

Some Remarks on Convex Analysis in Topological Groups*

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We discuss some key results from convex analysis in the setting of topological groups and monoids. These include separation theorems, Krein-Milman type theorems, and minimax theorems.

1. Introduction

1.1. Background

A topological group is a group which is also a topological space, such that the group operations are continuous. In this note we consider only commutative groups. Similarly, a topological monoid is a monoid (i.e., commutative semigroup with unit), which is also a topological space, such that the addition operation is continuous. In [2] the present authors proposed a natural convexity structure for groups and monoids that coincides with the classical notion when the underlying structure is a vector space. It is then natural to ask when known algebraic or topological results for vector spaces still hold true in a group or monoid. We should note that if a semigroup does not have a natural identity we simply add one.

Earlier related work is to be found in [2, 5, 11, 12, 9, 14, 15, 18], among other authors. It is appropriate to point out that Moreau [11, 12] studies the *infimal*

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convolution in a monoid. For extended real-valued functions f and g he defines the inf-convolution by

$$f \square g(x) = \inf_{y+z=x} [f(y) + g(z)], \quad (1)$$

and observes that for subadditive functions, the monoid provides the appropriate level of generality wherein to study infimal convolution.

Our own motivation is discussed in [2] where also various illustrative examples are given and which provided a variety of primarily algebraic results.

In this note we study two topological topics. First, we look at topological separation theorems and their consequences, including a group-theoretic version of the Krein-Milman theorem and Milman's converse theorem. To prove adequate separation theorems, we define and study a group version of the well-known gauge functional. We show that in many respects it behaves similarly to the case of locally convex topological vector spaces. (See Section 2.) Then in Section 3 we use the separation results to prove a version of the Krein-Milman theorem for locally convex topological groups. We note that some versions of the Krein-Milman theorem have also been studied in the case of topological monoids/lattice structures, e.g., in [15].

Second, we look at the classical minimax theorem. The first proof was due to Von-Neumann [13], and later generalisations and different proofs appeared in [1, 6, 17, 10, 4] and elsewhere. Herein we show that using the results of [6], one can easily deduce a satisfactory minimax theorem for appropriate topological monoids. (See Section 4.)

1.2. Locally convex topological groups basics

We begin with some basic definitions. For more about topological groups, see for example [7].

Definition 1.1 (Topological group). A group endowed with a topology is said to be a topological group if the group operations are continuous. That is, the function $(x, y) \mapsto x - y$ is continuous.

We also require a definition of a topological monoid.

Definition 1.2 (Topological monoid). A monoid which is also a topological space is said to be a topological monoid if the addition operation is continuous.

While the notion of convexity is usually studied in the context of vector spaces, it can be defined and studied in a very general setting. We refer the reader to [16] for more about abstract convexity, and to [2] for more about convexity in groups and monoids. In particular, given a space with a collection of convex sets, we can define the following notion.

Definition 1.3 (Locally convex topological space). A topological space is

said to be locally convex if its topology has a basis which contains only convex sets.

Remark 1.4. In Section 2.1, we define the notion of convexity in groups in monoids. However, as discussed in [16], convex sets can be defined in a completely axiomatic fashion.

In a topological group X we have that for every $x_0 \in X$, the map $x \mapsto x + x_0$ as well as its inverse $x \mapsto x - x_0$ are continuous (it suffices to assume only the latter). In particular, it follows that if U is a neighbourhood of $x \in X$, then by the continuity, $U - x$ is a neighbourhood of 0. Thus, any neighbourhood of x can be written as $x + U$, where U is a neighbourhood of 0. Note that this is *not* the case for arbitrary topological monoids. This is evident if we consider only the simple example $X = \mathbb{R}$ with the operation $x \wedge y$, that is taking the minimum.

It is known that if X is a topological group and the topology is Hausdorff, then singletons are closed sets. Indeed, in topological groups, the T_1 and Hausdorff properties are equivalent. In this note, for these and other reasons all topological groups will assumed to be Hausdorff.

By the maximum formula [2, Theorem 3], it follows that every finite convex function on a semidivisible group, as defined in Definition 2.2 below, is equal to the supremum over its additive minorants. Thus, locally convex topological groups admit ‘many’ additive functions. This is in contrast to the non-locally convex case, for example in the topological vector space $L_p([0, 1])$, $p \in (0, 1)$.

Proposition 1.5. *Assume that X is a locally convex, T_1 topological group. Then all singletons are convex, and no elements have finite order. In particular, the group has at most unique divisors.*

By considering the discrete topology, the previous result implies that in a locally convex group points are convex (resp. closed) iff the topology is Hausdorff.

Proof of Proposition 1.5. Let $x \in X$. If $y \neq x$ then since the topology is T_1 , there exists U open and convex, such that $x \in U$, $y \notin U$. Since $\text{conv}(\{x\}) \subseteq U$, the first assertion follows. To prove the second assertion, suppose that $x \neq 0$, and note that if x is of finite order, then there exists $m \in \mathbb{N}$ such that $mx = 0 = m \cdot 0$. Thus, $0 \in \text{conv}(\{x\})$, which can happen only if the set $\{x\}$ is not convex. $\square$

Also, recall the following definitions.

Definition 1.6 (Semidivisible monoid). Given $p \in \mathbb{N}$ prime, a monoid X is said to be p -semidivisible if $pX = X$. That is, for every $x \in X$ there exists $y \in X$ such that $x = py$. A monoid X is said to be semidivisible if it is p -semidivisible for some $p \in \mathbb{N}$ prime. A monoid is said to be divisible if it is p -semidivisible for every $p \in \mathbb{N}$ prime.

Note that the assumption in Definition 1.6 that p is prime is not necessary. However, if $m \in \mathbb{N}$ is such that $X = mX$ and $m = p_1^{n_1} \cdots p_k^{n_k}$ where $p_1, \dots, p_k \in \mathbb{N}$ are primes, then we have

$$X = mX = p_1 (p_1^{n_1-1} \cdots p_k^{n_k} X) \subseteq p_1 X \subseteq X.$$

Hence, $X = p_1 X$ and so X is p_1 -semidivisible. Similarly, X is p_i -semidivisible for $i = 1, \dots, k$. Also, note that if X is p -divisible and q -divisible then it is $p^n q^m$ -divisible for $m, n \in \mathbb{N}$.

Definition 1.7 (Uniquely divisible monoid). A monoid X is said to be uniquely divisible if it is divisible and for every $n \in \mathbb{N}$, the map $x \mapsto nx$ is injective.

It is known that torsion-free divisible abelian groups are modules over $\mathbb{Q}$ [2], and so the class of merely semidivisible monoids and groups is a much larger and potentially richer one. We recall the following examples of semidivisible groups. These examples appeared already in [2], but now they can be usefully considered in the context of locally convex topological groups.

Example 1.8 (σ -algebra with symmetric difference and a measure as a distance). Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, that is, Ω is a set, $\mathcal{F}$ is a σ -algebra of subsets of Ω , and μ is a positive measure on elements in $\mathcal{F}$. For $A, B \in \mathcal{F}$, define $A + B = A \triangle B$. Then it is known (see for example [2]) that under this operation, $A + B = B + A$, $A + \emptyset = \emptyset + A = A$ and $A + A = \emptyset$. Also, it is known that if $\mathcal{A} \subseteq \mathcal{F}$, then

$$\text{conv}(\mathcal{A}) = \left\{ A \subseteq X \mid A = \sum_{i=1}^n A_i, A_i \in \mathcal{A}, n \in \mathbb{N} \right\}. \quad (2)$$

Also, for $A, B \in \mathcal{F}$, define $d_\mu(A, B) = \mu(A \triangle B)$. Then it is known that $d_\mu(\cdot, \cdot)$ is a pseudo-metric on $\mathcal{F}$. Therefore, let $X = \mathcal{F} / \sim$, where $A \sim B \iff \mu(A \triangle B) = 0$. Assume that $A_n \rightarrow A$, $B_n \rightarrow B$ in X , that is $\mu(A_n \triangle A) \rightarrow 0$, $\mu(B_n \triangle B) \rightarrow 0$. Then we have

$$\mu((A_n \triangle B_n) \triangle (A \triangle B)) = \mu((A_n \triangle A) \triangle (B_n \triangle B)) \stackrel{(*)}{\leq} \mu(A_n \triangle A) + \mu(B_n \triangle B),$$

where in $(*)$ we used the fact that for every sets A, B , we have

$$\mu(A \triangle B) = \mu((A \cup B) \setminus (A \cap B)) \leq \mu(A \cup B) \leq \mu(A) + \mu(B).$$

Therefore, it follows that $\mu((A_n \triangle B_n) \triangle (A \triangle B)) \rightarrow 0$ and so $A_n + B_n \rightarrow A + B$. Since $B_n = -B_n$ and $B = -B$, we also have $A_n - B_n \rightarrow A - B$, which shows that X is indeed a topological group. On the other hand, in general X is not locally convex. To see this, consider the example where $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, that is, the Borell sets on $[0, 1]$ and μ is the Lebesgue measure. Then for $\varepsilon > 0$,

the set $\mathcal{A}_\varepsilon = \{A \in \mathcal{F} \mid \mu(A) < \varepsilon\}$ is a neighbourhood of $\emptyset$. However, by (2), it follows that if we choose $A_i = ((i-1)\varepsilon/2, i\varepsilon/2)$, $1 \leq i \leq n$, where $n = \lfloor \frac{\varepsilon}{2} \rfloor + 1$. Note that the sets A_i are disjoint. Thus, we have

$$A = \sum_{i=1}^n A_i \stackrel{(*)}{=} \bigcup_{i=1}^n A_i = [0, 1],$$

where in $(*)$ we used the fact that for disjoint sets $A_1, \dots, A_n$, we have $\sum_{i=1}^n A_i = \bigcup_{i=1}^n A_i$. In particular, we have $[0, 1] \in \text{conv}(\mathcal{A}_\varepsilon)$ but since $[0, 1] \notin \mathcal{A}_\varepsilon$, it follows that $\mathcal{A}_\varepsilon$ is not convex, and so X is not locally convex.

The group X is connected. Indeed, let $\mathcal{A} \subseteq X$ be the connected component that contains $\emptyset$. In particular, $\mathcal{A}$ is open. Taking

$$\mathcal{A}_\varepsilon = \{B \in \mathcal{F} \mid \mu(A \triangle B) < \varepsilon \text{ for some } A \in \mathcal{A}\},$$

since $d_\mu(\cdot, \cdot)$ is a distance on X , it follows that $\mathcal{A}_\varepsilon$ is open. Hence, we must have $\mathcal{A} = X$ and so X is connected. $\square$

Example 1.9 (Positive hyperbolic group). Let X be the commutative group of matrices of the form $M(\theta)$, where $\theta \in \mathbb{R}$, and

$$M(\theta) = \begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{bmatrix}.$$

This can be thought of the ‘positive’ branch of the group $X_{\mathbb{R}}$, as defined in [2]. See also Remark 1.10 below. The group operation is given by the matrix multiplication. It follows that we have $M(\theta_1) \cdot M(\theta_2) = M(\theta_1 + \theta_2)$. The topology on X is the topology induced by the euclidean metric in $\mathbb{R}^4$. The function $\theta \mapsto M(\theta)$ is continuous. Thus, the group X is connected. Note also that this group is divisible, as we have $M(\theta) = (M(\theta/n))^n$ for every $\theta \in \mathbb{R}$ and every $n \in \mathbb{N}$. X is also locally convex, since if $x = M(\theta) \in X$, let

$$U(\theta, \varepsilon) = \{M(\theta') \mid |\theta' - \theta| < \varepsilon\},$$

for some $\varepsilon > 0$. Then $U(\theta, \varepsilon)$ is an open and convex neighbourhood of x . To show that $U(\theta, \varepsilon)$ is open, let $\varepsilon' > 0$, $M(\theta') \in U(\theta, \varepsilon)$, and $M(\theta'') \in X$ such that

$$\text{dist}_{\mathbb{R}^4}(M(\theta'), M(\theta'')) \leq \varepsilon'. \quad (3)$$

We would like to show that if ε' is sufficiently small, $M(\theta'') \in U(\theta, \varepsilon)$. Indeed, (3) implies in particular that

$$|\sinh(\theta') - \sinh(\theta'')| \leq \varepsilon'. \quad (4)$$

and so we have If ε' is sufficiently small, then since $\sinh(\cdot)$ is continuous and injective, we have that $|\theta'' - \theta| < \varepsilon$. Altogether, we have $M(\theta'') \in U(\theta, \varepsilon)$ and so

$U(\theta, \varepsilon)$ is open. To show that $U(\theta, \varepsilon)$ is convex, let $M(\theta_1), \dots, M(\theta_n) \in U(\theta, \varepsilon)$, $m_1, \dots, m_n \in \mathbb{N}$ and assume

$$m(M(\theta)) = \sum_{i=1}^n m_i M(\theta_i), \quad m = \sum_{i=1}^n m_i. \quad (5)$$

Now, if we have that $M(\theta) = M(\theta')$ then by comparing all the entries of the two matrices, it follows that $\theta = \theta'$. Hence, (5) implies that $m\theta = \sum_{j=1}^n m_j \theta_j$. Since $\theta_1, \dots, \theta_n \in (\theta' - \varepsilon, \theta' + \varepsilon)$ and $(\theta' - \varepsilon, \theta' + \varepsilon)$ is a convex subset of $\mathbb{R}$, it follows that $\theta \in (\theta' - \varepsilon, \theta' + \varepsilon)$, which proves that $U(\theta, \varepsilon)$ is convex. Given any open neighbourhood U of $M(\theta)$, then again since the topology on X is the topology induced by the metric in $\mathbb{R}^4$ and since $\sinh(\cdot)$ and $\cosh(\cdot)$ are continuous, there exists $\varepsilon > 0$ such that $U(\theta, \varepsilon) \subseteq U$. This shows that the group X is locally convex. $\square$

Remark 1.10. If we consider the group

$$X_{\mathbb{R}} = \left\{ e^{it} M(\theta) \mid \theta, t \in \mathbb{R} \right\},$$

where $M(\theta)$ is as in Example 1.9. This is a group under the matrix multiplication. See [2] for the details. Let the topology on $X_{\mathbb{R}}$ be the topology induced by the euclidean metric on $\mathbb{C}^4$. Then $X_{\mathbb{R}}$ is connected as it is the image of the continuous map $(t, \theta) \mapsto e^{it} M(\theta)$. On the other hand, $X_{\mathbb{R}}$ is not locally convex. To see this, let U be an open neighbourhood of $M(0)$. Since the topology is the topology on $\mathbb{C}^4$, there must be $\varepsilon > 0$ such that $\{e^{it} \mid |t| < \varepsilon\} \subseteq U$. However, we have

$$\text{conv}(\{e^{it} \mid |t| < \varepsilon\}) = \{e^{it} \mid t \in [0, 2\pi)\}. \quad (6)$$

See [2] for a more detailed study of convexity in the circle group. In particular, (6) implies that there is no U' convex and open such that $M(0) \in U'$ and $U' \subseteq U$. Hence $X_{\mathbb{R}}$ is not locally convex. $\square$

2. Separation theorems in groups and monoids

2.1. Convexity in groups and monoids

For the sake of completeness, we present some basic facts that appeared in [2]. First, we define convex sets in monoids.

Definition 2.1 (Convex set). Let X be a monoid and $A \subseteq X$. A is said to be convex, if for every $x_1, \dots, x_n \in A$, every $m_1, \dots, m_n \in \mathbb{N} = \{1, 2, \dots\}$ such that $\sum_{i=1}^n m_i = m$, we have

$$mx = \sum_{i=1}^n m_i x_i \implies x \in A.$$

Next, we define some classes of functions on monoids.

Definition 2.2 (Convex and concave functions). Let X be a monoid. A function $f : X \rightarrow [-\infty, \infty]$ is said to be convex if for every $x_1, \dots, x_n \in X$, $m_1, \dots, m_n \in \mathbb{N}$ such that $m = \sum_{i=1}^n m_i$ and $mx = \sum_{i=1}^n m_i x_i$, we have

$$mf(x) \leq \sum_{i=1}^n m_i f(x_i).$$

A function $f : X \rightarrow [-\infty, \infty]$ is said to be concave if the function $-f$ is convex.

Here and in what follows, we let $\infty - \infty = +\infty$ when considering a convex function and $\infty - \infty = -\infty$ when considering a concave function.

Definition 2.3 (Generalised affine functions). Let X be a monoid. A function $f : X \rightarrow [-\infty, \infty]$ is said to be affine if it is both convex and concave.

Definition 2.4 (Subadditive functions). Let X be a monoid. A function $f : X \rightarrow [-\infty, \infty]$ is said to be subadditive if for every $x, y \in X$, we have

$$f(x + y) \leq f(x) + f(y).$$

Definition 2.5 ($\mathbb{N}$ -sublinear functions). Let X be a monoid. A function $f : X \rightarrow [-\infty, \infty]$ is said to be $\mathbb{N}$ -sublinear if it is subadditive and in addition it is positively homogeneous, i.e.,

$$f(kx) = kf(x), \quad k \in \mathbb{N} \cup \{0\}, \quad x \in X.$$

Note that all the classes of functions defined above can be defined when X is a group rather than a monoid. More generally, the above classes can be defined when X , as well as the range are semimodules. For the sake of concreteness, we do not include the most general case. See [2] for the definitions in generality. We now study some of the properties of the inf-convolution, defined in (1).

Proposition 2.6 (Inf-convolution of subadditive functions, Moreau).

Let X be a monoid and assume that $f, g : X \rightarrow [-\infty, \infty]$ are subadditive. Then $f \square g$ is subadditive.

Proof. Let $x_1, x_2 \in X$. Assume that we can write $x_1 = y_1 + z_1$, $x_2 = y_2 + z_2$. This is always possible since we can choose one of the elements to be 0. Thus we have $x_1 + x_2 = (y_1 + y_2) + (z_1 + z_2)$, and so

$$\begin{aligned} f \square g(x_1 + x_2) &\leq f(y_1 + y_2) + g(z_1 + z_2) \\ &\leq [f(y_1) + g(z_1)] + [f(y_2) + g(z_2)]. \end{aligned} \tag{7}$$

Taking the infimum over the right side of (7), the result follows. $\square$

Under the assumption that X is semidivisible, we can obtain much stronger convexity results regarding $f \square g$.

Proposition 2.7 (Inf-convolution of convex and $\mathbb{N}$ -sublinear functions).

Let X be a p -semidivisible monoid and assume that $f, g : X \rightarrow [-\infty, \infty]$ are convex (resp. $\mathbb{N}$ -sublinear). Suppose, moreover, that X has at most unique divisors as holds in the locally convex case. Then $f \square g$ is convex (resp. $\mathbb{N}$ -sublinear).

Proof. Assume first that f and g are convex. Let $x, x_1, \dots, x_n \in X$ and $m_1, \dots, m_n \in \mathbb{N}$ such that $p^k x = \sum_{i=1}^n m_i x_i$ and $p^k = \sum_{i=1}^n m_i$, where $k \in \mathbb{N}$. Let $y_1, \dots, y_n, z_1, \dots, z_n \in X$ be such that $x_i = y_i + z_i$, $1 \leq i \leq n$. Since X is p -semidivisible, there exist $y, z \in X$ such that $p^k y = \sum_{i=1}^n m_i y_i$ and $p^k z = \sum_{i=1}^n m_i z_i$. Thus, we have $p^k x = p^k(y + z)$. Since X is uniquely divisible, we have $x = y + z$. Thus,

$$\begin{aligned} p^k f \square g(x) &\leq p^k f(y) + p^k g(z) \\ &\stackrel{(*)}{\leq} \sum_{i=1}^n m_i f(y_i) + \sum_{i=1}^n m_i g(z_i) \\ &= \sum_{i=1}^n m_i [f(y_i) + g(z_i)], \end{aligned} \tag{8}$$

where in $(*)$ we used the convexity of f and g . Taking the infimum over the right side of (8), it follows that

$$p^k f \square g(x) \leq \sum_{i=1}^n m_i f \square g(x_i).$$

Now, use [2, Proposition 6] to deduce that $f \square g$ is convex. To prove the $\mathbb{N}$ -sublinear case, using Proposition 2.6 it is enough to prove that $f \square g$ is positively homogeneous. Assume that $px = y' + z'$. Then since X is p -semidivisible, there exist $y, z \in X$ such that $py = y'$ and $pz = z'$. This means that $px = p(y + z)$. Since X is uniquely divisible, it follows that $x = y + z$. Therefore, we have,

$$\begin{aligned} f \square g(px) &= \inf_{y' + z' = px} [f(y') + g(z')] \\ &= \inf_{py + pz = px} [f(py) + g(pz)] \\ &= p \inf_{py + pz = px} [f(y) + g(z)] \\ &= p \inf_{y + z = x} [f(y) + g(z)]. \end{aligned}$$

Now, using [2, Proposition 7], the result follows. □

We turn to the study of the gauge function.

2.2. Rational dilation of sets and the Minkowski functional

Given a set $A \subseteq X$ and $m \in \mathbb{N}$, let

$$\begin{aligned} mA &= \left\{ x \in X \mid x = \sum_{i=1}^m x_i, x_i \in A \right\} \\ &= \left\{ x \in X \mid x = \sum_{i=1}^n m_i x_i, x_i \in A, \sum_{i=1}^n m_i = m \right\}. \end{aligned} \quad (9)$$

In the case of convex sets, we can generalise (9) in the following way.

Definition 2.8 (Rational dilation of set). Let X be a monoid and $C \subseteq X$ be a convex set. Also, let $q \in \mathbb{Q}_+$ be a reduced fraction. Define

$$qC = \left\{ x \in X \mid lx = \sum_{i=1}^n m_i x_i, x_i \in C, \sum_{i=1}^n m_i = m, \frac{m}{l} = q \right\}.$$

Remark 2.9. Note that if $q = k \in \mathbb{N}$ then Definition 2.8 coincides with (9). Also, note that if C is not convex, we do not necessarily have $1C = C$. $\square$

Remark 2.10. If C is convex, then we have $1C = C$. $\square$

We have the following proposition.

Proposition 2.11 (Dilations of convex sets are monotone). Assume that X is a monoid and $C \subseteq X$ is convex and $0 \in C$. Assume that $q_1, q_2 \in \mathbb{Q}_+$ are reduced fractions with $q_1 \leq q_2$. Then $q_1 C \subseteq q_2 C$.

Proof. Write $q_1 = \frac{m}{l}$ and $q_2 = \frac{m'}{l'}$. Since $q_1 \leq q_2$, we have that $ml' \leq m'l$. Assume that $x \in q_1 C$. Then we can write $lx = \sum_{i=1}^n m_i x_i, x_i \in C, \sum_{i=1}^n m_i = m$. Thus, it follows that $l'l'x = \sum_{i=1}^n m_i l' x_i = \sum_{i=1}^n m_i l' x_i + (m'l - ml') \cdot 0$. Now, we have $\sum_{i=1}^n m_i l' = ml'$, and so $\sum_{i=1}^n m_i l' + (m'l - ml') = m'l$. Since $\frac{m'l}{l'l'} = \frac{m'}{l'}$, we have $x \in \frac{m'}{l'} C$, which completes the proof. $\square$

The first step in our proof of the Hahn-Banach separation theorem requires us to construct a group version of the Minkowski functional. For this we need the following result, which is an immediate consequence of Proposition 2.11.

Corollary 2.12. Assume that X is a monoid, $C \subseteq X$ is convex and $0 \in C$, and let $x \in C$. Then the set $\{q \in \mathbb{Q}_+ \mid x \in qC\}$ is of the form $[\lambda, \infty) \cap \mathbb{Q}_+$ or $(\lambda, \infty) \cap \mathbb{Q}_+$, where $\lambda \in \mathbb{R}_+$.

Using Corollary 2.12, it is natural to define the following.

Definition 2.13 (Minkowski functional for groups). Let X be a monoid and $C \subseteq X$. Define

$$\rho_C(x) = \inf \{q \in \mathbb{Q}_+ \mid x \in qC\}. \quad (10)$$

If there is no $q \in \mathbb{Q}_+$ that satisfies (10), define $\rho_C(x) = \infty$.

Remark 2.14. Note that Proposition 2.11, and consequently Corollary 2.12 and Definition 2.13, are purely algebraic, and do not require any topological structure. $\square$

Remark 2.15. By Definition 2.13, if $x \in C$ then $\rho_C(x) \leq 1$, and if $x \notin C$, then $\rho_C(x) \geq 1$. $\square$

Proposition 2.16 (Sublinearity of the Minkowski functional). *Assume that X is a monoid, and $C \subseteq X$ is convex. Then the functional defined by (10) is $\mathbb{N}$ -sublinear.*

Proof. We start by showing that ρ_C is subadditive. Indeed, let $x, y \in X$, and choose $m, m', l, l' \in \mathbb{N}$ such that $\frac{m}{l} < \rho_C(x) + \varepsilon$, $\frac{m'}{l'} \leq \rho_C(y) + \varepsilon$, and $lx = \sum_{i=1}^n m_i c_i$, $c_i \in C$, $\sum_{i=1}^n m_i = m$ and $l'y = \sum_{i=1}^{n'} m'_i c'_i$, $c'_i \in C$, $\sum_{i=1}^{n'} m'_i = m'$. Thus, we have $ll'(x + y) = \sum_{i=1}^n l'm_i c_i + \sum_{i=1}^{n'} lm'_i c'_i$, and by (10) we have

$$\begin{aligned} \rho_C(x + y) &\leq \frac{1}{ll'} \left(\sum_{i=1}^n l'm_i + \sum_{i=1}^{n'} lm'_i \right) = \frac{l'm + lm'}{ll'} \\ &= \frac{m}{l} + \frac{m'}{l'} \leq \rho_C(x) + \rho_C(y) + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the subadditivity of ρ_C follows. To show the positive homogeneity, note that since ρ_C is subadditive, we have $\rho_C(kx) \leq k\rho_C(x)$ for all $x \in X$, $k \in \mathbb{N}$. Thus, all we need to prove is $\rho_C(kx) \geq k\rho_C(x)$. Indeed,

$$\begin{aligned} \rho_C(kx) &\stackrel{(10)}{=} \inf \left\{ \frac{m}{l} \mid lkx = \sum_{i=1}^n m_i c_i, \sum_{i=1}^n m_i = m, c_i \in C \right\} \\ &= k \inf \left\{ \frac{m}{lk} \mid lkx = \sum_{i=1}^n m_i c_i, \sum_{i=1}^n m_i = m, c_i \in C \right\} \\ &\stackrel{(*)}{\geq} k \inf \left\{ \frac{m}{l} \mid lx = \sum_{i=1}^n m_i c_i, \sum_{i=1}^n m_i = m, c_i \in C \right\} \\ &= k\rho_C(x), \end{aligned}$$

where in $(*)$ we used the fact that we take an infimum over a larger set. Altogether, we have that $\rho_C(kx) = k\rho_C(x)$, and along with the subadditivity of ρ_C , this completes the proof. $\square$

2.3. Hahn-Banach separation theorem

Under no additional topological assumption, we can obtain the following group version of the Hahn-Banach separation theorem.

Theorem 2.17 (Hahn-Banach weak separation). *Assume that X is a semi-divisible topological group, and $C, D \subseteq X$ convex. Assume that $\text{int } C \neq \emptyset$ and $D \cap \text{int } C = \emptyset$. Then there exists $\varphi : X \rightarrow [-\infty, \infty]$ which is nonzero and affine such that*

$$\sup_{c \in C} \varphi(c) \leq \inf_{d \in D} \varphi(d).$$

Proof. Assume without loss of generality that $0 \in \text{int } C$. Let $f = \rho_C$ and $g = \iota_{\overline{D}} - 1$, where

$$\iota_{\overline{D}}(x) = \begin{cases} 0 & x \in \overline{D}, \\ \infty & x \notin \overline{D}. \end{cases}$$

We have that $f, g : X \rightarrow (-\infty, \infty]$ are convex and $-g \leq f$. Then by [2, Theorem 2], there exists a nonzero affine $\varphi : X \rightarrow \mathbb{R}$ such that $-g \leq \varphi \leq f$. Now, for every $c \in C$, $\varphi(c) \leq f(c) \leq 1$ and for every $d \in D$, $\varphi(d) \geq -g(d) = 1$, which completes the proof. $\square$

Many applications of the Hahn-Banach separation theorem, require *strict* separation: if C is convex and $x \notin C$ then there exists φ linear (additive, in our case), such that $\sup_{c \in C} \varphi(c) < \varphi(x)$. In order to prove such a result in groups, we need more topological structure. Under additional topological assumptions we draw a stronger conclusion about ρ_C . Recall the following result.

Theorem 2.18 (Theorem 7.4 in [7]). *Assume that X is a connected Hausdorff topological group, and that U is an open set containing 0. Then $X = \bigcup_{k=1}^{\infty} kU$.*

We have the following proposition.

Proposition 2.19. *Assume that X is a topological group, $C \subseteq X$ is convex, and $0 \in \text{int } C$. Then ρ_C is everywhere continuous on its domain. If, in addition, X is connected, then ρ_C is everywhere finite.*

Proof. Let U be a neighbourhood of 0. Then $V = \text{int } C \cap U$ is also a neighbourhood of 0. Let $l \in \mathbb{N}$. Since $V \subseteq C$, we have $\frac{1}{l}V \subseteq \frac{1}{l}C$, where the dilation of sets is as in Definition 2.8. Note also that if we define $\phi_l(x) = lx$, then ϕ_l is continuous and $\frac{1}{l}V = \phi_l^{-1}(V)$, which implies that $\frac{1}{l}V$ is open. Also, since $0 \in V$, we have that $0 \in \frac{1}{l}V$. Thus, V is again an open neighbourhood of 0. Assume that $x \in \frac{1}{l}C$, then $lx \in C$ and then by the positive homogeneity of ρ_C , $l\rho_C(x) = \rho_C(lx) \leq 1$. Thus $\rho_C(x) \leq \frac{1}{l}$. This means that ρ_C is continuous at 0. Now, if $x_0 \in X$, then since we have

$$\rho_C(x) - \rho_C(x_0) \leq \rho_C(x - x_0),$$

and

$$\rho_C(x_0) - \rho_C(x) \leq \rho_C(x_0 - x),$$

it follows that continuity at $x = 0$ implies continuity everywhere else on the domain. If X is connected, then by Theorem 2.18, for every $x \in X$ there exists $k \in \mathbb{N}$ such that we have $\rho_C(x) \leq k$. This means that ρ_C is everywhere finite and therefore the proof is complete. $\square$

The following is an easy but useful proposition.

Proposition 2.20. *Let X be a monoid. If $a : X \rightarrow [-\infty, \infty]$ is affine and everywhere finite, then we can write $a(x) = \alpha + \phi(x)$, where $\alpha \in \mathbb{R}$ and $\phi : X \rightarrow \mathbb{R}$ is additive. If $\alpha \geq 0$, then a is also subadditive.*

Proof. If a is affine, then it is both convex and concave. Then $\phi(x) = a(x) - a(0)$ is convex, concave, and $\phi(0) = 0$. Let $x_1, \dots, x_n \in X$. Then $n \sum_{i=1}^n x_i = \sum_{i=1}^n 1 \cdot (nx_i)$. Thus, since ϕ is both convex and concave, we have

$$n\phi\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n \phi(nx_i). \quad (11)$$

Letting $x_2 = \dots = x_n = 0$, it follows that ϕ is positively homogeneous. Thus, (11) gives

$$n\phi\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n n\phi(x_i).$$

which implies that ϕ is additive. Choosing $\alpha = a(0)$, the first assertion follows. To prove the second assertion, note that if $\alpha \geq 0$, we have

$$a(x+y) = \alpha + \phi(x+y) = \alpha + \phi(x) + \phi(y) \leq 2\alpha + \phi(x) + \phi(y) = a(x) + a(y),$$

which concludes the proof. $\square$

Another useful auxiliary result is the following early subadditive separation theorem due to Kaufman.

Theorem 2.21 (Kaufman, [8]). *Assume that X is a monoid and $f, g : X \rightarrow [-\infty, \infty)$ are subadditive, and $-g \leq f$. Then there exists a finite additive map a such that $-g \leq a \leq f$.*

We are now in a position to prove a strict separation theorem.

Theorem 2.22 (Hahn-Banach strict separation). *Assume that X is a connected, locally convex topological group, $C \subseteq X$ is closed and convex, while $x_0 \notin C$. Then there exists a continuous additive function $\varphi : X \rightarrow \mathbb{R}$ such that*

$$\sup_{c \in C} \varphi(c) < \varphi(x_0). \quad (12)$$

Proof. Assume without loss of generality that $x_0 = 0$. Since C is closed, there exists a convex neighbourhood U of 0 such that $U \cap C = \emptyset$. Since $C \cap U = \emptyset$, we have that $C \cap U \cap (-U) = \emptyset$, and so we may assume that U is symmetric, that is, $U = -U$. In particular, we have

$$\rho_U(x) = \rho_U(-x). \quad (13)$$

Let $f = \rho_U$ and $g = \iota_C - 1$. Then f, g are convex and $-g \leq f$. Thus, by [2, Theorem 2], there exists $a : X \rightarrow \mathbb{R}$ nonzero and affine such that $-g \leq a \leq f$. a can be assumed to be everywhere finite because ρ_U is everywhere finite (see [2, Corollary 3]). Since $-g \leq a \leq f$, we have $a(0) \leq 0$. Use Proposition 2.20 to write $a = \phi + \alpha$ where ϕ is additive and $\alpha \leq 0$.

Thus, again by Proposition 2.20, $-a$ is subadditive. Since by Proposition 2.16, ρ_U is subadditive, use Theorem 2.21 to deduce the existence of an additive $\varphi : X \rightarrow \mathbb{R}$, such that $a \leq \varphi \leq \rho_U$. Since $a \geq -g$ we have $\varphi \geq -g$, and we have $\varphi(c) \geq 1$ for every $c \in C$ and $\varphi(0) = 0$, which proves (12). To prove the continuity of φ , note that since φ is additive,

$$-\varphi(x) = \varphi(-x) \leq \rho_U(-x) \stackrel{(13)}{=} \rho_U(x),$$

and so $|\varphi(x)| \leq \rho_U(x)$. Thus, we have

$$|\varphi(x) - \varphi(y)| = |\varphi(x - y)| \leq \rho_U(x - y).$$

Since ρ_U is continuous (in fact it is enough that ρ_U is continuous at $x = 0$), it follows that φ is continuous. This concludes the proof. $\square$

Remark 2.23. In many classical topological vector space proofs of the separation theorem, one deduces separation between sets from separation between a set and a point by applying the latter to the a set of the form $A - B$, where A, B are convex. In general, however, if $A, B \subseteq X$ are convex subsets of a group, $A + B$ need not be convex. For example if $X = \mathbb{Z}^2$, and we choose $A = \{(0, 1), (2, 0)\}$, $B = \{(0, 2), (1, 0)\}$, then $A + B = \{(1, 1), (2, 2), (3, 0), (0, 3)\}$. Also, we have $3 \cdot (1, 2) = 2 \cdot (0, 3) + 1 \cdot (3, 0)$ but $(1, 2) \notin A + B$.

On the other hand, if the group is divisible then convexity is preserved under taking sums of sets. Indeed, if $a_1, \dots, a_n \in A$, $b_1, \dots, b_n \in B$ and $m_1, \dots, m_n \in \mathbb{N}$ such that $\sum_{i=1}^n m_i = m$, then there exist $a, b \in X$ such that $ma = \sum_{i=1}^n m_i a_i$ and $mb = \sum_{i=1}^n m_i b_i$. Since A, B are convex, we have $a \in A$ and $b \in B$. Thus, $\sum_{i=1}^n m_i(a_i + b_i) = m(a + b)$. Now, if we assume that $\sum_{i=1}^n m_i(a_i + b_i) = mx$, then if we assume further that we have unique divisibility (such as the case for locally convex topological groups, as shown in Proposition 1.5), then $x = a + b \in A + B$, which proves that $A + B$ is indeed convex. $\square$

Remark 2.24. While $A + B$ need not be convex for convex $A, B \subseteq X$, as shown in Remark 2.23, it is true that translations of convex sets are convex. Indeed, if $A \subseteq X$ is convex and $x_0 \in X$ then $x_0 + A$ is convex. To see this, let

$a_1, \dots, a_n \in A$, $m_1, \dots, m_n \in \mathbb{N}$, $m = \sum_{i=1}^n m_i$, and assume $mx = \sum_{i=1}^n m_i(x_0 + a_i) = mx_0 + \sum_{i=1}^n m_i a_i$. Then $m(x - x_0) = \sum_{i=1}^n m_i a_i$. Since A is convex, it follows that $x - x_0 \in A$, which means that $x \in x_0 + A$. Also, by [2, Proposition 3], if $T : Y \rightarrow X$ is additive and $A \subseteq X$ is convex, then $T^{-1}(A)$ is also convex in Y . $\square$

Remark 2.25. If A is convex and $A + U$ is convex for every convex neighbourhood of 0, then $\overline{A}$ is also convex. In particular, by Remark 2.23, the closure of a convex set in a divisible locally convex topological group is convex. $\square$

3. Extreme points in topological groups

3.1. The Krein-Milman theorem in topological groups

We begin with a few natural definitions.

Definition 3.1 (Extreme points). Let X be a group and $A \subseteq X$. A point $x \in A$ is said to be an extreme point of A , if whenever $mx = \sum_{i=1}^n m_i x_i$, $m_i \in \mathbb{N}$, $\sum_{i=1}^n m_i = m$, $x_i \in A$, we have $x_1 = \dots = x_n = x$. Denote the set of extreme points of A by $\mathcal{E}(A)$.

Definition 3.2 (Face of set). A subset $F \subseteq A$ is said to be a face of A if whenever $x_1, \dots, x_n \in A$, $m_1, \dots, m_n \in \mathbb{N}$, $m = \sum_{i=1}^n m_i$, $mx = \sum_{i=1}^n m_i x_i$ and $x \in F$, then $x_i \in F$ for all $1 \leq i \leq n$.

As in vector spaces we have the following.

Proposition 3.3 (The set of maximisers is a face). Assume that $A \subseteq X$ is a compact convex subset of a topological group. Let $\varphi : X \rightarrow \mathbb{R}$ be additive and continuous. Then the set

$$F_\varphi = \left\{ x \in A \mid \varphi(x) = \max_{x' \in A} \varphi(x') \right\},$$

is a compact face of A .

Proof. First, note that since A is compact and φ is continuous, then F_φ is a nonempty compact set. Assume that $mx = \sum_{i=1}^n m_i x_i$, where $x_1, \dots, x_n \in A$, $m_1, \dots, m_n \in \mathbb{N}$, $m = \sum_{i=1}^n m_i$, and $x \in F_\varphi$. We have

$$\begin{aligned} m \max_{x' \in A} \varphi(x') &= m\varphi(x) = \varphi\left(\sum_{i=1}^n m_i x_i\right) = \sum_{i=1}^n m_i \varphi(x_i) \\ &\leq \sum_{i=1}^n m_i \max_{x' \in A} \varphi(x') = m \max_{x' \in A} \varphi(x'). \end{aligned}$$

Hence, we must have $\varphi(x_i) = \max_{x' \in A} \varphi(x')$, or in other words $x_i \in F_\varphi$. This completes the proof of the proposition. $\square$

Proposition 3.4 (Existence of extreme points). *Assume that X is a semi-divisible, connected, locally convex group. Let $C \subseteq X$ be convex and compact. Then $\mathcal{E}(C) \neq \emptyset$.*

Proof. If C contains only one point, then since it is convex, we have $\mathcal{E}(C) = C \neq \emptyset$. Assume then that C contains at least two points $x \neq y$. By Proposition 1.5, $\{x\}$ is convex and since $y \notin \{x\}$, by Theorem 2.22 there exists $\varphi : X \rightarrow \mathbb{R}$ additive such that $\varphi(y) < \varphi(x)$. Thus, by Proposition 3.3, F_φ is a compact face of C and clearly $y \notin F_\varphi$.

Next, repeat the procedure for the set F_φ instead of for C . Altogether, we obtain a sequence of compact faces $\{F_\varphi\}_\varphi$, which is increasing, where the partial order is given by

$$F_\varphi \leq F_{\varphi'} \iff F_\varphi \supseteq F_{\varphi'}.$$

Thus, the chain has a nonempty upper bound, which is the intersection. Note that the intersection is non-empty because each F_φ is compact. Define the set

$$F = \{F_\varphi \mid \varphi : X \rightarrow \mathbb{R} \text{ additive and continuous}\}.$$

By Zorn's Lemma, $\mathcal{F}$ has a maximal element, call it F . Note that we can choose

$$F = \bigcap_{F_\varphi \in \mathcal{F}} F_\varphi.$$

In particular, F satisfies $F \subseteq F_\varphi$ for every $F_\varphi \in \mathcal{F}$. F is indeed a face, since if $mx = \sum_{i=1}^n m_i x_i$, $m_1, \dots, m_n \in \mathbb{N}$, $m = \sum_{i=1}^n m_i$, $x_1, \dots, x_n \in C$ and $x \in F$, then $x \in F_\varphi$ for all $F_\varphi \in \mathcal{F}$. Then, since F_φ is a face, we get that $x_i \in F_\varphi$ for all $F_\varphi \in \mathcal{F}$. Thus, $x_i \in F$. This means that F is also a face. The compactness of F follows from it being an intersection of compact sets. If F contains more than one point, we can repeat the same procedure and get a contradiction to the maximality of F . This completes the proof. $\square$

Theorem 3.5 (Krein-Milman theorem for groups). *Assume that X is a semidivisible, connected locally convex group. If $C \subseteq X$ is compact and convex, then*

$$C = \overline{\text{conv}}(\mathcal{E}(C)),$$

that is, C is equal to the closed convex hull of its extreme points.

Proof. Let B be the closed convex hull of the extreme points of C . We want to show $B = C$. Since C is convex and compact, we clearly have $B \subseteq C$. Assume to the contrary that there exists $x \in C \setminus B$. B is a compact convex set. Thus, by Theorem 2.22, there exists $\varphi : X \rightarrow \mathbb{R}$ additive and continuous such that $\sup_{b \in B} \varphi(b) < \varphi(x)$. Then construct the closed face F_φ as before. We have $B \cap F_\varphi = \emptyset$. By Proposition 3.4, F_φ has an extreme point, which is also an extreme point of C . This is a contradiction, and so $B = C$. $\square$

Remark 3.6. If X is a meet semilattice, so that with $\wedge$ as the monoid operation every element is an idempotent, then for an additive φ , $\varphi(ny) = \varphi(y)$ for $n \in \mathbb{N}$. This implies that the only finite value of φ is zero. Hence, a direct analogue of our results does not hold in this monoid. Note that the extreme points of a convex set are the minimal elements and a Krein-Milman theorem holds in this case in an appropriate order topology [15]. This can be derived from Stone's lemma for monoids as given in [2]. $\square$

Example 3.7 (Krein-Milman theorem for the positive hyperbolic group).

Let X be the positive hyperbolic group, as defined in Example 1.9. It was noted in Example 1.9, that this is a connected topological group, which is also locally convex. It was also noted that X is divisible. Let $C \subseteq X$ be a compact subset. Let $\Lambda : \mathbb{R} \rightarrow X$ be the map $\theta \mapsto M(\theta)$. Since $\sinh(\cdot)$ is strictly increasing, it follows that Λ is a bijection. More specifically, if $x = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in X$, then Λ^{-1} is given by

$$\Lambda^{-1}(x) = \operatorname{arcsinh}(b) = \ln \left(b + \sqrt{b^2 + 1} \right).$$

In particular, we have $C = \Lambda(\Lambda^{-1}(C))$. Also, as was shown in Example 1.9, if $U \subseteq X$ is open and $M(\theta) \in U$, then there exists $\varepsilon > 0$ such that $\Lambda((\theta - \varepsilon, \theta + \varepsilon)) \subseteq U$, and so Λ is a continuous map. In particular, if $C \subseteq X$ is compact, then $\Lambda^{-1}(C) \subseteq \mathbb{R}$ is compact. Therefore, we have $\operatorname{conv}(\Lambda^{-1}(C)) \subseteq [\alpha, \beta]$, where $\alpha = \min \{ \theta \mid \theta \in \Lambda^{-1}(C) \}$ and $\beta = \max \{ \theta \mid \theta \in \Lambda^{-1}(C) \}$. For $M(\theta) \in C$ to be an extreme point, we need that $mM(\theta) = \sum_{i=1}^n m_i M(\theta_i)$ implies $\theta = \theta_i$, $1 \leq i \leq n$. But if $mM(\theta) = \sum_{i=1}^n m_i M(\theta_i)$, then we have $m\theta = \sum_{i=1}^n m_i \theta_i$, and θ must be an extreme point of $\Lambda^{-1}(C)$. Altogether $\mathcal{E}(C) = \{M(\alpha), M(\beta)\}$, and by Theorem 3.5, we have that $C = \overline{\operatorname{conv}}(\{M(\alpha), M(\beta)\}) = M([\alpha, \beta])$, and C is a curve in $\mathbb{R}^4$. $\square$

Remark 3.8. The matrices $M(\theta)$ with $\theta \geq 0$ form a partially divisible submonoid, say H . Since it is known that a direct product of p -semidivisible structures is a similar structure, we have abundant other examples. For example, we may consider any of the groups $X \times \mathbb{R}$ or $X \times X$ or $H \times H$. $\square$

We observe in passing that we can use these extreme point ideas to study the structure of convex cones in topological groups. This allows one to use ordered groups to carefully analysis vector optimisation problems [2].

3.2. Milman converse theorem in groups

We should also like to have a group version for the Milman converse theorem [3]. This turns out to require additional restrictions on the underlying space. In topological vector spaces, one crucial requirement for the Milman converse theorem to be true is the fact that $\operatorname{conv}(A \cup B)$ is compact whenever A and B are compact. In the context of topological groups this is not always the case.

For example, if $X = \mathbb{Q}$ with the standard metric, then

$$\text{conv}(\{0\} \cup \{1\}) = \mathbb{Q} \cap [0, 1],$$

which is not compact. However, in groups where compactness is preserved under finite unions and convex hulls, we have the following.

Theorem 3.9 (Milman converse theorem for groups). *Assume that X is a locally convex topological group, such that the convex hull of finite unions of compact sets are compact (as holds in a locally convex vector space over $\mathbb{R}$ or $\mathbb{C}$). Let $C \subseteq X$ be a compact set such that $\overline{\text{conv}}(C)$ is compact. Then*

$$\mathcal{E}(\overline{\text{conv}}(C)) \subseteq C.$$

Proof. Let $x \in \mathcal{E}(\overline{\text{conv}}(C))$. Let U be a convex neighbourhood of 0. Since C is compact, there exist finitely many $x_1, \dots, x_n \in C$ such that

$$C \subseteq \bigcup_{i=1}^n (x_i + \overline{U}).$$

For $i = 1, \dots, n$, define the following sets,

$$A_i = \overline{\text{conv}}(C \cap (x_i + \overline{U})).$$

Since $\overline{\text{conv}}(C)$ is assumed compact, it follows that A_i is compact for each $1 \leq i \leq n$. Also, for each $1 \leq i \leq n$, since $C \cap (x_i + \overline{U}) \subseteq C$, we have $A_i \subseteq \overline{\text{conv}}(C)$, and so $\text{conv}(\bigcup_{i=1}^n A_i) \subseteq \overline{\text{conv}}(C)$. On the other hand, since $C \subseteq \bigcup_{i=1}^n (x_i + \overline{U})$, it follows that $\text{conv}(\bigcup_{i=1}^n A_i) \supseteq C$.

Finally, by the assumption on X , we have that $\text{conv}(\bigcup_{i=1}^n A_i)$ is compact and therefore closed. Altogether, we have $\text{conv}(\bigcup_{i=1}^n A_i) = \overline{\text{conv}}(C)$. Hence, there exist $m_1, \dots, m_n \in \mathbb{N} \cup \{0\}$ and $x_i \in A_i$ such that $mx = \sum_{i=1}^n m_i x_i$, and $m = \sum_{i=1}^n m_i$. Since $x \in \mathcal{E}(\overline{\text{conv}}(C))$, we must have $x_1 = \dots = x_n = x$. Thus, $x \in A_i \subseteq x_i + \overline{U} \subseteq C + \overline{U}$. Since C is closed and U is arbitrary, it follows that $x \in C$. This concludes the proof. $\square$

4. Minimax theorem for monoids

We turn to the proof of a minimax theorem in monoids.

Definition 4.1. Let X be a monoid. A function $f : X \rightarrow \mathbb{R}$ is said to be convex-like, if for every $x, y \in X$ and for every $\mu \in [0, 1]$, there exist $z \in X$ such that

$$f(z) \leq \mu f(x) + (1 - \mu)f(y).$$

Again, g is said to be concave-like exactly when $-g$ is convex-like.

The next result most satisfactorily connects convexity of a function in a monoid to abstract convex-likeness.

Proposition 4.2. *Assume that X is a p -semidivisible topological monoid such that for every $x, y \in X$, the set $\text{conv}(\{x, y\})$ is precompact. Assume that $f : X \rightarrow [-\infty, \infty)$ is convex and lower semicontinuous. Then f is convex-like. If, instead, we assume that f is concave and upper semicontinuous, then f is concave-like.*

Proof. Let $x, y \in X$ and $\mu \in [0, 1]$. For every $k \in \mathbb{N}$, we can find $m_k \in \mathbb{N}$ and $z_k \in X$ such that $p^k z_k = m_k x + (p^k - m_k)y$, and $\mu \leq \frac{m_k}{p^k} \leq \mu + \frac{1}{p^k}$. Such a z_k exists because X is p -semidivisible. Now, since f is convex, we have

$$p^k f(z_k) \leq m_k f(x) + (p^k - m_k)f(y),$$

and so

$$f(z_k) \leq \frac{m_k}{p^k} f(x) + \left(1 - \frac{m_k}{p^k}\right) f(y) \leq \left(\mu + \frac{1}{p^k}\right) f(x) + (1 - \mu)f(y).$$

Since $\text{conv}(\{x, y\})$ is assumed to be precompact, by passing to a subsequence, we may assume without loss of generality that $z_k \rightarrow z$. Then, by the lower semicontinuity of f , we get

$$f(z) \leq \liminf_{k \rightarrow \infty} f(z_k) \leq \mu f(x) + (1 - \mu)f(y),$$

which completes the proof of the first assertion. The proof of the second assertion follows from replacing f by $-f$. The proof is therefore complete. $\square$

The following theorem was proved in [4] by easy Lagrange multiplier techniques.

Theorem 4.3 (Fan's Theorem A in [4]). *Suppose that X and Y are non-empty sets and f is a convex-concave-like function on $X \times Y$. Suppose that X is compact and $f(\cdot, y)$ is lower semicontinuous on X for each $y \in Y$. Then*

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y).$$

Thus, using Proposition 4.2, we immediately obtain the following.

Theorem 4.4 (Minimax formula for partially divisible topological monoids). *Assume that X is a convex and compact subset of a p -divisible topological monoid, and Y is a subset of a q -divisible topological monoid such that for every $y, y' \in Y$, $\text{conv}(\{y, y'\})$ is precompact (as holds if, for example, Y is compact and convex). Assume that $f : X \times Y \rightarrow \mathbb{R}$ is such that for every $y \in Y$, $f(\cdot, y)$ is convex and lower semicontinuous on X , and for every $x \in X$, $f(x, \cdot)$ is concave and upper semicontinuous. Then*

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y). \quad (14)$$

Example 4.5 (Minimax theorem in the positive hyperbolic group).

Let X be the (positive) hyperbolic group, as defined in Example 1.9. Let $\Lambda : \mathbb{R} \rightarrow X$ be the map defined in Example 3.7. Then if $\alpha, \beta \in \mathbb{R}$, we have $\text{conv}(\{M(\alpha), M(\beta)\}) \subseteq \Lambda([\alpha, \beta])$. Since it was shown in Example 3.7 that Λ is continuous, it follows that $\Lambda([\alpha, \beta])$ is compact and therefore $\text{conv}(\{M(\alpha), M(\beta)\})$ is precompact. Hence, if $C \subseteq X$ is compact and convex and $f : C \times X \rightarrow \mathbb{R}$ is such that for every $y \in Y$, $f(\cdot, y)$ is convex and lower semicontinuous on X , and for every $x \in X$, $f(x, \cdot)$ is concave and upper semicontinuous, then by Theorem 4.4, equation (14) holds. $\square$

Remark 4.6. Continuing with the notation of Example 3.7, we note that in general we do not have $\text{conv}(\{M(\alpha), M(\beta)\}) = \Lambda([\alpha, \beta])$. For example, if $\alpha = 0$ and $\beta = 1$, then

$$\text{conv}(\{M(\alpha), M(\beta)\}) = \{M(\theta) \mid \theta \in \mathbb{Q} \cap [0, 1]\}.$$

Thus, when the underlying scalars are incomplete we cannot hope for the convex hull of a pair of points to be anything better than a precompact set. $\square$

Example 4.7 (Saddle functions on the positive hyperbolic group). Let X be the (positive) hyperbolic group, as defined in Example 1.9. Using [2], it is known that if $g : \mathbb{R} \rightarrow \mathbb{R}$ is convex, then $f : X \rightarrow \mathbb{R}$, given by $f(x) = g(\Lambda^{-1}(x))$, is also convex. Similarly, if $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is convex in the first variable and concave in the second variable, then $f : X \times X \rightarrow \mathbb{R}$ given $f(x, y) = g(\Lambda^{-1}(x), \Lambda^{-1}(y))$ will have the same saddle properties. Also, since both Λ and Λ^{-1} are continuous, the continuity properties of g will be inherited by f . So if we choose for example $f(x, y) = (\Lambda^{-1}(x))^2 - (\Lambda^{-1}(y))^2$ (i.e., $g(\theta_1, \theta_2) = \theta_1^2 - \theta_2^2$), and $C = \Lambda([-1, 1])$, then

$$\min_{x \in C} \sup_{y \in X} [(\Lambda^{-1}(x))^2 - (\Lambda^{-1}(y))^2] = \sup_{y \in X} \min_{x \in C} [(\Lambda^{-1}(x))^2 - (\Lambda^{-1}(y))^2].$$

Note, however, that not all convex functions on X are of the explicit form $g(\Lambda^{-1}(x))$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ is convex. $\square$

5. Conclusion

We hope that the results we have presented make the case well that it is useful to study locally convex groups and monoids. In our opinion it both opens up new pathways and sheds new light on old structures.

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