Journal of Convex Analysis
Volume 23 (2016), No. 1, 227–236

Intersection of a Set with a Hyperplane

Maxim V. Balashov
Department of Higher Mathematics, Moscow Institute of Physics and Technology, Institutskii pereulok 9, Dolgoprudny, Moscow region, Russia 141700
balashov73@mail.ru

Dedicated to the memory of Jean Jacques Moreau.

Received: February 1, 2015
Revised manuscript received: February 14, 2015
Accepted: February 14, 2015

In the present work we consider the set-valued mapping whose images are intersections of a fixed closed convex bounded set with nonempty interior from a real Hilbert space with shifts of a closed linear subspace. We characterize such strictly convex sets in the Hilbert space, that the considered set-valued mapping is Hölder continuous with the power $\frac{1}{2}$ in the Hausdorff metric. We also consider the question about intersections of a fixed uniformly convex set with shifts of a closed linear subspace. We prove that the modulus of continuity of the set-valued mapping in this case is the inverse function to the modulus of uniform convexity and vice versa: the modulus of uniform convexity of the set is the inverse function to the modulus of continuity of the set-values mapping.

Keywords: Hilbert space, strongly convex set of radius R, Hausdorff metric, Lipschitz continuous set-valued mapping, Hölder continuous set-valued mapping, modulus of uniform convexity, modulus of continuity

2010 Mathematics Subject Classification: Primary 49J52, 46C05, 26B25; Secondary: 46B20, 52A07

1. Introduction and main notations

Let $\mathcal{H}$ be a real Hilbert space (and $\mathbb{R}^n$ be the usual n-dimensional Euclidean space) with the *inner product* (x, y) and the ball $B_r(a) = \{x \in \mathcal{H} \mid \|x - a\|^2 = (x - a, x - a) \le r^2\}$, $a \in \mathcal{H}$, $r > 0$.

For a set $A \subset \mathcal{H}$ denote by ∂A, $\operatorname{int} A$ and $\operatorname{cl} A$ the *boundary, the interior and the closure* of the set A, respectively.

For bounded sets $A, B \subset \mathcal{H}$ we denote by $h(A, B)$ the *Hausdorff distance* between the sets A and B, i.e.

$$h(A, B) = \inf\{r > 0 \mid A \subset B + B_r(0),\ B \subset A + B_r(0)\}.$$

ISSN 0944-6532 / $ 2.50

Also if the sets A and B are bounded closed and convex, then

$$h(A,B) = \sup_{\|p\|=1} |s(p,A) - s(p,B)|, \tag{1}$$

here $s(p,A) = \sup_{x\in A}(p,x)$ [8, Lemma 1.11.4].

A set $A \subset \mathcal{H}$ is called *strongly convex of radius* $R > 0$ [7], [8], if $A = \bigcap_{x\in X} B_R(x)$, where $X \subset \mathcal{H}$.

For a closed convex set $A \subset \mathcal{H}$ the *modulus of convexity* $\delta_A : [0, \operatorname{diam} A) \to [0,+\infty)$ is the function defined by

$$\delta_A(\varepsilon) = \sup\left\{\delta \geq 0 \;\middle|\; B_\delta\left(\frac{x_1+x_2}{2}\right) \subset A,\ \forall x_1, x_2 \in A:\ \|x_1 - x_2\| = \varepsilon\right\}.$$

A set $A \subset \mathcal{H}$ is called *uniformly convex* if $\delta_A(\varepsilon) > 0$ for all permissible $\varepsilon > 0$.

Let $A \subset \mathcal{H}$ be a closed convex bounded set. The polar set for the set A is the set $A^\circ = \{p \in \mathcal{H} \mid (p,x) \leq 1,\ \forall x \in A\}$. For a point $a \in A$ we denote by $N(A,a)$ the *normal cone* to the set A at the point a, $N(A,a) = \{p \in \mathbb{R}^n \mid (p, a-x) \geq 0,\ \forall x \in A\}$. A point $a \in \partial A$ is a *point of smoothness* of the set A, if dimension of $N(A,a)$ is 1. We shall call a closed convex set $A \subset \mathcal{H}$ *strictly convex,* if for any vector $p \in \mathcal{H}$, $p \neq 0$, the set $\{x \in A \mid (p,x) = s(p,A)\}$ is singleton.

Define the *distance* from the point $x \in \mathcal{H}$ to the set $A \subset \mathcal{H}$ as follows

$$\varrho(x,A) = \inf_{a\in A} \|x - a\|.$$

Suppose that $A \subset \mathcal{H}$ (or $A \subset \mathbb{R}^n$) is a closed convex bounded set with nonempty interior, $L \subset \mathcal{H}$ (or $L \subset \mathbb{R}^n$) is an arbitrary closed linear subspace, $L \neq \mathcal{H}$. Throughout this paper we shall consider the set-valued mapping

$$A \ni x \to F(x) = A \bigcap (L + x)\,. \tag{2}$$

The set-valued mapping (2) has valuable number of applications in set-valued analysis [10], [2, Theorems 2.6, 2.7], [11] and in control theory [9, Formula (19)]. Note that in the case $\mathbb{R}^n$ the mapping (2) is continuous for any subspace $L \subset \mathbb{R}^n$ if and only if the set A is a P-set, see [1, 2].

The question which is considered in the paper is to characterize the class of sets for which the set-valued mapping (2) has a given modulus of continuity for any closed subspace L.

We shall characterize strictly convex closed bounded sets with nonempty interior in $\mathcal{H}$ for which F is Hölder continuous with the power $\frac{1}{2}$ in the Hausdorff metric. The latter means that there exists a constant $C > 0$ (depending on A) such that for any closed subspace L and any points $x_1, x_2 \in A$ we have

$$h(F(x_1), F(x_2)) \leq C \cdot \|x_1 - x_2\|^{\frac{1}{2}}.$$

The set A should be strongly convex of some radius $R > 0$.

We also shall characterize convex sets for which F is uniformly continuous with the modulus of continuity w. The considered set should be uniformly convex with the modulus of convexity $\delta_A(t) \asymp w^{-1}(t)$, $t \to +0$ (w^{-1} is the inverse function for w). And if the set A is uniformly convex with the modulus of convexity δ_A, then $w(t) \asymp \delta_A^{-1}(t)$, $t \to +0$.

This work was stimulated by the nice result of D. W. Walkup and R. J.-B. Wets [14] about characterization convex sets $A \subset \mathbb{R}^n$ for which F is Lipschitz continuous in the Hausdorff metric, i.e. there exists a constant $C > 0$ (depending on A and L) such that for any points $x_1, x_2 \in A$ we have

$$h(F(x_1), F(x_2)) \le C \cdot \|x_1 - x_2\|.$$

In the appendix we shall give another proof (using duality in linear programming) of the fact that for every convex polyhedron from $\mathbb{R}^n$ the mapping F is Lipschitz continuous. We also consider one application of this fact.

The main results of the present paper establish the fact of duality between properties of convexity of the set and the properties of modulus of continuity for the mapping (2). I would like to dedicate these results to the memory of Jean Jacques Moreau – one of the pioneers in the field of duality concepts.

2. Holder continuity with the power $\frac{1}{2}$ in $\mathcal{H}$

Theorem 2.1. *Let $A \subset \mathcal{H}$ be a closed strictly convex bounded set with nonempty interior. Then the set-valued mapping F (2) is Hölder continuous with the power $\frac{1}{2}$ for any closed subspace $L \subset \mathcal{H}$ if and only if the set A is a strongly convex set of some radius $R > 0$.*

Proof. If the set A is strongly convex of radius R, then the modulus of convexity for the set A satisfies the inequality

$$\delta_A(t) \ge \delta_{B_R(0)}(t) \ge \frac{t^2}{8R}, \quad \forall t \in [0, \operatorname{diam} A).$$

Further, it follows from [3, Theorem 3.1] (see also [4, §2, Lemma 1]), that the modulus of continuity for F is of the order $\delta_A^{-1}(t) \le \sqrt{8Rt}$. Thus, F satisfies the Hölder continuity condition with the power $\frac{1}{2}$.

Vice versa, let the mapping F be Hölder continuous with the power $\frac{1}{2}$. Then, by Theorem 3.2, $\delta_A(t) \asymp t^2$, $t \to +0$. Following A. Weber and G. Reißig [15] this means that the set A is strongly convex. □

3. Intersection with a uniformly convex set

Proposition 3.1 ([5, Remark, p. 58], (The Kadec theorem)). *Let $\varepsilon = \varepsilon_X(\delta)$ (X — a real Banach space) be the least upper bound of the diameters* *of sets cut off from the unit sphere $S(X)$ by hyperplanes determined by norm one functionals at distance $1-\delta$ from the origin, $\delta \in (0,1)$. Then the function inverse to $\varepsilon_X(\delta)$ is the modulus of convexity $\delta_X(\varepsilon)$ of X.*

Let B be a ball (not necessary Euclidean) in $\mathcal{H}$ and define a particular norm as $\|\cdot\|_B$. Let $A \subset \mathcal{H}$ be a closed convex bounded set. We define the modulus

$$\delta_A^B(\varepsilon) = \sup\left\{\delta \geq 0 \;\Big|\; \frac{x_1+x_2}{2} + \delta B \subset A,\ \forall x_1, x_2 \in A:\ \|x_1 - x_2\|_B = \varepsilon\right\}.$$

The function δ_A^B strictly increases (if it is strictly larger than zero), see [3, Corollary 2.1].

Theorem 3.2. *Let $A \subset \mathcal{H}$ be a closed strictly convex bounded subset with $B_r(0) \subset A \subset B_R(0)$. Suppose that the set-valued mapping F has the modulus of continuity $w(t)$ ($0 = w(0) = \lim_{\tau\to+0} w(\tau)$ and w is an increasing function) for any subspace $L \subset \mathcal{H}$ of codimension 1. Then the set A is uniformly convex with the modulus $\delta_A(\tau) \geq 2rh^{-1}\left(\frac{\tau}{2R}\right)$, where $h(\tau) = \frac{4}{r}w\left(\frac{4R^2}{r}\tau\right)$.*

Note that for any closed convex set A there exists a constant $C_A > 0$ such that $\delta_A(\tau) \leq C_A\tau^2$, see [3]. Hence Theorem 3.2 gives nontrivial estimate for the modulus of convexity from below.

Proof. Let $B = A + (-A)$, $B_{2r}(0) \subset B \subset B_{2R}(0)$. Let $\|\cdot\|$ be the Euclidean norm in $\mathcal{H}$ and $\|\cdot\|_B$ be another norm in $\mathcal{H}$ with the unit ball B. It is easy to see that for all $x \in \mathcal{H}$ we have

$$\frac{1}{2R}\|x\| \leq \|x\|_B \leq \frac{1}{r}\|x\|. \tag{3}$$

Choose $\varepsilon \in (0,r)$, $p \in \partial B^\circ$ and define the sets

$$H_p^0 = \{x \in \mathcal{H} \mid (p,x)_B = s(p,A)_B\}, \qquad H_p^\varepsilon = \{x \in \mathcal{H} \mid (p,x)_B = s(p,A)_B - \varepsilon\}.$$

Here $(p,x)_B$ is the value of the functional p at the point x, $s(p,A)_B = \sup_{x\in A}(p,x)_B$.

Choose $L = \{x \in \mathcal{H} \mid (p,x)_B = 0\}$.

Let $\{x_1\} = A \cap H_p^0$, $\{x_2\} = [0,x_1] \cap H_p^\varepsilon$, $t = \|x_1 - x_2\|$.

Let $l = \{\lambda a \mid \lambda \in \mathbb{R}\}$ be such a one dimensional subspace, $\|a\| = 1$, that $h := |(a, x_1 - x_2)|$ is the Euclidean distance between the planes H_p^0 and H_p^ε. Put $z = l \cap H_p^\varepsilon$.

From the similarity we see that $\frac{t}{\|x_2\|} = \frac{h}{\|z\|}$ and $\frac{t}{h} = \frac{\|x_2\|}{\|z\|} \leq \frac{R}{r}$. By Formula (3) we have

$$\frac{1}{2R}h \leq \varepsilon \leq \frac{1}{r}h,$$

i.e. $r\varepsilon \le h \le 2R\varepsilon$. Hence

$$t \le \frac{R}{r}h \le \frac{R}{r}2R\varepsilon = \frac{2R^2}{r}\varepsilon. \tag{4}$$

For any closed convex set $D \subset \mathcal{H}$, vector $q \in \partial B^\circ$ and $\alpha \ge 0$ define the sets

$$\Sigma_q^D(\alpha) = \{x \in \mathcal{H} \mid (q,x)_B \ge s(q,D)_B - \alpha\} \cap D,$$
$$S_q^D(\alpha) = \{x \in \mathcal{H} \mid (q,x)_B = s(q,D)_B - \alpha\} \cap D.$$

Let $a, b \in \Sigma_p^A(\varepsilon)$. This means that there exist points $x, y \in [x_1, x_2]$ such that $a \in F(x)$, $b \in F(y)$. Then, using that $F(x_1) = (x_1 + L) \cap A = \{x_1\}$, we get

$$\begin{aligned}\|a-b\| &\le \|a-x_1\| + \|x_1-b\| \le h(F(x_1),F(x)) + h(F(x_1),F(y))\\ &\le w(\|x-x_1\|) + w(\|y-x_1\|) \le 2w(\|x_1-x_2\|).\end{aligned}$$

Thus $\operatorname{diam}\Sigma_p^A(\varepsilon) \le 2w(t)$. In the same way $\operatorname{diam}\Sigma_p^{-A}(\varepsilon) \le 2w(t)$. By Formula (3) we have

$$\operatorname{diam}\Sigma_p^{\pm A}(\varepsilon) \ge r\operatorname{diam}_B \Sigma_p^{\pm A}(\varepsilon),$$

and

$$\sup_{\|p\|_{B^\circ}=1} \operatorname{diam}_B \Sigma_p^{\pm A}(\varepsilon) \le \frac{2}{r}w(t). \tag{5}$$

By the inclusion

$$S_p^B\left(\frac{\varepsilon}{2}\right) \subset \Sigma_p^A(\varepsilon) + \Sigma_p^{-A}(\varepsilon)$$

and using (5) we obtain that

$$\sup_{\|p\|_{B^\circ}=1} \operatorname{diam}_B S_p^B\left(\frac{\varepsilon}{2}\right) \le \frac{4}{r}w(t). \tag{6}$$

From Proposition 3.1 we have

$$\left(\delta_B^B\right)^{-1}\left(\frac{\varepsilon}{2}\right) = \sup_{\|p\|_{B^\circ}=1} \operatorname{diam}_B S_p^B\left(\frac{\varepsilon}{2}\right)$$

and taking into account (4), (6) we have the next estimate

$$\left(\delta_B^B\right)^{-1}\left(\frac{\varepsilon}{2}\right) \le \frac{4}{r}w(t) \le \frac{4}{r}w\left(\frac{2R^2}{r}\varepsilon\right).$$

Taking in the last formula $\tau = \left(\delta_B^B\right)^{-1}\left(\frac{\varepsilon}{2}\right)$, or $\varepsilon = 2\delta_B^B(\tau)$ (by the Kadec theorem), we have the following inequality

$$\tau \le \frac{4}{r}w\left(\frac{4R^2}{r}\delta_B^B(\tau)\right), \tag{7}$$

i.e. $\delta_B^B(\tau) \geq h^{-1}(\tau)$. From the equality $A = \bigcap_{a\in A}(B+a)$ we obtain that $\delta_A^B(\tau) \geq \delta_B^B(\tau)$ and hence

$$\delta_A^B \geq \delta_B^B(\tau) \geq h^{-1}(\tau).$$

From the definition of δ_A^B we obtain that for any points $a_1, a_2 \in A$, $\|a_1 - a_2\|_B = \tau$, the next inclusion holds

$$\frac{a_1+a_2}{2} + \delta_A^B(\tau)B \subset A.$$

Then

$$\frac{a_1+a_2}{2} + 2r\delta_A^B(\tau)B_1(0) \subset A,$$

and $\|a_1 - a_2\| \leq 2R\tau$, see (3). Thus $\delta_A(2R\tau) \geq 2r\delta_A^B(\tau) \geq 2rh^{-1}(\tau)$, $\delta_A(\tau) \geq 2rh^{-1}\left(\frac{\tau}{2R}\right)$. □

In fact Theorem 3.2 is a generalization of the Kadec theorem from the case of the ball on the case of an arbitrary bounded convex closed set. Note that the above proof is still valid if we replace the Hilbert space $\mathcal{H}$ by an arbitrary uniformly convex space E (we did not use properties of Euclidean space in the proof).

It is interesting to mention that from the inequality $\delta_A(\tau) \leq C_A\tau^2$ we obtain that for uniformly convex set A the "uniform" modulus of continuity $w(\tau)$ from Theorem 3.2 is not better than of the order $\tau^{\frac{1}{2}}$.

Theorem 3.3. *Let $A \subset \mathcal{H}$ be a closed uniformly convex set with the modulus of convexity δ_A. Then the set-valued mapping F is uniformly continuous for any closed subspace $L \subset \mathcal{H}$ with the modulus of continuity w such that $w(t) \asymp (\delta_A)^{-1}(t)$, $t \to +0$.*

Proof. The proof follows from [3, Theorem 3.1]. □

What can we say about the modulus of continuity for F if the closed convex bounded set $A \subset \mathcal{H}$ is not uniformly convex? Unfortunately, not so much. The next two theorems slightly clarify the situation.

Theorem 3.4. *Let $A \subset \mathcal{H}$ be a closed convex bounded set. Let P be a 2-dimensional subspace and π be a linear projector on P. Suppose that there exist a number $\varepsilon > 0$ and a point x_0 from the relative boundary of the set $\pi A \subset P$ such that $B_\varepsilon(x_0) \cap \pi A$ is a strictly convex plane set (in P). Then there exists a subspace $L \subset \mathcal{H}$ of codimension 1 such that $F(x) = (x+L) \cap A$ is not better than Hölder continuous with the power $\frac{1}{2}$.*

Proof. Let Γ be the connected part of the relative boundary of the set $B_\varepsilon(x_0) \cap \pi A$, $x_0 \in \Gamma$ and the intersection $\Gamma \cap \partial B_\varepsilon(x_0)$ consist of two endpoints of Γ. By Stechkin's result [13] there exists a point a from the relative interior of the set $B_\varepsilon(x_0) \cap \pi A$ (a is sufficiently close to Γ and sufficiently far from the endpoints of

Γ) such that the distance from a to the relative boundary of the set $B_\varepsilon(x_0)\cap\pi A$ is realized at some point $z\in\Gamma$ (and z is not an endpoint of Γ).

Put $r=\|a-z\|>0$, $B_r(a)\cap P\subset B_\varepsilon(x_0)\cap\pi A$. Let l, $l\subset P$, be the supporting line at the point z to the set $B_\varepsilon(x_0)\cap\pi A$, $l_0=l-z$. The mapping

$$\pi A\ni x\to G(x)=(x+l_0)\cap\pi A$$

satisfies the condition (for $x_1=z$ and $x_2(t)=z+t\frac{a-z}{\|a-z\|}$, $t\in(0,r)$)

$$h(G(x_1),G(x_2(t)))\geq\sqrt{2rt-t^2}=\sqrt{2r\|x_1-x_2(t)\|-\|x_1-x_2(t)\|^2}.$$

Let $L=l_0+\ker\pi$. Then, the codimension of L is 1 and thus for any $x\in\pi A$ there exists $y\in A$ with $\pi y=x$ and

$$G(x)=\pi A\cap\pi(L+y).$$

The linear projector π is Lipschitz continuous, hence F (with L) not better than Hölder continuous with the power $\frac{1}{2}$. $\square$

In the same way one can prove another theorem.

Theorem 3.5. *Let $A\subset\mathcal{H}$ be a closed convex bounded set. Let P be a 2-dimensional affine subspace. Suppose that there exist a number $\varepsilon>0$ and a point x_0 from the relative boundary of the set $A\cap P$ such that $B_\varepsilon(x_0)\cap A\cap P$ is a strictly convex plane set (in P). Then there exists a subspace $L\subset\mathcal{H}$ of dimension 1 such that $F(x)=(x+L)\cap A$ is not better than Hölder continuous with the power $\frac{1}{2}$.*

Consider an example which shows essentiality of uniform (strict) convexity in the above results. Fix $p\in(1,2)$. On the Euclidean plane (x,y) consider the next set

$$A=\operatorname{cl}\operatorname{co}\left\{\{(\pm x_k,x_k^p)\}_{k=1}^\infty\cup\{(0,0)\}\right\},$$

where $x_1=1$, $x_k>0$, $x_{k+1}<x_k$ for all k and $\lim_{k\to\infty}x_k=0$, $\lim_{k\to\infty}\frac{x_k-x_{k+1}}{x_k^p}=0$.

If the subspace L does not coincide with $\{t(1,0)\mid t\in\mathbb{R}\}$, then $F(x)=(x+L)\cap A$ is Lipschitz (with the Lipschitz constant depending on L). This takes place because for the supporting to A line l, $l||L$, there exists $\varepsilon>0$ such that $(l+B_\varepsilon((0,0)))\cap A$ is a polyhedron (see Appendix).

If $L=\{t(1,0)\mid t\in\mathbb{R}\}$, then $F((0,0))=(0,0)$. If $m\in(x_{k+1}^p,x_k^p]$ for particular $k=k(m)\in\mathbb{N}$, then $F((0,m))=[(-y_k,m),(y_k,m)]$, $y_k\in(x_{k+1},x_k]$.

$$h(F((0,0)),F((0,m))\sim x_k,\quad m\to+0,$$

$$\|(0,0)-(0,m)\|=m\sim x_k^p,\quad m\to+0,$$

and $k=k(m)\to\infty$, $m\to+0$. So, F is Hölder continuous with the power $\frac{1}{p}\in(\frac{1}{2},1)$.

A. Appendix. Lipschitz continuity in $\mathbb{R}^n$ for convex polyhedra and the splitting problem for selections

Theorem A.1 (originally proved in [14, Theorem 1]). *Let $A \subset \mathbb{R}^n$ be a convex (bounded) polyhedron. Then for any closed subspace $L \subset \mathbb{R}^n$ the set-valued mapping F (2) is Lipschitz continuous if the set A is a polyhedron.*

Remark A.2. The boundedness of A is not necessary in the previous theorem, we use it for simplicity.

Proof. Without loss of generality we may assume that $0 \in \operatorname{int} A$. Otherwise we can consider the problem in the affine hull of the set A.

Suppose that the set A is a polyhedron. Then $A = \{x \in \mathbb{R}^n \mid (p_k, x) \leq \alpha_k,\ k = 1, \ldots, M\}$, $\alpha_k \geq 0$ for all k, and $L = \{x \in \mathbb{R}^n \mid (\pm q_k, x) \leq 0,\ k = 1, \ldots, K\}$, $\|p_k\| = \|q_j\| = 1$ for all k, j. Thus the Lipschitz continuity of F is equivalent to the Lipschitz continuity for the mapping

$$G(x_0) = \{x \in \mathbb{R}^n \mid (p_k, x) \leq \alpha_k - (p_k, x_0),\ k = 1, \ldots, M,\\ (\pm q_k, x) \leq 0,\ k = 1, \ldots, K\}$$

with respect to the point $x_0 \in A$. The last assertion follows from the formula $F(x_0) = x_0 + (A - x_0) \cap L$.

Further we shall consider the set-valued mapping

$$G(\alpha) = \{x \in \mathbb{R}^n \mid (p_k, x) \leq \alpha_k,\ k = 1, \ldots, N\},\\ \alpha = (\alpha_1, \ldots, \alpha_N), \quad \|p_k\| = 1,\ \forall k,$$

and prove that it is Lipschitz continuous on any set $S \subset \mathbb{R}^N$ such that for any $\alpha \in S$ we have $\alpha \geq 0$, $G(\alpha) \neq \emptyset$, and the set $\{G(\alpha) \mid \alpha \in S\}$ is bounded.

Fix parameters α, β from the set S. Fix $p \in \partial B_1(0)$. Suppose that $s(p, G(\beta)) \leq s(p, G(\alpha))$. It is well known from the duality in linear programming (see also [12, Corollary 17.1.6], [8, Corollary 1.14.3]) that

$$s(p, G(\beta)) = \inf \left\{ \sum_{k=1}^{N} \lambda_k \beta_k \;\middle|\; \lambda_k \geq 0,\ \sum_{k=1}^{N} \lambda_k p_k = p \right\}. \tag{8}$$

The problem of linear programming (8) has a solution at some extreme point of the set

$$R(p) = \left\{ \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N \;\middle|\; \lambda_k \geq 0,\ \sum_{k=1}^{N} \lambda_k p_k = p \right\}.$$

It is well known (see [8, Theorem 2.4.3]) that a point $\lambda \in R(p)$ is extreme for the set $R(p)$ if and only if we can separate the set of vectors $\{p_k\}_{k=1}^N$ on two subsets: $\{p_{k_m}\}_{m=1}^n$ and $\{p_k\}_{k=1}^N \backslash \{p_{k_m}\}_{m=1}^n$, with $\lambda_k = 0$, $k \in \{1, \ldots, N\} \backslash \{k_m\}_{m=1}^n$, vectors $\{p_{k_m}\}_{m=1}^n$ are linearly independent and $\sum_{m=1}^n \lambda_{k_m} p_{k_m} = p$ for $\lambda_{k_m} \geq 0$, $m = 1, \ldots, n$.

Consider all convex polyhedron cones of the form

$$\mathbb{K} = \mathrm{cone}\{p_{k_m}\}_{m=1}^n,$$

where vectors $\{p_{k_m}\}_{m=1}^n$ are linearly independent. We shall call such cones *sharp.*

For any sharp cone $\mathbb{K}$ the number

$$\sup \left\{ \|\lambda\|_{R^n} \;\middle|\; p \in \mathbb{K} \cap \partial B_1(0),\ \lambda_m \geq 0,\ \sum_{m=1}^n \lambda_m p_{k_m} = p \right\}$$

is finite, because the distance from the vertex of the cone $\mathbb{K}$ (zero) to the affine hull of the points $\{p_{k_m}\}_{m=1}^n$ is strictly positive. The number of all sharp cones is finite, hence

$$C = \sup \left\{ \|\lambda\|_{R^n} \;\middle|\; \mathbb{K} \text{ is sharp},\ p \in \mathbb{K} \cap B_1(0),\ \lambda_m \geq 0,\ \sum_{m=1}^n \lambda_m p_{k_m} = p \right\} < +\infty. \tag{9}$$

Let $s(p, G(\beta)) = \sum_{m=1}^n \lambda_m \beta_{k_m}$. Then $s(p, G(\alpha)) \leq \sum_{m=1}^n \lambda_m \alpha_{k_m}$ and, using (9), we have for any unit vector p

$$\begin{aligned} &s(p, G(\alpha)) - s(p, G(\beta)) \\ &\leq \sum_{m=1}^n \lambda_m (\alpha_{k_m} - \beta_{k_m}) \leq \sqrt{\sum_{m=1}^n \lambda_m^2} \sqrt{\sum_{m=1}^n (\alpha_{k_m} - \beta_{k_m})^2} \leq C \cdot \|\alpha - \beta\|. \end{aligned}$$

By Formula (1) we obtain the Lipschitz continuity of mappings G and hence F. □

The converse statement is also true. It was proved in [14] on the base of characterization for convex polyhedra by V. Klee [6, Theorems 4.1, 4.7].

Consider a variant of the splitting problem for selection [10, 2]. Let $A, B \subset \mathbb{R}^n$ be convex polyhedra and $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$: $L(x_1, x_2) = x_1 + x_2$. For any $c \in A + B$ put

$$L^{-1}(c) = \{(x_1, x_2) \mid L(x_1, x_2) = c\}.$$

Let $A \times B$ be the direct sum of A and B in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. This is also a polyhedron. Then by the previous theorem the set-valued mapping $F(c) = (A \times B) \cap L^{-1}(c)$ is Lipschitz continuous. Then taking the Steiner point $s(F(c))$ of the set $F(c)$ we get Lipschitz continuous functions

$$(a(c), b(c)) = s\,(F(c))$$

with $a(c) + b(c) = c$, for all $c \in A + B$.

Acknowledgements. The work was supported by grant RFBR 16-01-00259. The author is grateful to the referee for useful comments.

References

[1] M. V. Balashov: On the P-property of convex compacts, Math. Notes 71(3) (2002) 295–304.

[2] M. V. Balashov, D. Repovš: On the splitting problem for selections, J. Math. Anal. Appl. 355 (2009) 277–287.

[3] M. V. Balashov, D. Repovš: Uniform convexity and the spliting problem for selections, J. Math. Anal. Appl. 360(1) (2009) 307–316.

[4] V. I. Berdyšev: Continuity of a multivalued mapping connected with the problem of minimizing of a functional, Math. USSR, Izv. 16(3) (1981) 431–456.

[5] J. Diestel: Geometry of Banach Spaces. Selected Topics, Springer, Berlin (1975).

[6] V. Klee: Some characterizations of convex polyhedra, Acta Math. 102 (1959) 79–107.

[7] H. Frankowska, C. Olech: R-convexity of the integral of the set-valued functions, in: Contributions to Analysis and Geometry, D. N. Clark et al. (ed.), John Hopkins Univ. Press, Baltimore (1981) 117–129.

[8] E. S. Polovinkin, M. V. Balashov: Elements of Convex and Strongly Convex Analysis, 2nd Ed., Fizmatlit, Moscow (2007) (in Russian).

[9] L. S. Pontryagin, A. S. Mishchenko: The linear differential game of pursuit (analytic theory), Math. USSR, Sb. 59(1) (1988) 129–154.

[10] D. Repovš, P. V. Semenov: Sections of convex bodies and splitting problem for selections, J. Math. Anal. Appl. 334 (2007) 646–655.

[11] D. Repovš, P. V. Semenov: Continuous selections of multivalued mappings, in: Recent Progress in General Topology III (Prague, 2001), K. P. Hart, Jan van Mill, P. Simon (eds.), Atlantis Press, Amsterdam (2014) 711–749.

[12] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[13] S. B. Stechkin: Approximative properties of sets in linear normed spaces, Acad. Républ. Popul. Roum., Rev. Math. Pur. Appl. 8(1) (1963) 5–18.

[14] D. W. Walkup, R. J.-B. Wets: A Lipschitz characterization of convex polyhedra, Proc. Amer. Math. Soc. 23(1) (1969) 167–173.

[15] A. Weber, G. Reißig: Local characterization of strongly convex sets, J. Math. Anal. Appl. 400(2) (2012) 743–750.