F(x)\}. \tag{14}$$

Its *inverse focus hypermap* $\widehat{F}^{-1} : \mathcal{P}(Y) \mapsto \mathcal{P}(X)$ of F is thus defined by

$$\forall\, L \in \mathcal{P}(Y),\;\; \widehat{F}^{-1}(L) := \bigcap_{y\in L} F^{-1}(y) = \{x \in X \text{ such that } L \subset F^{-1}(y)\}. \tag{15}$$

The focus and inverse focus transform unions into intersections:

$$\widehat{F}\left(\bigcup_{i\in I} K_i\right) = \bigcap_{i\in I} \widehat{F}(K_i) \quad \text{and} \quad \widehat{F^{-1}}\left(\bigcup_{i\in I} L_i\right) = \bigcap_{i\in I} \widehat{F^{-1}}(L_i). \tag{16}$$

The focus and inverse focus are decreasing:

$$\begin{array}{c} \text{if } K_1 \subset K_2 \subset X, \;\; \text{then } \widehat{F}(K_2) \subset \widehat{F}(K_1) \\ \text{and} \quad \text{if } L_1 \subset L_2 \subset Y, \;\; \text{then } \widehat{F}^{-1}(L_2) \subset \widehat{F}^{-1}(L_1). \end{array} \tag{17}$$

Furthermore

$$\widehat{F^{-1}}\widehat{F}(\widehat{F^{-1}}\widehat{F}(K)) = \widehat{F^{-1}}\widehat{F}(K) \quad \text{and} \quad \widehat{F}\widehat{F^{-1}}(\widehat{F}\widehat{F^{-1}}(L)) = \widehat{F}\widehat{F^{-1}}(L). \tag{18}$$

Consequently, both $\widehat{F^{-1}}\widehat{F} : \mathrm{Dom}(\widehat{F}) \mapsto \mathrm{Dom}(\widehat{F})$ and $\widehat{F}\widehat{F^{-1}} : \mathrm{Dom}(\widehat{F^{-1}}) \mapsto \mathrm{Dom}(\widehat{F^{-1}})$ are idempotent and decreasing.

Focuses and inverse focuses enjoy a very useful graphical characterization:

Proposition 2.8 (Graphical characterization of focus). *The following statements are equivalent:*

$$\left\{\begin{array}{ll} (i) & K \subset \widehat{F^{-1}}(L) \\ (ii) & L \subset \widehat{F}(K) \\ (iii) & K \times L \subset \mathrm{Graph}(F). \end{array}\right. \tag{19}$$

The subset $K \times L \subset \mathrm{Graph}(F)$ *is maximal if* $K := \widehat{F^{-1}}(L)$ *and* $L = \widehat{F}\widehat{F^{-1}}(L)$ *or, equivalently, if* $K := \widehat{F^{-1}}(\widehat{F}(K))$ *and* $L := \widehat{F}(K)$.

In some sense, the focus and inverse focus hypermaps solve the system of the system of lower and upper inclusions (19) (i) and (ii).

2.4. Atypical Logics

In logics, we associate with properties P on elements $x \in X$ the subset $V(P) \subset$X of elements $x \in V(P)$ which are true[5] for P. The set of elements which are false is the $\complement V(P)$. Atypical logic are associated with hypermaps $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ with which we associate the following concepts

Definition 2.9 (Real and Potential Truth). We associate with an hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ the following subsets

1. $\Omega(V(P)) = \complement\Omega^{\ddagger}(\complement V(P))$ is the set of "really" true elements;
2. $\Omega(\complement V(P)) = \complement\Omega^{\ddagger}(V(P))$ is the set of "really" false elements;
3. $\Omega^{\ddagger}(V(P)) = \complement\Omega(\complement V(P))$ is the set of "potentially true";
4. $\Omega^{\ddagger}(\complement V(P)) = \complement\Omega(V(P))$ is the set of "potentially" false (or falsifiable in the sense of *Karl Popper*) elements.

so that really true elements are non potentially false, really false elements are not potentially true, that potentially true elements are not really false and potentially false elements are not really true.

Partition (6), p. 190, states that the Aristotle's principle of excluded third $X = V(P) \cup \complement V(P)$ becomes the "principle of excluded fifth":

$$X = \Omega(V(P)) \cup_{\emptyset} (V(P) \cap \Omega^{\ddagger}(\complement V(P)) \cup_{\emptyset} (\complement V(P) \cap \Omega^{\ddagger}(V(P)) \cup_{\emptyset} \Omega(\complement V(P)) \tag{20}$$

so that statements can be either really true, really false, true and falsifiable, or false and potentially true, the "law of contradiction" translating the disjoint disjunctions.

By using the powers of Ω, we can "quantize truth" of x by interpreting the internal marche $\partial_{\Omega}^{n} V(P)$ as the set of elements of truth degree n and the set $\partial_{\Omega}^{-n} V(P)$ as the of elements of falsification degree n, as in n-valued logics.

[5]By the way, if there exists the verb "to lie" associated with a lie, there is no verb "to truth" associated with truth. We may "tell the truth", but think the opposite, since "lying" is easier than "truthing". The same is true in French, where the verb "vraire" needs to be coined for telling the "vrai".

Using the contingent and tychastic stratifications, we may define contingent and tychastic temporal logic.

Remark (Locologies and colocologies). We associate with the inverse image (shadow) and core of a set-valued map $F : X \mapsto Y$ the families

$$\begin{cases} (i) & \mathcal{L}_F^{-1}(X) := \{F^{-1}(L)\}_{L \in \mathcal{P}(Y)} \\ (ii) & \mathcal{L}_F^{\ominus 1}(X) := \{F^{\ominus 1}(L)\}_{L \in \mathcal{P}(Y)} \\ (iii) & \widehat{\mathcal{L}}_F^{-1}(X) := \left\{\widehat{F}^{-1}(L)\right\}_{L \in \mathcal{P}(Y)} \end{cases} \tag{21}$$

called respectively *locology* and *colocology* associated with the set-valued map F through their inverse hypermaps. Locologies are stable under union and colocologies under intersection. However, thanks to *Galois transform*, we can define Galois intersection $\sqcap$ and union $\sqcup$ by formulae

$$\begin{cases} (i) & \forall\, K_1, K_2 \in \mathcal{L}_F^{-1}(X), \quad K_1 \sqcap K_2 := \displaystyle\bigcup_{M \in \mathcal{L}_F^{-1}(X),\ M \subset K_1 \cap K_2} M \\ (ii) & \forall\, K_1, K_2 \in \mathcal{L}_F^{\ominus 1}(X), \quad K_1 \sqcup L := \displaystyle\bigcap_{M \in \mathcal{L}_F^{\ominus 1}(X),\ M \supset K_1 \cup K_2} M. \end{cases} \tag{22}$$

In other words, for any $K_1, K_2 \in \mathcal{L}_F^{-1}(X)$, $K_1 \sqcap K_2$ is the largest subset of $\mathcal{L}_F^{-1}(X)$ contained in the intersection $K_1 \cap K_2$ and for any $K_1, K_2 \in \mathcal{L}_F^{\ominus 1}(X)$, $K_1 \sqcup K_2$ is the smallest subset of $\mathcal{L}_F^{\ominus 1}(X)$ containing the union $K_1 \cup K_2$. Naturally, the reflection operation is a bijection from the locology $\mathcal{L}_F^{-1}(X)$ to the colocology $\mathcal{L}_F^{\ominus 1}(X)$.

This is not the place to translate here the properties of intensive hypermap in the language of logics. It suffices to state that there are many atypical logics as intensive hypermaps. Among so numerous references, we refer to [23], [27] for atypical logics, to Section 8.9, p. 305, of *Viability Theory. New Directions*, [13], to *Building models by games*, [35], and to *Independence-Friendly Logic: A Game-Theoretic Approach*, [40].

2.5. Relations and Set-Valued Maps

Set-valued analysis and viability theory use systematically the graphical approach for dealing with set-valued maps (or the epigraphical approach for studying extended numerical functions). An abundant literature concerns this view point (see among many monographs *Variational Analysis*, [51], *Set-valued analysis*, [19], *Viability Theory. New Directions*, [13], etc.). This is the reason why we shall define *relations*, generalizing graphs of set-valued maps, regarded as *binary relations*. The idea of maps is so entrenched in our minds that it seems difficult to go from the number two to higher numbers of elements, without separating variables in two categories only, inputs and outputs, top and down, left or right. The graphical approach of maps allows us to overcome this difficulty by introduces relations other than binary ones:

Definition 2.10 (Relations of a Product of Sets). Let us consider a set $\mathbb{I}$ of indices and the product $X := \prod_{i\in\mathbb{I}} X_i$ of sets X_i. A *relation* is a subset $\mathcal{R} \subset \prod_{i\in\mathbb{I}} X_i$.
We denote by $\mathcal{P}(X_i)$ the family of subsets of the space X_i. A relation $\mathcal{R}$ induces its *focus*, which is the *hyperrelation* $\overbrace{\mathcal{R}} \subset \prod_{i\in\mathbb{I}} X_i$ made of families $(A_i)_{i=1,\ldots,\ell}$ of subsets $A_i \subset X_i$ such that $\prod_{i\in\mathbb{I}} A_i \subset \mathcal{R}$.

Since the closedness of the graph of the set-valued map is the "degree 0" of continuity relations, closedness of relations are as important, so that the closure of a relation is a the standard operation on relations. We mention several other operations on relations which adapt corresponding operations on set-valued maps (and thus, single-valued maps):

Definition 2.11 (Operations on Relations).

1. *Permutation of a Relation.* If $\sigma : \mathbb{I} \mapsto \mathbb{I}$ is a permutation, we set $X^\sigma := \prod_{i\in\mathbb{I}} X_{\sigma(i)}$ and the relation $\mathcal{R}^\sigma := \{(x_{\sigma(i)})_{i\in\mathbb{I}}\}_{x\in\mathcal{R}}$ is called the *permutation of a relation* $\mathcal{R}$.
2. *Partition of a Relation.* We associate with any *partition* $\pi := (\mathbb{I}_k)_{k\in\mathbb{K}}$ of $\mathbb{I} = \bigcup_{k\in\mathbb{K}} \mathbb{I}_k$ by disjoint subsets $\mathbb{I}_K$ the corresponding organisation $X := \prod_{k\in\mathbb{K}} \prod_{i\in\mathbb{I}_k} X_i$ so that any sequence $x = (x_i)_{i\in\mathbb{I}} \in \mathcal{R}$ can be written in the form of a relation $\mathcal{R}^\pi$ of elements $x = (\widehat{x}_k)_{k\in\mathbb{K}}$ of sequences $\widehat{x}_k := (x_i)_{i\in\mathbb{I}_k}$.
3. *Projection of a Relation.* For any $x \in \mathcal{R}$ and for any $k \in \mathbb{K}$, the sequence $x_k := (x_i)_{i\in\mathbb{I}_k} \in \pi_k(\mathcal{R})$ is called its k-th *projection* π_k.
4. *Composition of Two Relations.* Let us consider two relations $\mathcal{R} \in \prod_{i\in\mathbb{I}} X_i$ and $\mathcal{G} \in \prod_{k\in\mathbb{K}} X_k$ such that $\mathbb{L} := \mathbb{I} \cap \mathbb{K}$ is not empty. Let us consider the partitions $\mathbb{I} := (\mathbb{I} \setminus \mathbb{L}) \cup_\emptyset \mathbb{L}$ and $\mathbb{K} := \mathbb{L} \cup_\emptyset (\mathbb{K} \setminus \mathbb{L})$.
   Then the composition $\mathcal{G} \circ \mathcal{R}$ of the relations $\mathcal{G}$ and $\mathcal{R}$ is the set of sequences $(\pi_{(\mathbb{I}\setminus\mathbb{L})}x, \pi_{(\mathbb{K}\setminus\mathbb{L})}y)$ when $x \in \mathcal{R}$, $y \in \mathcal{G}$ and $\pi_\mathbb{L} x = \pi_\mathbb{L} y$.

If $\mathbb{I} := \{1, 2\}$, then a relation $\mathcal{R} \subset X_1 \times X_2$ is the graph $\mathrm{Graph}(F) := \mathcal{R}$ of set-valued map $F : X_1 \rightsquigarrow X_2$. We recover the familiar operations on set-valued maps:

1. *Inverse of a Set-Valued Map.* If $\sigma : (1, 2) \mapsto (2, 1)$, then the relation $\mathcal{R}^\sigma \subset X_2 \times X_1$ is the graph $\mathrm{Graph}(F^{-1}) := \mathcal{R}$ of set-valued map $F^{-1} : X_2 \rightsquigarrow X_1$ of the inverse F^{-1} of F.
2. *Domain and Images of a Set-Valued Map.* If $\pi : \{1, 2\} = \{1\} \cup \{2\}$ and if $\mathcal{R} := \mathrm{Graph}(F)$, then $\pi_1(\mathcal{R}) := \mathrm{Dom}(F)$ and $\pi_2(\mathcal{R}) := \mathrm{Im}(F)$.
3. *Composition of Two Relations.* If $\mathbb{I} := \{1, 2\}$ and $\mathbb{K} := \{2, 3\}$ with $\{1, 2\} \cap \{2, 3\} = \{2\}$ and if $\mathcal{R} := \mathrm{Graph}(F) \subset X_1 \times X_2$ where $F : X_1 \rightsquigarrow X_2$ and $\mathcal{G} := \mathrm{Graph}(G) \subset X_2 \times X_3$ where $G : X_2 \rightsquigarrow X_3$, then $\mathcal{G} \circ \mathcal{F} = \mathrm{Graph}(G \circ F)$.

2.6. Solving Hyperinclusions

We recall that concepts of focus and inverse focus hypermaps allowed us to solve lower and upper inclusions (see Proposition 2.8, p. 193). We extend this concept for a class of hypermaps for *selecting* the minimum and maximum solutions of the hyperinclusions and relating them by the Galois transform (see for instance [4], [3], [1], [2]). We summarize Section 7.3, p. 343, of *Mutational and Morphological Analysis*, [9], among many references on these topic:

Definition 2.12 (Lower and Upper Sections of an Hypermap). We associate with a

1. hypermap $\Omega : \mathcal{P}(X) \rightsquigarrow \mathcal{P}(Y)$ and any $L \subset Y$ its *lower section*, the family of *subinverses* $K \subset X$

$$\Omega^{\subset}(L) := \{K \in \mathcal{P}(X) \text{ such that } \Omega(K) \subset L\}. \tag{23}$$

2. hypermap $\Delta : \mathcal{P}(Y) \rightsquigarrow \mathcal{P}(X)$ and any $K \subset X$ its *upper section*, the family of *supinverses* $L \subset Y$

$$\Delta^{\supset}(K) := \{L \in \mathcal{P}(Y) \text{ such that } \Delta(L) \supset K\}. \tag{24}$$

A natural question is to select one subinverse and one supinverse in these families[6]. The idea is to select the minimum of the supinverses and the maximum of the subinverses, if any, and to relate them, a king of "lattice minimax". Their existence requires assumptions described by the following definitions:

Definition 2.13 (Galois Hypermaps). We shall say that

1. an increasing hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$ is an *upper Galois hypermap* if its lower sections are stable for the union;
2. an increasing hypermap $\Delta : \mathcal{P}(Y) \mapsto \mathcal{P}(X)$ is a a *lower Galois hypermap* if its upper sections are stable for the intersection.

Consequently,

1. the lower section of an upper Galois intensive hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$ has a maximum subinverse

$$\Omega^{\subset\sharp}(L) := \bigcup_{K \in \Omega^{\subset}(L)} K. \tag{25}$$

2. the upper section of an lower Galois extensive hypermap $\Delta : \mathcal{P}(Y) \mapsto \mathcal{P}(X)$ has a minimum supinverse

$$\Delta^{\flat\supset}(K) := \bigcap_{L \in \Delta^{\supset}(K)} L. \tag{26}$$

Combining those two definitions, we obtain the concepts of *Galois inverse* of an hypermap Ω:

[6] In vector spaces structures, the standard choice is to choose elements of the sets with minimal norm, among a manifold of selection procedures for defining orthogonal left and right inverse and pseudo-inverses in Hilbert spaces.

Definition 2.14 (Galois Transform of Galois Hypermaps). Let $\Omega : \mathcal{P}(X) \rightsquigarrow \mathcal{P}(Y)$. The maximal selection $\Omega^{\subset\sharp}(K)$ of a lower section Ω of an upper Galois hypermap is called its *Galois maximum subinverse* and the hypermap $L \mapsto \Omega^{\subset\sharp}(L)$ its *Galois transform* $\Omega^{\subset\sharp} : \mathcal{P}(Y) \mapsto \mathcal{P}(X)$.
Let $\Delta : \mathcal{P}(Y) \mapsto \mathcal{P}(Y)$. The maximal selection $\Omega^{\flat\supset}(K)$ of the upper section Δ of a lower Galois hypermap is called its *Galois minimum supinverse* and the hypermap $K \mapsto \Delta^{\flat\supset}(K)$ its *Galois transform* $\Delta^{\flat\supset} : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$.

Proposition 7.3.4, p. 346, of [9], implies that a hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$ is upper Galois if and only if Ω is an increasing dilation satisfying $\Omega(\emptyset) = \emptyset$, and Proposition 7.3.2, p. 345 that the Galois transform are inverse hypermaps:

Proposition 2.15 (Properties of Galois Transform). *If $\Omega : \mathcal{P}(X) \rightsquigarrow \mathcal{P}(Y)$ is an upper Galois hypermap, its Galois transform $\Omega^{\subset\sharp}$ is the unique lower Galois hypermap $\Delta :\rightsquigarrow \mathcal{P}(Y)$ satisfying*

$$\begin{cases} (i) & \forall\, K \in \mathcal{P}(X),\ K \subset \Delta(\Omega(K)) \\ (ii) & \forall\, L \in \mathcal{P}(Y),\ \Omega(\Delta(L)) \subset L. \end{cases} \tag{27}$$

The pair of hypermaps (Ω, Δ) satisfying (27) are called Galois connections *Therefore the correspondence $\Omega \mapsto \Omega^{\subset\sharp}$ is a bijection between the space of upper Galois hypermaps from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ and the space of lower Galois hypermaps from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$, called the* Galois transform*:*

$$\left(\Omega^{\subset\sharp}\right)^{\flat\supset} = \Omega. \tag{28}$$

It satisfies

$$\begin{cases} (i) & \forall\, K \in \mathcal{P}(X),\ \ \Omega(\Omega^{\subset\sharp}(\Omega(K))) = \Omega(K) \\ (ii) & \forall\, L \in \mathcal{P}(Y),\ \ \Omega^{\subset\sharp}(\Omega(\Omega^{\subset\sharp}(L))) = \Omega^{\subset\sharp}(L). \end{cases} \tag{29}$$

If $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(Y)$ is upper Galois, then $\Omega^{\subset\sharp}\Omega$ is a closing and $\Omega\Omega^{\subset\sharp}$ is an opening.

Example. The hypermap associated with a set-valued map $F : X \mapsto Y$ (again denoted by F) is an upper Galois map and its Galois transform is $F^{\subset\sharp} = F^{\ominus 1}$ is its core. □

2.7. Legacy of Underlying Space Structures

Even though the lattice structure is appropriate for hyperspaces of any underlying space devoid of structures, the question arises as to what are the underlying space structures that can be inherited by hyperspaces. This is useful for governing the evolution of sets or choosing those that minimize a criterion, among many other problems. Algebraic structures will inherit little, topological structures more easily, but hypersets inherit structures to adapt a differential calculus (called “mutational calculus”). The mutational structure does not require linearity. This is important for proving theorems in the hyperspace of non-empty compact of a finite dimensional vector space.

2.7.1. Minkowski Sum and Difference

Until now, we did not impose any assumption on the underlying space X restricting our operations to the Boolean ones. The question arises where structures on the underlying space X can be “extended” to the hyperspaces.

For instance, let us consider a vector space X and the operation $+ : X \times X \mapsto X$. Any subset $S \subset X$ can be used to generate the hypermaps associated with the addition the image and focus hypermaps which extend the concepts of “sum” and “differences” on the hyperspace $\mathcal{P}(X)$:

Definition 2.16 (Minkowski Sum and Difference). Let X be a vector space. We extend the addition of vectors to sum and difference of subsets in the following way:

$$\begin{cases} S\oplus & : K \mapsto S \oplus K := \bigcup_{y \in S} (K + y) \\ S\ominus & : K \mapsto S \ominus K := \bigcap_{y \in S} (K + y) \end{cases} \tag{30}$$

where $S \oplus K$ is called the *Minkowski sum* and $S \ominus K$ *Minkowski difference*, the largest subset R such that $S \oplus R \subset K$. The *Minkowski boundary* of K with respect to S is the abrasion

$$\partial_S(K) := (S \oplus K) \setminus (S \ominus K) \tag{31}$$

of the Minkowski sum by the Mikowski difference.

Whenever subset $0 \in S$, the Minkowski sum is extensive and the Minkowski difference hypermap $S\ominus$ intensive. Such a subset S is called a *structuring element* in mathematical morphology (see Section 5, p. 220).

2.7.2. Hausdorff-Pompeiu Space

There are numerous topologies on hyperspaces: see for instance the monographs [22], [51] on closed sets and [54], [28] on measurable sets for more information on the problem of set topologies, etc.

In this survey, we single out the simplest example of the hyperset $\mathcal{K}(X)$ of nonempty compact subsets of a metric space X, which is itself a metric space supplied with the Pompeiu-Hausdorff distance defined by

$$d\!\!l(L, M) := \max \left(\sup_{y \in L} \inf_{z \in M} d(y, z), \sup_{z \in M} \inf_{y \in L} d(y, z) \right). \tag{32}$$

It is complete whenever X is complete and compact whenever it is compact.

2.7.3. Mutational Analysis

Differential Inclusions were introduced during the same period, the second half of the 1970's, when Jean Jacques Moreau, in [43], introduced sweeping processes(see also *Differential Inclusions in Nonsmooth Mechanical Problems*, [42], and [21]).

Differential calculus uses both topology and the linear structure of vector spaces for defining the directional derivative of a map $U : X \mapsto Y$:

$$u := DU(x)(v) := \lim_{h \mapsto 0} \frac{U(x+hv) - U(x)}{h} \tag{33}$$

It is a quite a restrictive definition which excluded too many maps. This paved the way to the relaxation of topologies either "functionwise" (i.e., in function and distribution spaces) or pointwise, replacing the pointwise limit by all possible combinations of $limitinf$ and *limitsup*, leading to convex, non smooth and set-valued analysis (see Section 18.9, *The Grail of the Ultimate Derivative*, p. 765, and Chapter 18 of *Viability Theory. New Directions*, [13], and, among many other books, *Set-valued analysis*, [19], *Variational Analysis*, [51]).

However, the linear structure was mandatory, whereas there was and still is a pressing need to differentiable maps on metric space, in particular, in the hyperspace $\mathcal{K}(X)$ on nonempty compact subsets of X. For instance, as it was initiated by *Jean Céa* (see *Lectures on Optimization. Theory and Algorithms*, [25] *Optimization Techniques. Modeling and Optimization in the Service of Man*, [26], and *Lectures on Optimization. Theory and Algorithms*, [25]), and next by *Jean-Paul Zolésio* and *Michel Delfour*, for defining by *shape derivatives.* Mutational Analysis initiated in [10], was developed by *Anne Bru-Gorre* ([34]), *Luc Doyen*, *Laurent Najman*, *Juliette Mattioli* ([31], [20], [32] and *Mutational and Morphological Analysis*, [9]), and, above all, by *Thomas Lorenz* in his *Mutational Analysis*, [39], for studying numerous problems, the motivating one being the design of evolutionary system governing the evolution of "tubes" $t \rightsquigarrow K(t) \subset X$.

The idea was very simple! Regarding the directions $v \in X$ as "half lines" $\varphi_v : (h,x) \mapsto := \varphi_v(h,x) := x + hv$ they generate, we observe that we can define directional derivatives in the form

$$u \in DU(x)(v) \;\text{ if and only if }\; \lim_{h \mapsto 0} \frac{d(U(\varphi_v(h,x)), \varphi_u(h, U(x)))}{h} = 0. \tag{34}$$

Part of the linear structure of the underlying space is removed. In metric spaces, it is enough to replace the "half-lines φ_v when v ranges over X by any family $\Theta(X)$ of *transitions* $\varphi : (h,x) \mapsto := \varphi(h,x)$ satisfying an adequate set of axioms. This family defines a *mutational structure.* It allows us to adapt the above formula for maps $U : X \mapsto Y$ from X to Y supplied with mutational structures $\Theta(X)$ and $\Theta(Y)$. We define the "directional mutations" $\overset{\circ}{U}(x) : \Theta(X) \rightsquigarrow \Theta(Y)$ of U at x associating with any transition (replacing directions) $\vartheta \in \Theta(X)$ the subset $\overset{\circ}{U}(x)(\vartheta)$ of transitions $\tau \in \Theta(Y)$ defined by

$$\tau \in \overset{\circ}{U}(x)(\vartheta) \text{ if and only if } \lim_{h \mapsto 0} \frac{d(U(\vartheta(h,x)), \tau(h, U(x)))}{h} = 0. \tag{35}$$

Almost of theorems of differential calculus on vector spaces can be adapted to the "mutational calculus.

We describe in few words the mutational calculus on the hyperset $\mathcal{K}(X)$ of nonempty compact subsets of a finite dimensional vector space X, which is a metric space supplied with the Pompeiu-Hausdorff distance. The "natural" choice of the hyperspace $\Theta(\mathcal{K}(X)) := \mathrm{Lip}(X,X)$ of transitions associated with transitions $h \mapsto \vartheta_\varphi(h,K)$ associated with bounded Lipschitz maps[7] $\varphi \in \mathrm{Lip}(X,X)$ where $\varphi : X \mapsto X$ is a Lipschitz map, $\vartheta_\varphi(h,x)$ denotes the value at time h of the solution to the differential equation $z' = \varphi(z)$ starting at $z(0) = x$ at time 0 and $\vartheta_\varphi(h,K) := \{\vartheta_\varphi(h,x)\}_{x \in K}$ is the *reachable set* from K at time h under φ.

In particular, this allows us to define *morphological mutational equations*

$$\overset{\circ}{K}(t) \ni f(K(t))$$

governing the evolutions of sets $K(t)$, as differential equations $x'(t) = g(x(t))$ govern the evolutions of vectors $x(t)$. They have been used in many problems, in particular, to a dynamic approach of fuzzy sets ([38, 37]).

2.8. Societies Associated with Cultural and Economic Regulons

Focuses of set-valued maps appear naturally in several fields. For instance, societies are made of agents sharing subsets of cultural regulons (or codes) and economic classes are economic agents behaving according to classes of dynamic economic regulons (transactions of commodities, fluctuation of prices and of credit). They can be defined by focuses and their co-evolution can be governed by mutational equations. This is the most important section motivating further open mathematical investigation for designing at least mathematical metaphors, maybe, possible applications, if, however, they make sense in sciences of life.

Societies and Cultural Codes

The brains of human beings perceive the outside world, build "cultures", subsets of "cultural regulons" (or codes), intercommunicate in various ways, and share some of them thanks to their ability to believe in cultural codes. But prophets and dissidents often challenge this consensus, while high priests and guardians of the ideology tend to dogmatize it and impose it on the members of the social group (quite often prophets may become the high priests and guardians of the ideology, and very rarely if ever, the other way around[8]).

[7] Actually, with bounded Lipschitz set-valued maps Φ for increasing the set of transitions and include morphological transitions $\vartheta_S(h,K) : K \mapsto K \oplus hS$.

[8] This could be called the *third principle of social entropy.*

It is the *consensus* on the cultures that defines societies of agents who share them. In other words, agent produces cultural regulons and associate with each culture the society of agents believing in the cultural codes of this culture. Each agent may belong to several societies defined by cultures, which co-evolve with the evolutions of agents[9].

Mathematically, assume that n agents in $N := \{i = 1, \ldots, n\}$ choose actions $x_i \in X^i$ in spaces X_i. We introduce a space $U := \mathbb{R}^m$ of *cultural regulons.* A *culture* is described by subset $B \subset U$ of cultural regulons, and range over the hyperspace of cultural regulons. The *cultural knowledge* of each agent $i \in N$ is described by a set-valued map $H_i : X_i \rightsquigarrow U$ associating with any action $x_i \in X_i$ a subset $H_i(x_i) \subset U$ of cultural regulons.

A *culture* $B \subset U$ is said a *consensus* on a collective action $x \in X^S$ of a coalition $S \subset N$ of agents if

$$\forall\, i \in S,\;\; B \subset H_i(x_i) \text{ or } B \subset \widehat{H}(x) := \bigcap_{i=1}^{n} H_i(x_i).$$

The *society* $S_H(B, x)$ associated with a culture $B \subset U$ and a collective action $x \in X^S$ is the coalition of agents $i \in N$ such that $B \subset H_i(x_i)$. The culture B becomes the *"reality"* perceived by agents of the society $S_H(B, x)$.

We associate with any coalition $S \subset N$ of agents the subspace $X^S := \prod_{i\in S} X_i$ and interpret the set-valued maps $\widehat{H_S} : X^S \rightsquigarrow U$ associating with any action $x_i \in X_i$ of the agents belonging to the coalition S the subset $\widehat{H_S}(x) := \bigcap_{i\in S} H_i(x_i)$ as the *cultural map* of the coalition S. It is the culture needed to implement a collective action of coalition S.

The *inverse focus* $\widehat{H_S^{-1}}(B) \subset S$ of a culture by a coalition $S \subset N$ is the subset

$$\begin{aligned}\widehat{H_S^{-1}}(B) &:= \{x \in X^S \text{ such that } \forall\, i \in S,\; B \subset H_i(x_i)\}\\ &:= \{x \in X^S \text{ such that } B \subset \widehat{H_S}(x)\}.\end{aligned}$$

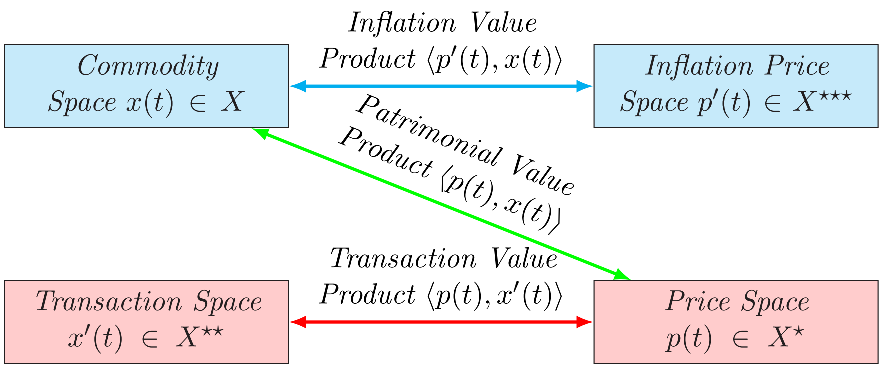
Consequently, *a culture B is a consensus on a collective action $x \in X^S$ for a coalition S of agents if and only if $x \in \widehat{H_S^{-1}}(B)$, or, equivalently, if and only if the coalition S is contained in the society $S_H(B, x)$.*

Using mutational analysis, one can study the viable joint co-evolution of societies and cultures.

Transactions of Commodities, Fluctuation of Prices and Credit Flows

In economics, $X := \mathbb{R}^\ell$ can be regarded as a *commodity space* of commodities $x := (x_h)_{1\leq h\leq \ell}$ of amounts $x_h \in \mathbb{R}$ of units e^h of goods or services labelled

[9]See chapters 4 and of *La mort du devin, l'émergence du démiurge*, [6] for more details about this approach.



$h = 1, \ldots, \ell$. Its dual $X^\star := \mathbb{R}^{\ell\star}$ is the space of *prices* $p : x \mapsto \langle p, x\rangle \in \mathbb{R}$ associating with any commodity $x \in X$ its *duality product* $\langle p, x\rangle := \sum_{h=1}^{\ell} p^h x_h$ which is interpreted as a *(patrimonial) value*. The velocity $x'(t)$ at time t of the evolution of a commodity $x(\cdot)$ is regarded as a *transaction* (actually, an infinitesimal one). Transactions range over the bidual $X^{\star\star}$. In the same way, the *price fluctuations* $p'(t)$ range over the tridual $X^{\star\star\star}$. Even though these four finite dimensional vector spaces are all isomorphic, they play different roles in economics[10].

We study the evolution of the following (economic) variables:

$$\begin{array}{c} x(t) \in X \ (\textit{commodities}), \qquad p(t) \in X^\star \ (\textit{prices}) \\ \text{and} \quad v(t) := \langle p(t), x(t)\rangle \in \mathbb{R} \ (\textit{values}), \end{array} \tag{36}$$

the derivatives of which are denoted by

$$x'(t) \in X^{\star\star} \ (\textit{transactions}) \quad \text{and} \quad p'(t) \in X^{\star\star\star} \ (\textit{price fluctuations}) \tag{37}$$

and their *value flow* $v'(t) = \langle p(t), x'(t)\rangle + \langle p'(t), x(t)\rangle \in \mathbb{R}$, which is the sum of the transaction value $\langle p(t), x'(t)\rangle$ and the inflation value $\langle p'(t), x(t)\rangle$ (see *Time and Money. How Long and How Much Money is Needed to Regulate a Viable Economy*, [12] and [14] for alternative definition of Fisher's "velocity of money" by the velocity of value $v'(t)$.)

Remark (Dynamic Economic Behaviors). For each commodity $h = 1, \ldots, \ell$, the *dynamic behavior of an economic agent* with respect to a commodity h can be described by the following velocities:

1. a consumer if $x_h'(t) \le 0$ and a producer if $x_h'(t) \ge 0$;
2. a buyer if $p_h'(t) \le 0$ and a seller[11] if $p_h'(t) \ge 0$;

[10] As well as in other domains, such that the presentation of classical mechanics by Jean Jacques Moreau from which these ideas emerged: $X := \mathbb{R}^\ell$ can be regarded as a *position or configuration space*, its dual $X^\star$ as the space of *forces* $p : x \mapsto \langle p, x\rangle \in \mathbb{R}$ associating with any position $x \in X$ its *duality product* $\langle p, x\rangle$ which is interpreted as a *work*. The bidual $X^{\star\star}$ is regarded as the space of velocities v and the duality product $\langle p, v\rangle$ is interpreted as a *power*.
[11] Note that in this case, for any commodity $x_h \ge 0$, the price fluctuation increases for a buyer and decreases for a seller.

3. a debtor if $v'_h(t) \leq 0$ and a creditor if $v'_h(t) \geq 0$.

By doing so, we classify the dynamic behavior of an agent by one of the 8ℓ elements $\beta(x,p,v)$ (behavior) of the *economic classes* of the qualitative set $\{-\infty,+\infty\}^{3\ell}$ according the two possible signs of those three components (see for instance *Analyse qualitative*, [30] and *Neural Networks and Qualitative Physics: a Viability Approach*, [11]). *economic class* by their dynamic economic behavior with respect to commodities, price and values[12].

Hence, when they evolve, at each time t, the dynamic behavior of the economic agent $i = 1,\ldots,\ell$ ranges over the set of economic codes $\beta_i(x(t),p(t),v(t))$. This is how we can define evolving economic classes depending of the evolution of and the mutational equations governing their evolution depending upon $(x(t),p(t),v(t))$ on which the agents act themselves (*Mutational and Morphological Analysis: Tools for Shape Regulation and Morphogenesis*, [9]). □

3. Galois Stratifications of Sets and their Galois Quotations

3.1. From Characteristic Functions to Galois Quotation

We start our analysis by the simplest example when the hypermap $\Omega(K) = K$ is the identity, so that $\Omega^\ddagger(K) = K$ and the partition (6), p. 190, boils down to $X = K \cup_\emptyset \complement K$.

Before introducing the Galois quotations, we recall that the concept of *characteristic function* $\chi_K : X \mapsto \{0,1\}$ of a subset $K \subset X$ is defined by

$$\chi_K(x) = \begin{cases} +1 & \text{if } x \in K \\ 0 & \text{if } x \in \complement K. \end{cases}$$

In this setting, the map $\chi : \mathcal{P}(X) \mapsto \{0,1\}^X$ associating with any subset $K \in \mathcal{P}(X)$ its characteristic function $\chi_K \in \{0,1\}^X$ is a *bijection* between $\mathcal{P}(X)$ and $\{0,1\}^X$. In particular, if $X := \{1,2,\ldots,m\}$ is a finite set of m elements, then $\{0,1\}^X$ is the set of the 2^m vertices of the cube $[0,1]^m \subset \mathbb{R}^m$. This was a convenient way to characterize subsets with two numbers, 0 and 1, translating logical operations into simple operations on characteristic functions, suitable to computer operations. The convex hull of $\overline{\text{co}}(\{0,1\}^X) = [0,1]^X$ is the family of *membership functions* of *fuzzy sets* that *Lofti Zadeh* introduced in 1965. If we restrict the characteristic map to σ-algebra $\mathcal{A}$ of a measurable space $\mathcal{O}$ into the set $L^1(\mathcal{O},\{0,1\})$ for a measure μ, its convex hull can be interpreted as the family of *measurable fuzzy sets.* When the measure is nonatomic, Charles Castaing

[12]If the conservative behavior described by derivatives equal to 0 is taken into account, we could obtain 27ℓ economic classes of the qualitative set $\{-\infty,0,+\infty\}^{3\ell}$. We could also have distinguish the signs of the transaction values and the price fluctuations impacts to have a finer analysis instead of their sum. One could also take into account not only the velocities of values, but also the values itself. For practical purposes, we should decrease the number of economic classes by regrouping commodities into few sectors. On the mathematical side, there is no limit to the number of behavioral economic classes of this nature.

derived from the "Lyapunov Convexity Theorem" that the set $L^1(\Omega, \{0,1\})$ of measurable sets is dense in the set $L^1(\Omega, [0,1])$ of measurable fuzzy sets when $L^1(\Omega, \mathbf{R})$ is supplied with the weakened topology (see *Convex Analysis and measurable multifunctions*, [24]). This theorem is at the origin of important results on non atomic and fuzzy cooperative games (see for instance [8]).

The choice of the scaling $\{0, +1\}$ could appear (and does appear) quite arbitrary. It played an important historical role in Lebesgue integration for which it has been invented. However, this choice has several drawbacks in other frameworks. Another choice could have been

$$\widehat{\chi}_K(x) = \begin{cases} +1 & \text{if } x \in K \\ -1 & \text{if } x \in \complement K \end{cases}$$

for which we obtain the interesting property $\widehat{\chi}_{\complement K}(x) = -\widehat{\chi}_K(x)$ to "quote" both the set and its complement. The embedding of subsets through characteristic functions is not convenient when convexity plays a key role. Indeed, convexity of a subset of a vector space cannot be characterized by some properties of its characteristic functions χ_K (see for instance *Analyse qualitative*, [30] and [18]).

We advocate another scaling, $\{-\infty, +\infty\}$, which allows us to replace characteristic functions by its *Galois quotation*

$$\alpha_K(x) := \begin{cases} +\infty & \text{if } x \in K \\ -\infty & \text{if } x \notin K. \end{cases} \tag{38}$$

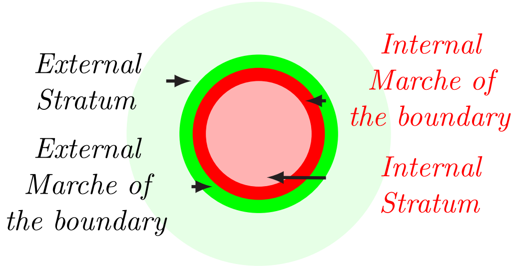
We observe that K *is convex (respectively closed) if and only if its Galois quotation is concave (respectively upper semicontinuous)*[13] and that $\alpha_{\complement K}(x) = -\alpha_K(x)$.

3.2. Thick Boundaries

Recall that the (topological) boundary ∂K of a subset K of a topological space is the subset $\partial K := \overline{\complement K} \cap \overline{K}$. It is the "meeting place" of the limits of interior and exterior sequences, a "thin" boundary because the interior and the exterior are "thick" subsets of K and $\complement K$ since they are dense in the set and its complement respectively. Limits is derived from the Latin *limes*, meaning also a boundary. The boundary of K is an example of a "demarcation" between the topological interior $F = \text{Int}(K) \subset K$ and the topological exterior (defined as the interior of its complement) $G = \text{Int}(\complement K) \subset \complement K$ in the following sense:

If we replace the topological interior and exterior by smaller subsets of $F \subset \text{Int}(K) \subset K$ and $G \subset \text{Int}(\complement K)$ respectively, their demarcation will be larger than

[13]In convex analysis, it is usual to replace the function α_K by the indicator ψ_K, regarded as a "cost function", defined by $\psi_K = 0$ if $x \in K$ and $\psi_K = +\infty$ if $x \notin K$, which is convex whenever K is convex. It plays in convex analysis a role analogous to the one that characteristic functions play in integration theory.



the topological boundary, and thus, examples of "thick boundaries". Hence, we are led to define thick boundaries without topological structures by introducing $F \subset K$ and a subset $G \subset \complement K$ and take their demarcation:

Definition 3.1 (Thick Boundaries). Let $F \subset K$, the *internal stratum*, and $G \subset \complement K$, *external stratum*. Their demarcation, equal to

$$F \mho G := \complement G \cap \complement F = (K \setminus F) \cup (\complement K \setminus G) \tag{39}$$

is, called a *thick boundary* of K, where $K \setminus F$ is its *internal marche*, and $\complement K \setminus G$, its *external marche*.

Hence, for any $K \subset X$,

$$X = F \cup_{\emptyset} ((K \setminus F) \cup (\complement K \setminus G)) \cup_{\emptyset} G. \tag{40}$$

Taking $F = K$ and $G = \complement K$, their thick boundary is ... empty, and taking $F = \emptyset$ and $G = \emptyset$, their thick boundary is really thick, since it the whole space!

Thick boundaries allow the construction of mathematical metaphors of many concepts introduced outside mathematics for delimitating an "inside" from "outside". They could have been called *membranes* in biology, since they are the subset in which take places chemical "exchanges" between cells through ionic pumps, for example, economic barriers or tolls, market places where economic exchanges take place, which are transformed into political marches, thicker than the political border, where peaceful or less peaceful exchanges are active (customs zone), demilitarized zones in international relations, contours in image processing, "weak inferences" in logics (such as vicarious temporal logic), etc. One of the main motivation of this study was the definition given by *Cesare Beccaria* in his fundamental book *Of Crimes and Punishments*: *"Thus it was necessity that forced men to give up apart of their liberty. It is certain, then, that every individual would choose to put into the public stock the smallest portion possible, as much only as was sufficient to engage others to defend it. The aggregate of these, the smallest portions possible, forms the right of punishing; all that extends beyond this, is abuse, not justice."*. In other words, the "individual freedom" internal strata is the subset of the "freedom set" of an individual made of the "portion" that he is inclined to place at the disposal of others. He tries to minimize it whereas, at the same time, he is tempted to increase the one of the others.

3.3. The Galois Trichotomy

The question arises how to construct *systematically* the internal and external strata: this is at this point that we need to introduce an *intensive* hypermaps Ω and their reflection $\Omega^{\ddagger}$ for designing a factory producing thick boundaries by choosing $F := \Omega(K) \subset K$, $G := \Omega(\complement K) = \complement\Omega^{\ddagger}(K) \subset \complement K$ and their demarcation $\Omega(K) \mho \Omega(\complement K) = \Omega^{\ddagger}(K) \setminus \Omega(K)$ as their thick boundary:

Definition 3.2 (Galois Boundary and Quotation). We associate with any intensive hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ and *any* subset $K \subset X$ the

1. *internal stratum* $\Omega(K)$;
2. *external stratum* $\Omega(\complement K) = \complement\Omega^{\ddagger}(K)$;
3. *internal marche* $\partial^{0+}_{\Omega}(K) := K \setminus \Omega(K)$;
4. *external marche* $\partial^{0-}_{\Omega^{\ddagger}}(K) := \Omega^{\ddagger}(K) \setminus K = \complement K \setminus \Omega(\complement K)$;
5. *Galois boundary* $\partial^{0}_{\Omega}(K) := \Omega(K) \mho \Omega(\complement K) = \Omega^{\ddagger}(K) \setminus \Omega(K) = \partial^{0+}_{\Omega}(K) \cup \partial^{0-}_{\Omega^{\ddagger}}(K)$.

The *Galois trichotomy* of X is the partition

$$X = \Omega(K) \cup_{\emptyset} (\partial^{0+}_{\Omega}(K) \cup \partial^{0-}_{\Omega^{\ddagger}}(K)) \cup_{\emptyset} \complement\Omega^{\ddagger}(K). \tag{41}$$

Hence *there are as many concepts of Galois boundaries* $\Omega(\complement K) \setminus \Omega(K)$ *as intensive hypermaps* Ω *from* $\mathcal{P}(X)$ *to* $\mathcal{P}(X)$.

Example (Thick boundary associated with an abrasion hypermap). Let $L \subset K$ and consider the abrasion hypermap $\Lambda_L(K) := K \setminus L = K \cap \complement L$. The internal stratum is $\Lambda_L(K) := K \setminus L$, the external stratum $\Lambda_L(\complement K) = \complement K$, the internal marche $\partial^{0+}_{\Lambda_L}(K) := L$ and the external marche $\partial^{0-}_{\Lambda_L^{\ddagger}}(K) := \emptyset$. □

Other more sophisticated examples derived from viability theory are provided in Section 4, p. 211.

Once Galois canonical stratification is defined, we can quantize it by associating with it its *Galois quotation*:

Definition 3.3 (Galois Quotation). The *Galois quotation* associates with Ω, K and x

$$\alpha_{\Omega}(K,x) := \begin{cases} +\infty & \text{if } x \in \Omega(K) \\ 0 & \text{if } x \in \partial^{0}_{\Omega}(K) \\ -\infty & \text{if } x \in \complement\Omega^{\ddagger}(K) = \Omega(\complement K) \end{cases} \tag{42}$$

which can be refined by introducing

$$\alpha_{\Omega}(K,x) := \begin{cases} 0+ & \text{if } x \in \partial^{0+}_{\Omega}(K) \\ 0- & \text{if } x \in \partial^{0-}_{\Omega^{\ddagger}}(K) \end{cases} \tag{43}$$

for quoting the internal and external marches.

3.4. The Galois Plurichotomies

Powers Ω^n of intensive maps are more and more intensive since, for any $K \subset X$, $\Omega^n(K) \subset \Omega^{n-1}(K) \ldots \subset \ldots K$. Therefore, we can introduce Galois plurichotomies (or transchotomies) by associating with each Ω^n the trichotomies that leads us to a family of thinner and thinner plurichotomies "refining" the initial one. They induce the concept of *Galois stratification* from which we define *Galois quotation.*

Except for idempotent intensive hypermaps which cannot be refined, such as the topological one, which cannot be refined since, for any $K \subset X$ and any positive integer n, $\Omega^n(K) = K$.

Proposition 3.4 (Marches of Order n). *We associate with any intensive hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(X)$, its reflection $\Omega^\ddagger$ and* any *subset $K \subset X$ the*

1. *n-th order internal marches*

$$\begin{cases} \partial_\Omega^{0+}(K) =: K \setminus \Omega(K) \\ \partial_\Omega^{n}(K) =: \Omega^n(K) \setminus \Omega^{n+1}(K) = \Omega^{\ddagger^{n+1}}(\complement K) \setminus \Omega^{\ddagger^n}(\complement K) \\ \partial_\Omega^{\infty}(K) =: \bigcap_{n \geq 1} \Omega^n(K). \end{cases} \tag{44}$$

2. *n-th order external marches*

$$\begin{cases} \partial_{\Omega^\ddagger}^{0-}(K) =: \Omega^\ddagger(K) \setminus K \\ \partial_{\Omega^\ddagger}^{-n}(K) =: \Omega^{\ddagger^{n+1}}(K) \setminus \Omega^{\ddagger^n}(K) = \Omega^n(\complement K) \setminus \Omega^{n+1}(\complement K) \\ \partial_{\Omega^\ddagger}^{-\infty}(K) =: \complement\left(\bigcup_{n \geq 1} \Omega^{\ddagger^n}(K)\right). \end{cases} \tag{45}$$

The Galois stratification *of X generated by K is the partition*

$$\begin{aligned} X = \partial_\Omega^{+\infty}(K) \cup_\emptyset \left(\bigcup_{n \geq 1} \partial_\Omega^{n}(K)\right) \cup_\emptyset \left[\partial_\Omega^{0+}(K) \cup_\emptyset \partial_{\Omega^\ddagger}^{0-}(K)\right] \\ \cup_\emptyset \left(\bigcup_{n \geq 1} \partial_{\Omega^\ddagger}^{-n}(K)\right) \cup_\emptyset \partial_{\Omega^\ddagger}^{-\infty}(K). \end{aligned} \tag{46}$$

Proof. The powers of an intensive hypermap $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ associate with any $K \subset M$ the decreasing sequence

$$\begin{aligned} \partial_\Omega^{+\infty}(K) := \bigcap_{n \geq 0} \Omega^n(K) \cdots \subset \Omega^{(n+1)}(K) \\ \subset \Omega^n(K) \cdots \subset \cdots \Omega(K) \subset K =: \Omega^0(K). \end{aligned} \tag{47}$$

The sequence of their *internal marches* is defined by

$$\forall\, n \geq 1,\;\; \partial_{\Omega}^{n}(K) := \Omega^{n}(K) \setminus \Omega^{(n+1)}(K) = \partial_{\Omega}^{0+}(\Omega^{n}(K)). \tag{48}$$

We infer that

$$K = \partial_{\Omega}^{+\infty}(K) \cup_{\emptyset} \left(\bigcup_{n\geq 1} \partial_{\Omega}^{n}(K) \right) \cup_{\emptyset} \partial_{\Omega}^{0+}(K). \tag{49}$$

Replacing K by its complement, we observe that

$$\begin{cases} \partial_{\Omega^{\ddagger}}^{0-}(K) =: \partial_{\Omega}^{0+}(\complement K) \\ \partial_{\Omega^{\ddagger}}^{-n}(K) =: \partial_{\Omega}^{n}(\complement K) = \Omega^{\ddagger^{n+1}}(K) \setminus \Omega^{\ddagger^{n}}(K) \\ \partial_{\Omega^{\ddagger}}^{-\infty}(K) =: \complement \partial_{\Omega}^{+\infty}(\complement K) = \complement \left(\bigcup_{n\geq 1} \Omega^{\ddagger^{n}}(K) \right). \end{cases} \tag{50}$$

Therefore

$$\complement K = \partial_{\Omega^{\ddagger}}^{-\infty}(K) \cup_{\emptyset} \left(\bigcup_{n\geq 1} \partial_{\Omega^{\ddagger}}^{-n}(K) \right) \cup_{\emptyset} \partial_{\Omega^{\ddagger}}^{0-}(K) \tag{51}$$

and thus, the partition (46), p. 207, ensues. □

By construction:

Corollary 3.5 (Idempotent Envelopes). *The hypermaps* $\partial_{\Omega}^{+\infty} : K \mapsto \partial_{\Omega}^{+\infty}(K)$ *and* $\partial_{\Omega^{\ddagger}}^{-\infty} : K \mapsto \partial_{\Omega^{\ddagger}}^{-\infty}(K)$ *are idempotent reflections:*

$$\partial_{\Omega^{\ddagger}}^{-\infty} = \left(\partial_{\Omega}^{+\infty}\right)^{\ddagger} \tag{52}$$

and invariant in the sense that

$$\Omega\left(\partial_{\Omega}^{+\infty}(K)\right) = \partial_{\Omega}^{+\infty}(K) \quad \text{and} \quad \Omega^{\ddagger}\left(\partial_{\Omega^{\ddagger}}^{-\infty}(K)\right) = \partial_{\Omega^{\ddagger}}^{-\infty}(K). \tag{53}$$

They play, so to speak, the role of idempotent envelopes.

Naturally, the idempotent envelope $\partial_{\Omega}^{+\infty}$ of an intensive map can take empty values. Further assumptions forbid this case. For instance, in topological spaces, if the intensive hypermap Ω

1. maps non empty compact subsets to non empty compact subsets, then $\partial_{\Omega}^{+\infty}$ has non empty values;
2. maps closed subsets to closed subsets, if X is complete metric space and if the diameter of the strata $\Omega^{n}(K)$ converges to 0, then $\partial_{\Omega}^{+\infty}$ is single-valued: the values $\partial_{\Omega}^{+\infty}(K)$ are singleta.

3.5. Galois Quotations and Measures

The Galois stratification allows us to quantize the "position of an element with respect to a subset":

(Galois Connection). Let $\mathcal{I}(X)$ denote the set of intensive hypermaps $\Omega : \mathcal{P}(X) \mapsto \mathcal{P}(X)$. The *Galois connection* $\alpha : \mathcal{I}(X) \times \mathcal{P}(X) \times X \mapsto \mathbb{Z}$ associates with any intension $\Omega \in \mathcal{I}(X)$, any subset $K \in \mathcal{P}(X)$ and any element $x \in X$ the integer $\alpha_\Omega(K, x) \in \mathbb{Z}$ defined by

$$\alpha_\Omega(K, x) := n \text{ if and only if } x \in \partial^n_\Omega(K). \tag{54}$$

For any fixed intension $\Omega \in \mathcal{I}(X)$, its *Galois quotation* $\alpha_\Omega : \mathcal{P}(X) \times X \mapsto \mathbb{Z}$ associates with any pair (K, x) the integer $\alpha_\Omega(K, x)$. Similarly, the map

$$\begin{cases} (\Omega, K) : x \mapsto \alpha_\Omega(K, x) \in \mathbb{Z}^X, \\ \quad \text{is a membership function of the Galois stratification} \\ (\Omega, x) : K \mapsto \alpha_\Omega(K, x) \in \mathbb{Z}^{\mathcal{P}(X)}, \\ \quad \text{plays the role of a "Dirac" measure} \\ (K, x) : \Omega \mapsto \alpha_\Omega(K, x) \in \mathbb{Z}^{\mathcal{I}(X)}, \\ \quad \text{is an intension functional associating Galois quotations.} \end{cases} \tag{55}$$

We observe[14] that

$$\forall\, p \geq n \geq 0, \;\; \Omega^n(K) \setminus \Omega^p(K) = \bigcup_{j=0}^{j=p-n+1} \partial^{n+j}_\Omega(K) \tag{56}$$

is the union of the $p-n$ mutually disjoint internal marches $\partial^{n+j}_\Omega(K)$. The *Galois width* of the slice $\Omega^n(K) \setminus \Omega^p(K)$ is equal to $p - n$.

On the other hand, we note that

$$\begin{cases} \alpha_\Omega(K, x) \geq n & \text{if and only if } x \in \Omega^n(K) \\ \alpha_\Omega(K, x) \leq p + 1 & \text{if and only if } x \in \complement\Omega^p(K). \end{cases} \tag{57}$$

Since $K \mapsto \alpha_\Omega(K, x) \in \mathbb{Z}^{\mathcal{P}(X)}$ can be regarded as kind of "Dirac" measure, we can extend it by associating with Ω measures M inspired by the concept of Maslov measure, when the arithmetic operations are replaced by the lattice operations on $\mathbb{R}$:

[14]The slice $\Omega^n(K) \setminus \Omega^p(K)$ could be denoted by $\Omega^n(K) \setminus \Omega^p(K) = \int_{j=0}^{j=p-n+1} \partial^{n+j}_\Omega(K)$, equal to $p - n$.

Definition 3.6 (Galois Measures). The lower and upper Maslov-Galois measures are defined by

$$\forall\, M \in \mathcal{P}(X), \quad \begin{cases} \alpha^\flat_\Omega(K,M) := \inf_{x\in M} \alpha_\Omega(K,x) \\ \alpha^\sharp_\Omega(K,M) := \sup_{x\in M} \alpha_\Omega(K,x) \end{cases} \tag{58}$$

quantize the “location” of subsets $K \subset X$ versus the subset M.
The *Galois width* of M with respect to Ω associates with any $K \subset X$ the integer $\gamma_\Omega(K,M) = \alpha^\flat_\Omega(K,M) - \alpha^\sharp_\Omega(K,M)$.

The upper and lower Galois measures allow us to quantize solutions of lower and upper hyperinclusions

Proposition 3.7 (Quantization of Solutions to Hyperinclusions). *Let Ω an intensive hypermap and a subset M, a datum of lower hyperinclusion, and its complement, the datum of the upper hyperinclusion. Then*

$$\begin{cases} \alpha^\flat_\Omega(K,M) = n & \text{if and only if } M \subset \Omega^n(K) \\ & \qquad \text{(solution to an upper hyperincusion)} \\ \alpha^\sharp_\Omega(K,M) = p+1 & \text{if and only if } \complement M \supset \Omega^p(K) \\ & \qquad \text{(solution to a lower hyperincusion).} \end{cases} \tag{59}$$

This is equivalent to say that M is contained in the slice

$$M \subset \Omega^{\alpha^\flat_\Omega(K,M)}(K) \setminus \Omega^{\alpha^\sharp_\Omega(K,M)}(K) \tag{60}$$

of width $\alpha^\sharp_\Omega(K,M) - \alpha^\flat_\Omega(K,M)$.

Pictorially, we can interpret

1. the subset $\partial^{+\infty}_\Omega(K)$ as the “paradise” of K;
2. subset $\partial^{-\infty}_{\Omega^\ddagger}(K)$ of K as the “hell”;
3. their demarcation as being the “purgatory” of K.

Purgatory is itself partitioned into *paradisiac purgatorial strata* $\partial^n_\Omega(K)$ indexed by positive integers, in *hellish purgatorial strata* $\partial^{-n}_\Omega(K)$ for negative integers and the *styx*[15] $\partial^0_\Omega(K) = \partial^{0+}_\Omega(K) \cup \partial^{0-}_\Omega(K)$.

[15]This terminology is not faithful to the mythology, because this fatal river separates the Earth and the Underworld, crossed by the ferryman *Charon*, for an obol placed in the mouth of the dead. Hell and Purgatory did not exist for the Greeks nor the Hebrews, who preferred to inflict punishment in real finite time rather than abstract asymptotic time. In Jerusalem, Gehenna (Ge-Hinnom Valley), a place where were buried and cremated deads, designated Hell, place of punishment in the afterlife. Purgatory was invented by the Catholic Church in the XII-th century for allowing financiers of the time to loan without risking eternal hell, at the price of *indulgences* (see *L'usurier de Liège*, p. 407, de *La naissance du Purgatoire*, [36], by *Jacques Le Goff*).

4. Contingent and Tychastic Quotations

Let $F : x \in X \rightsquigarrow F(x) \in \mathcal{P}(X)$ be a set-valued map from X to itself (assumed to be non extensive). We associate with it the "discrete nondeterministic evolutionary" system

$$\forall\, n \geq 0, \;\; x_{n+1} \in F(x_n) \tag{61}$$

governing the evolution of sequences $\overrightarrow{x} := (x_n)_{n\geq 0}$ starting at x_0.

Adding to the set-valued map F a closed subset K and taking for target $C := \partial K = \partial \overline{\complement K}$, "viability theory" deals with the study of viable evolutions in K for some duration, finite or infinite, or to evolutions reaching C after some finite duration. It provides ultimately the regulation map, the "mother of all the feedbacks", and their domains (viability kernels and capture basins). We do not mention other generalizations (differential games, mutational equations, viability multipliers, etc.). At the beginnings of viability theory, viability kernels and capture basins capture underly *contingent uncertainty* ("there exists [...]"), whereas invariance kernels and absorption basins could translate mathematically *tychastic uncertainty* ("for all [...]") associated with the quantifiers $\exists$ and $\forall$. They interplay in case of "dynamical games" regulated by the player "Eloise" who chooses (contingently) one evolution "*against*" all available evolutions played by the player "Abelard[16]" for guaranteeing viability.

The purpose of this section is to relate the abstract concept of Galois stratification to (discrete) "viability concepts" associated with this set-valued map F, regarded as the engine of a discrete dynamical system (61).

Associating with the inverse F^{-1} of F their intensions $F^{-1}(K)\cap K$ and $F^{\ominus 1}(K)\cap K$, we obtain the following contingent and tychastic stratification processes.

4.1. Contingent Stratification

We recall that the intension $F^{-1}(K) \cap K$ enjoys viability interpretations:

1. $\Omega(K) := \mathrm{Viab}_F^{[1]}(K) := F^{-1}(K) \cap K$ is the subset of elements $x \in K$ such that all *at least one successor* $y \in F(x)$ belongs to K. In other words, $\mathrm{Viab}_F^{[1]}(K)$ is the duration-1 viability kernel of K under F.
2. $\Omega^{\ddagger}(K) := \mathrm{Abs}_F^{[1]}(K)$ is the subset of elements x such that *all successor* $y \in F(x)$ belongs reach K, i.e., the "one step absorption basin" of K, the

[16] The logician *Wilfred Hodges* and specialist of game theory used the sad story of the logician monk of the XIIth century, *Pierre Abélard*, the "tychastic" player, nicknamed "$\forall$belard", who was cruelly punished for seducing the future abbess of Argenteuil, Eloise (*Héloïse*), the "contingent regulator" nicknamed "$\exists$loise". She was at that time his young student attracted by an inclination that logic does not know, but biology does. While working on logics, the young Pierre-Astrolabe was born as the fruit of their love. His punishment was the reverse retaliation of his "sin" and the transgression of ecclesiastical canons. Driven from their paradise by a singular punishment, $\exists$loise and $\forall$belard could have given way to "$\forall$dam" and "$\exists$va", also punished for wanting to "know" and being the culprits for the sins of all of us. The authors thank *Allen Mann* for this information

subset of elements outside K from which all successors reach K in one step.

Definition 4.1 (The Contingent Stratification). We associate with a set-valued map $F : X \rightsquigarrow X$ the two following hypermaps:

1. The hypermap $\Omega := \mathrm{Viab}_F^{[1]} : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ defined by

$$\forall\, K \in \mathcal{P}(X), \quad \mathrm{Viab}_F^{[1]}(K) := K \cap F^{-1}(K) \tag{62}$$

called the the *duration*-1 *contingent hypermap* associated with F;

2. the hypermap $\Omega^{\ddagger} := \mathrm{Abs}_F^{[1]} : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ defined by

$$\forall\, K \in \mathcal{P}(X), \quad \mathrm{Abs}_F^{[1]}(K) := K \cup F^{\ominus 1}(K) \tag{63}$$

called the the *duration*-1 *absorption hypermap* associated with F.

related by

$$\mathrm{Viab}_F^{[1]}(K) = \complement \mathrm{Abs}_F^{[1]}(\complement K) \quad \text{or} \quad \mathrm{Abs}_F^{[1]}(K) = \complement \mathrm{Viab}_F^{[1]}(\complement K) \tag{64}$$

The hypermap $\mathrm{Viab}_F^{[1]}$ is increasing and intensive and consequently the hypermap $\mathrm{Abs}_F^{[1]}$ is increasing and extensive.

By iterating the duration 1 viability kernel and absorption basin hypermaps, we infer that

$$\left(\mathrm{Viab}_F^{[n]}\right)^{\ddagger}(K) = \mathrm{Abs}_F^{[n]}(K) \tag{65}$$

and that the reflection of the viability kernel map Viab_F defined by

$$\mathrm{Viab}_F = \bigcap_{n \geq 1} \mathrm{Viab}_F^{[n]}(K) \tag{66}$$

is the absorption basin hypermap Abs_F which associates with any K

$$\mathrm{Abs}_F(K); := \complement \mathrm{Viab}_F(\complement K) = \bigcup_{n \geq 1} \mathrm{Abs}_F^{[n]}(K). \tag{67}$$

Therefore, we translate the Galois structuration for the viability kernel hypermap:

Definition 4.2 (Contingent Strata and Marches).

1. *Internal contingent strata and marches*:
   (a) The contingent internal strata of duration n are the subsets $\mathrm{Viab}_F^{[n]}(K)$;
   (b) The contingent internal marche of duration $0+$ is the subset $\partial^{0+}_{Viab}(K) := K \setminus \mathrm{Viab}_F^{[1]}(K)$;

   (c) The contingent internal marches of duration n are the subsets $\partial^n_{Viab}(K) := \text{Viab}^{[n]}_F(K) \setminus \text{Viab}^{[n+1]}_F(K) = \text{Abs}^{[n+1]}_F(\complement K) \setminus \text{Abs}^{[n]}_F(\complement K)$ of elements such that all successors remains in K during n steps but one successor leaves K at time $n+1$;
   (d) The contingent internal marche of duration $+\infty$ is the *viability kernel* $\text{Viab}(K) := \partial^{+\infty}_{Viab}(K) := \bigcap_{n\geq 1} \text{Viab}^{[n]}_F(K)$ of elements such at least a sequence of successors remains in K forever.
2. *External contingent strata and marches*:
   (a) The contingent external strata of duration n are the subsets $\text{Abs}^{[n]}_F(K)$;
   (b) The contingent external marche of duration 0+ is the subset $\partial^{0+}_{Viab}(K) := \text{Abs}^{[1]}_F(K) \setminus K$,
   (c) The contingent external marches of duration $-n$ is are the subsets $\partial^{-n}_{Viab}(K) := \text{Abs}^{[n+1]}_F(K) \setminus \text{Abs}^{[n]}_F(K) = \text{Viab}^{[n]}_F(\complement K) \setminus \text{Viab}^{[n+1]}_F(\complement K)$ of elements such that at least a sequence of successors remains outside K during n steps but all successors reach K at time $n+1$;
   (d) The contingent external marche of duration $-\infty$ is the complement of the *absorption basin* $\complement\text{Abs}(K) := \partial^{-\infty}_{Viab}(K) := \complement \bigcup_{n\geq 1} \text{Abs}^{[n]}_F(K)$ of elements such that all evolutions reach K in finite duration.

The contingent (or viability) quotation is thus defined by

$$\begin{cases} +\infty & \text{if } x \in \text{Viab}_F(K) \\ n & \text{if } x \in \text{Viab}^{[n]}_F(K) \setminus \text{Viab}^{[n+1]}_F(K) \\ 0+ & \text{if } x \in K \setminus \text{Viab}^{[1]}_F(K) \\ 0- & \text{if } x \in \text{Abs}^{[1]}_F(K) \setminus K \\ -n & \text{if } \in \text{Abs}^{[n+1]}_F(K) \setminus \text{Abs}^{[n]}_F(K) \\ -\infty & \text{if } x \notin \text{Abs}_F(K). \end{cases} \tag{68}$$

4.2. Tychastic Stratification

In the same way, we recall that the intension $F^{\ominus 1}(K) \cap K$ enjoys viability interpretations:

1. $\Omega(K) := \text{Inv}^{[1]}_F(K) := F^{\ominus 1}(K) \cap K$ is the subset of elements $x \in K$ such that all *all successors* $y \in F(x)$ belongs to K. In other words, $\text{Inv}^{[1]}_F(K)$ in the duration-1 tychastic kernel of K under F.
2. $\Omega^\ddagger(K) := \text{Capt}^{[1]}_F(K)$ is the subset of elements x such that *at least one successor* $y \in F(x)$ reaches K, i.e., the "one step capture basin" of K, the subset of elements outside K which can reach K in one step.

We also recall that the duration 1 capture basin hypermap is the reflection of the duration 1 of the tychastic kernel hypermap. Hence the reflection of the duration 1 of the invariance kernel hypermap is the duration 1 capture basin:

Definition 4.3 (The Tychastic Stratification). We associate with a set-valued map $F : X \rightsquigarrow X$ the two following hypermaps:

1. The hypermap $\Omega := \mathrm{Inv}_F^{[1]} : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ defined by

$$\forall\, K \in \mathcal{P}(X),\;\; \mathrm{Inv}_F^{[1]}(K) := K \cap F^{\ominus 1}(K) \tag{69}$$

called the the *duration*-1 *tychastic hypermap* associated with F;

2. the hypermap $\Omega^{\ddagger} := \mathrm{Capt}_F^{[1]} : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ defined by

$$\forall\, K \in \mathcal{P}(X),\;\; \mathrm{Capt}_F^{[1]}(K) := K \cup F^{-1}(K) \tag{70}$$

called the the *duration*-1 *capture hypermap* associated with F.

related by

$$\mathrm{Inv}_F^{[1]}(K) = \complement \mathrm{Capt}_F^{[1]}(\complement K) \quad \text{or} \quad \mathrm{Capt}_F^{[1]}(K) = \complement \mathrm{Inv}_F^{[1]}(\complement K) \tag{71}$$

The hypermap $\mathrm{Inv}_F^{[1]}$ is an increasing intensive erosion and consequently the hypermap $\mathrm{Capt}_F^{[1]}$ is an increasing extensive dilation.

By iterating the duration 1 invariance kernel and capture basin hypermaps, we infer that

$$\left(\mathrm{Inv}_F^{[n]}\right)^{\ddagger}(K) = \mathrm{Capt}_F^{[n]}(K) \tag{72}$$

and that the reflection of the invariance kernel map Inv_F defined by

$$\mathrm{Inv}_F = \bigcap_n \mathrm{Inv}_F^{[n]}(K) \tag{73}$$

is the capture basin hypermap Capt_F which associates with any K

$$\mathrm{Capt}_F(K); := \complement \mathrm{Inv}_F(\complement K) == \bigcup_{n \geq 1} \mathrm{Capt}_F^{[n]}(K) \tag{74}$$

Therefore, we translate the Galois structuration for the tychastic kernel hypermap:

Definition 4.4 (Tychastic Strata and Marches).

1. *Internal tychastic strata and marches*:
   (a) The tychastic internal strata of duration n are the subsets $\mathrm{Inv}_F^{[n]}(K)$;
   (b) The tychastic internal marche of duration 0+ is the subset $\partial_{Inv}^{0+}(K) := K \setminus \mathrm{Inv}_F^{[1]}(K)$;
   (c) The tychastic internal marches of duration n are the subsets $\partial_{Inv}^{n}(K) := \mathrm{Inv}_F^{[n]}(K) \setminus \mathrm{Inv}_F^{[n+1]}(K) = \mathrm{Capt}_F^{[n+1]}(\complement K) \setminus \mathrm{Capt}_F^{[n]}(\complement K)$ of elements such that all successors remains in K during n steps but one successor leaves K at time $n+1$;

(d) The tychastic internal marche of duration $+\infty$ is the *invariance kernel* $\text{Inv}(K) := \partial^{+\infty}_{Inv}(K) := \bigcap_{n\geq 1}\text{Inv}^{[n]}_F(K)$ of elements such that all successors remains in K forever.

2. *External tychastic strata and marches*:
 (a) The tychastic external strata of duration n are the subsets $\text{Capt}^{[n]}_F(K)$;
 (b) The tychastic external marche of duration $0+$ is the subset $\partial^{0+}_{Inv}(K) := \text{Capt}^{[1]}_F(K) \setminus K$,
 (c) The tychastic external marches of duration $-n$ is are the subsets $\partial^{-n}_{Inv}(K) := \text{Capt}^{[n+1]}_F(K) \setminus \text{Capt}^{[n]}_F(K)$ of elements such that all successors remain outside K during n steps but one successor reaches K at time $n+1$;
 (d) The tychastic external marche of duration $-\infty$ is the *capture basin* $\complement\text{Capt}(K) := \partial^{-\infty}_{Capt}(K) := \complement\bigcup_{n\geq 1}\text{Capt}^{[n]}_F(K)$ of elements such that all evolutions reach K in finite duration.

4.3. Viability Reliability

The viability hypermap is increasing and intensive, but neither an erosion nor a dilation. However, the invariant hypermap is an intensive erosion, so that the capture hypermap is an extensive dilation.

Since the invariant hypermap is contained in the viability hypermap, they induce the following partitions: for all $n \geq 0$,

$$K = \text{Inv}^{[n]}_F(K) \cup_\emptyset [\text{Viab}^{[n]}_F(K) \cup \text{Capt}^{[n]}_F(K)] \cup_\emptyset \text{Abs}^{[n]}_F(\complement K). \tag{75}$$

The subsets $\text{Inv}^{[n]}_F(K)$ (white zone) and $\text{Abs}^{[n]}_F(\complement K)$ (black zone) are *reliable* since all sequence of successors are viable in K in the first case, or leave K for n units of duration in the second one.

The subset $[\text{Viab}^{[n]}_F(K) \cup \text{Capt}^{[n]}_F(K)]$ (gray zone) is *unreliable* since for all initial condition in this set, there exist on sequence of n successors viable in K *and* one sequence of n successors which leave K before $n+1$.

Remark. This study deals with discrete quantization. Il can be extended to continuous time valuation when we use the viability concepts for continuous time valuation (see *Viability Theory. New Directions*, [13] for continuous time systems). The concepts of viability, capturability, invariance and absorption "tubes" replace the sequences of viability internal and external continent maps tychastic marches by viability, invariance, capture and absorption tubes. The marches are the boundaries of those tubes at each instant, and provide a continuous time stratification, the Galois quotations being exit time functions. They play the role of continuous time tychastic measure of the viability risk. □

4.4. Viability Algorithms

The viability algorithms compute at each iteration viability and invariance kernels on one hand, absorption and capture basins on the other one. Viability algorithms handle at each iteration subsets of a grid instead of vectors, and, thus, are prone to Bellman's *curse of dimensionality*. They have been developed since their discovery in [52] to the latest developments in [29] by several investigators. They are intimately related to the contingent and tychastic stratifications, since the iterate combination of iterations of the form

$$\begin{cases} \mathrm{Inv}_F^{[n]}(K) := \mathrm{Inv}_F^{[1]}(\mathrm{Inv}_F^{[n-1]}(K)) \\ \mathrm{Capt}_F^{[n]}(K, \complement K) := \mathrm{Capt}_F^{[1]}(K, \mathrm{Capt}_F^{[n-1]}(K, \complement K)) \\ \mathrm{Viab}_F^{[n]}(K) := \mathrm{Viab}_F^{[1]}(\mathrm{Viab}_F^{[n-1]}(K)) \\ \mathrm{Abs}_F^{[n]}(K, \complement K) := \mathrm{Abs}_F^{[1]}(K, \mathrm{Abs}_F^{[n-1]}(K, \complement K)) \end{cases} \tag{76}$$

which are related by reflection relations.

These algorithms depend on three elements: the set-valued map defining the discrete evolutionary system, the constraints and the targets. Any combination of these three components provides a specific response. Changing one of these three data is changing the problem.

The viability algorithms use the Boolean operations, inverse images and cores, in various combinations. Other algorithms specify *a priori* classes of subsets for approximating subsets, such as (ellipsoids, hypercubes) or regularize them (vector machines). However, these operations are not stable by the Boolean operations used in viability algorithms: neither ellipsoidal nor hypercubes, are stable under unions, bringing useless complications and causing damaging losses of information.

Viability algorithms compute kernels and basins in two versions:

1. the viability and invariance algorithms of kernels, using the internal stratification, provide approximations *contained into* the viability and tychastic kernels (with target);
2. the capture and absorption algorithms of basins, using the external approximations, provide approximations *containing* the capture and absorption basins.

Using them pairwise provide *a posteriori* errors, the only ones that matter.

Providing regulation maps, there is no need to track nominal trajectories. However, the viability algorithms provide them by simulation. One can use trajectory tracking algorithms to remain viable and/or reach a target. Again, there will be a loss of information because the computed nominal trajectories depend on these three elements, which are not explicitly taken into account by unconnected trajectory tracking algorithms. Provided in addition that they do not deviate too much of the nominal path, that, whenever they leave the viability kernel, all evolutions violate the constraints in finite time.

Among the problems to be solved, some are "direct" in the sense that these three components are given directly. Others are "indirect", such as solutions of first-order partial differential equations (conservation laws, Hamilton Jacobi): they are camouflaged viability issues, for which the underlying dynamic is "the characteristic system" mentioned when speaking of the method of characteristics. Characteristic systems are indeed the keys for solving these problems. Boundary conditions (Cauchy, Dirichlet), or internal conditions are described by targets. The tangential conditions provided by viability theorem are translated in those partial differential equations. Indeed, since Fermat, the graph of the derivative of a function is the tangent to the function. Since Leibniz and Newton, this approach was forgotten until the (re) discovery of set-valued analysis, where the graphs of the derivatives of set-valued maps are defined by tangent cones to their graphs and where the epigraphs of the "epiderivatives" of extended numerical functions are defined by tangent cones to their epigraphs. The graphs or hypographs of solutions to first-order partial differential equations being combinations of kernels or basins of adequate sets and targets under the characteristic system, viability algorithms provide these solutions in a stable way, providing *a posteriori* errors if needed, to the price of handling subsets instead of vectors.

Conversely, level sets of solutions to adequate first-order partial differential equations are sometimes kernels of basins. Using those partial differential equations for computing them is a waste of one dimension. Furthermore, finite difference schemes for computing them often undergo instabilities.

4.5. Polygamy and Polyandry

Let X be any set and $F : X \rightsquigarrow X$ be a set-valued map defined on X. Let $L \subset X$ and $M \subset X$ be two subsets of X. We set

$$\mathrm{Viab}_F^{[1]}(L, M) := L \cap F^{-1}(M) \subset L.$$

Lemma 4.5. *We observe that*

$$\mathrm{Viab}^{[1]}_{\bigcup_k F_k}\left(\bigcup_i L_i, \bigcup_j M_j\right) = \bigcup_k \bigcup_i \bigcup_j \mathrm{Viab}^{[1]}_{F_k}\left(L_i, M_j\right). \tag{77}$$

Setting

$$R(i,j) := \left\{k \textit{ such that } \mathrm{Viab}^{[1]}_{F_k}\left(L_i, M_j\right) \neq \emptyset\right\}$$

then

$$\begin{aligned}\mathrm{Viab}^{[1]}_{\bigcup_k F_k}\left(\bigcup_i L_i, \bigcup_j M_j\right) &= \bigcup_k \bigcup_{(i,j)\in R^{-1}(k)} \mathrm{Viab}^{[1]}_{F_k}\left(L_i, M_j\right) \\ &= \bigcup_{(i,j)} \bigcup_{k\in R(i,j)} \mathrm{Viab}^{[1]}_{F_k}\left(L_i, M_j\right).\end{aligned}$$

Each parameter k can be regarded as a lineage, indexes i of L_i referring to the name of the fathers, indexes j of M_i the names of the mothers, so that $\mathrm{Viab}^{[1]}_{F_k}(L_i, M_j) \subset L_i$ is the lineage k of their offsprings (hence the title of the section!).

This approach has been generalized for designing algorithms for discrete mutational "genetic" evolution of tissues under sequences of genetic regulons for studying biological morphogenesis by *Alexandra Fronville* (see for instance [33], [32]) and *Olivier Dordan & Fleur-Isadora Dordan*).

Observe that if for any pair (k, l), $k \neq l$, $F_k^{-1}(M) \cap F_l^{-l}(M) = \emptyset$, then,

$$\mathrm{Viab}^{[1]}_{F_k}(L, M) \cap \mathrm{Viab}^{[1]}_{F_l}(L, M) = \emptyset.$$

Remark (Parallelization of the viability kernel algorithm). Consider the case when $F := \bigcup_k F_k$. Using formula (77), p. 217, the viability kernel algorithm defines the decreasing sequence of subsets K_n by

1. Initial step : $K_0 := K$
2. First step
$$K_1 = \bigcup_k \mathrm{Viab}^{[1]}_{F_k}(K_0, K_0).$$
3. Other steps
$$K_n = \bigcup_k \bigcup_{(i,j)\in R^{-1}(k)} \mathrm{Viab}^{[1]}_{F_k}(\mathrm{Viab}^{[1]}_{F_i}(K_{n-1}, K_{n-1}), \mathrm{Viab}^{[1]}_{F_j}(K_{n-1}, K_{n-1})).$$

This is a parallel or distributed algorithm (the terms overlap ...), since, at each iteration $n-1$, knowing K_{n-1}, the "cells" $\mathrm{Viab}^{[1]}_{F_k}(\mathrm{Viab}^{[1]}_{F_i}(K_{n-1}, K_{n-1}),$ $\mathrm{Viab}^{[1]}_{F_j}(K_{n-1}, K_{n-1}))$ can be computed independently of each other and, next, collected to obtain K_n.

4.6. Minkowski-Bouligand Dimension of Viability Kernels

The Minkowski-Bouligand dimension appeared in 1925 and formally introduced by in 1928 by *Georges Bouligand*. He called it the *Cantor-Minkowski order*, since he used basic ideas of his two predecessors. Since, it has been called *fractional dimension*, *logarithmic density*, *fractal dimension* (by *Benoït Mandelbrot*), *entropy*, *capacity* (with adequate adjectives), etc. According to *Benoït Mandelbrot*, a *fractal* is a set the fractal dimension is larger than or equal to the dimension d of the vector space.

We present the Minkowski-Bouligand concept in the framework of a vector space $X := \mathbb{R}^d$ and of a covering of the vector space $X := \mathbb{R}^d$ by the cubes $B(jh) := jh + hB$ where $B \subset X$ (for instance, $B := [0, 1]^d$), where $j := (j_1, \ldots, j_d)$ ranges over $\mathbb{Z}^d$, of dimension $N(h)$ (for instance, $h_n := \lambda^n$ for $\lambda < 1$ in fractal studies).

We associate with a subset K its

1. *restriction hypermap* from $\mathbb{R}^d$ to $\mathbb{Z}^d$ defined by

$$r(K,h) := \left\{j \in \mathbb{Z}^d \text{ such that } B(jh) \cap K \neq \emptyset\right\}. \tag{78}$$

2. *internal restriction* hypermap $\overset{\circ}{r}$ defined by

$$\overset{\circ}{r}(K,h) := \left\{j \in \mathbb{Z}^d \text{ such that } B(jh) \subset K\right\}. \tag{79}$$

3. *prolongation hypermap* $p(S,h)$ defined by

$$\forall\, S \subset \mathbb{Z}^d, \;\; p(S,h) := \bigcup_{j \in S} B(jh) \tag{80}$$

so that

$$p(\overset{\circ}{r}(K,h),h) \subset K \subset p(r(K,h),h) = \bigcup_{j \in r(K,h)} (K \cap B(jh)). \tag{81}$$

We observe the partition property:

$$r(K \cup L, h) = \overset{\circ}{r}(K \setminus L, h) \cup r(K \cap L, h) \cup \overset{\circ}{r}(L \setminus K, h). \tag{82}$$

The associated "box counting function" $N(h)$ is defined by

$$N(K,h) := |r(K,h)| \;\; \text{(the cardinal of } r(K,h)) \tag{83}$$

and the *h-fractal dimension of* K by

$$\delta(K,h) := -\frac{\log(N(K,h))}{\log(h)} \tag{84}$$

or, equivalently, $N(K,h) := h^{-\delta(K,h)}$. The *Minkowski-Bouligand dimension* is equal to

$$\delta(K) := \liminf_{h \to 0+} \delta(K,h) \tag{85}$$

For viability kernels, the viability kernel algorithm associated with a set-valued map F involves the computation of subsets

$$K \cap F^{-1}(L).$$

Therefore, on a grid

$$K \cap F^{-1}(L) = \bigcup_{p \in r(K,h)} \bigcup_{q \in S(L,h)} \left[K \cap B(ph) \cap F^{-1}(L \cap B(qh))\right]. \tag{86}$$

Fractals are obtained as viability kernels of set-valued map F obtained as the union of p single-valued maps φ_i:

$$F(x) := \bigcup_{i=1}^{p} \varphi_i(x). \tag{87}$$

We consider the case of *disconnecting maps* F defined by

$$\forall\, K, \forall\, i \neq j, \;\; \text{meas}(\varphi_i^{-1}(K) \cap \varphi_j^{-1}(K)) = 0. \tag{88}$$

Then the number of boxes is equal to $N(F^{-1}(K), h) := \sum_{i=1}^{p} N(\varphi_j^{-1}(K), h)$. The map φ is self-similar if there exists $\lambda < 1$ such that

$$\forall\, K, \;\; N(\varphi^{-1}(K, \lambda)) = N(K, 1). \tag{89}$$

Consequently, if all the maps φ_i are self-similar for λ,

$$N(F^{-1}(K, \lambda)) = p$$

so that $N(F^{-n}(K, \lambda^n)) = p^n$.

Taking $h_n := \lambda^n$, we infer that the fractal dimension of viability kernels is equal to

$$\delta(N(F^{-n}(K, \lambda^n))) = \frac{\log(p)}{\log(\lambda)} \tag{90}$$

(see Section 2.9, p. 71, of *Viability Theory. New Directions*, [13] for more details).

5. Morphological Mathematics

We end this survey by a glimpse to *mathematical morphology* pioneered by *Georges Matheron*, and thoroughly used in image processing[17]. He used the Minkowski sum and difference hypermaps on vector spaces together with the lattice operations for defining exotic algebras of any kind.

It allows a connection between the temporal interpretation of the contingent and tychastic Galois quotations associated with dynamic systems and the spatial interpretation of the Minkowski operations.

Remark (Mathematical morphological definitions). If $0 \in S$, the Minkowski difference hypermap $S\ominus$ is intensive. Such a subset S is called a *structuring element* in mathematical morphology. It is regarded as a shape, used to probe or interact with other subsets K, regarded as images, studying how this shape fits or misses this shapes in the image. For deriving more results, it is assumed to be convex, closed and, sometimes, symmetric (in this case, it is a ball). The subset $K \oplus S$ is called the *dilation* of K by S, $K \ominus S$ the *erosion* of K by S and $\partial_S(K) := (K \oplus S) \setminus (K \ominus S)$ the morphological boundary of K. The dilation of an erosion is called an *opening* and the erosion of the dilation a *closing*. When S is a cone, the fixed-sets S such that $K \oplus S = K$ are called S-*exhaustive* subsets.Therefore, when S is a convex cone, the Minkowski sum hypermap is idempotent since $S \oplus S = S$. □

[17]see for instance [41], *Éléments de morphologie mathématique*, [53], *Morphologie mathématique: Tome 1, Approches déterministes*, [46] and *Mathematical Morphology*, [45], among many monographs on this topic.

Subsets describe "black and white" images, whereas we would like to deal with grey (or color images). They are described by functions, either characteristic functions can be regarded as associating intensities the two colors on the scale $\{0,1\}$ (0 for black and 1 for white). Actually, the real computer scale being $\{0,256\}$, 256 shades of grey are used for representing images by functions associating with each point of the shape its intensity in grey levels: a single pixel occupies six bits (one bythe). Hence we need to transfer the operations on the hyperspace $\mathcal{P}(X)$ of a vector space to operations on spaces of functions. Fortunately, any numerical function can be characterized by a set, whether its graphs, intersection of its epigraph and hypograph, all subsets belonging to $\mathcal{P}(X \times \mathbb{R})$. We just therefore need a short dictionary to pass from the language on sets to the one on functions:

Definition 5.1 (Graphs and Semigraphs). Let $\mathbf{v} : X \mapsto \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ be an *extended function*. We associate with it the four *semigraphs*, which are the following subsets of $X \times \mathbb{R}$:

1. the *epigraph* $\mathcal{E}p(\mathbf{v}) := \{(x,\lambda)$ such that $\mathbf{v}(x) \leq \lambda\}$ and the *strict epigraph* $\overset{\circ}{\mathcal{E}p}(\mathbf{v}) := \{(x,\lambda)$ such that $\mathbf{v}(x) < \lambda\}$;
2. the *hypograph* $\mathcal{H}yp(\mathbf{v}) := \{(x,\lambda)$ such that $\mathbf{v}(x) \geq \lambda\}$ and the *strict hypograph* $\overset{\circ}{\mathcal{H}yp}(\mathbf{v}) := \{(x,\lambda)$ such that $\mathbf{v}(x) > \lambda\}$.

We shall say that the hypermap $\Omega : \mathcal{P}(X \times \mathbb{R}) \mapsto \mathcal{P}(X \times \mathbb{R})$ is a semigraphical hypermap if it maps semigraphs into semigraphs.

We just mention that the abrasion $\mathcal{E}p(\mathbf{v}) \setminus \overset{\circ}{\mathcal{E}p}(\mathbf{w})$ is the graph of the interval map $x \rightsquigarrow [\mathbf{v}(x), \mathbf{w}(x)]$ whenever $\mathbf{v}(x) \leq \mathbf{w}(x)$ and $\emptyset$ if not. We thus apply the morphological operations (30), p. 198: we define the functions $\mathbf{v} \oplus \mathbf{w}$ and $\mathbf{v} \ominus \mathbf{w}$ through their epigraphs:

$$\mathcal{E}p(\mathbf{v} \oplus \mathbf{w}) = \mathcal{E}p(\mathbf{v}) \oplus \mathcal{E}p(\mathbf{w}) \quad \text{and} \quad \mathcal{E}p(\mathbf{v} \ominus \mathbf{w}) = \mathcal{E}p(\mathbf{v}) \ominus \mathcal{E}p(\mathbf{w}) \tag{91}$$

their morphological boundary

$$\mathcal{E}p(\mathbf{v} \oplus \mathbf{w}) \setminus (\overset{\circ}{\mathcal{E}p}\,(\mathbf{v} \ominus \mathbf{w})) = \text{Graph}(x \rightsquigarrow [(\mathbf{v} \oplus \mathbf{w})(x), (\mathbf{v} \ominus \mathbf{w})(x)]). \tag{92}$$

For example, we choose for function $\mathbf{w} := \psi_S$ the function associated with of a convex compact structuring set $S \subset X$ defined by $\psi_S(x) = 0$ when $x \in S$ and $+\infty$ when $x \notin S$. It is then easy to observe that, under adequate semicontinuity assumptions, that

$$\begin{cases} (\mathbf{v} \oplus \psi_S)(x) = \inf_{\xi \in S} \mathbf{v}(x+\xi) \\ (\mathbf{v} \ominus \psi_S)(x) = \sup_{\xi \in S} \mathbf{v}(x+\xi). \end{cases} \tag{93}$$

In this case, the *morphological gradient* is defined as the middle of the morpho-

Figure 5.1: Example of Morphological Boundary. This figure represents the painting "Rolla" by *Henri Gervex* (musée d'Orsay) and its morphological gradient computed by *Laurent Najman*, used in the cover of the text-book *Initiation à l'analyse appliquée*, [5].

logical boundary:

$$x \mapsto \frac{\sup_{\xi \in S} \mathbf{v}(x+\xi) + \inf_{\xi \in S} \mathbf{v}(x+\xi)}{2}.$$

This is the basic method in mathematical morphology for obtaining contours of images represented by their gray-scale functions.

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