

About Moreau-Yosida Regularization of the Minimal Time Crisis Problem*

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We study an optimal control problem where the cost functional to be minimized represents the so-called *time of crisis*, i.e. the time spent by a trajectory solution of a control system outside a given set K . This functional can be expressed using the characteristic function of K that is discontinuous preventing the use of the standard Maximum Principle. We consider a regularization scheme of the problem based on the Moreau-Yosida approximation of the indicator function of K . We prove the convergence of an optimal sequence for the approximated problem to an optimal solution of the original problem. We then investigate the convergence of the adjoint vector given by Pontryagin’s Principle when the regularization parameter goes to zero. Finally, we provide an example illustrating the convergence property and we compute explicitly an optimal feedback policy and the value function.

Keywords: Optimal control, Pontryagin maximum principle, hybrid maximum principle, regularization

1. Introduction

We consider the following optimal control problem with state constraints:

$$\begin{cases} \dot{x} = f(x, u) & \text{a.e. } t \in [0, T], \\ x(0) = x_0, \\ x(t) \in K \quad \forall t \in [0, T], \end{cases} \quad (1)$$

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where x is the state, u is the control, $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the dynamics, and K is a non-empty subset of $\mathbb{R}^n$. Such state constraints in optimal control problems appear naturally in many fields of applied science such as in robotics or bio-engineering. Optimal control problems under state constraints such as (1) have been extensively studied in the literature (see e.g. [5, 14, 15, 17, 24] and references therein). One essential feature is a so-called *inward pointing condition* on the velocity set:

$$f(x, U) \cap T_K(x) \neq \emptyset \quad \forall x \in \partial K, \quad (2)$$

where $U \subset \mathbb{R}^m$ is the admissible control set and $T_K(x)$ denotes the contingent cone to K at x . When this condition is not satisfied, one may use viability theory and study properties of the *viability kernel* of K , i.e. the largest set of initial conditions in K from which there exists a solution of problem (1) (see e.g. [1, 2, 3]).

When the set K is not viable by the dynamics f or when x_0 is not in the viability kernel of K , one may be interested in finding a control u for which the time spent by the associated trajectory outside the set K is minimal. This approach has been developed in [13] using viability theory and consists in minimizing with respect to (w.r.t., for short) the control u the so-called *time crisis function* defined by:

$$\int_0^\infty \mathbb{1}_{K^c}(x_u(t)) \, dt, \quad (3)$$

where $x_u(\cdot)$ is a solution of $\dot{x} = f(x, u)$ starting from K , and $\mathbb{1}_{K^c}$ denotes the characteristic function of the complement of K , i.e. $\mathbb{1}_{K^c}(x) = 1$ if $x \notin K$ and $\mathbb{1}_{K^c}(x) = 0$ if $x \in K$. The crisis is characterized by the set of times $t \geq 0$ for which the control system violates the state constraint, i.e. $x(t) \notin K$. This problem has been also considered in [6, 7] in the case where the state of the system is governed by a linear parabolic equation.

The objective of this work is to investigate necessary optimality conditions for the time crisis problem over a finite horizon $[0, T]$. As the characteristic function of the complement of the set K is discontinuous at the boundary of K , the usual Lipschitz regularity assumptions on the data are not satisfied. Therefore, the application of the Pontryagin Maximum Principle (PMP) to find an optimal control is not straightforward (see [10, 12, 23, 26]). A possible way to overcome this difficulty is to use the so-called *hybrid maximum principle* with the partition of $\mathbb{R}^n$ such that $\mathbb{R}^n = K \cup K^c$ (see e.g. [18, 19] and references therein). To avoid Zeno's phenomena, one usually requires a *transverse assumption* on a trajectory when crossing K (i.e. optimal trajectories cannot enter or leave K tangentially).

In this paper, we propose a regularization scheme of the time crisis problem using the Moreau envelope [20, 21, 4] of the indicator function of K assuming this set to be convex (see e.g. [6, 7] for different regularization approaches in the PDE setting, and [25] for a similar regularization scheme in the context of reflecting boundary control problems). As the regularized problem is smooth, we can apply the PMP in a standard way without assuming a transverse condition

on optimal trajectories when crossing K . We first prove that optimal solutions of the regularized problem converge (up to a sub-sequence) to an optimal solution of the original one. We then investigate the convergence of solutions of the adjoint system for the regularized problem when the regularization parameter goes to zero, assuming a transverse condition on optimal trajectories for the limiting problem. This allows to apply the Hybrid Maximum Principle (HMP) on the time crisis problem. We then prove the convergence (up to a sub-sequence) of solutions of the adjoint system to a solution of the adjoint system associated to the original problem.

The paper is organized as follows. In Section 2, we state the minimal time crisis problem, and we derive necessary optimality conditions using the HMP. In Section 3, we introduce a regularization scheme via the Moreau envelope of the indicator function of the set K , and we prove that (up to a sub-sequence) optimal solutions for the regularized problem converge to an optimal solution of the time crisis problem. In Section 4, we apply the PMP on the regularized problem, and we study the convergence of the adjoint vector associated to the regularized problem when the regularization parameter goes to zero. Theorem 4.8 is the main result of the paper and summarizes the aforementioned properties. We provide in the last section an example where we explicitly compute an optimal control and the value function for the time crisis problem. We illustrate numerically the convergence of the value function of the regularized problem to the value function of the time crisis problem.

2. Background and problem statement

2.1. Main assumptions and existence of an optimal control

We consider a dynamical controlled system:

$$\dot{x} = f(x, u), \quad (4)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the dynamics, x is the state, and u is the control. Given a terminal time $T > 0$ and a non-empty subset $U \subset \mathbb{R}^m$, we define $\mathcal{U}$ as the set of admissible control functions that are measurable w.r.t. t over $[0, T]$ and take values in U :

$$\mathcal{U} := \{u : [0, T] \rightarrow U; u \text{ meas.}\}.$$

We then define the extended velocity set associated to f as the set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by:

$$F(x) := \{f(x, u); u \in U\}.$$

We consider a non-empty subset $K \subset \mathbb{R}^n$, and we denote by $\text{Int}(K)$ its interior, ∂K its boundary, and K^c the complement of K in $\mathbb{R}^n$. Recall that the characteristic function of the set K^c is defined by:

$$\mathbb{1}_{K^c}(x) := \begin{cases} 0, & x \in K, \\ 1, & x \notin K. \end{cases}$$

We recall that $\mathbf{1}_{K^c}$ is lower semi-continuous (l.s.c., for short) whenever K is closed. The *minimal time crisis problem* over the finite horizon $[0, T]$ is the following optimal control problem (see [13]):

$$\inf_{u \in \mathcal{U}} J^T(u) \text{ with } J^T(u) := \int_0^T \mathbf{1}_{K^c}(x_u(t)) \, dt, \quad (\text{TC})$$

where x_u is a solution of the Cauchy problem

$$\begin{cases} \dot{x} = f(x, u), \\ x(0) = x_0, \end{cases} \quad (5)$$

defined over $[0, T]$, for given initial condition x_0 and control u . The functional J^T is called the *time of crisis function* over $[0, T]$. In other words, the objective consists in finding an optimal control $u \in \mathcal{U}$ such that the time spent by a solution of (4) outside the set K is minimal. Next, we make the following assumptions on the system:

- (H1) The set U is a non-empty compact convex set of $\mathbb{R}^m$.
- (H2) The dynamics f is continuous w.r.t. (x, u) , locally Lipschitz w.r.t. x and satisfies the linear growth condition: there exist $c_1 > 0$ and $c_2 > 0$ such that for all $x \in \mathbb{R}^n$ and all $u \in U$, one has:

$$\|f(x, u)\| \leq c_1 \|x\| + c_2. \quad (6)$$

- (H3) For any $x \in \mathbb{R}^n$, the set $F(x)$ is a non-empty convex set.

- (H4) The set K is a non-empty closed compact convex set of $\mathbb{R}^n$.

Under (H2), one can show that for any $x_0 \in \mathbb{R}^n$, there exists a unique solution x_u of the Cauchy problem (5). Following [13], one can easily verify that there exists an optimal control for problem (TC). For sake of completeness, we provide a proof of this result.

Proposition 2.1. *There exists an optimal control of problem (TC).*

Proof. Set $\alpha := \inf_{u \in \mathcal{U}} J^T(u)$ and let $u_n \in \mathcal{U}$ be a minimizing sequence, x_n the associated solution of (5), and $\alpha_n := J^T(u_n)$ so that one has $\alpha_n \rightarrow \alpha$ as n goes to infinity. Let us define the function $y_n : [0, T] \rightarrow \mathbb{R}$ by $y_n(t) := \int_0^t \mathbf{1}_{K^c}(x_n(s)) \, ds$. So one has $\dot{y}_n(t) = \mathbf{1}_{K^c}(x_n(t))$, a.e. $t \in [0, T]$ and $y_n(0) = 0$. Now, consider the set-valued map G from $\mathbb{R}^{n+1}$ into the subsets of $\mathbb{R}^{n+1}$ defined by:

$$G(z) := \begin{cases} f(x, U) \times \{0\} & \text{if } x \in \text{Int}(K), \\ f(x, U) \times [0, 1] & \text{if } x \in \partial K, \\ f(x, U) \times \{1\} & \text{if } x \in K^c, \end{cases}$$

where $z := (x, y) \in \mathbb{R}^n \times \mathbb{R}$. From (H1) and (H3), we have that $G(z)$ is a non-empty compact convex set of $\mathbb{R}^{n+1}$ for any $z \in \mathbb{R}^{n+1}$. Moreover, using

the compactness of U and the continuity of f , one can show that G is upper semi-continuous. Finally, for any $w \in G(z)$, one has

$$\|w\| \leq c_1\|x\| + c_2 + 1 \leq c_1\|z\| + c_2 + 1.$$

One has for every $n \in \mathbb{N}$, $\dot{z}_n \in G(z_n)$ a.e., where $z_n := (x_n, y_n)$. Hence, Theorem 1.11 in [12] implies that there exists a sub-sequence $z_i = (x_i, y_i)$ that converges uniformly over $[0, T]$ to a solution $z^* = (x^*, y^*)$ of $\dot{z}^*(t) \in G(z^*(t))$, a.e. $t \in [0, T]$, and whose derivatives converge weakly to $\dot{z}^*$ in $L^2([0, T])$. We then obtain $\dot{x}^*(t) \in F(x^*(t))$ for a.e. $t \in [0, T]$, i.e. x^* is a solution of (4), and moreover $y_i(T) = \int_0^T \mathbb{1}_{K^c}(x_i(t)) dt$ converges to $y^*(T)$. Let us denote by u^* a parametrization of the inclusion $\dot{x}^*(t) \in F(x^*(t))$ for a.e. $t \in [0, T]$, i.e. $\dot{x}^*(t) = f(x^*(t), u^*(t))$ for a.e. $t \in [0, T]$. We then obtain $\alpha \geq \int_0^T \mathbb{1}_{K^c}(x^*(t)) dt = J^T(u^*)$ by Fatou's Lemma and the l.s.c. of $\mathbb{1}_{K^c}$, thus $J^T(u^*) = \alpha$ as was to be proved. $\square$

Our aim is now to give necessary optimality conditions on optimal trajectories of (TC). As $\mathbb{1}_{K^c}$ is not continuous over ∂K , one cannot apply directly the PMP on (TC). However, both the dynamics and the cost are smooth whenever the trajectory is either in $\text{Int}(K)$ or in K^c . Therefore, we can treat problem (TC) as an hybrid control problem.

2.2. Hybrid Maximum Principle

We now state optimality conditions for the problem (TC) using the Hybrid Maximum Principle by considering the partition of $\mathbb{R}^n$ such that $\mathbb{R}^n = K \cup K^c$ (see e.g. [18, 19] and references therein). Let us underline that the hybrid formulation is used to treat the cost function $J^T(u)$ only and does not affect the dynamics f which is the same in K and K^c . Let us recall the notion of radial cone, tangent cone and normal cone. For $x \in K$, the radial cone $R_K(x)$, the tangent cone $T_K(x)$ and the normal cone $N_K(x)$ to K at x are defined respectively by:

$$\begin{aligned} R_K(x) &:= \{h \in \mathbb{R}^n; \exists \bar{\alpha} \text{ s.t. } \forall \alpha \in [0, \bar{\alpha}], x + \alpha h \in K\}, \\ T_K(x) &:= \overline{R_K(x)}, \\ N_K(x) &:= \{h^* \in \mathbb{R}^n; h^* \cdot (y - x) \leq 0, \text{ for all } y \in K\}, \end{aligned}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^n$.

We now apply the Hybrid Maximum Principle as in [11] to state optimality conditions. First, we define the notion of *regular crossing time* [19] adapted to our context.

Definition 2.2. We say that a time $t_c \in [0, T]$ is a regular crossing time from K into K^c for a solution x of (5) if the following properties hold at time $t = t_c$:

- (i) The point $x(t_c)$ is in ∂K , and there exists $\eta > 0$ such that for any $t \in [t_c - \eta, t_c)$ (resp. $t \in (t_c, t_c + \eta]$), one has $x(t) \in K$ (resp. $x(t) \in K^c$).

- (ii) The control u associated to the solution x is left- and right-continuous at t_c .
- (iii) The trajectory is transverse to K at $x(t_c)$, i.e. for any $h^* \in N_K(x(t_c))$ such that there exists $h \in T_K(x(t_c)) \setminus R_K(x(t_c))$ with $h^* \cdot h = 0$, then one has:

$$h^* \cdot f(x(t_c), u(t_c)) \neq 0, \quad (7)$$

where u is the control associated to the solution x .

Similarly, we define a regular crossing time t_c from K^c into K for a solution x when properties (i'), (ii) and (iii) hold, with

- (i') The point $x(t_c)$ is in ∂K , and there exists $\eta > 0$ such that for any $t \in [t_c - \eta, t_c]$ (resp. $t \in (t_c, t_c + \eta]$), one has $x(t) \in K^c$ (resp. $x(t) \in K$).

Remark 2.3. (i) The condition (7) means that a trajectory cannot hit K tangentially.

(ii) Let us fix $n_I \in \mathbb{N}$ and m functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^1 . Suppose that the set K is given by $K := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, 1 \leq i \leq n_I\}$. Then, (7) is equivalent to write $\nabla g_i(x(t_c)) \cdot f(x(t_c), u(t_c)) \neq 0$ for $1 \leq i \leq n_I$ such that $g_i(x(t_c)) = 0$.

We introduce the following assumption (called *transverse condition*) that is required to state the Hybrid Maximum Principle (HMP). It will also be used in Section 4.2.

- (H') An optimal trajectory of (TC) has no ($m = 0$) or a finite number $m \geq 1$ of regular crossing times $\{t_1, \dots, t_m\}$ over $[0, T]$.

The HMP provides the following necessary conditions. Let $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ the Hamiltonian associated to the system defined by:

$$H(x, p, p^0, u) := p \cdot f(x, u) + p^0 \ell(x, u).$$

Let $u \in \mathcal{U}$ be an optimal control, and suppose that the solution x of (5) satisfies (H'). Then, the following conditions are satisfied:

- There exists $p^0 \leq 0$ and a piece-wise absolutely continuous map $p : [0, T] \rightarrow \mathbb{R}^n$ called *adjoint vector* (row vector in $\mathbb{R}^n$) such that $(p^0, p(\cdot)) \neq (0, 0)$. Moreover, $p(\cdot)$ is absolutely continuous on each interval (t_i, t_{i+1}) ($i = 0, \dots, m$) where we posit $t_0 = 0$, $t_{m+1} = T$, and satisfies the *adjoint equation*:

$$\dot{p}(t) = -p(t) \cdot D_x f(x(t), u(t)) \quad \text{a.e. } t \in (t_i, t_{i+1}), \quad i = 0, \dots, m-1. \quad (8)$$

- The Hamiltonian satisfies the maximization condition:

$$u(t) \in \arg \max_{\alpha \in U} p(t) \cdot f(x(t), \alpha) \quad \text{a.e. } t \in [0, T]. \quad (9)$$

- At every regular crossing time $t_c \in \{t_1, \dots, t_m\}$, one has the jump condition

on the adjoint vector p :

$$\begin{aligned} \exists h^* \in N_K(x(t_c)), \\ p(t_c^+) = p(t_c^-) + \frac{p(t_c^-) \cdot (f(x(t_c), u(t_c^-)) - f(x(t_c), u(t_c^+))) + \sigma p_0}{h^* \cdot f(x(t_c), u(t_c^+))} h^*, \end{aligned} \quad (10)$$

where $\sigma = -1$, resp. $\sigma = +1$ if t_c is a regular crossing time from K into K^c , resp. from K^c into K .

- The adjoint vector satisfies the transversality condition

$$p(T) = 0, \quad (11)$$

as $x(T)$ is free.

As the pair $(p^0, p(\cdot))$ is non-null, we may suppose that $p^0 = -1$ using the transversality condition. We call *extremal trajectory* a triple $(x(\cdot), p(\cdot), u(\cdot))$ satisfying (5)–(8)–(9).

Remark 2.4. (i) Notice that (10) follows from the transversality condition in the HMP at the point $t = t_c$ [11] (Theorem 22.20). In fact, this condition implies the constancy of the Hamiltonian H along any extremal trajectory (using that t_c is free in $[0, T]$ and that the system is autonomous). Moreover, the jump condition on the adjoint vector $p(t_c^+) - p(t_c^-) \in N_K(x(t_c))$ follows also from [11]. The constancy of H and the jump condition then imply (10).

(ii) We can modify (H') by supposing that the trajectory can enter or leave K tangentially when the number of crossing times is finite (thus any crossing time is an isolated point excluding Zeno's phenomena). However, if a trajectory hits K tangentially at a time t_c , then (10) should be written $p(t_c^+) - p(t_c^-) \in N_K(x(t_c))$.

In order to state the HMP, one has to set the number of regular crossing times of an optimal trajectory. Therefore, it is convenient to introduce a regularized scheme for which we can derive optimality conditions in a standard way (see [23]), without any a priori knowledge on the number of crossing times. In particular, hypothesis (H') will not be required to state the PMP on the regularized problem.

3. Regularized problem and convergence property

In this section, we introduce the regularized scheme, and we show in Proposition 3.2 that, up to a sub-sequence, an optimal solution of the regularized problem converges to an optimal solution of (TC).

In the following, we denote by $d(\cdot, K)$ the distance function to the set K defined for $x \in \mathbb{R}^n$ by $d(x, K) := \inf_{y \in K} \|x - y\|$ and let ψ_K be the indicator function of the set K defined by:

$$\psi_K(x) := \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{cases}$$

Using the hypothesis that K is closed and convex, ψ_K is a convex and lower semi-continuous function. We can then consider the *Moreau envelope* $e_\varepsilon(\cdot)$ of ψ_K with the parameter ε , defined by (see [20, 21, 4]):

$$e_\varepsilon(x) := \frac{1}{2\varepsilon} d(x, K)^2.$$

As K is a closed convex subset of $\mathbb{R}^n$, $x \mapsto e_\varepsilon(x)$ is of class $C^{1,1}$ on $\mathbb{R}^n$ (see [22]) and for $x \in \mathbb{R}^n$, we have $\nabla e_\varepsilon(x) = \frac{1}{\varepsilon}(x - P_K(x))$, where $P_K : \mathbb{R}^n \rightarrow K$ is the projection onto the set K (P_K is well-defined as K is a closed convex subset of $\mathbb{R}^n$). When ε goes to zero, one has:

$$\lim_{\varepsilon \rightarrow 0} e_\varepsilon(x) = \psi_K(x),$$

for any $x \in \mathbb{R}^n$. Now, set $\gamma(v) := 1 - e^{-v}$ so that for any $x \in \mathbb{R}^n$, one has:

$$\mathbb{1}_{K^c}(x) = \gamma(\psi_K(x)).$$

We then consider the regularized optimal control problem:

$$\inf_{u \in \mathcal{U}} J_\varepsilon^T(u), \quad (\text{TC}_\varepsilon)$$

where J_ε^T is defined for $u \in \mathcal{U}$ by:

$$J_\varepsilon^T(u) := \int_0^T \gamma(e_\varepsilon(x_u(t))) \, dt,$$

and x_u is the unique solution of (5). The proof of existence of an optimal control for (TC_ε) is a standard routine using Filipov's Theorem [9]. In fact, the mapping $x \mapsto \gamma(e_\varepsilon(x))$ is continuous over $\mathbb{R}^n$, bounded by 1, and for any $x \in \mathbb{R}^n$, the set $f(x, U)$ is compact and convex. Hereafter, we denote by u_ε a minimizer of problem (TC_ε) and x_ε (in place of x_{u_ε}) the associated trajectory.

Lemma 3.1. *For any control $u \in \mathcal{U}$, one has $J_\varepsilon^T(u) \rightarrow J^T(u)$ as ε tends to zero.*

Proof. For any $t \in [0, T]$, one has $\gamma(e_\varepsilon(x_u(t))) \rightarrow \mathbb{1}_{K^c}(x_u(t))$. Moreover, the sequence $(\gamma(e_\varepsilon(x_u(\cdot))))_\varepsilon$ is uniformly bounded by 1. The result follows from Lebesgue's Theorem. $\square$

It follows that for each $\varepsilon > 0$, there exists a pair $(u_\varepsilon, x_\varepsilon)$ such that for any $u \in \mathcal{U}$:

$$J_\varepsilon^T(u_\varepsilon) \leq J_\varepsilon^T(u). \quad (12)$$

By using Theorem 1.11 in [12], we may assume that there exists a pair (x^*, u^*) with $u^* \in \mathcal{U}$ that is a solution of (5) and such that (up to a sub-sequence) $x_\varepsilon(\cdot)$ converges uniformly to $x^*(\cdot)$ over $[0, T]$ and $\dot{x}_\varepsilon(\cdot)$ converges weakly in $L^2([0, T])$ to $\dot{x}^*(\cdot)$ as ε goes to 0. We now show that x^* is a minimum of problem (TC).

Proposition 3.2. *The trajectory x^* is a minimum of problem (TC).*

The proof of Proposition 3.2 relies on Lemma 3.3 given below. Let $g : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}$ be defined by:

$$g(x, v) := \begin{cases} 0 & \text{if } x \in \text{Int}(K), \\ v & \text{if } x \in \partial K, \\ 1 & \text{if } x \notin K, \end{cases}$$

and let us consider two sequences (ε_i) and (λ_i) such that $\varepsilon_i > 0$, $\lambda_i > 0$, $\varepsilon_i \rightarrow 0$, $\lambda_i \rightarrow 0$, and $\frac{\lambda_i^2}{\varepsilon_i} \rightarrow +\infty$ as i goes to infinity.

Lemma 3.3. *Let $x(\cdot)$ be a solution of (4). Then, there exists three measurable functions $a_i : [0, T] \rightarrow \mathbb{R}^n$, $b_i : [0, T] \rightarrow \mathbb{R}$, and $v_i : [0, T] \rightarrow [0, 1]$ such that for any i , one has:*

$$\gamma(e_{\varepsilon_i}(x(t))) = g(x(t) + a_i(t), v_i(t)) + b_i(t) \quad \text{a.e. } t \in [0, T]. \quad (13)$$

Moreover, (a_i) and (b_i) converge to zero and satisfy the inequalities

$$0 \leq \|a_i(t)\| \leq \lambda_i \quad \text{and} \quad 0 \leq |b_i(t)| \leq e^{-\frac{\lambda_i^2}{2\varepsilon_i}} \quad \text{a.e. } t \in [0, T].$$

Proof. First, if t is such that $x(t) \in K$, then we take $a_i(t) := 0$, $b_i(t) := 0$, and $v_i(t) := 0$, and (13) is straightforward. We denote by $\bar{B}(0, \lambda_i)$ the closed ball of center 0 and radius λ_i , and we consider the set K_{λ_i} defined by:

$$K_{\lambda_i} := \{x \in \mathbb{R}^n; x = a + b, a \in K, b \in \bar{B}(0, \lambda_i)\}. \quad (14)$$

Suppose that $x(t) \notin K_{\lambda_i}$. We set $a_i(t) := 0$, $b_i(t) := -e^{-\frac{d(x(t), K)^2}{2\varepsilon_i}}$, and $v_i(t) := 0$. Hence, we have $\gamma\left(\frac{1}{2\varepsilon_i}d(x(t), K)^2\right) = \mathbb{1}_{K^c}(x(t)) - e^{-\frac{1}{2\varepsilon_i}d(x(t), K)^2} = g(x(t) + a_i(t), v_i(t)) + b_i(t)$ as in (13). Moreover, as $d(x(t), K) \geq \lambda_i$, we obtain that $0 \leq |b_i(t)| \leq e^{-\frac{\lambda_i^2}{2\varepsilon_i}}$. Now, suppose that $x(t) \in K_{\lambda_i} \setminus K$. Set $a_i(t) := P_K(x(t)) - x(t)$, $b_i(t) := 0$, and $v_i(t) := \gamma\left(\frac{d(x(t), K)^2}{2\varepsilon_i}\right)$. As we have $x(t) + a_i(t) \in \partial K$, we have $g(x(t) + a_i(t), v_i(t)) = v_i(t) = \gamma\left(\frac{d(x(t), K)^2}{2\varepsilon_i}\right)$ as was to be proved. Notice that in this case, one has $\|a_i(t)\| \leq \lambda_i$. To conclude, the sets $\{t \in [0, T]; x(t) \in \text{Int}(K)\}$ and $\{t \in [0, T]; x(t) \in K_{\lambda_i}\}$ are measurable as $x(\cdot)$ is absolutely continuous. This proves that the sequences a_i , b_i , and v_i are measurable. $\square$

Proof of Proposition 3.2. Let us set $u_i := u_{\varepsilon_i}$ so that we have $J_{\varepsilon_i}^T(u_i) \leq J_{\varepsilon_i}^T(u)$ for any $u \in \mathcal{U}$. Without any loss of generality, up to a sub-sequence, we may assume that $x_i \rightarrow x^*$ uniformly over $[0, T]$ and $\dot{x}_i \rightarrow \dot{x}^*$ as i goes to infinity. We can apply Lemma 3.3 with x_i , and we get:

$$\gamma\left(\frac{d(x(t), K)^2}{2\varepsilon_i}\right) = g(x_i(t) + a_i(t), v_i(t)) + b_i(t). \quad (15)$$

Notice that the bounds over a_i and b_i in Lemma 3.3 do not depend on the trajectory $x(\cdot)$. Let us set $\eta_i := \int_0^T b_i(t) dt$. As b_i is bounded and converges to zero, Lebesgue's Theorem implies that $\eta_i \rightarrow 0$, as i goes to infinity. Set $f_i(t) := g(x_i(t) + a_i(t), v_i(t))$ and consider the sets:

$$\begin{aligned} A &:= \{t; x^*(t) \in \text{Int}(K)\}, & B &:= \{t; x^*(t) \in \partial K\}, \\ C &:= \{t; x^*(t) \notin K\}. \end{aligned}$$

By integrating (15), we obtain:

$$J_{\varepsilon_i}^T(u_i) := \int_A f_i(t) dt + \int_B f_i(t) dt + \int_C f_i(t) dt + \eta_i.$$

First, suppose that $t \in A$. As (a_i) converges to zero and $x_i(t)$ goes to $x^*(t)$ as i goes to infinity, one has $f_i(t) = 0$ if i is large enough. Thus, $f_i(\cdot)$ converges to $\mathbb{1}_{K^c}(x^*(\cdot))$ over the set A . As f_i is bounded, we deduce that $\int_A f_i(t) dt$ goes to zero as i goes to infinity.

Now, suppose that $t \in C$. If i is large enough, we deduce that $x_i(t) + a_i(t) \notin K$, therefore $f_i(t) = 1$. Hence, $f_i(\cdot)$ converges to $\mathbb{1}_{K^c}(x^*(\cdot))$ over the set C . Therefore, one has $\int_C f_i(t) dt \rightarrow \int_C \mathbb{1}_{K^c}(x^*(t)) dt = J^T(u^*)$. Recall that we have $J_{\varepsilon_i}^T(u_i) \leq J_{\varepsilon_i}^T(u^*)$ and $\int_B f_i \geq 0$. It follows that:

$$\int_A f_i + \int_C f_i + \eta_i \leq J_{\varepsilon_i}^T(u_i) \leq J_{\varepsilon_i}^T(u^*). \quad (16)$$

When i goes to infinity, we obtain from (16) that $\lim_{i \rightarrow +\infty} J_{\varepsilon_i}^T(u_i) = J^T(u^*)$. To conclude that u^* is optimal, observe that for any $u \in \mathcal{U}$, one has:

$$J_{\varepsilon_i}^T(u_i) \leq J_{\varepsilon_i}^T(u),$$

which gives $J^T(u^*) \leq J^T(u)$ (by letting i go to infinity). This ends the proof. $\square$

Remark 3.4. Proposition 3.2 implies straightforwardly that for any $0 \leq t_0 \leq T$ and $x_0 \in \mathbb{R}^n$, the value function $V_\varepsilon(t_0, x_0)$ associated to the regularized problem converges point-wise to the value function $V(t_0, x_0)$ of the minimal time crisis problem.

4. Optimality conditions

4.1. Pontryagin Maximum Principle on the regularized problem

In this section, we study the application of the Pontryagin Maximum Principle to the regularized problem (TC_ε) . To do so, we will consider an auxiliary problem (TC'_ε) . Let us consider the admissible control set $\mathcal{V}$ defined by:

$$\mathcal{V} := \{v : [0, T] \rightarrow K; v \text{ meas.}\},$$

and let $\mathcal{W} := \mathcal{U} \times \mathcal{V}$. Next, we call $\mathbf{u} = (u, v)$ an element of $\mathcal{W}$. We introduce the augmented system:

$$\begin{cases} \dot{x} = f(x, u), \\ \dot{y} = \gamma \left(\frac{1}{2\varepsilon} \|x - v\|^2 \right), \end{cases} \quad (17)$$

with initial conditions $x(0) = x_0$, $y(0) = 0$, and $(u, v) \in \mathcal{W}$. We denote by $z_{\mathbf{u}} := (x_{\mathbf{u}}, y_{\mathbf{u}})$ the unique solution of (17) associated to the control $\mathbf{u}$ such that $x(0) = x_0$ and $y(0) = 0$. We consider the Mayer problem:

$$\inf_{\mathbf{u} \in \mathcal{W}} y_{\mathbf{u}}(T) \quad (\text{TC}'_{\varepsilon})$$

Lemma 4.1. *Problems $(\text{TC}_{\varepsilon})$ and $(\text{TC}'_{\varepsilon})$ are equivalent.*

Proof. First, suppose that (u_{ε}) is an optimal solution of $(\text{TC}_{\varepsilon})$, and let $x_{\varepsilon}(\cdot)$ be the associated solution of (5). Then, for any $u \in \mathcal{U}$, one has $J_{\varepsilon}^T(u_{\varepsilon}) \leq J_{\varepsilon}^T(u)$, therefore we deduce using the convexity of K that for any function $v : [0, T] \rightarrow K$ one has:

$$\int_0^T \gamma \left(\frac{1}{2\varepsilon} d(x_{\varepsilon}(t), K)^2 \right) dt \leq J_{\varepsilon}^T(u) \leq \int_0^T \gamma \left(\frac{1}{2\varepsilon} \|x(t) - v(t)\|^2 \right) dt,$$

where x is the solution of (5) associated to u . Hence, $(u_{\varepsilon}(\cdot), P_K(x_{\varepsilon}(\cdot))) \in \mathcal{W}$ is an optimal solution of $(\text{TC}'_{\varepsilon})$. Suppose now that we are given a solution $(u_{\varepsilon}, v_{\varepsilon}) \in \mathcal{W}$ of $(\text{TC}'_{\varepsilon})$, and let $x_{\varepsilon}(\cdot)$ be the associated trajectory solution of (5). If there exists a set $I \subset [0, T]$ of positive measure such that $v_{\varepsilon}(t) \neq P_K(x_{\varepsilon}(t))$ for any $t \in I$, then we have:

$$\|x_{\varepsilon}(t) - P_K(x_{\varepsilon}(t))\| < \|x_{\varepsilon}(t) - v_{\varepsilon}(t)\|, \quad \forall t \in I,$$

using the inclusion $v_{\varepsilon}(t) \in K$. As γ is strictly increasing, we obtain

$$\int_0^T \gamma \left(\frac{1}{2\varepsilon} \|x_{\varepsilon}(t) - P_K(x_{\varepsilon}(t))\|^2 \right) dt < \int_0^T \gamma \left(\frac{1}{2\varepsilon} \|x_{\varepsilon}(t) - v_{\varepsilon}(t)\|^2 \right) dt,$$

which is a contradiction with the optimality of $(u_{\varepsilon}, v_{\varepsilon})$. Therefore, one has $v_{\varepsilon}(t) = P_K(x_{\varepsilon}(t))$ for a.e. $t \in [0, T]$. By optimality of $(u_{\varepsilon}, v_{\varepsilon})$, we get:

$$J_{\varepsilon}^T(u_{\varepsilon}) \leq \int_0^T \gamma \left(\frac{1}{2\varepsilon} \|x(t) - v(t)\|^2 \right) dt,$$

where $u \in \mathcal{U}$, $v \in \mathcal{V}$ and x is the solution of (5) associated to u . Let us take $v(t) := P_K(x(t))$ for any $t \in [0, T]$. We then obtain $J_{\varepsilon}^T(u_{\varepsilon}) \leq J_{\varepsilon}^T(u)$ which proves that u_{ε} is a solution of $(\text{TC}_{\varepsilon})$. $\square$

As for $(\text{TC})_\varepsilon$, the proof of the existence of an optimal solution for problem (TC'_ε) is a straightforward routine. We are now in position to apply the Pontryagin Maximum Principle on (TC'_ε) . The Hamiltonian $H_\varepsilon : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ associated to (17) is defined by:

$$H_\varepsilon(x, y, p, q, u, v) := p \cdot f(x, u) + q\gamma \left(\frac{1}{2\varepsilon} \|x - v\|^2 \right).$$

Let an optimal control $\mathbf{u}_\varepsilon = (u_\varepsilon, v_\varepsilon) \in \mathcal{W}$ of (TC'_ε) be given, and let $z_\varepsilon = (x_\varepsilon, y_\varepsilon)$ the associated trajectory. Then, the following conditions are satisfied:

- There exists a constant $q_\varepsilon \leq 0$ and an absolutely continuous function $p_\varepsilon : [0, T] \rightarrow \mathbb{R}^n$ called adjoint (row) vector such that $(p_\varepsilon(t), q_\varepsilon) \neq (0, 0)$ for any t . Moreover, $p_\varepsilon(\cdot)$ satisfies the *adjoint equation*:

$$\begin{aligned} \dot{p}_\varepsilon(t) = & -p_\varepsilon(t) \cdot D_x f(x_\varepsilon(t), u_\varepsilon(t)) \\ & - \frac{q_\varepsilon}{\varepsilon} \gamma' \left(\frac{1}{2\varepsilon} \|x(t) - v_\varepsilon(t)\|^2 \right) (x_\varepsilon(t) - v_\varepsilon(t)) \quad \text{a.e. } t \in [0, T]. \end{aligned} \quad (18)$$

- We have the following maximization condition. For a.e. $t \in [0, T]$ one has:

$$\begin{cases} u_\varepsilon(t) \in \arg \max_{\alpha \in U} p_\varepsilon(t) \cdot f(x_\varepsilon(t), \alpha), \\ v_\varepsilon(t) \in \arg \max_{w \in K} q_\varepsilon \gamma \left(\frac{1}{2\varepsilon} \|x_\varepsilon(t) - w\|^2 \right). \end{cases} \quad (19)$$

- We have the transversality condition $p_\varepsilon(T) = 0$ and $q_\varepsilon = -1$.

We call *extremal trajectory* a triple $(z_\varepsilon, p_\varepsilon, u_\varepsilon)$ satisfying (5)–(18)–(19). In particular any optimal trajectory corresponds to a normal extremal trajectory (i.e. $q_\varepsilon \neq 0$). Notice that the transversality condition follows directly from the condition $(p_\varepsilon(\cdot), q_\varepsilon) \neq (0, 0)$ and the fact that $x_\varepsilon(T)$ and $y_\varepsilon(T)$ are free (see [12] p. 231). Using the convexity of K and (19), an extremal control v_ε necessarily satisfies:

$$v_\varepsilon(t) = P_K(x_\varepsilon(t)),$$

for a.e. $t \in [0, T]$. As the system is autonomous, the value of the Hamiltonian H_ε along any extremal trajectory is conserved. For $t = T$, one has $H_\varepsilon = -\gamma \left(\frac{1}{2\varepsilon} \|x_\varepsilon(T) - v_\varepsilon(T)\|^2 \right)$, hence we have $|H_\varepsilon| \leq 1$ for any $\varepsilon > 0$.

Remark 4.2. From a numerical point of view, the formulation (TC'_ε) can be useful when the quantities $d(x_\varepsilon(t), K)$ and $P_K(x_\varepsilon(t))$ are not explicit.

4.2. Convergence for the adjoint system

In this section, we suppose that hypotheses (H1)–(H4) are in force. We show that when the regularization parameter $\varepsilon > 0$ goes to zero, then an extremal of the regularized problem converges (up to a sub-sequence) to an extremal of the original problem (see Proposition 4.6 and Theorem 4.8).

For convenience, let us write (u_n, v_n) the solution of (TC'_ε) for $\varepsilon = \varepsilon_n$, x_n the unique solution of (5) for the control $u = u_n$, and p_n the unique solution of the Cauchy problem:

$$\begin{cases} \dot{p}_n(t) = -p_n(t) \cdot D_x f(x_n(t), u_n(t)) \\ \quad + \frac{1}{\varepsilon_n} \gamma' \left(\frac{1}{2\varepsilon_n} \|x_n(t) - v_n(t)\|^2 \right) (x_n(t) - v_n(t)) \text{ a.e. } t \in [0, T], \\ p_n(T) = 0. \end{cases} \quad (20)$$

The next lemma provides a uniform bound in $L^\infty([0, T])$ for the sequence p_n and is crucial to show that its convergence (up to a sub-sequence) to a solution of (8)–(9)–(10)–(11). Its proof is rather technical and follows arguments that are used in the more general context of the regularization of hybrid systems [19]. For sake of completeness, we provide in the Appendix a proof of Lemma 4.3 adapted to our context.

Lemma 4.3. *The sequence (p_n) is bounded in $L^\infty([0, T])$: there exists $C \geq 0$ such that for any $n \in \mathbb{N}$*

$$\|p_n\|_{L^\infty([0, T])} \leq C. \quad (21)$$

Remark 4.4. (i) The proof of Lemma 4.3 is based on needle variations and regularization of variation vectors as in the proof of the Hybrid Maximum Principle (see Appendix and [19]). It appears that (20) does not provide enough information on (p_n) in order to obtain (21).

(ii) Let us introduce the function $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\varphi_n(x) := \frac{x}{\varepsilon_n} e^{-\frac{x^2}{2\varepsilon_n}}$. Then, one can immediately check that $\int_0^{+\infty} \varphi_n(x) dx = 1$ and $\sup_{x \geq 0} \varphi_n(x) = \frac{1}{\sqrt{\varepsilon_n}}$ (achieved for $x_n := \sqrt{\varepsilon_n}$) for any $n \in \mathbb{N}$. Equation (20) can be written:

$$\dot{p}_n(t) = -p_n(t) \cdot D_x f(x_n(t), u_n(t)) + \varphi_n(\|x_n(t) - v_n(t)\|) \text{ a.e. } t \in [0, T].$$

Hence, we can expect that (p_n) is bounded (according to Lemma 4.3), but $(\dot{p}_n)$ may be unbounded.

Under hypotheses, (H1)–(H4), we know that there exists a solution x^* of (5) defined over $[0, T]$ such that (up to a sub-sequence) the sequence $x_n(\cdot)$ converges uniformly to $x^*(\cdot)$ over $[0, T]$ and $\dot{x}_n(\cdot)$ converges weakly in $L^2([0, T])$ to $\dot{x}^*(\cdot)$. Next, we assume that $x^*(\cdot)$ satisfies (H'), and we denote by $(t_k)_{1 \leq k \leq m}$ the regular crossing times of x^* (note that (t_{2l+1}) are crossing times from K into K^c). By convention, we set $t_0 = 0$ and $t_{m+1} = T$. Let $\eta^0 > 0$ be such that $\eta^0 < \min_{1 \leq k \leq n} \frac{1}{2}(t_{k+1} - t_k)$. For $0 < \eta < \eta_0$, we denote by I_η the set:

$$I_\eta := [0, t_1 - \eta] \bigcup_{1 \leq k \leq n-1} [t_k + \eta, t_{k+1} - \eta] \bigcup [t_n + \eta, T].$$

Lemma 4.5. *For any $\eta \in (0, \eta^0]$, there exists an absolutely continuous function $p_\eta^* : I_\eta \rightarrow \mathbb{R}^n$ such that up to a sub-sequence, $p_n(\cdot)$ uniformly converges to $p_\eta^*(\cdot)$*

over I_η and $\dot{p}_n(\cdot)$ converges weakly in $L^2(I_\eta)$ to $\dot{p}_\eta^*(\cdot)$. Moreover, p_η^* satisfies

$$\dot{p}_\eta^*(t) = -p_\eta^*(t) \cdot D_x f(x^*(t), u^*(t)) \quad \text{a.e. } t \in I_\eta. \quad (22)$$

Proof. Combining the uniform convergence of $x_n(\cdot)$, and the fact that each crossing time is transverse, there exists $N \in \mathbb{N}$ and $\rho_\eta > 0$ such that for any $n \geq N$ one has $d(x_n(t), \partial K) \geq \rho_\eta$ for any $t \in I_\eta$. Using that f is continuous, that the sequence $(x_n(\cdot), u_n(\cdot))$ is uniformly bounded over $[0, T]$, and that K is compact, there exists $A > 0$ such that for any $t \in [0, T]$ one has $\|D_x f(x_n(t), u_n(t))\| \leq A$ and $\|x_n(t) - P_K(x_n(t))\| \leq A$. Now, for $t \in I_\eta$ and $n \geq N$, one has:

$$\|\dot{p}_n(t)\| \leq A\|p_n(t)\|,$$

if $x_n(t) \in K$, and

$$\|\dot{p}_n(t)\| \leq A\|p_n(t)\| + \frac{A}{\varepsilon_n} \gamma' \left(\frac{1}{2\varepsilon_n} \rho_\eta \right),$$

if $x_n(t) \notin K$. The sequence (ξ_n) defined by $\xi_n := \frac{A}{\varepsilon_n} \gamma' \left(\frac{1}{2\varepsilon_n} \rho_\eta \right)$ clearly goes to zero as n goes to infinity. Therefore, we obtain for any $n \geq N$:

$$\|\dot{p}_n(t)\| \leq A\|p_n(t)\| + \xi_n \quad \text{a.e. } t \in I_\eta. \quad (23)$$

By Lemma 4.3, we conclude that the sequence (p_n) is uniformly Lipschitz continuous. By Arzelà-Ascoli Theorem, there exists a function $p_\eta^* : I_\eta \rightarrow \mathbb{R}^n$ such that up to a sub-sequence $p_n(\cdot)$ uniformly converges to $p^*(\cdot)$. As $\dot{p}_n(\cdot)$ is bounded in $L^2(I_\eta)$, the weak convergence is straightforward. Now (22) follows from Theorem 1.11 in [12] using that the mapping $t \mapsto r_n(t) := \frac{1}{\varepsilon_n} \gamma' \left(\frac{1}{2\varepsilon_n} \|x_n(t) - v_n(t)\|^2 \right)$ ($x_n(t) - v_n(t)$) uniformly converges to 0 over I_η . $\square$

Proposition 4.6. *There exists a function $p^* : [0, T] \setminus \{t_1, \dots, t_m\} \rightarrow \mathbb{R}^n$ satisfying the following properties:*

- (i) *Up to a sub-sequence, one has $p_n(t) \rightarrow p^*(t)$ a.e. $t \in [0, T]$ when n goes to infinity.*
- (ii) *The function p^* is (locally) absolutely continuous on each interval (t_i, t_{i+1}) , $0 \leq i \leq m$ (where $t_{m+1} = T$).*
- (iii) *The function p^* is bounded over $[0, T]$ and satisfies:*

$$\begin{cases} \dot{p}^*(t) = -p^*(t) \cdot D_x f(x^*(t), u^*(t)) & \text{a.e. } t \in [0, T], \\ p^*(T) = 0. \end{cases} \quad (24)$$

Proof. To prove (i), we argue first that if $\eta' < \eta \leq \eta^0$, then we have $p_{\eta'}^*(t) = p_\eta^*(t)$ for any $t \in I_\eta$. Indeed, Lemma 4.5 ensures the existence of a sequence $(p_{\varphi_1(n)})$ such that we have $p_{\varphi_1(n)} \rightarrow p_\eta^*$ uniformly over I_η and $p_{\varphi_1(n)} \rightharpoonup p_\eta^*$ weakly

in $L^2(I_\eta)$. Now, given $\eta' < \eta$, there exists a sub-sequence $(p_{\varphi_1 \circ \varphi_2(n)})$ such that we have $p_{\varphi_1 \circ \varphi_2(n)} \rightarrow p_\eta^*$ uniformly over $I_{\eta'}$ and $p_{\varphi_1 \circ \varphi_2(n)} \rightharpoonup p_\eta^*$ weakly. The result then follows using that $I_\eta \subset I_{\eta'}$. Consider now a decreasing sequence η_n toward 0 such that for all $n \in \mathbb{N}$ we have $\eta_n \in \mathbb{Q}$ and $\eta_n < \eta^0$. By repeating the application of Lemma 4.5 on each set I_{η_n} , we can consider an extraction $\varphi_n : \mathbb{N} \rightarrow \mathbb{N}$ and a diagonal sub-sequence $p_{\varphi_1 \circ \dots \circ \varphi_n(n)}(\cdot)$ which by construction converges uniformly to $p_{\eta_k}^*(\cdot)$ on each interval I_{η_k} , $k \in \mathbb{N}$. We then define p^* as the uniform limit of $(p_{\varphi_1 \circ \dots \circ \varphi_n(n)})$ on each interval I_{η_k} which concludes the proof.

Let us now prove (ii) and (iii). First, we write $p_n := p_{\varphi_1 \circ \dots \circ \varphi_n(n)}$. From Lemma 4.5, the function p^* coincides with p_η^* on I_η , therefore it satisfies (22) on each set I_η . We then obtain (24) using that $\eta > 0$ is arbitrary and $p_n(T) = 0$ for all n . Now, similarly as for (23), one obtains:

$$\| \dot{p}^*(t) \| \leq A \| p^*(t) \| \quad \text{a.e. } t \in [0, T]. \quad (25)$$

By Gronwall's Lemma, one obtains $\| p^*(t) \| \leq \| p^*(\tau_i) \| e^{AT}$ for any $1 \leq i \leq m-1$ and any $t \in (t_i, t_{i+1})$, where $\tau_i \in (t_i, t_{i+1})$. This shows that p^* is bounded over $[0, T]$. $\square$

Proposition 4.7. *Let t_c be a regular crossing time. The following properties hold true.*

(i) *Let (t_n^1) and (t_n^2) be two sequences such that for any n , one has $t_n^1 < t_c$ and $t_n^2 > t_c$. Suppose in addition that $t_n^1 \rightarrow t_c$ and $t_n^2 \rightarrow t_c$ as n goes to infinity. Then, we have:*

$$\lim_{n \rightarrow +\infty} (p_n(t_n^2) - p_n(t_n^1)) \in N_K(x^*(t_c)).$$

(ii) *Adjoint vectors $p^*(t_c^+)$ and $p^*(t_c^-)$ exist and satisfy:*

$$p^*(t_c^+) - p^*(t_c^-) \in N_K(x^*(t_c)). \quad (26)$$

(iii) *The function p^* satisfies (10) at each crossing time.*

Proof. Let us take $\eta < \eta_0$. As $x_n(\cdot)$ uniformly converges to x^* over $[0, T]$ as n goes to infinity, there exists $\tilde{t}_n \in [t_c - \eta, t_c + \eta]$ such that $x_n(\tilde{t}_n) \in \partial K$ (without any loss of generality, we can suppose that $\tilde{t}_n$ is the first time $t \in [t_c - \eta, t_c + \eta]$ such that $x_n(\tilde{t}_n) \in \partial K$). It follows that $\tilde{t}_n \rightarrow t_c$ as n goes to infinity. For $n \in \mathbb{N}$, let ζ_n be defined by

$$\zeta_n := \int_{t_n^1}^{t_n^2} -p_n(t) D_x f(x_n(t), u_n(t)) \, dt.$$

Lemma 4.3 implies that $\zeta_n \rightarrow 0$ as n goes to infinity. By integrating (20) over $[t_n^1, t_n^2]$, one has:

$$p_n(t_n^2) - p_n(t_n^1) = \zeta_n + \frac{1}{\varepsilon_n} \int_{t_n^1}^{t_n^2} \gamma' \left(\frac{1}{2\varepsilon_n} \|x_n(t) - v_n(t)\|^2 \right) (x_n(t) - v_n(t)) \, dt.$$

It follows that there exists $\hat{t}_n \in (t_n^1, t_n^2)$ such that

$$p_n(t_n^2) - p_n(t_n^1) = \zeta_n + \frac{t_n^2 - t_n^1}{\varepsilon_n} \gamma' \left(\frac{1}{2\varepsilon_n} \|x_n(\hat{t}_n) - v_n(\hat{t}_n)\|^2 \right) (x_n(\hat{t}_n) - v_n(\hat{t}_n)). \quad (27)$$

Let us set $\rho_n := \frac{t_n^2 - t_n^1}{\varepsilon_n} \gamma' \left(\frac{1}{2\varepsilon_n} \|x_n(\hat{t}_n) - v_n(\hat{t}_n)\|^2 \right) \|x_n(\hat{t}_n) - v_n(\hat{t}_n)\|$ and $w_n := \frac{x_n(\hat{t}_n) - v_n(\hat{t}_n)}{\|x_n(\hat{t}_n) - v_n(\hat{t}_n)\|}$ so that (27) can be written:

$$p_n(t_n^2) - p_n(t_n^1) = \zeta_n + \rho_n w_n.$$

The left member of the previous inequality is bounded, so the sequence $(\rho_n w_n)$ is bounded. As $w_n \in \mathcal{S}^{n-1}$ (where $\mathcal{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^{n-1}$), there exists $w \in \mathcal{S}^{n-1}$ such that (up to a sub-sequence) $w_n \rightarrow w$. By taking again a sub-sequence, we can assume that there exists $\rho \geq 0$ such that $\rho_n \rightarrow \rho$ as n goes to infinity. Recall now that $v_n(t) = P_K(x_n(t))$ for any t so that for any $y \in K$ one has:

$$w_n \cdot (y - P_K(x_n(\hat{t}_n))) \leq 0.$$

By letting n go to infinity, we find that $w \cdot (y - P_K(x^*(t_c))) \leq 0$ which shows that $w \in N_K(x^*(t_1))$. The result follows.

To prove (ii), we know from the previous proposition that $\dot{p}^*$ is bounded over $[0, T] \setminus \{t_1, \dots, t_m\}$. Therefore, there exists $C \geq 0$ such that for any $0 \leq i \leq m-1$, we have $\|p^*(t) - p^*(t')\| \leq C|t - t'|$ for any $(t, t') \in [t_i, t_{i+1}]$. Hence, p^* satisfies the Cauchy criterion at the point t_c^- , which proves that $\lim_{t \rightarrow t_c^-} p^*(t)$ exists. Similarly $\lim_{t \rightarrow t_c^+} p^*(t)$ exists and property (26) is fulfilled as n goes to infinity.

Finally, let us prove (iii). As the system is autonomous, the Hamiltonian H_n is constant along any extremal solution of $(\text{TC}'_{\varepsilon_n})$. Hence, we may assume (by taking a sub-sequence) that there exists $h \in \mathbb{R}$ such that $H_n \rightarrow h$ as n goes to infinity. Now, consider two times t, t' such that $t < t_c < t'$. If n is large enough, one has (using the uniform convergence of $x_n(\cdot)$ to $x^*(\cdot)$):

$$p_n(t) \cdot f(x_n(t), u_n(t)) = p_n(t') \cdot f(x_n(t'), u_n(t')) - \gamma(x_n(t')).$$

If we let $n \rightarrow \infty$, we find $p^*(t_c^-) \cdot f(x^*(t_c), u^*(t_c^-)) = p^*(t_c^+) \cdot f(x^*(t_c), u^*(t_c^+)) - 1$, and the result follows from (26). $\square$

The next Theorem is a rephrasing of Propositions 3.2, 4.6, and 4.7.

Theorem 4.8. *Suppose that (H1)–(H4) hold true. Let $(\varepsilon_n) \downarrow 0$, (x_n, u_n) a solution of $(\text{TC}'_{\varepsilon_n})$, and p_n the unique solution of (20). Then, up to a sub-sequence, the following properties are satisfied:*

- (i) *There exists a solution (x^*, u^*) of (TC) such that $x_n(\cdot)$ uniformly converges to $x^*(\cdot)$ over $[0, T]$, and $\dot{x}_n(\cdot)$ weakly converges to $\dot{x}^*(\cdot)$ in $L^2([0, T])$.*
- (ii) *The value function associated to $(\text{TC}'_{\varepsilon_n})$ converges point-wise on $[0, T] \times \mathbb{R}^n$ to the value function of (TC).*

- (iii) *If in addition x^* satisfies (H'), then there exists a function $p^* : [0, T] \setminus \{t_1, \dots, t_m\} \mapsto \mathbb{R}^n$ satisfying (8)–(9)–(10)–(11) such that $p_n(\cdot)$ converges to $p^*(\cdot)$ a.e. in $[0, T]$. This convergence is uniform on every compact set C of $[0, T]$ not containing a crossing time of x^* , and $\dot{p}_n(\cdot)$ weakly converges to $\dot{p}^*(\cdot)$ in $L^2(C)$.*

5. Illustration of an optimal synthesis for the time crisis problem

In this section, we consider an example for which we explicit an optimal control policy for both the original and the regularized problem. We also illustrate Proposition 3.2 (see also Remark 3.4) by depicting numerically the convergence of the value function of the regularized problem to the original one. Consider the planar dynamics:

$$\begin{cases} \dot{x}_1 = -x_2(2 + u), \\ \dot{x}_2 = x_1(2 + u), \end{cases} \quad (28)$$

where $u : [0, T] \rightarrow [-1, 1]$ is a measurable control function. We shall consider (28) on the compact domain defined as a disk:

$$\mathcal{D} := \{x \in \mathbb{R}^2; x_1^2 + x_2^2 \leq R^2\},$$

that is clearly invariant by (28), and the compact convex subset K is defined by:

$$K := \{x \in \mathcal{D}; x_2 \geq 0\}.$$

The minimal time crisis problem (from $t_0 < T$ to T) then becomes:

$$\inf_{u(\cdot)} \int_{t_0}^T \mathbf{1}_{\mathcal{D} \setminus K}(x_u(t)) \, dt, \quad (29)$$

where $x_u(\cdot)$ is a solution of (28) starting from $x_0 \in \mathcal{D}$ at time t_0 . Straightforwardly, trajectories lie on circles centered on the origin of radius r , whatever is the control $u(\cdot)$. The control u acts on the angular velocity of the trajectories. Therefore, system (28) can be equivalently written in polar coordinates (r, θ) with

$$\dot{\theta} = r(2 + u), \quad (30)$$

and $r := \|x_0\| = \sqrt{x_1^2 + x_2^2}$ constant.

Hereafter, the notation $u[\cdot]$ denotes a feedback control (depending on the state), while the notation $u(\cdot)$ denotes the open-loop control (as a function of time). Consider the feedback $\mathbf{u}_m[\cdot]$ that consists in minimizing the angular velocity when the state x belongs to K and on the contrary maximizing it outside:

$$\mathbf{u}_m[x] := \begin{cases} -1 & \text{if } x \in K, \\ 1 & \text{if } x \notin K, \end{cases} \quad (31)$$

that we shall call the *myopic* strategy. Given the angle $\theta_0 \in [0, 2\pi)$ of the initial condition x_0 and $T > 0$, let us now define $\theta_m(\cdot)$ as the unique solution of

$$\begin{cases} \dot{\theta}_m(t) = 2 + u_m(t) & \text{a.e. } t \in [0, T], \\ \theta(0) = \theta_0, \end{cases}$$

with $u_m(t) := \mathbf{u}_m[x(t)]$. We will prove hereafter that this intuitive strategy is optimal for (29) (but non-unique).

5.1. Study of the original problem (29)

In this part, we compute an optimal control for (29) using the Hybrid Maximum Principle. It will be convenient to introduce the two subsets A, B of $\mathbb{R}$ defined by:

$$A := \bigcup_{k \in \mathbb{Z}} [2k\pi, (2k+1)\pi] \quad \text{and} \quad B := \mathbb{R} \setminus A.$$

Hereafter, we suppose that $t_0 = 0$, $r = 1$. Let us fix $\theta_0 \in [0, 2\pi)$ and a measurable control $u : [0, T] \rightarrow [-1, 1]$, and denote by θ_u the unique solution of $\dot{\theta} = 2 + u(t)$ on $[0, T]$ such that $\theta(0) = \theta_0$.

Proposition 5.1. *Let u^* be an optimal control for (29) and x^* the unique solution of (28) starting from x_0 . Then, the following results hold:*

(1) *If $x^*(T) \in K$, then:*

$$u^*(t) = \begin{cases} v \in [-1, 1] & \text{if } \theta_{u^*}(t) \in A, \\ 1 & \text{if } \theta_{u^*}(t) \in B. \end{cases} \quad (32)$$

(2) *If $x^*(T) \in K^c$, then:*

$$u^*(t) = \begin{cases} -1 & \text{if } \theta_{u^*}(t) \in A, \\ v \in [-1, 1] & \text{if } \theta_{u^*}(t) \in B. \end{cases} \quad (33)$$

Proof. The Hamiltonian $H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ associated to the system is defined by:

$$H = H(\theta, \lambda, u) := \lambda(2 + u) - \mathbb{1}_B(\theta).$$

Let u be an optimal control and let us denote by θ_u the associated trajectory. According to the Hybrid Maximum Principle, there exists a piece-wise absolutely continuous function $\lambda : [0, T] \rightarrow \mathbb{R}$ satisfying $\lambda(T) = 0$ and the adjoint equation $\dot{\lambda}(t) = 0$ for a.e. $t \in [0, T]$. Moreover, we have the following maximization condition: $u = +1$, resp. $u = -1$ whenever $\lambda > 0$, resp. $\lambda < 0$. If $\lambda(t) = 0$, then one has $u(t) \in [-1, 1]$. From [11, Theorem 22.20], the Hamiltonian H is constant along any extremal trajectory. As we have $\dot{\theta} \geq 1$ for any time t , the number of crossing times (from K to K^c or from K^c to K) is finite and each

crossing time is transverse. Therefore, Hypothesis (H') is fulfilled. The Hybrid Maximum Principle then implies that λ has a discontinuity at a crossing time t_c and that $\lambda(t_c^+) - \lambda(t_c^-)$ is finite.

First case: suppose that $\theta_u(T) \in A$. Using $\lambda(T) = 0$, we obtain that $H = 0$. Whenever $\theta_u(t) \notin A$, we have $H = 0 = 2\lambda + |\lambda| - 1$, therefore, the only possibility is to have $\lambda = 1/3$. Now, if $\theta_u(t) \in A$, then $\lambda = 0$. We deduce that $u(t) = 1$ whenever $\theta_u(t) \in B$ and $u(t) \in [-1, 1]$ whenever $\theta_u(t) \in A$ which proves (1) and (32).

Second case: suppose that $\theta_u(T) \in B$. Using $\lambda(T) = 0$, we obtain that $H < 0$. Whenever $\theta_u(t) \in A$, one has $H = 2\lambda + |\lambda|$, hence the only possibility is to have $\lambda < 0$. We deduce that $\lambda = H < 0$ whenever $\theta_u(t) \in A$ and $u(t) = -1$. On the other hand, suppose that $\theta_u(t) \in B$. We then have $H = 2\lambda + |\lambda| - 1$ implying that λ is constant in B with $\lambda \geq 0$. Indeed, suppose that $\lambda < 0$ in B . As λ is constant in B , we then obtain $\lambda = H + 1 < 0$. Hence, we would have a contradiction with $\lambda(T) = 0$ as we should have $\lambda < 0$ for any time $t \in [0, T]$. Now, as $\lambda \geq 0$, we have two sub-cases depending if $\lambda = 0$ or not in B . Suppose that $\lambda = 0$ in B . We then have $u(t) \in [-1, 1]$ in B . If now $\lambda > 0$, then we have $u(t) = 1$ in B . This proves (2) and (33). $\square$

We then obtain the following optimality result for the myopic strategy.

Corollary 5.2. *Consider a control u satisfying (32) or (33) and such that $\theta_u(0) = \theta_0$. Then, one has:*

$$J^T(u) \geq J^T(u_m). \quad (34)$$

Therefore the myopic strategy (31) is optimal for any initial condition.

Proof. As $d\theta_m/dt \geq 1$ there is a one-to-one correspondence between time t and $\theta = \theta_m(t)$. One can then change t into θ and obtain straightforwardly

$$J^T(u_m) = \int_0^T \mathbb{1}_B(\theta_m(t)) \, dt = \frac{1}{3} \int_{\theta_0}^{\theta_m(T)} \mathbb{1}_B(\theta) \, d\theta.$$

Suppose first that $\theta_u(\cdot)$ ends in A . Thus, we are in case (1) of Proposition 5.1 and u is given by (32). Similarly as for u_m , we get $J^T(u) = \frac{1}{3} \int_{\theta_0}^{\theta_u(T)} \mathbb{1}_B(\theta) \, d\theta$. It follows that:

$$J^T(u) - J^T(u_m) = \frac{1}{3} \int_{\theta_m(T)}^{\theta_u(T)} \mathbb{1}_B(\theta) \, d\theta.$$

Now, it can be easily verified that $\theta_u(t) \geq \theta_m(t)$ for any $t \in [0, T]$ (as $\theta_u(0) = \theta_m(0) = \theta_0$ and when both trajectories are in A , resp. in B , one has $\dot{\theta}_u - \dot{\theta}_m \geq 0$, resp. $\dot{\theta}_u = \dot{\theta}_m = 3$). Thus we have $\theta_u(T) \geq \theta^*(T) \geq 0$ and the result follows.

Suppose now that $\theta_u(\cdot)$ ends in B . Thus, we are in case (2) of Proposition 5.1 and u is given by (33). As $t \mapsto \theta_u(t)$ is one-to-one from $[0, T]$ into $[\theta_0, \theta_u(T)]$,

we can change t into θ and we obtain:

$$\theta_m(t) - \theta_u(t) = \int_{t_0}^t (u_m(s) - u(s)) \, ds = \int_{\theta_0}^{\theta_u(t)} \frac{u_m(\theta_u^{-1}(\theta)) - u(\theta_u^{-1}(\theta))}{2 + u(\theta_u^{-1}(\theta))} \, d\theta.$$

Therefore, we deduce using (33) that:

$$\theta_m(t) - \theta_u(t) = \int_{\theta_0}^{\theta_u(t)} \mathbb{1}_B(\theta) \left(\frac{3}{2 + u(\theta_u^{-1}(\theta))} - 1 \right) \, d\theta.$$

It follows that the cost $J^T(u)$ can be written:

$$J^T(u) = \frac{1}{3} \left(\theta_m(T) - \theta_u(T) + \int_{\theta_0}^{\theta_u(T)} \mathbb{1}_B(\theta) \, d\theta \right).$$

We then obtain using $J^T(u_m) = \frac{1}{3} \int_{\theta_0}^{\theta_m(T)} \mathbb{1}_B(\theta) \, d\theta$:

$$J^T(u) = J^T(u_m) + \frac{1}{3} \left(\theta_m(T) - \theta_u(T) - \int_{\theta_u(T)}^{\theta_m(T)} \mathbb{1}_B(\theta) \, d\theta \right). \quad (35)$$

As in the first case, we can check that $\theta_u(t) \leq \theta_m(t)$ for any $t \in [0, T]$ (as $\theta_u(0) = \theta_m(0) = \theta_0$ and when both trajectories are in A , resp. in B , one has $\dot{\theta}_u = \dot{\theta}_m = -1$, resp. $\dot{\theta}_u - \dot{\theta}_m \leq 0$). It follows that $J^T(u) \geq J^T(u_m)$ which ends the proof. $\square$

Remark 5.3. Optimal trajectories for problem (29) are non-unique. If the trajectory ends in K^c or in $\text{Int}(K)$, then the cost is unchanged whatever is the control $u(\cdot)$ in a neighborhood of $t = T$. For instance, if $\theta_u(T)$ is close to $\theta_m(T) \in B$, then (35) implies straightforwardly that $J^T(u) = J^T(u_m)$ as $\mathbb{1}_B(\theta) = 1$ for $\theta \in B$. Nevertheless, the feedback control (31) is still optimal.

5.2. Study of the regularized problem

We use Pontryagin's Principle to tackle the regularized optimal control problem. As before, we shall consider without any loss of generality $r = 1$. In view of the definition of K , one has $d(x(t), K) = -\min(\sin \theta(t), 0)$, therefore the regularized control problem becomes:

$$\inf_{u(\cdot)} J_\varepsilon^T(u) := \int_0^T 1 - e^{-\frac{\min(\sin \theta_u(t), 0)^2}{2\varepsilon}} \, dt, \quad (36)$$

where $u : [0, T] \rightarrow [-1, 1]$ is a measurable control and θ_u is the unique solution of $\dot{\theta} = 2 + u(t)$ on $[0, T]$ such that $\theta(0) = \theta_0$ with $\theta_0 \in \mathbb{R}$. Given $\theta_1 \in (0, \pi/2]$, we define two subsets A_{θ_1} , B_{θ_1} of $\mathbb{R}$ as:

$$A_{\theta_1} := \bigcup_{k \in \mathbb{Z}} [2k\pi - \theta_1, (2k+1)\pi + \theta_1] \quad \text{and} \quad B_{\theta_1} := \mathbb{R} \setminus A_{\theta_1},$$

and the *stretched myopic* strategy as

$$u_{\theta_1}[\theta] := \begin{cases} -1 & \text{if } \theta \in A_{\theta_1}, \\ 1 & \text{if } \theta \in B_{\theta_1}. \end{cases} \quad (37)$$

The next proposition provides optimality conditions for the regularized control problem (36).

Proposition 5.4. *Let x_ε be an optimal solution for (36). Then, the following results hold:*

- (1) *If $x_\varepsilon(T) \in K$, then the optimal control satisfies the condition (32).*
- (2) *If $x_\varepsilon(T) \in K^c$, then the optimal control is given by the stretched myopic strategy (37) for a unique $\theta_1^\varepsilon \in (0, \pi/2]$.*

Proof. The Hamiltonian $H_\varepsilon : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ associated to the system can be written:

$$H_\varepsilon = H_\varepsilon(\theta, \lambda, u) = \lambda(2 + u) - \left(1 - e^{-\frac{\min(\sin \theta, 0)^2}{2\varepsilon}}\right),$$

where $\lambda \in \mathbb{R}$ is the adjoint vector. Let $u_\varepsilon(\cdot)$ an optimal control and $\theta_{u_\varepsilon}(\cdot)$ the unique solution of $\dot{\theta} = 2 + u_\varepsilon$ such that $\theta(0) = \theta_0$. Then, there exists an absolutely continuous function $\lambda_\varepsilon : [0, T] \rightarrow \mathbb{R}$ satisfying the adjoint equation:

$$\dot{\lambda}_\varepsilon(t) = \frac{1}{\varepsilon} e^{-\frac{\min(\sin \theta_{u_\varepsilon}(t), 0)^2}{2\varepsilon}} \min(\sin \theta_{u_\varepsilon}(t), 0) \cos \theta_{u_\varepsilon}(t) \quad \text{a.e. } t \in [0, T].$$

The maximization condition provides the following control law. One has $u_\varepsilon(t) = 1$ if $\lambda_\varepsilon(t) > 0$ and $u_\varepsilon(t) = -1$ if $\lambda_\varepsilon(t) < 0$. Moreover, if $\lambda_\varepsilon(t) = 0$, then $u_\varepsilon(t) \in [-1, 1]$. The function $t \mapsto \lambda_\varepsilon(t)$ plays the role of the switching function and the transversality condition implies $\lambda_\varepsilon(T) = 0$. Notice also that $\lambda_\varepsilon(\cdot)$ is increasing, resp. decreasing whenever $\theta_\varepsilon(\cdot)$ goes from π to $3\pi/2$ (mod. 2π), resp. from $3\pi/2$ to 2π (mod. 2π).

As the Hamiltonian is constant along an extremal trajectory, one has:

$$H_\varepsilon = 2\lambda_\varepsilon + |\lambda_\varepsilon| - \left(1 - e^{-\frac{\min(\sin \theta_\varepsilon, 0)^2}{2\varepsilon}}\right). \quad (38)$$

First case. Suppose that $\theta_{u_\varepsilon}(T) \in A$. Using that $\lambda_\varepsilon(T) = 0$, we obtain $H_\varepsilon = 0$. If for $t \in [0, T]$, one has $\lambda_\varepsilon(t) > 0$ or $\lambda_\varepsilon(t) < 0$ whenever $\theta_{u_\varepsilon}(t) \in A$, then we obtain a contradiction from (38) (as one has $H_\varepsilon = 0$). Thus, the only possibility is to have $\lambda_\varepsilon(t) = 0$ for any $t \in [0, T]$ such that $\theta_{u_\varepsilon}(t) \in A$. In this case, the optimal control $u_\varepsilon(t)$ can take any value in $[-1, 1]$. Similarly, we necessarily have $\lambda_\varepsilon(t) > 0$ for any $t \in [0, T]$ such that $\theta_{u_\varepsilon}(t) \notin A$, and thus $u(t) = 1$. This proves (1).

Second case. Suppose that $\theta_{u_\varepsilon}(T) \notin A$. Using (38) and the transversality condition, we have that $H_\varepsilon < 0$. Moreover, it is straightforward to check that

whenever $\theta_{u_\varepsilon}(t) \in A$, then $\lambda_\varepsilon(t)$ is constant and $\lambda_\varepsilon(t) < 0$ (otherwise we would have a contradiction with $H_\varepsilon < 0$). Therefore we have $u_\varepsilon(t) = -1$ for any time $t \in [0, T]$ such that $\theta_{u_\varepsilon}(t) \in A$ and $\lambda_\varepsilon(t) = H_\varepsilon$. At the terminal time we have $\lambda_\varepsilon(T) = 0$ and $\theta_{u_\varepsilon}(T) \notin A$, hence, $H_\varepsilon = -(1 - \exp(-\frac{\sin^2 \theta_\varepsilon(T)}{2\varepsilon}))$. This implies that $1 + H_\varepsilon > 0$. Thus, there exists a unique $\theta_1^\varepsilon \in (0, \pi/2]$ such that:

$$\sin(\pi + \theta_1^\varepsilon) := -\sqrt{-2\varepsilon \ln(1 + H_\varepsilon)}.$$

From the expression of the Hamiltonian, we have that H_ε is unchanged if θ is replaced by $\pi - \theta$. We deduce that for any $k \in \mathbb{Z}$, the function $t \mapsto \lambda_\varepsilon(t)$ admits exactly two zeros $\theta_1^\varepsilon + (2k+1)\pi < (2k+1)\pi + \pi/2 - \theta_1^\varepsilon$ on $[(2k+1)\pi, 2(k+1)\pi]$ (otherwise we would have $\lambda < 0$ for any time t and a contradiction with the transversality condition). At the terminal time, we necessarily have $\theta_{u_\varepsilon}(T) = \theta_1^\varepsilon + (2k+1)\pi$, $k \in \mathbb{Z}$ or $\theta_{u_\varepsilon}(T) = \pi/2 - \theta_1^\varepsilon + (2k'+1)\pi$, $k' \in \mathbb{Z}$. Using the monotonicity of λ_ε , we deduce that $u_\varepsilon(t) = -1$ whenever $\theta_{u_\varepsilon}(t) \in [(2k+1)\pi, \theta_1^\varepsilon + (2k+1)\pi)$, and $u_\varepsilon(t) = +1$ whenever $\theta_{u_\varepsilon}(t) \in (\theta_1^\varepsilon + (2k+1)\pi, (2k+1)\pi + \pi/2]$. By symmetry w.r.t. $3\pi/2$, we have the same property for u_ε whenever the trajectory is such that $\theta_{u_\varepsilon}(t) \in [3\pi/2 + 2k\pi, 2(k+1)\pi]$. This proves (2). $\square$

Remark 5.5. It is interesting to notice that the optimal controls in case (1) of Propositions 5.4 and 5.1 are the same.

In case (1) of Proposition 5.4, we can prove that the myopic strategy is optimal for (36).

Corollary 5.6. *Consider an initial condition with angle θ_0 and a control u_ε satisfying (32) with $\theta_{u_\varepsilon}(0) = \theta_0$. Then one has:*

$$J_\varepsilon^T(u_\varepsilon) \geq J_\varepsilon^T(u_m). \quad (39)$$

Therefore the myopic strategy (31) is optimal for such an initial condition.

Proof. As $\theta_{u_\varepsilon}(\cdot)$ ends in A , the control u_ε is given by (32) (see Proposition 5.4) and a similar computation as in the proof of Corollary 5.2 shows that

$$J_\varepsilon^T(u_\varepsilon) - J_\varepsilon^T(u_m) = \frac{1}{3} \int_{\theta_m(T)}^{\theta_{u_\varepsilon}(T)} \mathbb{1}_B(\theta) \left(1 - e^{-\frac{\sin^2 \theta}{2\varepsilon}}\right) d\theta.$$

Using that $\theta_m(t) \leq \theta_{u_\varepsilon}(t)$ for any time $t \in [0, T]$ (see Corollary 5.2), the previous equality implies straightforwardly (39). $\square$

Remark 5.7. In case (2) of Proposition 5.4, the structure of the optimal control (37) slightly differs with (33) as we can have $u_\varepsilon = -1$ in B . A natural question is to know if the feedback (37) converges to (31) as ε goes to zero, or equivalently if θ_1^ε goes to zero (recall that θ_1^ε necessarily converges as $\varepsilon \downarrow 0$ as $\theta_{u_\varepsilon}(\cdot)$ uniformly converges to an optimal solution of (29)). However, this is not true in general as optimal trajectories of (29) are non-unique (see Remark 5.3).

5.3. Numerical computation

In this section we compute the value function of problems (29) and (36) provided by (31) whenever the optimal trajectory for (29) ends in K . We illustrate numerically the convergence property of the value function (Proposition 3.2 and Remark 3.4).

We denote by $\mathcal{A}(x_0)$ the angle between a point $x_0 \in \mathbb{R}^2$ and the horizontal axis, by $E(x) \in \mathbb{Z}$, resp. $\text{Fr}(x)$ the integer part of a real x , resp. the fractional part of x such that $E(x) \leq x < E(x) + 1$ and $\text{Fr}(x) = x - E(x)$. For $x \in \mathbb{R}$, we write $x^- := \min(x, 0)$ and $x^+ := \max(x, 0)$.

Lemma 5.8. *The value function associated to criterion (29) on $\mathcal{D} \setminus \{0\}$ is given by the following expression:*

$$V(t_0, x_0) = W \left(t_0 + \pi - \mathcal{A}(x_0) - \frac{2[\pi - \mathcal{A}(x_0)]^-}{3} \right) + \frac{[\pi - \mathcal{A}(x_0)]^-}{3},$$

where the function $W(\cdot)$ is defined as follows

$$W(t) = \frac{\pi}{3} \left[E \left(\frac{3[T - t]^+}{4\pi} \right) + \min \left(4\text{Fr} \left(\frac{3[T - t]^+}{4\pi} \right), 1 \right) \right].$$

Proof. The value function associated to the criterion (29) can be first computed when $\mathcal{A}(x_0) = \pi$. In order to show that $V(t_0, x_0) = W(t_0)$, we notice that the number of times the trajectory turns around the circle of radius r (at angular velocity 3 in K^c and 1 in K) between $t = t_0$ and $t = T$ is exactly $E \left(\frac{3[T - t_0]^+}{4\pi} \right)$.

This gives the first part in the expression of W . Next, we obtain the second part in W by discussing if the trajectory ends in K or in K^c .

Now, suppose that $\mathcal{A}(x_0) \neq \pi$. Then, we obtain the expression of V as follows. When $0 < \mathcal{A}(x_0) < \pi$, resp. $\pi < \mathcal{A}(x_0) < 2\pi$, we have to change t_0 into $t_0 + \pi - \mathcal{A}(x_0)$, resp. t_0 into $t_0 - (\mathcal{A}(x_0) - \pi)$ to obtain the desired expression. $\square$

In order to compute the value function of (36) in case (1) of Proposition 5.4, we define the function

$$\zeta_\varepsilon(r, \theta) = \frac{1}{3} \int_0^\theta \left(1 - e^{-\frac{r^2 \sin^2(\pi+s)}{2\varepsilon}} \right) ds.$$

Lemma 5.9. *The value function on $\mathcal{D} \setminus \{0\}$ associated to criterion (36) in case 1 (for which the strategy (31) is optimal) is given by the following expression:*

$$\begin{aligned} & V_\varepsilon(t_0, x_0) \\ &= W_\varepsilon \left(\|x_0\|, t_0 + \pi - \mathcal{A}(x_0) - \frac{2[\pi - \mathcal{A}(x_0)]^-}{3} \right) - \zeta_\varepsilon(\|x_0\|, [\mathcal{A}(x_0) - \pi]^+), \end{aligned}$$

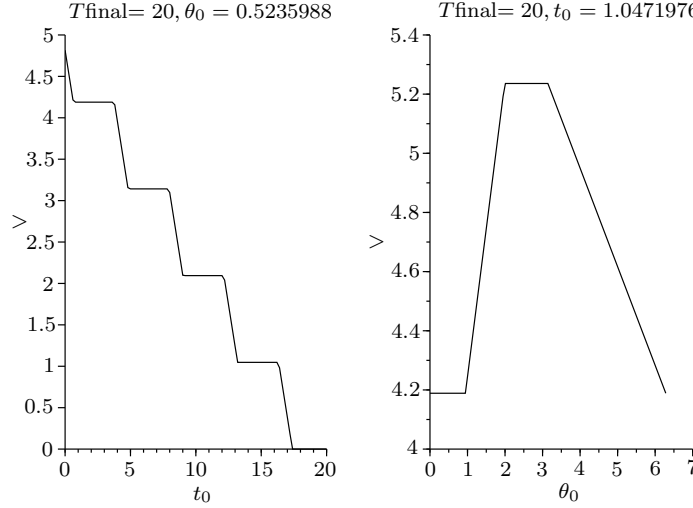


Figure 5.1: *Picture left:* value function as function of $t_0 \in [0, T]$. *Picture right:* value function as function of $\mathcal{A}(x_0) \in [0, 2\pi]$.

with

$$W_\varepsilon(r, t) = \mathbb{E} \left(\frac{3[T - t]^+}{4\pi} \right) \zeta_\varepsilon(r, \pi) + \zeta_\varepsilon \left(r, \min \left(4\text{Fr} \left(\frac{3[T - t]^+}{4\pi} \right), 1 \right) \pi \right)$$

Proof. The same argumentation than for the computation of the function V in the previous Lemma leads to the computation of V_ε . $\square$

Remark 5.10. Recall that when ε goes to zero, $\theta_{u_\varepsilon}(\cdot)$ uniformly converges to $\theta^*(\cdot)$ on $[0, T]$ (up to a sub-sequence). It follows from Corollary 5.6 that if $\theta^*(T) \in \text{Int}(A)$, an optimal control u_ε for problem (36) is given by (31) provided that ε is small enough.

Straightforwardly, one can check $\zeta_\varepsilon(r, \theta) \rightarrow \frac{\theta}{3}$ for any $\theta \in \mathbb{R}$ as $\varepsilon \downarrow 0$, hence, we verify that

$$V_\varepsilon(t_0, x_0) \rightarrow V(t_0, x_0),$$

as ε goes to zero as in Proposition 3.2. Notice also that given $0 \leq t_0 \leq T$, the mapping $\theta_0 \mapsto \tilde{V}(t_0, \theta_0) := V(t_0, x_0)$ and $\theta_0 \mapsto \tilde{V}_\varepsilon(t_0, \theta_0) := V_\varepsilon(t_0, x_0)$ (where $\theta_0 := \mathcal{A}(x_0)$) are periodic of period 2π .

Fig. 5.1 depicts the value function $V(t_0, x_0)$ as a function of $t_0 \in [0, T]$ for a fixed value of $\mathcal{A}(x_0)$ (picture left) and as a function of $\mathcal{A}(x_0)$ for a fixed value of $t_0 \in [0, T]$. Fig. 5.2 depicts the convergence of V_ε to V on $[t_0, T]$ for a fixed value of $\mathcal{A}(x_0)$ (picture left), and the convergence of V_ε to V when $\mathcal{A}(x_0) \in [0, 2\pi]$ for a fixed value of t_0 . On this picture the convergence toward the function V is observed whatever are initial t_0, θ_0 i.e. for both cases 1 and 2 (although V_ε is not the value function of the regularized problem in case 2).

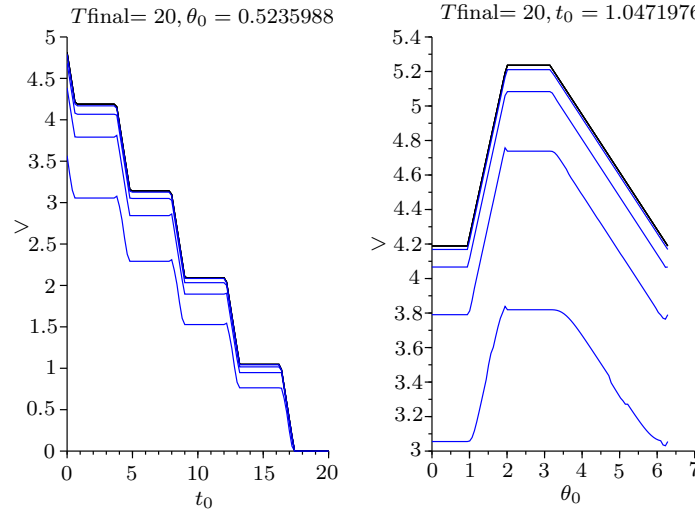


Figure 5.2: *Picture left:* convergence of V_ϵ to V for a fixed value of $\mathcal{A}(x_0)$. *Picture right:* convergence of V_ϵ to V for a fixed value of t_0 .

6. Conclusions and Perspectives

We have proposed a regularization scheme for the problem of minimizing a discontinuous functional given by the characteristic function of a convex set K . Our main result is that an extremal of the regularized problem converges (up to a sub-sequence) to an extremal of the original problem. This regularization allows the application of the Pontryagin's Maximum Principle, without requiring an apriori knowledge of the number of switching times for the original problem. Several points could deserve further studies. First, it could be interesting to study the case when K is no longer convex (for instance if K is the union of convex sets), and to address the question of the finite number of crossing times (to exclude Zeno's phenomena). One other objective would be to find a class of controlled systems (such as in [16]) for which we can determine an optimal feedback control for the time crisis model based on the *myopic* strategy which consists in minimizing the time spent outside K and maximizing it outside. These points are out of the scope of the paper.

7. Appendix

In this section, we prove Lemma 4.3 following the proof of Pontryagin's Principle (see e.g. [8, 19]). Let $\tau_1 \in [0, T]$ and $u_1 \in U$. The extended dynamics $\tilde{f}$ associated to the system is defined by $\tilde{f}(x, u) := (f(x, u), \gamma(x))$ and the *variation vector* $\tilde{v}_n = (v_n, v_n^0) : [0, T] \rightarrow \mathbb{R}^{n+1}$ for the regularized problem $(\text{TC}'_{\epsilon_n})$ as the unique solution of the Cauchy problem:

$$\begin{cases} \dot{\tilde{v}}_n(t) = \frac{\partial \tilde{f}}{\partial x}(x_n(t), u_n(t)) \tilde{v}_n(t) & \text{a.e. } t \in [0, T], \\ \tilde{v}_n(t_1) = \tilde{f}(x_n(t_1), u_1) - \tilde{f}(x_n(t_1), u_n(t_1)). \end{cases}$$

We then obtain for a.e. $t \in [0, T]$:

$$\begin{cases} \dot{v}_n(t) = \frac{\partial f}{\partial x}(x_n(t), u_n(t))v_n(t), \\ \dot{v}_n^0(t) = \nabla \gamma(x_n(t)) \cdot v_n(t), \end{cases} \quad (40)$$

together with the initial conditions $v_n(\tau_1) = f(x_n(\tau_1), u_1) - f(x_n(\tau_1), u_n(\tau_1))$ and $v_n^0(\tau_1) = 0$ (in fact recall that $\tilde{f}_{n+1}$ does not depend on the control). Using that $x_n(\cdot)$ is uniformly bounded over $[0, T]$ and the continuity of f , we easily obtain that the sequence $v_n(\cdot)$ is uniformly bounded over $[0, T]$ by a constant C . Let $A \geq 0$ be such that for all $n \in \mathbb{N}$, $\|x_n(t) - P_K(x_n(t))\| \leq A$ for any $t \in [0, T]$.

Let us now prove that $v_n^0(\cdot)$ is uniformly bounded over $[0, T]$. Suppose that t_n^1 is a crossing time for $x_n(\cdot)$ (i.e. $x_n(t_n^1) \in \partial K$) and that $t_n^1 \rightarrow t_1$ where t_1 is the first crossing time for $x^*(\cdot)$. As t_1 is an isolated crossing time for x^* , we may assume without any loss of generality that $x_n(t) \in K$ for $0 \leq t \leq t_n^1$ and $x_n(t) \notin K$ for $t > t_n^1$, t close to t_n^1 . Let $t \in [t_n^1, t_n^1 + \varepsilon_n]$. By integrating the second equation of (40) over $[t_n^1, t]$, we find that:

$$v_n^0(t) = \frac{1}{\varepsilon_n} \int_{t_n^1}^t e^{-\frac{\|x_n(t) - P_K(x_n(t))\|^2}{2\varepsilon_n}} (x_n(t) - P_K(x_n(t))) \cdot v_n(t) dt.$$

As $t \in [t_n^1, t_n^1 + \varepsilon_n]$, we find that $|v_n^0(t)| \leq AC$, and thus $v_n^0(\cdot)$ is bounded in a neighborhood of the first crossing time t_1 of x^* . We prove similarly that $v_n(\cdot)$ is bounded near each crossing time of x^* . Now, $v_n^0(\cdot)$ is also bounded on each compact interval I that does not contain a crossing time for x^* . Indeed, given such a time interval there exists $\gamma_I > 0$ such that for any $t \in I$ one has

$d(x_n(t), K) \geq \gamma_I$ if n is large enough. We then obtain $\dot{v}_n^0(t) \leq \frac{AC}{\varepsilon_n} e^{-\frac{\gamma_I^2}{2\varepsilon_n}}$ for a.e. $t \in I$ if n is large enough, and we obtain the result by integration.

To prove that $p_n(\cdot)$ is bounded, we consider the mapping $t \mapsto p_n(t) \cdot v_n(t)$. By differentiating w.r.t t , one obtains $\frac{d}{dt}(p_n(t) \cdot v_n(t)) = \nabla \gamma(x_n(t)) \cdot v_n(t) = v_n^0(t)$ for a.e. $t \in [0, T]$. By integrating and using that $p_n(T) = 0$, we find that:

$$p_n(t) \cdot v_n(t) = v_n^0(t) - v_n^0(T). \quad (41)$$

As $v_n^0(\cdot)$ is uniformly bounded over $[0, T]$, we obtain that the sequence $t \mapsto p_n(t) \cdot v_n(t)$ is uniformly bounded over $[0, T]$.

Now, for $t \in [0, T]$, the Pontryagin's cone $K(t) \subset \mathbb{R}^n$ is defined as the smallest cone that contains all variation vectors $v(t)$ (corresponding to needle-like variations of the control u_n) for all Lebesgue points $0 < \tau_1 < t$. Using that $(x_n(\cdot), p_n(\cdot), u(\cdot))$ is a normal extremal, Lemma 2.6 of [19] implies that $K(t) = \mathbb{R}^n$ for any $t \in [0, T]$. We deduce that $p_n(\cdot)$ is uniformly bounded. Otherwise, there would exist a sequence of time (t'_n) such that $\|p_n(t'_n)\| \rightarrow +\infty$. By taking a normalized variation vector that is co-linear to $p_n(t'_n)$, we would get a contradiction as the right member of (41) is uniformly bounded. This ends the proof.

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