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# A Skorokhod Problem Governed by a Closed Convex Moving Set

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*Dedicated to the memory of Jean Jacques Moreau.*

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We present several results in Skorokhod problems governed by a sweeping process with application to a Skorokhod differential equation in both deterministic and stochastic cases. In particular, we study a Skorokhod evolution inclusion involving the normal cone of a closed convex moving set with nonempty interior via Young measures.

*Keywords:* Bounded variation, differential measure, normal cone, Skorokhod problem, sweeping process, stochastic differential equation, martingale solution, Young measure

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## 1. Introduction

The pioneering work concerning the existence of continuous BV solutions for the perturbation of sweeping process (Moreau's process) [25, 26, 28] of the form

$$-Du \in N_{C(t)}(u(t)) + F(t, u(t))$$

where $C$ is a closed convex valued continuous multifunction from $[0, T]$ to $\mathbb{R}^d$ and $F : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a convex compact valued upper semicontinuous multifunction goes back to [6, 7]. In this vein, we present here some Skorokhod

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problems in the line of Tanaka, Lions and Sznitman, Saisho [31, 21, 30] (see also Frankowska [12] using a viability approach), but with a convex moving set in a Hilbert space. Further applications to Skorokhod differential equations in both deterministic and stochastic cases are also provided.

The paper is organized as follows. In Section 2, we recall some basic results in sweeping process, in particular, we focus on the existence and uniqueness of continuous BV solutions for a sweeping process $\mathcal{SW}(C;a)$ associated with a closed convex moving set $C$ of a Hilbert space $\mathbb{H}$ with $a \in C(0)$ and also some stability results for the sweeping process. We consider a closed convex valued multifunction $C : [0,T] \rightrightarrows \mathbb{H}$ and we assume that $C$ is continuous for the Hausdorff distance $d_H$ and has nonempty interior. Our first Skorokhod problem in the BV continuous case (Theorem 2.3) is the following. Let $a \in C(0)$. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of continuous mappings $h_n : [0,T] \to \mathbb{H}$ satisfying: (i) $h_n(0) = 0, \forall n \in \mathbb{N}$, (ii) $(h_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,T]$ to a continuous mapping $h : [0,T] \to \mathbb{H}$. Then the decomposition $v_{h_n} = w_{h_n} + h_n$ and $v_h = w_h + h$ with $v_{h_n}, v_h$ continuous and $w_{h_n}, w_h$ continuous and BV associated with $h_n$ and $h$ hold: $v_{h_n}(t) \in C(t), \forall t \in [0,T]$, $v_{h_n}(0) = a \in C(0)$, $v_{h_n}(t) = w_{h_n}(t) + h_n(t), \forall t \in [0,T]$, $-\frac{dDw_{h_n}}{d|dDw_{h_n}|}(t) \in N_{C(t)}(v_{h_n}(t))$ ($|Dw_{h_n}|$-a.e.), $w_{h_n}(0) = a$ and $v_h(t) \in C(t), \forall t \in [0,T]$, $v_h(0) = a \in C(0)$, $v_h(t) = w_h(t) + h(t), \forall t \in [0,T]$, $-\frac{dDw_h}{d|dDw_h|}(t) \in N_{C(t)}(v_h(t))$, $|Dw_h|$-a.e. $w_h(0) = a \in C(0)$. Furthermore $v_{h_n}$ and $w_{h_n}$converge uniformly on $[0,T]$ to $v_h$ and $w_h$, respectively.

In Section 3, we prove stability results for the Skorokhod problem in both Lipschitz and absolutely continuous settings (Theorem 3.2, Theorem 3.3).

In Section 4, using the results obtained in the above sections, we provide some existence results for Skorokhod differential equations in both BV, Lipschitz and absolutely continuous settings (Theorems 4.1–4.2–4.3) in the deterministic case.

Finally, in Section 5, using Tonelli's scheme, we solve in the finite dimensional setting a Skorokhod stochastic differential equation involving the normal cone of the closed convex moving set $C$ with non empty interior (Theorem 5.2). Assuming only continuity assumptions on the coefficients of the stochastic equation, we construct a weak solution (i.e., a solution defined on an enlarged probability space, such a solution is sometimes called a martingale solution). Our construction is done with the help of Young measures, using a method which was initiated by Pellaumail [29]. When the coefficients are Lipschitz, this solution is a strong solution.

## 2. A Skorokhod problem associated with a convex sweeping process: The BV continuous case

In all the paper, $\mathbb{H}$ is a Hilbert space and $C : [0,T] \rightrightarrows \mathbb{H}$ is a multifunction with closed convex values. We denote by cc($\mathbb{H}$) (respectively ck($\mathbb{H}$)) the set of nonempty closed convex (respectively nonempty compact convex) subsets of $\mathbb{H}$.

Let us recall and summarize a fundamental result in the sweeping process [24, 32].

**Proposition 2.1.** *Let $C : [0,T] \rightrightarrows \mathbb{H}$ be a closed convex valued continuous multifunction with respect to the Hausdorff distance. Assume that*

$(i)$ *there exist $x_0 \in \mathbb{H}$ and $r_0 > 0$ such that $\overline{B}_{\mathbb{H}}(x_0, r_0) \subset C(t), \forall t \in [0,T]$.*

*Let $a \in C(0)$. Then there exists a unique $BV$ continuous solution to the following sweeping ($\mathcal{SW}$) process:*

$$\mathcal{SW}(C,a) : \begin{cases} -\dfrac{dDu}{d|Du|}(t) \in N_{C(t)}(u(t)), & |Du|\text{-}a.e., \\ u(0) = a \in C(0). \end{cases}$$

*Here $Du$ is the differential vector measure associated with the $BVC$ function $u$ and $|Du|$ is its variation measure, and $N_{C(t)}(u(t))$ denotes the normal cone of $C(t)$ at the point $u(t) \in C(t)$. Further, the solution $u$ satisfies*

$$||Du|| = \int_0^T |Du| \leq l(r_0, ||a - x_0||)$$

*where the function $l : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is given by*

$$l(s,S) := \begin{cases} \max\left\{0, \dfrac{S^2 - s^2}{2s}\right\} & \text{if } d \geq 2, \\ \max\{0, S - s\} & \text{if } d = 1, \end{cases}$$

*where $d$ is the dimension (possibly $+\infty$) of $\mathbb{H}$.*

In the rest of the paper, we assume that $C$ is continuous for the Hausdorff distance $d_H$, that is:

$(*)$ For all $t \in [0,T]$, $\lim_{s \to t} d_H(C(s), C(t)) = 0$.

We are mainly interested in the case when $C(t)$ has nonempty interior for every $t \in [0,T]$:

$(**)$ $\forall t \in [0,T]$, $\mathrm{Int}(C(t)) \neq \emptyset$.

**Lemma 2.2.** *Assume $(*)$. Then Condition $(**)$ is equivalent to the following condition:*

$(**')$ *We can find times $t_0 = 0 < t_1 < \cdots < t_N = T$, points $x_1, \ldots, x_N \in \mathbb{H}$, and $R > 0$ such that, for each $i = 1, \ldots, N$, $\overline{B}_{\mathbb{H}}(x_i, R) \subset C(t)$, $\forall t \in [t_{i-1}, t_i]$.*

*If furthermore $C$ takes separable values, then $(**)$ and $(**')$ are equivalent to*

$(**'')$ *There exists a continuous selection $\gamma$ of $C$ and a real number $R > 0$ such that, for every $t \in [0,T]$, $\overline{B}_{\mathbb{H}}(\gamma(t), R) \subset C(t)$.*

**Proof.** The implications $(**')\Rightarrow(**)$ and $(**'')\Rightarrow(**)$ are clear. Assume $(**)$: For each $t \in [0,T]$, there exists $a(t) \in C(t)$ and $r(t) > 0$ such that $\overline{B}(a(t), r(t)) \subset \mathrm{Int}(C(t))$, where $\mathrm{Int}(C(t))$ denotes the interior of $C(t)$. By continuity of $C$, using [24, Lemma 2.2.3 (b)], there exists for each $t \in [0,T]$ an open interval $J_t$ containining $t$ such that that $\overline{B}(a(t), r(t)) \subset \mathrm{Int}(C(s))$ for all $s \in J_t$. By compactness of $[0,T]$, we can extract a finite family $(J_{t_i})_{1\le i\le n}$ which covers $[0,T]$. Then we can take $r = \min_{1\le i\le N} r_i$, and the proof of $(**')$ is complete.

Let us prove $(**'')$, under the assumption that $C$ takes separable values. Following Michael [22], a closed convex subset $S$ of a closed convex set $A$ is said to be a *supporting set* of $A$ if $S \neq A$ and, if whenever the interior of a segment of $A$ intersects $S$, then the whole segment lies in $S$. The *inside* $\mathrm{Inside}(A)$ of a closed convex set $A$ is the set of points of $A$ which are not in any supporting set of $A$. If $A$ has nonempty interior, $\mathrm{Inside}(A) = \mathrm{Int}(A)$, see [22, page 372]. Similarly, arguing in the affine subspace generated by $A$, we see that, if $A$ has nonempty relative interior $\mathrm{Relint}(A)$, then $\mathrm{Inside}(A) = \mathrm{Relint}(A)$. By a result of Michael [23, Corollary 1.3], if $\mathfrak{X}$ is a metric space and $\mathfrak{Y}$ a Banach space, any lower semicontinuous mutifunction $\Phi : \mathfrak{X} \to \mathrm{cc}(\mathfrak{Y})$ with separable values has a continuous selection $\varphi$ such that $\varphi(x) \in \mathrm{Inside}(\Phi(x))$ for every $x \in \mathfrak{X}$ (this proves in particular that the inside of a closed convex separable subset of a Banach space is always nonempty). Let $\mathrm{ccs}(\mathbb{H})$ denote the set of closed convex separable subsets of $\mathbb{H}$, let us take $\mathfrak{X} = \mathrm{ccs}(\mathbb{H})$ (endowed with the Hausdorff distance) and $\mathfrak{Y} = \mathbb{H}$, and define $\Phi : \mathfrak{X} \rightrightarrows \mathfrak{Y}$ by $\Phi(A) = A$. By [23, Corollary 1.3], there exists a continuous mapping $\varphi :\ \mathrm{ccs}(\mathbb{H}) \to \mathbb{H}$ such that $\varphi(A) \in \mathrm{Inside}(A)$ for every $A \in \mathrm{ccs}(\mathbb{H})$. We can set $\gamma(t) = \varphi(C(t))$ for every $t \in [0,T]$. By $(**)$, we have $\gamma(t) \in \mathrm{Int}(C(t))$. Then the function

$$\rho(t) = d\left(\gamma(t), \mathbb{H} \setminus C(t)\right) = \inf\left\{\|x - \gamma(t)\|\,;\, x \notin C(t)\right\}$$

is continuous and takes positive values. We only need to set

$$r = \inf_{t\in[0,T]} \rho(t). \qquad \square$$

The following stability result and its proof have direct applications to the Skorokhod problem that we develop in Sections 4 and 5.

We need first to fix some notations and terminology. Given a triple $(C, a, h)$ where $C$ is a closed moving set with nonempty interior, $a \in C(0)$, and $h : [0,T] \to \mathbb{H}$ a continuous mapping such that $h(0) = 0$, we denote $C_h = C - h$. A pair $(v_h, w_h)$ associated with $(C, a, h)$ is a *Skorokhod decomposition* (or *decomposition* for simplicity) if it satisfies the equation $v_h = w_h + h$ with the properties

$$\begin{cases} v_h(t) = w_h(t) + h(t), \\ v_h(0) = w_h(0) = a, \\ v_h(t) \in C(t), \quad \forall t \in [0,T], \\ -\dfrac{dDw_h}{d|Dw_h|}(t) \in N_{C_h(t)}(w_h(t)), \quad |Dw_h|\text{-a.e.}, \end{cases} \tag{1}$$

where $w_h$ is BVC and $v_h$ is continuous. We will often write as $v_h = w_h + h$ such a decomposition whose properties depends on those given for $(C, a, h)$.

**Theorem 2.3.** *Let $C : [0,T] \to \mathrm{cc}(\mathbb{H})$ be a closed convex valued continuous mapping satisfying Conditions $(*)$ and $(**)$. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of continuous mappings $h_n : [0,T] \to \mathbb{H}$ satisfying:*

(*i*) $h_n(0) = 0$, $\forall n \in \mathbb{N}$,

(*ii*) $(h_n)_{n\in\mathbb{N}}$ *converges uniformly on* $[0,T]$ *to a continuous mapping* $h_\infty : [0,T] \to \mathbb{H}$.

*With the notations of* (1)*, let us consider the continuous decompositions $v_{h_n} = w_{h_n} + h_n$ $(n \in \mathbb{N} \cup \{\infty\})$ associated with $C$, $a$ and $h_n$. Then there is a positive constant $K$ such that*

$$\sup_{n\in\mathbb{N}} ||Dw_{h_n}|| \leq K < \infty. \tag{2}$$

*Furthermore, $v_{h_n}$ and $w_{h_n}$ converge uniformly on $[0,T]$ to $v_{h_\infty}$ and $w_{h_\infty}$, respectively.*

**Proof.** Let us first prove (2). The closed convex valued mappings $C_{h_n} = C - h_n$ are continuous with nonempty interior. Then by [24, Theorem 2.2.1], the sweeping process $\mathcal{SW}(C_{h_n}; a)$ with initial value $a \in C_{h_n}(0) = C(0)$ has a unique BV continuous solution $w_{h_n}$ with $w_{h_n}(0) = a$, $w_{h_n}(t) \in C_{h_n}(t)$. Note that $C_{h_n} = C - h_n$ satisfies uniformly Condition $(**')$: There exist $t_0 = 0 < t_1, \dots < t_N = T, x_1, \dots, x_N \in \mathbb{H}$ and $r > 0$ such that, for $n$ large enough and for $i = 1, \dots, N$, we have

$$\overline{B}_{\mathbb{H}}(x, r) \subset C_{h_n}(t), \quad \forall t \in [t_{i-1}, t_i].$$

Indeed, by Lemma 2.2, we can find times $t_0 = 0 < t_1 < \dots < t_N = T$, points $x_1, \dots, x_N \in \mathbb{H}$, and $r > 0$ such that

$$\overline{B}_{\mathbb{H}}(x_i, 2r) \subset C_h(t)$$

for each $t \in [t_{i-1}, t_i]$, $i = 1, \dots, N$. Let $n_0$ such that

$$\sup_{0\leq t\leq T} ||h_n(t) - h(t)|| < r$$

for $n \geq n_0$. Then for $n \geq n_0$, we have

$$\overline{B}_{\mathbb{H}}(x_i, r) \subset C_{h_n}(t)$$

for each $t \in [t_{i-1}, t_i]$, $i = 1, \dots, N$. By [24, Lemma 2.3.2-(b)], we have on each interval $[t_{i-1}, t_i]$, $i = 1, \dots, N$,

$$|Dw_h| \leq \sup_{1\leq n\leq N} l(r, ||a - x_i||)$$

and

$$|Dw_{h_n}| \leq \sup_{1\leq n\leq N} l(r, ||a - x_i||), \quad \forall n \geq n_0.$$

Then to prove the required estimate we only need to take

$$K = \sup_{1 \le n \le N} l(r, ||a - x_i||).$$

We now prove that the sequences $(v_n)_n \in \mathbb{N}$ and $(w_n)_{n\in\mathbb{N}}$ satisfy uniformly on $[0,T]$ the Cauchy criterion. Firstly, using classical properties of differential measures for BV continuous functions $u : [0,T] \to \mathbb{H}$, $v : [0,T] \to \mathbb{H}$, we have

$$\int_0^t \langle u - v, D(u - v)\rangle = \frac{1}{2}||u(t) - v(t)||^2 - \frac{1}{2}||u(0) - v(0)||^2.$$

Secondly, for every $n \in \mathbb{N}$ we have

$$-\frac{dDw_{h_n}}{d|Dw_{h_n}|}(t) \in N_{C_{h_n}(t)}(w_{h_n}(t)) = N_{C(t)}(v_{h_n}(t)), \quad |Dw_{h_n}|\text{-a.e.}$$

Let $\nu$ be a Radon positive measure such that $|Dw_{h_n}| \preccurlyeq \nu$, $\forall n \ge n_0$, then

$$-\frac{dDw_{h_n}}{d\nu}(t) \in N_{C_{h_n}(t)}(w_{h_n}(t)) = N_{C(t)}(v_{h_n}(t)), \quad \nu\text{-a.e.} \tag{3}$$

because $N_{C(t)}$ is a cone. Since $x \mapsto N_{C(t)}(x)$ is monotone, (3) implies

$$\left\langle \left(-\frac{dDw_{h_m}}{d\nu}(t)\right) - \left(-\frac{dDw_{h_n}}{d\nu}(t)\right), v_{h_m}(t) - v_{h_n}(t) \right\rangle \ge 0, \quad \nu\text{-a.e.}$$

Equivalently

$$\left\langle \frac{dD(w_{h_m} - w_{h_n})}{d\nu}(t), v_{h_m}(t) - v_{h_n}(t) \right\rangle \le 0, \quad \nu\text{-a.e.} \tag{4}$$

Integrating (4) over $[0,t]$ with respect to the measure $\nu$ gives

$$\int_0^t \langle v_{h_m} - v_{h_n}, D(w_{h_m} - w_{h_n})\rangle \le 0. \tag{5}$$

We have

$$v_{h_m} - v_{h_n} = (w_{h_m} - w_{h_n}) + (h_m - h_n).$$

Therefore

$$||v_{h_m} - v_{h_n}||^2 = ||w_{h_m} - w_{h_n}||^2 + 2\langle h_m - h_n, w_{h_m} - w_{h_n}\rangle + ||h_m - h_n||^2.$$

On the other hand, we have also

$$\int_0^t \langle w_{h_m} - w_{h_n}, Dw_{h_m} - Dw_{h_n}\rangle = \frac{1}{2}||w_{h_m}(t) - w_{h_n}(t)||^2.$$

Let us write

$$\begin{aligned}
&\langle h_m(t) - h_n(t), w_{h_m}(t) - w_{h_n}(t)\rangle \\
&= \langle h_m(t) - h_n(t), w_{h_m}(t) - w_{h_n}(t)\rangle - \langle h_m(t) - h_n(t), w_{h_m}(0) - w_{h_n}(0)\rangle \\
&= \int_0^t \langle h_m(t) - h_n(t) - h_m(s) + h_n(s), Dw_{h_m}(s) - Dw_{h_n}(s)\rangle \\
&\qquad + \int_0^t \langle h_m(s) - h_n(s), Dw_{h_m}(s) - Dw_{h_n}(s)\rangle.
\end{aligned}$$

We get

$$\begin{aligned}
&||v_{h_m}(t) - v_{h_n}(t)||^2 \\
&= ||h_m(t) - h_n(t)||^2 + 2\int_0^t \langle w_{h_m}(s) - w_{h_n}(s), Dw_{h_m}(s) - Dw_{h_n}(s)\rangle \\
&\qquad + 2\int_0^t \langle h_m(t) - h_n(t) - h_m(s) + h_n(s), Dw_{h_m}(s) - Dw_{h_n}(s)\rangle \\
&\qquad + 2\int_0^t \langle h_m(s) - h_n(s), Dw_{h_m}(s) - Dw_{h_n}(s)\rangle.
\end{aligned}$$

Since $v_{h_m} = w_{h_m} + h_m$ and $v_{h_n} = w_{h_n} + h_n$, by (5), we get the estimate

$$\begin{aligned}
&||v_{h_m}(t) - v_{h_n}(t)||^2 \\
&\le ||h_m(t) - h_n(t)||^2 \\
&\qquad + 2\int_0^t \langle h_m(t) - h_n(t) - h_m(s) + h_n(s), Dw_{h_m}(s) - Dw_{h_n}(s)\rangle.
\end{aligned}$$

Let $H := \sup ||Dw_{h_n}||$. Then we get the final estimate

$$||v_{h_m}(t) - v_{h_n}(t)||^2 = ||h_m(t) - h_n(t)||^2 + 8H \sup_{s\in[0,t]} ||h_m(s) - h_n(s)||$$

which proves that $(v_{h_n})$ converges uniformly to some continuous functions $v$ and that $(w_{h_n})$ converges uniformly to some continuous functions $w$.

There remains to show that $v = v_{h_\infty}$ and $w = w_{h_\infty}$. By the Banach-Helly theorem [24, Theorem 0.2.1], $(Dw_{h_n})$ converges weakly to $Dw$ in the space of bounded vector measures $\mathcal{M}([0,T];\mathbb{H})$ of bounded variation

$$\lim_{n\to\infty} \int_{]s,t]} \left\langle \frac{dDw_n}{d|Dw_{h_n}|}(\tau), f(\tau)\right\rangle |Dw_{h_n}|(\tau) = \int_{]s,t]} \left\langle \frac{dDw}{d|Dw|}(\tau), f(\tau)\right\rangle |Dw|(\tau)$$

for all $0 \le s < t \le T$ and for all continuous mappings $f : [0,T] \to \mathbb{H}$. As the functions $w_{h_n}$ are BV continuous solutions to the sweeping process $\mathcal{SW}(C_{h_n}; a)$, we have

$$-\frac{dDw_{h_n}}{d|Dw_{h_n}|}(t) \in N_{C(t)-h_n(t)}(w_{h_n}(t)) \quad |Dw_{h_n}||\text{-a.e.}$$

Equivalently

$$\int_{]s,t]} \delta^* \left(-\frac{dDw_{h_n}}{d|Dw_{h_n}|}(\tau), C_{h_n}(\tau)\right) |Dw_{h_n}|(\tau) + \frac{1}{2}[||w_{h_n}(t)||^2 - ||w_{h_n}(s)||^2] \leq 0 \quad (6)$$

for all $0 \leq s < t \leq T$. Since $h_n$ converges uniformly to $h$, it implies the well-known Kuratowski limit of sets: for any sequence $(t_n)$ in $[0,T]$ converging to $t$, $C_{h_\infty}(t) \subset \mathrm{Li}\, C_{h_n}(t_n)$, where $\mathrm{Li}\, C_{h_n}(t_n)$ denotes the limit inferior. Then, from an epilower version of Reshetnyak's theorem, e.g., [19, Cor. 3.4], [8] and the weak limit $Dw_{h_n} \to Dw$, we have, by taking the liminf in the inequality (6),

$$\begin{aligned} 0 \geq\ & \liminf_n \left[\int_{]s,t]} \delta^* \left(-\frac{dDw_{h_n}}{d|Dw_{h_n}|}(\tau), C_{h_n}(\tau)\right) |Dw_{h_n}|(\tau) \right. \\ & \left. \quad + \frac{1}{2}[||w_{h_n}(t)||^2 - ||w_{h_n}(s)||^2 \right] \\ \geq\ & \int_{]s,t]} \delta^* \left(-\frac{dDw}{d|Dw|}(\tau), C_h(\tau)\right) |Dw|(\tau) + \frac{1}{2}[||w(t)||^2 - ||w(s)||^2], \end{aligned}$$

therefore proving that $w$ is solution of the sweeping process $\mathcal{SW}(C_{h_\infty}; a)$ and by uniqueness $w = w_{h_\infty}$. Hence we may conclude that $(v_{h_n})$ and $(w_{h_n})$ converge uniformly to $v_{h_\infty}$ and $w_{h_\infty}$ respectively. □

## 3. A Skorokhod problem associated with a convex sweeping process: The Lipschitz case

In this section $\mathbb{H}$ is a Hilbert space and $C : [0,T] \to \mathrm{ck}(\mathbb{H})$ is Lipschitz with respect to the Hausdorff distance, i.e., there is a constant $L > 0$ such that

$$d_H(C(t), C(\tau)) \leq L|t - \tau|, \quad \forall\, t, \tau \in [0,T].$$

Given a $\mathrm{ck}(\mathbb{H})$-valued $L$-Lispschitz function, then, from classical result in the sweeping process [28], given $x \in C(0)$, there is a unique $L$-Lipschitz function $u : [0,T] \to \mathbb{H}$ such that

$$-\dot{u}(t) \in N_{C(t)}(u(t)), \quad t \in [0,T]$$

$$u(0) = x \in C(0).$$

We state and summarize a stability result in $(\mathcal{SW})$ in the Lipschitzean case.

**Theorem 3.1.** *Let $(C_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence of $L$-Lipschitz mappings from $[0,T]$ to the set $\mathrm{ck}(\mathbb{H})$ of nonempty convex compact subsets of $\mathbb{H}$. Assume that the following assumptions are fulfilled:*

*(i) for any sequence $(t_n)$ in $[0,T]$ converging to $t$, $C_\infty(t) \subset \mathrm{Li}(C_n(t_n))$,*

*(ii) for any $t \in [0,T]$, $\mathrm{Ls}(C_n(t)) \subset C_\infty(t)$.*

*Let $(u_k)_{k\in\mathbb{N}}$ be the L-Lipschitz solutions of the sweeping process*

$$-\dot{u}_k(t) \in N_{C_k(t)}(t)(u_k(t)), \qquad u_k(0) = a_k \in C_k(0).$$

*If $(a_k)_{k\in\mathbb{N}}$ converges to $a_\infty \in \mathbb{H}$, and if $(u_k(t))$ is relatively compact for each $t \in [0,T]$, then $(u_k)_{k\in\mathbb{N}}$ converges uniformly to the unique L-Lipschitz solution $u_\infty$ of the sweeping process*

$$-\dot{u}_\infty(t) \in N_{C_\infty(t)}(t)(u_\infty(t)), \qquad u_\infty(0) = a_\infty \in C_\infty(0).$$

**Proof.** Here one can adapt the proof of stability result for sweeping process in the BV continuous case [6, Theorem 4.4]. For convenience we provide the proof in the Lispschitz case that is based on an epi-lower convergence result of integral convex functional on the space of vector measures and the definition of normal cone for closed convex set in the sense of Convex Analysis. Since $(u_k)_{k\in\mathbb{N}}$ is equi-Lipschitz and $(u_k(t))$ is relatively compact for each $t \in [0,T]$, it is relatively compact in $C_{\mathbb{H}}([0,T])$, therefore $(u_k)_{k\in\mathbb{N}}$ converges uniformly to an $L$-Lipschitz function $u_\infty \in C_{\mathbb{H}}([0,T])$ with $\dot{u}_k \to \dot{u}_\infty$ with respect to the $\sigma(\mathrm{L}^1_{\mathbb{H}}, \mathrm{L}^\infty_{\mathbb{H}})$ topology and $u_\infty(0) = a_\infty \in C_\infty(0)$. It remains to check that

$$-\dot{u}_\infty(t) \in N_{C_\infty(t)}(u_\infty(t)) \quad \text{a.e.}$$

**Main fact.** We have

$$\liminf_k \int_0^T \delta^*(-\dot{u}_k(t), C_k(t))dt \geq \int_0^T \delta^*(-\dot{u}_\infty(t), C_\infty(t))dt.$$

Let $\delta^*(x, C_k(t))$ denote the support function of $C_k(t)$. From $(i)$ and $(ii)$, it is not difficult to check that $\delta^*(., C_k(.))$ lower epiconverges to $\delta^*(., C_\infty(.))$, that is, for any sequence $(t_k, x_k)$ in $[0,T] \times \mathbb{H}$ converging to $(t,x) \in [0,T] \times \mathbb{H}$ we have that $\liminf_k \delta^*(x_k, C_k(t_k)) \geq \delta^*(x, C_\infty(t))$. Let $\varphi_\infty$ be a continuous selection of $C_\infty$. By [6, Proposition 3.9], there are continuous selections $\varphi_k$ of $C_k$ which converge uniformly to $\varphi_\infty$. Let us set $\Gamma_k = C_k - \varphi_k$, $k \in \mathbb{N} \cup \{\infty\}$. Then $\delta^*(., \Gamma_k(.))$ is nonnegative and lower epiconverges to $\delta^*(., \Gamma_\infty(.))$. Now let $m_k = -\dot{u}_k dt$ be the vector measure of density $-\dot{u}_k$ ($k \in \mathbb{N} \cup \{\infty\}$). Then $(m_k)$ converges weakly to $m_\infty = -\dot{u}_\infty dt$ in the space of bounded vector measures $\mathcal{M}^b([0,T], \mathbb{H})$. Hence, invoking a lower epi-convergence version of Reshetnyak's theorem [6, Theorem 3.4], we get

$$\liminf_k \int_0^T \delta^*\left(\frac{dm_k}{d|m_k|}(t), \Gamma_k(t)\right) d|m_k|(t) \geq \int_0^T \delta^*\left(\frac{dm_\infty}{d|m_\infty|}, \Gamma_\infty(t)\right) d|m_\infty|(t).$$

This gives

$$\begin{aligned} &\liminf_k \int_0^T \delta^*\left(\frac{dm_k}{d|m_k|}(t), C_k(t)\right) d|m_k|(t) \\ &\geq \int_0^T \delta^*\left(\frac{dm_\infty}{d|m_\infty|}(t), C_\infty(t)\right) d|m_\infty|(t). \end{aligned} \tag{7}$$

Since $m_k = -\dot{u}_k dt$ we have for a.e. $t \in [0,T]$ and for all $k \in \mathbb{N} \cup \{\infty\}$

$$dm_k = \frac{dm_k}{d|m_k|}\frac{d|m_k|}{dt}dt = \frac{dm_k}{d|m_k|}||\dot{u}_k||dt,$$

hence

$$\begin{aligned}\delta^*\left(\frac{dm_k}{dt}(t), C_k(t)\right) &= \delta^*\left(\frac{dm_k}{d|m_k|}(t)||\dot{u}_k(t)||, C_k(t)\right)\\ &= ||\dot{u}_k(t)||\delta^*\left(\frac{dm_k}{d|m_k|}(t), C_k(t)\right).\end{aligned}$$

By integrating, we get

$$\begin{aligned}\int_0^T \delta^*\left(\frac{dm_k}{dt}(t), C_k(t)\right)dt &= \int_0^T \delta^*\left(\frac{dm_k}{d|m_k|}(t), C_k(t)\right)||\dot{u}_k(t)||dt \\ &= \int_0^T \delta^*\left(\frac{dm_k}{d|m_k|}(t), C_k(t)\right)d|m_k|(t).\end{aligned} \tag{8}$$

Using (8) and coming back to (7) we get finally

$$\liminf_k \int_0^T \delta^*(-\dot{u}_k(t), C_k(t))dt \geq \int_0^T \delta^*(-\dot{u}_\infty(t), C_\infty(t))dt.$$

From $-\dot{u}_k(t) \in N_{C_k(t)}(t)(u_k(t))$ we have

$$\delta^*(-\dot{u}_k(t), C_k(t)) + \langle u_k(t), \dot{u}_k(t)\rangle \leq 0.$$

Integrating on $[0,T]$ yields

$$\int_0^T \delta^*(-\dot{u}_k(t), C_k(t))dt + \int_0^T \langle u_k(t), \dot{u}_k(t)\rangle dt \leq 0. \tag{9}$$

It is clear that

$$\lim_k \int_0^T \langle u_k(t), \dot{u}_k(t)\rangle dt = \int_0^T \langle u_\infty(t), \dot{u}_\infty(t)\rangle dt.$$

Taking the lim inf in (9) and applying the main fact gives

$$\int_0^T \delta^*(-\dot{u}_\infty(t), C_\infty(t))dt + \int_0^T \langle u_\infty(t), \dot{u}_\infty(t)\rangle dt \leq 0.$$

Whence

$$\delta^*(-\dot{u}_\infty(t), C_\infty(t)) + \langle u_\infty(t), \dot{u}_\infty(t)\rangle \leq 0 \quad \text{a.e.}$$

thus proving the desired inclusion

$$-\dot{u}_\infty(t) \in N_{C_\infty(t)}(t)(u_\infty(t)) \quad \text{a.e.}$$

□

**Theorem 3.2.** *Let $C : [0,T] \to \mathrm{ck}(\mathbb{H})$ be a convex compact valued $L$-Lipschitz mapping. Let $a \in C(0)$. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of $M$-Lipschitz mappings $h_n : [0,T] \to \mathbb{H}$ satisfying:*

(*i*) $h_n(0) = 0,\ \forall n \in \mathbb{N}$,

(*ii*) $(h_n)_{n\in\mathbb{N}}$ *converges uniformly on* $[0,T]$ *to an* $M$-*Lipschitz mapping* $h_\infty : [0,T] \to \mathbb{H}$.

*Let us consider the Lipschitz decompositions $v_{h_n} = w_{h_n} + h_n$ ($n \in \mathbb{N} \cup \{\infty\}$) associated with $(C, a, h_n)$ with $-\dot{w}_{h_n}(t) \in N_{C_{h_n}(t)}(w_{h_n}(t))$. Then $w_{h_n}$ and $v_{h_n}$ converge uniformly to $w_{h_\infty}$ and $v_{h_\infty}$, respectively.*

**Proof.** Since the mappings $C_{h_n} = C - h_n(t)$ are $(L+M)$-Lipschitz, there is a unique $(L+M)$-Lipschitz solution $w_{h_n}$ to $\mathcal{SW}(C_{h_n}; a)$, thus the decomposition $(v_{h_n}, w_{h_n})$ exists. It is clear that $v_{h_n}$ is $(L+2M)$-Lipschitz. Since $(h_n)$ converges uniformly to $h_\infty$, we have

$$\lim_{n\to\infty} d_H(C_{h_n}(t), C_{h_\infty}(t)) = \lim_{n\to\infty} ||h_n(t) - h_\infty(t)|| = 0, \quad \forall t \in [0,T].$$

As $v_{h_n}(t) \in C(t) \subset C([0,T])$ and $C([0,T])$ is compact, $(v_{h_n})$ converges uniformly to an $(L+M)$-Lipschitz mapping $v$. As $v_{h_n}(t) = w_{h_n}(t) + h_n(t)$, $\forall t \in [0,T]$ and $(h_n)$ converges uniformly to $h_\infty$, $w_{h_n}$ converges uniformly to the $(L+M)$-Lipschitz mapping $w = v - h$, therefore one may apply the stability result for sweeping process in Theorem 3.1 which shows that the $(L+M)$-Lipschitz solutions $w_{h_n}$ to the sweeping process $\mathcal{SW}(C_{h_n}; a)$

$$-\dot{w}_{h_n}(t) \in N_{C_{h_n}}(t)(w_{h_n}(t)), \ \text{ a.e.}, \qquad v_{h_n}(0) = w_{h_n}(0) = a \in C(0)$$

converges uniformly to the unique $(L+M)$-Lipschitz solution $w = w_{h_\infty}$ to the sweeping process $\mathcal{SW}(C_{h_\infty}; a)$

$$-\dot{w}_{h_\infty}(t) \in N_{C_{h_\infty}(t)}(w_{h_\infty}(t)), \quad \text{a.e.}, \qquad v_{h_\infty}(0) = w_{h_\infty}(0) = a \in C(0).$$

As, for $n \in \mathbb{N} \cup \{\infty\}$,

$$\begin{gathered} v_{h_n}(t) = w_{h_n}(t) + h_n(t) \\ v_{h_n}(0) = w_{h_n}(0) = a \\ v_{h_n}(t) \in C(t), \quad \forall t \in [0,T] \\ -\dot{w}_{h_n}(t) \in N_{C_{h_n}(t)}(w_{h_n}(t)), \quad \text{a.e.}, \end{gathered}$$

we have

$$\begin{aligned} v(t) &= \lim_{n\to\infty} v_{h_n}(t) = \lim_{n\to\infty} w_{h_n}(t) + h_n(t) \\ &= w_{h_\infty}(t) + h_\infty(t) = v_{h_\infty}(t), \quad \forall t \in [0,T], \end{aligned}$$

which proves that $v_{h_n}$ converges uniformly to $v_{h_\infty}$ and $w_{h_n}$ converges uniformly to $w_{h_\infty}$. □

**The absolutely continuous case.** We need to recall some notations. A convex compact valued mapping $C : [0,T] \rightrightarrows \mathbb{H}$ is absolutely continuous in the sense $d_H(C(t), C(\tau)) \leq a(t) - a(\tau) = \int_\tau^t \dot{a}(s)ds$, $\forall \tau \leq t, \in [0,T]$, where $\dot{a} \in L^1_{\mathbb{R}^+}([0,T])$. Taking into account the existence of an absolutely continuous solution for a convex compact valued absolutely continuous mapping and the techniques given in Theorem 3.1 and Theorem 3.2 we have the following absolutely continuous version of the results given therein.

**Theorem 3.3.** *Let $C : [0,T] \to \mathrm{ck}(\mathbb{H})$ be a convex compact valued absolutely continuous mapping. Let $a \in C(0)$. Let $(h_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence of equi-absolutely continuous mappings, i.e., $h_n(t) = \int_0^t \dot{h}_n(s)ds, \forall t \in [0,T]$ for $n \in \mathbb{N}\cup\{\infty\}$, with $|\dot{h}_n| \leq g$ for some positive integrable function $g \in \mathrm{L}^1_{\mathbb{R}}([0,T])$, such that $(h_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,T]$ to the absolutely continuous mapping $h_\infty$. Let us consider the absolutely continuous decompositions $v_{h_n} = w_{h_n} + h_n$ ($n \in \mathbb{N}\cup\{\infty\}$) associated with $(C, a, h_n)$ with $-\dot{w}_h(t) \in N_{C_h(t)}(w_h(t))$. Then $w_{h_n}$ and $v_{h_n}$ converge uniformly to $w_{h_\infty}$ and $v_{h_\infty}$, respectively.*

## 4. Solvability of a Skorokhod differential equation associated with a convex sweeping process

Let $C : [0,T] \rightrightarrows \mathbb{H}$ be a closed convex continuous mapping. We aim to study a Skorokhod differential equation related to the sweeping process associated with the closed convex moving set $C$ and the Borel perturbation $b : [0,T] \times \mathbb{H} \to \mathbb{H}$ satisfying:

(a) $||b(t,x)|| \leq M, \forall t \in [0,T], \forall x \in \mathbb{H}$, for some positive constant $M$,
(b) $||b(t,x) - b(t,y)|| \leq K||x-y||, \forall t \in [0,T], \forall x \in \mathbb{H}, \forall y \in \mathbb{H}$,

for some positive constant $K$.

Given $a \in C(0)$, the problem is to find $(X, Y)$ satisfying:

$$X(t) = \int_0^t b(s, X(s))ds + Y(t), \quad \forall t \in [0,T] \tag{10}$$

where $X$ is continuous mapping on $[0,T]$ to $\mathbb{H}$ with $X(t) \in C(t)$ for all $t \in [0,T]$ and $Y$ is a BV continuous mapping from $[0,T]$ to $\mathbb{H}$ satisfying

$$-\frac{dDY}{d|DY|}(t) \in N_{C(t)}(X(t)), \quad |DY|\text{-a.e.}$$

$$X(0) = Y(0) = a \in C(0)$$

Before going further, it is easy to see that, when $b = 0$, then (10) is reduced to the existence of a continuous BV solution to the $\mathcal{SW}(C, a)$

$$-\frac{dDX}{d|DX|}(t) \in N_{C(t)}(X(t)), \quad |DX|\text{-a.e.}$$

$$X(0) = a \in C(0)$$

so that the differential equation (10) can be considered as an extension of the sweeping process $\mathcal{SW}(C,a)$ with BV continuous solution where $C$ satisfies the required conditions $(*)$ and $(**)$ of Section 2, i.e., $C$ is continuous with respect to the Hausdorff distance and $\mathrm{Int}(C(t))$ is non empty for all $t \in [0,T]$. In order to solve (10), it is necessary to know the structure of the BVC solution sets of the associated sweeping process under consideration. Here (10) is viewed as a perturbation problem for the specific sweeping process $\mathcal{SW}(C,a)$ and cannot be deduced from general results in differential inclusion.

**Theorem 4.1.** *Let $C : [0,T] \to \mathrm{ck}(\mathbb{R}^d)$ be a convex compact valued mapping satisfying* $(*)$ *and* $(**)$ *of Section* 2*, with the Borel perturbation $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying* (a) *and* (b)*. Let $a \in C(0)$. Then there exist a continuous function $X : [0,T] \to \mathbb{R}^d$ with $X(t) \in C(t)$ for all $t \in [0,T]$ and a BV continuous function $Y : [0,T] \to \mathbb{R}^d$ satisfying*

$$\begin{gathered} X(t) = \int_0^t b(s, X(s))ds + Y(t), \quad \forall t \in [0,T], \\ -\frac{dDY}{d|DY|}(t) \in N_{C(t)}(X(t)), \quad |DY|\text{-}a.e., \\ X(0) = Y(0) = a \in C(0). \end{gathered}$$

**Proof.** By Lemma 2.2, we can assume that there exist times $t_0 = 0 < t_1 < \cdots < t_N = T$, points $x_1, \ldots, x_N \in \mathbb{R}^d$, and $r > 0$ such that, for each $i = 1, \ldots, N$, $\overline{B}_{\mathbb{R}^d}(x_i, r) \subset C(t)$, $\forall t \in [t_{i-1}, t_i]$. Furthermore, dividing if necessary the intervals $[t_{i-1}, t_i]$, we can assume without loss of generality that $t_i - t_{i-1} \leq r/M$ for $i = 1, \ldots, N$, where $M$ is the constant of Condition $(i)$. Let

$$R = MT + \max_{1 \leq i \leq N} \|x_i\| .$$

By Proposition 2.1, there exists a unique BV continuous solution $X_a$ to

$$\begin{cases} -\dfrac{dDX_a}{d|DX_a|}(t) \in N_{C(t)}(X_a(t)), \quad |DX_a|\text{-}a.e., \\ X_a(0) = a \in C(0), \end{cases}$$

and we have, for $i = 1, \ldots, N$,

$$\int_{t_{i-1}}^{t_i} |DX_a| \leq l(r, x_i) \leq l(r, R),$$

which implies $||DX_a|| \leq Nl(r,R)$. Using Proposition 2.1, we define a sequence $(X^n)_{n \geq 0}$ of continuous mappings as follows. Let us set

$$\begin{gathered} X^0(t) = X_a(t), \quad \forall t \in [0,T], \\ h_1(t) = \int_0^t b(s, X^0(s))ds, \quad \forall t \in [0,T], \\ C_{h_1}(t) = C(t) - h_1(t), \quad \forall t \in [0,T]. \end{gathered}$$

We have $||h_1(t)|| \leq MT, \forall t \in [0,T]$ and, for $i = 1, \ldots, N$ and $t \in [t_{i-1}, t_i]$,

$$M(t_i - t_{i-1})\overline{B}_{\mathbb{R}^d}(h(t_{i-1}), 1) \subset r\overline{B}_{\mathbb{R}^d}(h(t_{i_1}), 1) \subset C_{h_1}(t).$$

By Proposition 2.1 applied on each interval $[t_{i-1}, t_i]$, there is a unique BVC solution $Y^1$ to

$$-\frac{dDY^1}{d|DY^1|}(t) \in N_{C_{h_1}(t)}(Y^1(t)) \quad |DY^1|\text{-a.e.}$$

$$Y^1(0) = a \in C_{h_1}(0) = C(0)$$

and we have, for $i = 1, \ldots, N$,

$$\int_{t_{i-1}}^{t_i} |DY^1| \leq l(r, h(t_i)) \leq l(r, R),$$

which implies $||DY^1|| \leq Nl(r, R)$. Let us set

$$X^1(t) = h_1(t) + Y^1(t) = \int_0^t b(s, X^0(s))ds + Y^1(t), \quad \forall t \in [0,T]$$

so that $X^1$ is continuous with $X^1(t) \in C(t), \forall t \in [0,T]$ and that

$$-\frac{dDY^1}{d|DY^1|}(t) \in N_{C(t)}(X^1(t)) \quad |DY^1|\text{-a.e.}$$

Now we construct $X^n$ by induction as follows. Let

$$h_n(t) = \int_0^t b(s, X^{n-1}(s))ds, \quad \forall t \in [0,T],$$
$$C_{h_n}(t) =, C(t) - h_n(t), \quad \forall t \in [0,T].$$

We have $||h_n(t)|| \leq MT$, $\forall t \in [0,T]$, and, for $i = 1, \ldots, N$ and $t \in [t_{i-1}, t_i]$,

$$r\overline{B}_{\mathbb{R}^d}(h_n(t_{i_1}), 1) \subset C_{h_n}(t).$$

By Proposition 2.1 applied on each interval $[t_{i-1}, t_i]$, there is a unique BVC solution $Y^n$ to

$$-\frac{dDY^n}{d|DY^n|}(t) \in N_{C_{h_n}}(Y^n(t)) \quad |DY^n|\text{-a.e.}$$

$$Y^n(0) = a \in C_{h_n}(0) = C(0)$$

with $||DY^n|| \leq Nl(r, R)$. Let us set

$$X^n(t) = h_n(t) + Y^n(t) = \int_0^t b(s, X^{n-1}(s))ds + Y^n(t), \quad \forall t \in [0,T]$$

so that $X^n$ is continuous with $X^n(t) \in C(t)$, $\forall t \in [0,T]$ and that

$$-\frac{dDY^n}{d|DY^n|}(t) \in N_{C(t)}(X^n(t)) \quad |DY^n|\text{-a.e.}$$
$$Y^n(0) = a \in C_{h_n}(0) = C(0).$$

By our assumption we infer that $(Y^n)$ is bounded and uniformly bounded in variation: $||DY^n|| \leq Nl(r,R)$, and so is $(X^n)$. By Helly's theorem, e.g., [24, Theorem 0.2.1] and by extracting a subsequence, we may assume that $(X^n)$ pointwise converges to a BV function $Z : [0,T] \to \mathbb{R}^d$. Then $\int_0^t b(s, X^{n-1}(s))ds \to \int_0^t b(s, Z(s))ds$ uniformly in $[0,T]$. Indeed we have

$$\sup_{t\in[0,T]} \left\| \int_0^t b(s, X^{n-1}(s))ds - \int_0^t b(s, Z(s))ds \right\|$$
$$\leq \sup_{t\in[0,T]} \int_0^t K\|X^{n-1}(s) - Z(s)\|ds \to 0$$

as $n \to \infty$. Here one may invoke a general fact, in view of a lemma due to Grothendieck [13, Ch. 5 §4 no. 1 Prop. 1 and exercice], on bounded subsets of $\mathrm{L}^\infty$, the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e., relatively weakly compact subsets, alias the Mackey topology, in other terms it converges to 0 with respect to the Mackey topology $\tau(L^\infty, \mathrm{L}^1)$, see also [5] for a more general result concerning the Mackey topology for bounded sequences in $\mathrm{L}^\infty_{\mathbb{H}^*}$. Invoking Theorem 2.3 via the above construction, shows that $(X^n)$ converges uniformly to a continuous function $X : [0,T] \to \mathbb{R}^d$ with $X(t) \in C(t)$ for all $t \in [0,T]$ and $(Y^n)$ converges uniformly to a BVC function $Y : [0,T] \to \mathbb{R}^d$ satisfying

$$X(t) = \int_0^t b(s, Z(s))ds + Y(t), \quad \forall t \in [0,T]$$
$$-\frac{dDY}{d|DY|}(t) \in N_{C(t)}(Z(t)), \quad t \in [0,T] \quad |DY|\text{-a.e.}$$
$$Y(0) = a \in C_h(0) = C(0),$$

where $h(t) = \int_0^t b(s, Z(s))ds$, $\forall t \in [0,T]$ and $C_h(t) = C(t) - h(t)$, $\forall t \in [0,T]$. Identifying the limits shows that

$$\lim_{n\to\infty} X^n(t) = X(t) = Z(t), \quad \forall t \in [0,T]$$
$$\lim_{n\to\infty} Y^n(t) = Y(t), \quad \forall t \in [0,T]$$

with

$$-\frac{dDY}{d|DY|}(t) \in N_{C(t)}(X(t)), \quad t \in [0,T] \quad |DY|\text{-a.e.} \qquad \square$$

Theorem 4.1 is completely different from a similar result in Adly, Haddadh and Thibault [1], dealing with a convex moving set of bounded variation.

The next two theorems have connections with results of Edmond and Thibault [10] dealing with the perturbation of the sweeping process associated with a closed convex moving set.

Now we present a Lipschitz version of a Skorokhod differential equation associated with a sweeping process. Our technique is based again on the stability of sweeping process.

**Theorem 4.2.** *Let $C : [0,T] \to \mathrm{ck}(\mathbb{H})$ be a convex compact valued $L$-Lipschitzean mapping. Let $b : [0,T] \times \mathbb{H} \to \mathbb{H}$ satisfying:*

*(i)* $t \mapsto b(t,x)$ *is measurable on* $[0,T]$ *for every fixed* $x \in \mathbb{H}$,
*(ii)* $||b(t,x)|| \leq M, \forall t \in [0,T], \forall x \in C(t)$,
*(iii)* $||b(t,x) - b(t,y)|| \leq M||x-y||, \forall t \in [0,T], \forall x \in C(t), \forall y \in C(t)$,

*for some positive constant $M$.*

*Let $a \in C(0)$. Then there is a unique pair $(X,Y)$ where $X$ is an $(L+2M)$-Lipschizean mapping from $[0,T]$ into $\mathbb{H}$ with $X(t) \in C(t)$ for all $t \in [0,T]$ and $Y$ is an $(L+M)$-Lipschizean mapping from $[0,T]$ into $\mathbb{H}$ satisfying*

$$\begin{cases} X(t) = a + \int_0^t b(s, X(s))ds + Y(t), \quad \forall t \in [0,T] \\ -\dot{Y}(t) \in N_{C(t)}(X(t)), \quad a.e. \;\; t \in [0,T] \\ X(0) = Y(0) = a \in C(0). \end{cases}$$

**Proof.** Let $X_a$ be the unique $L$-Lipschitz solution to

$$\begin{cases} X_a(0) = a \\ -\dot{X}_a(t) \in N_{C(t)}(X_a(t)) \;\; a.e. \;\; t \in [0,T]. \end{cases}$$

The existence and uniqueness of such a solution is clearly ensured by [28, 24]. Let us set

$$X^0(t) = X_a(t), \quad \forall t \in [0,T],$$
$$h_1(t) = a + \int_0^t b(s, X^0(s))ds, \quad \forall t \in [0,T],$$
$$C_{h_1}(t) = C(t) - h_1(t), \quad \forall t \in [0,T].$$

$h_1$ is $M$-Lipschitz because

$$||b(s, X^0(t)|| \leq M$$

for all $t \in [0,T]$ so that $C_{h_1}$ is an $(L+M)$-Lipschitz mapping. By Theorem 3.2, there is a unique $(L+M)$–Lipschitz solution $Y^1$ to

$$\begin{gathered} -\dot{Y}^1(t) \in N_{C_{h_1}(t)}(Y^1(t)) \ \text{ a.e.} \\ Y^1(0) = a \in C_{h_1}(0) = C(0) \\ ||\dot{Y}^1(t)|| \le (L+M) \ \text{ a.e.} \end{gathered}$$

Set

$$X^1(t) = h_1(t) + Y^1(t) = \int_0^t b(s, X^0(s))ds + Y^1(t), \quad \forall t \in [0,T]$$

so that $X^1$ is $(L+2M)$-Lipschitz with $X^1(t) \in C(t)$, $\forall t \in [0,T]$ and that

$$-\dot{Y}^1(t) \in N_{C(t)}(X^1(t)) \ \text{ a.e.}$$

with $||\dot{X}^1(t)|| \le (L+2M)$ a.e. Now we construct $X^n$ by induction as follows. Let

$$\begin{gathered} h_n(t) = \int_0^t b(s, X^{n-1}(s)ds, \quad t \in [0,T], \\ C_{h_n}(t) = C(t) - h_n(t), \quad t \in [0,T] \end{gathered}$$

with $X^{n-1}(t) \in C(t)$ for all $t \in [0,T]$ so that $||b(s, X^{n-1}(t))|| \le M$ for all $t \in [0,T]$ and that $||\dot{h}_n|| \le M$. By Theorem 3.2, there is an $(L+M)$–Lipschitz solution $Y^n$ to

$$\begin{gathered} -\dot{Y}^n(t) \in N_{C_{h_n}(t)}(Y^n(t)) \ \text{ a.e.} \\ Y^n(0) = X_a(t) \in C_{h_n}(0) = C(0) \\ ||\dot{Y}^n(t)|| \le (L+M) \ \text{ a.e.} \end{gathered}$$

Let us set

$$X^n(t) = h_n(t) + Y^n(t) = a + \int_0^t b(s, X^{n-1}(s))ds + Y^n(t), \quad \forall t \in [0,T]$$

so that $X^n$ is $(L+2M)$-Lipschitz with $X^n(t) \in C(t), \forall t \in [0,T]$ and that

$$-\dot{Y}^n(t) \in N_{C(t)}(X^n(t)) \ \text{ a.e.}$$

with $||\dot{X}^n(t)|| \le (L+2M)$ a.e. So we can infer that $(X^n)$ is relatively compact in $C_{\mathbb{H}}([0,T])$. By extracting a subsequence, we may assume that $(X^n)$ converges uniformly to a continuous function $Z : [0,T] \to \mathbb{H}$. Next by using $(ii)$–$(iii)$, it is not difficult to check that $\int_0^t b(s, X^{n-1}(s))ds \to \int_0^t b(s, Z(s))ds$ uniformly in $t \in [0,T]$ so that by invoking Theorem 3.2 we can assert that $X_n$ converges uniformly to $X$ and $Y^n$ converges uniformly to $Y$ where $X$ is $(2L+M)$-Lipschitz and $Y$ is $(L+M)$-Lipschitz such that

$$-\dot{Y}(t) \in N_{C(t)}(X(t)), \quad \text{a.e. } \ t \in [0,T].$$

Therefore we have that $X(t) = Z(t)$ for all $t \in [0,T]$ and

$$\lim_{n\to\infty} X^n(t) = X(t) = a + \lim_{n\to\infty}\left[\int_0^t b(s, X^{n-1}(s))ds + Y^n(t)\right]$$
$$= a + \lim_{n\to\infty}\int_0^t b(s, X^{n-1}(s))ds + Y(t) = a + \int_0^t b(s, X(s))ds + Y(t). \qquad \square$$

Now we present an absolutely continuous version of a Skorokhod problem.

**Theorem 4.3.** *Let $C : [0,T] \to \mathrm{ck}(\mathbb{H})$ be a convex compact valued absolutely continuous mapping. Let $b : [0,T]\times\mathbb{H} \to \mathbb{H}$ satisfying:*

(*i*) $t \mapsto b(t,x)$ *is measurable on* $[0,T]$ *for every fixed* $x \in \mathbb{H}$,
(*ii*) $||b(t,x)|| \le K(1+||x||), \forall t \in [0,T], \forall x \in C(t)$,
(*iii*) $||b(t,x) - b(t,y)|| \le K||x-y||, \forall t \in [0,T], \forall x \in C(t), \forall y \in C(t)$,

*for some positive constant $K$.*

*Let $a \in C(0)$. Then there exist an absolutely continuous function $X : [0,T] \to \mathbb{H}$ with $X(t) \in C(t)$ for all $t \in [0,T]$ and an absolutely continuous function $Y : [0,T] \to \mathbb{H}$ satisfying*

$$X(t) = \int_0^t b(s, X(s))ds + Y(t), \quad \forall t \in [0,T]$$
$$-\dot{Y}(t) \in N_{C(t)}(X(t)), \quad a.e.$$
$$X(0) = Y(0) = a \in C(0).$$

**Proof.** We follow the same lines as the proof of Theorem 4.2, using now Theorem 3.3. Let $a \in C(0)$. Let $X_a$ be the unique absolutely continuous solution to

$$\begin{cases} X_a(0) = a \\ -\dot{X}_a(t) \in N_{C(t)}(X_a(t)) \ \text{ a.e. } t \in [0,T]. \end{cases}$$

The existence and uniqueness of such a solution is clearly ensured by [28, 24] Let us set $C := \sup_{t\in[0,T]} |C(t)| < \infty$, $M = K(1+C)$ and $X^0(t) = X_a(t), \forall t \in [0,T]$, $h_1(t) = a + \int_0^t b(s, X^0(s))ds, \forall t \in [0,T]$, $C_{h_1}(t) = C(t) - h_1(t), \forall t \in [0,T]$.
It is clear that $C_{h_1}$ is an absolutely continuous mapping because

$$||b(t, X^0(t))|| \le K(1 + X^0(t)) \le K(1+C) := M > 0$$

for all $t \in [0,T]$. Hence there is a unique absolutely continuous solution $Y^1$ to

$$-\dot{Y}^1(t) \in N_{C_{h_1}(t)}(Y^1(t)) \ \text{ a.e.}$$
$$Y^1(0) = a \in C_{h_1}(0) = C(0)$$
$$||\dot{Y}^1(t)|| \le \dot{a}(t) + M \ \text{ a.e.}$$

Set

$$X^1(t) = h_1(t) + Y^1(t) = \int_0^t b(s, X^0(s))ds + Y^1(t), \quad \forall t \in [0, T],$$

so that $X^1$ is absolutely continuous with $X^1(t) \in C(t), \forall t \in [0, T]$ and that

$$-\dot{Y}^1(t) \in N_{C(t)}(X^1(t)) \text{ a.e.}$$

with $||\dot{X}^1(t)|| \leq \dot{a}(t) + 2M$ a.e. Now we construct $X^n$ by induction as follows. Let

$$h_n(t) = \int_0^t b(s, X^{n-1}(s)ds, \quad \forall t \in [0, T],$$
$$C_{h_n}(t) = C(t) - h_n(t), \quad \forall t \in [0, T],$$

with $X^{n-1}(t) \in C(t)$ for all $t \in [0, T]$ so that $||b(s, X^{n-1}(t))|| \leq K(1+||X^{n-1}(t)||) \leq K(1+C) := M$ for all $t \in [0, T]$ and that $||\dot{h}_n|| \leq M$. By Theorem 3.3, there is a unique absolutely continuous solution $Y^n$ to

$$-\dot{Y}^n(t) \in N_{C_{h_n}(t)}(Y^n(t)) \text{ a.e.}$$
$$Y^n(0) = X_a(t) \in C_{h_n}(0) = C(0)$$
$$||\dot{Y}^n(t)|| \leq \dot{a}(t) + M \text{ a.e.}$$

Let us set

$$X^n(t) = h_n(t) + Y^n(t) = a + \int_0^t b(s, X^{n-1}(s))ds + Y^n(t), \quad \forall t \in [0, T]$$

so that $X^n$ is absolutely continuous with $X^n(t) \in C(t), \forall t \in [0, T]$ and that

$$-\dot{Y}^n(t) \in N_{C(t)}(X^n(t)) \text{ a.e.}$$

with $||\dot{X}^n(t)|| \leq \dot{a}(t)+2M$ a.e. As $(X^n)$ is equiabsolutely continuous and $C(t)$ is compact, by Ascoli's theorem, by extracting a subsequence we may assume that $(X^n)$ converges uniformly on $[0, T]$ to a continuous mapping $Z : [0, T] \to H$. Then $\int_0^t b(s, X^{n-1}(s))ds \to \int_0^t b(s, Z(s))ds$ uniformly in $[0, T]$. By invoking Theorem 3.3 we can assert that $(X^n)$ converges uniformly to an absolutely continuous mapping $X : [0, T] \to \mathbb{H}$ and $(Y^n)$ converges uniformly to an absolutely continuous mapping $Y$ such that

$$-\dot{Y}(t) \in N_{C(t)}(X(t)), \quad \text{a.e. } t \in [0, T].$$

Therefore we have that $X(t) = Z(t)$ for all $t \in [0, T]$ and

$$\lim_{n\to\infty} X^n(t) = X(t) = a + \lim_{n\to\infty} \left[\int_0^t b(s, X^{n-1}(s))ds + Y^n(t)\right]$$
$$= a + \lim_{n\to\infty} \int_0^t b(s, X^{n-1}(s))ds + Y(t) = a + \int_0^t b(s, X(s))ds + Y(t). \qquad \square$$

## 5. A stochastic Skorokhod differential equation associated with a convex sweeping process in $\mathbb{R}^d$

In this section, we present a stochastic version in the finite dimensional setting of the Skorokhod problem given in Section 4. This result is similar to that of Bernicot and Venel [4], with different hypotheses and a different method.

In the sequel, $B$ is a standard $\mathbb{R}^m$-valued Brownian motion, defined on a stochastic basis $\mathfrak{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathrm{P})$, and $\mathcal{M}_{d\times m}$ denotes the space of linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^d$ (or the space of $d \times m$ matrices).

From now on, $C : [0,T] \times \Omega \to \mathrm{cc}(\mathbb{R}^d)$ is a closed convex valued random set. As usual with stochastic processes, $C(t,\omega)$ is also denoted by $C(t)$, and we use similar conventions for other random variables. Measurability of a multifunction is meant in the classical Effros sense, see, e.g., [14] or [15].

**Lemma 5.1.** *Let $C : [0,T] \times \Omega \to \mathrm{cc}(\mathbb{R}^d)$ be a closed convex valued random set such that $C(.,\omega)$ satisfies $(*)$ for every $\omega \in \Omega$ and $C(t,.)$ is $\mathcal{F}_t$-measurable for each $t \in [0,T]$. Let $a$ be an $\mathcal{F}_0$-measurable selection of the interior of $C(0)$.*

$(a)$ *The following conditions are equivalent:*

$(C1)$ *$C$ satifies $(**)$ $\mathrm{P}$-a.e., i.e., $\mathrm{P}$-a.e., for every $t \in [0,T]$, $C(t)$ has nonempty interior.*

$(C2)$ *There exists an adapted continuous selection $\gamma$ of $C$, with $\gamma(0) = a$, and an adapted continuous real-valued process $r > 0$ such that, $\mathrm{P}$-a.e., for every $t \in [0,T]$,*

$$\overline{B}(\gamma(t), r(t)) \subset C(t). \tag{11}$$

$(C3)$ *There exists an adapted continuous selection $\gamma$ of $C$, with $\gamma(0) = a$, such that, for some (or equivalently, for any) sequence $(r_n)$ of positive numbers which converges to $0$, there exists a sequence $(\tau_n)$ of stopping times with values in $[0,T]$ which converges a.e. to $T$, such that, $\mathrm{P}$-a.e., for every $t \in [0,T]$, and for every $n$,*

$$t \le \tau_n \Rightarrow \overline{B}(\gamma(t), r_n) \subset C(t). \tag{12}$$

$(b)$ *If furthermore, for some $p > 0$,*

$$\mathrm{E}\left(\sup_{t\in[0,T]} \inf_{x\in C(t)} \|x\|^p\right) < +\infty \quad \textit{and} \quad \mathrm{E}\left(\|a\|^p\right) < +\infty, \tag{13}$$

*then we can choose $\gamma$ in $(C2)$ and $(C3)$ such that*

$$\mathrm{E} \sup_{t\in[0,T]} \|\gamma(t)\|^p < +\infty. \tag{14}$$

**Proof.** $(a)$ We prove $(C1)\Rightarrow(C2)\Rightarrow(C3)\Rightarrow(C1)$.

Assume $(C1)$, and let us prove $(C2)$. Let us denote by $\mathrm{Int}(A)$ the interior of a closed convex subset $A$ of $\mathbb{R}^d$, and by $\mathrm{cci}(\mathbb{R}^d)$ the set of closed convex subsets of $\mathbb{R}^d$ with nonempty interior. Applying Michael's selection theorem ([23, Corollary 1.3], or [22, Theorem 3.1‴]), to the multifunction $\Phi : \mathrm{cci}(\mathbb{R}^d) \rightrightarrows \mathbb{R}^d$ defined by $\Phi(A) = A$ in the same manner as in the proof of Lemma 2.2, we deduce that there exists a continuous mapping $\varphi : \mathrm{cci}(\mathbb{R}^d) \to \mathbb{R}^d$ such that $\varphi(A) \subset \mathrm{Int}(A)$ for every $A \in \mathrm{cci}(\mathbb{R}^d)$. Then the mapping $\widetilde{\gamma}(t) = \varphi(C(t))$ is a continuous adapted selection of $\mathrm{Int}(C)$. We define a continuous adapted selection $\gamma$ starting from $a$ by taking a barycenter of $\widetilde{\gamma}(t)$ and $a$: For every $t \in [0,T]$, let

$$\rho(t) = d\left(a, \mathbb{R}^d \setminus C(t)\right) = \inf\left\{\|x-a\| \,;\, x \notin C(t)\right\},$$
$$\alpha(t) = \frac{\min\left\{\rho(0), \rho(t)\right\}}{\rho(0)}, \qquad \beta(t) = 1-\alpha(t).$$

Then $\alpha$ and $\beta$ are continuous, adapted, with values in $[0,1]$ such that $\alpha+\beta = 1$, $\alpha(0) = 1$, and $\beta(t) = 1$ whenever $a \notin \mathrm{Int}(C(t))$. We only need to set

$$\gamma(t) = \alpha(t)a + \beta(t)\widetilde{\gamma}(t).$$

Now, the process $r = d\left(\gamma(t), \mathbb{R}^d \setminus C(t)\right)$ has continuous trajectories with positive values and satisfies (11), which proves $(C2)$.

Assume $(C2)$, and let $(r_n)$ be any sequence of positive numbers which converges to 0. Let $\gamma$ and $r$ be as in (11). Let us denote $\rho = \min_{t\in[0,T]} r(t)$, and let $\Omega_0$ be the almost sure subset of $\Omega$ on which (11) holds. Set

$$\tau_n = \inf\left\{t \in [0,T];\, \overline{B}(\gamma(t), r_n) \not\subset C(t)\right\}$$

(with $\inf\emptyset = T$). The random variable

$$m = \min\{r(t);\, t \in [0,T]\}$$

takes positive values on $\Omega_0$, thus, for every $\omega \in \Omega_0$, there exists an integer $N(\omega)$ such that, for all $n \geq N(\omega)$,

$$\overline{B}(\gamma(t,\omega), r_n) \subset \overline{B}(\gamma(t,\omega), m(\omega)) \subset C(t,\omega),$$

i.e., $\tau_n(\omega) = T$ for $n \geq N(\omega)$. This proves $(C3)$.

Assume $(C3)$, for some sequence $(r_n)$. For every $t \in [0,T]$, we have, P-a.e.,

$$\mathrm{P}\left(\mathrm{Int}(C(t)) \neq \emptyset\right) \geq \mathrm{P}\left(\cup_n \left\{\overline{B}(\gamma(t), r_n) \subset C(t)\right\}\right) = 1,$$

because, for $n$ large enough (depending on the variable $\omega \in \Omega$), we have $\tau_n \geq t$. By continuity of $C$, we can swap the quantifiers "P-a.e." and "for all $t \in [0,T]$" (because it is sufficient to consider only the values taken by $C(t)$ for rational $t$), which proves $(C1)$.

$(b)$ In the case when (13) is satisfied, let us define, for every $t \in [0,T]$,

$$\rho(t) = \sup_{s\in[0,t]} \inf_{x\in C(s)} \|x\| + \|a\| + 1,$$
$$\widetilde{C}(t) = C(t) \cap \overline{B}(0,\rho(t)).$$

By [27, §9, Proposition] (or [14, Proposition 3.3.1]), the multifunction $\widetilde{C}$ is measurable. Furthermore, $\widetilde{C}$ has convex values with nonempty interior, thus, for every $\omega \in \Omega$, by [9, Proposition 2.3], the multifunction $\widetilde{C}(.,\omega)$ is continuous for the Hausdorff distance. Furthermore, we have

$$\mathrm{E}\left(\sup_{t\in[0,T]}\left(d_H(0,\widetilde{C}(t)^p\right)\right) \leq \mathrm{E}\rho(T)^p < +\infty.$$

As $a \in \mathrm{Int}(\widetilde{C}(0))$, we can thus construct $\gamma$ and $r$ as in $(a)$, replacing $C$ by $\widetilde{C}$. $\square$

**Remark.** The proof of Lemma 5.1 shows that, in Condition $(C3)$, we can choose the stopping times $\tau_n$ such that, for almost every $\omega \in \Omega$, $\tau_n(\omega) = T$ for $n$ large enough.

**Theorem 5.2.** *Let $C : [0,T] \times \Omega \to \mathrm{cc}(\mathbb{R}^d)$ be a closed convex valued random set such that $C(.,\omega)$ satisfies $(*)$ and $(**)$ for* P*-almost every $\omega \in \Omega$ and $C(t,.)$ is $\mathcal{F}_t$-measurable for each $t \in [0,T]$. Let $a$ be an $\mathcal{F}_0$-measurable selection of $C(0)$. We assume that $C$ and $a$ satisfy* (13) *with $p = 4$. Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}_{d\times m}$ be continuous mappings satisfying, for some $\mathfrak{c} > 0$,*

$$\|b(t,x)\| + \|\sigma(t,x)\| \leq \mathfrak{c}(1 + \|x\|^{1/2}), \tag{15}$$

*for every $t \in [0,T]$ and every $x \in \mathbb{R}^d$.*

$(a)$ *There exists a pair $(X,Y)$ of $\mathbb{R}^d$-valued stochastic processes, possibly defined on an extension of $\mathfrak{B}$, such that $Y$ has BV continuous trajectories and, almost surely,*

$$\begin{cases} X_0 = Y_0 = a, \\ X(t) = \displaystyle\int_0^t b(s,X(s))\,ds + \int_0^t \sigma(s,X(s))\,dB(s) + Y(t), \quad \forall t \in [0,T], \\ X(t) \in C(t), \quad \forall t \in [0,T], \\ \dfrac{dDY}{d\,|DY|}(t) \in -N_{C(t)}(X(t)), \quad |DY|\text{-a.e.} \end{cases} \tag{16}$$

$(b)$ *If furthermore $b$ and $\sigma$ satisfy*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq \mathfrak{d}\,\|x - y\| \tag{17}$$

*for some $\mathfrak{d} > 0$, and for every $t \in [0,T]$ and all $x, y \in \mathbb{R}^d$, then the pair $(X,Y)$ is unique and defined on $\mathfrak{B}$.*

Before proving Theorem 5.2, let us state some useful results.

**Lemma 5.3.** *Let us denote by $\Phi$ the mapping $h \mapsto v_h$ of* (1) *and Theorem* 2.3*, and by $\Psi$ the mapping $h \mapsto w_h$ (these mappings are random because $C$ is random too). Let $h$ be an adapted continuous stochastic process with $h_0 = 0$.*

($a$) *The processes $v_h = \Phi(h)$ and $w_h = \Psi(h)$ are adapted, $\Phi(h)$ has continuous trajectories, and $\Psi(h)$ has BVC trajectories.*

($b$) *Let $\gamma$ be a continuous adapted selection of the interior of $C$, and let $r$ be an adapted positive continuous process such that, for every $t \in [0,T]$,*

$$\overline{B}(\gamma(t), r(t)) \subset C(t) \text{ P-}a.e.$$

*Let us denote $\|Dw_h\|_{[s,t]}$ the total variation of $w_h$ on an interval $[s,t]$. Let $\sigma$ and $\tau$ be random times such that $0 \le \sigma \le \tau \le T$. We have*

$$\|Dw_h\|_{[\sigma,\tau]} \le \sup_{s\in[\sigma,\tau]} l(r(s), \|\gamma(s) - h(s)\|). \tag{18}$$

**Proof.** ($a$) This a direct consequence of the construction of $(v_h, w_h)$ in (1) and Theorem 2.3.

($b$) The estimation (18) is deterministic in nature, so we argue with fixed $\omega \in \Omega$.

By uniform continuity of $C(.,\omega)$, $\gamma(.,\omega)$ and $h(.,\omega)$ on $[0,T]$, there exists a partition $\sigma(\omega) = t_0 < t_1 < \cdots < t_n = \tau$ of $[\sigma(\omega), \tau(\omega)]$ such that, for $i = 0, \ldots, n-1$ and, for every $t \in [t_i, t_{i+1}]$,

$$\overline{B}\left(\gamma(t_i,\omega) - h(t_i,\omega), r(t_i,\omega)\right) \subset C_h(t,\omega).$$

By Proposition 2.1, we have

$$\|Dw_h(.,\omega)\|_{[t_i,t_{i+1}]} \le l(r(t_i,\omega), \|\gamma(t_i,\omega) - h(t_i,\omega)\|),$$

which yields (18). □

**Proof of Theorem 5.2.** *I. Proof of* ($a$) *under an extra condition* ($C4$). Let us prove Part ($a$) of Theorem 5.2 in the case when the following condition is satisfied, which is slightly stronger than Conditions ($C1$)–($C3$) of Lemma 5.1, because it assumes that the process $r(t)$ is bounded from below by a constant $r > 0$:

($C4$) There exist $r > 0$ and an adapted continuous selection $\gamma$ of $C$ such that, P-a.e. and for every $t \in [0,T]$,

$$\overline{B}(\gamma(t), r) \subset C(t) \tag{19}$$

and

$$\mathrm{E}\left(\sup_{t\in[0,T]} \|\gamma(t)\|^2\right) < +\infty. \tag{20}$$

*First step: construction of a tight sequence of approximating solutions via Tonelli'scheme.*

For each integer $n \geq 1$, we can construct a process $X_n$ by setting, with the notations of Lemma 5.3, $X_n(t) = \Phi(0)(t)$ if $0 \leq t \leq 1/n$ and, for $1/n \leq t \leq T$,

$$X_n(t) = \Phi\left(\int_0^\cdot b(s, X_n(s))\, ds + \int_0^\cdot \sigma(s, X_n(s))\, dB(s)\right)(t - 1/n). \tag{21}$$

Note that, once $X_n$ is known on an interval $[0,t] \subset [0,T]$, Equation (21) defines $X_n$ on $[0, t+1/n]$, thus $X_n$ is well defined and continuous on $[0,T]$. Let us denote $Z_n(t) = 0$ if $t \in [0, 1/n]$ and, for $t \geq 1/n$,

$$Z_n(t) = \int_0^{t-1/n} b(s, X_n(s))\, ds + \int_0^{t-1/n} \sigma(s, X_n(s))\, dB(s).$$

With the notations of (1) and Lemma 5.3, we thus have $X_n = Y_n + Z_n$, where $X_n(t) = v_{Z_n}(t) = \Phi(Z_n)(t)$ and $Y_n(t) = w_{Z_n}(t) = \Psi(Z_n)(t)$. By Lemma 5.3 we have, for each $n$,

$$\|DY_n\|_{[0,t]} \leq \rho_n(t), \tag{22}$$

where $\rho_n$ is adapted and defined by

$$\rho_n(t) = \sup_{s\in[0,t]} l(r, \|\gamma(s) - Z_n(s)\|). \tag{23}$$

In particular, we have

$$\sup_{s\in[0,t]} \|Y_n(s)\| \leq \|a\| + \rho_n(t) \in \mathrm{L}^2(\Omega, \mathcal{F}, \mathrm{P}). \tag{24}$$

This entails, using the inequality $x \leq 1 + x^2$ with $x = \|Z(s)\|$,

$$\begin{aligned} \sup_{s\in[0,t]} \|X_n(s)\| &\leq \|a\| + \rho_n(t) + \sup_{s\in[0,t]} \|Z_n(s)\| \\ &\leq \|a\| + \sup_{s\in[0,t]} \frac{\|Z_n(s)\|^2}{2r} + \sup_{s\in[0,t]} \frac{\|\gamma(s)\|^2}{2r} + \sup_{s\in[0,t]} \|Z_n(s)\| \\ &\leq \left(\frac{1}{2r} + 1\right) \sup_{s\in[0,t]} \|Z_n(s)\|^2 + A, \end{aligned} \tag{25}$$

where, by (14), we can take

$$A = \|a\| + 1 + \sup_{t\in[0,T]} \frac{\|\gamma(t)\|^2}{2r} \in \mathrm{L}^2(\Omega, \mathcal{F}, \mathrm{P}). \tag{26}$$

Let $p \geq 2$. Using Burkholder-Davis-Gundy and Jensen's inequalities, we get, for some constant $\kappa_p$,

$$\begin{aligned}
&\mathrm{E}\left(\sup_{s\in[0,t]}\|Z_n(s)\|^p\right)\\
&\leq 2^{p-1}\mathrm{E}\left(\sup_{s\in[0,t]}\left\|\int_0^{(s-1/n)_+} b(u,X_n(u))du\right\|^p\right)\\
&\qquad + 2^{p-1}\mathrm{E}\left(\sup_{s\in[0,t]}\left\|\int_0^{(s-1/n)_+} \sigma(u,X_n(u))\,dB(u)\right\|^p\right)\\
&\leq 2^{p-1}t^{p-1}\mathrm{E}\int_0^{(t-1/n)_+}\sup_{u\in[0,s]}\|b(u,X_n(u))\|^p\,du\\
&\qquad + 2^{p-1}\kappa_p\mathrm{E}\left(\int_0^{(t-1/n)_+}\|\sigma(s,X_n(s))\|^2\,ds\right)^{p/2}\\
&\leq 2^{p-1}t^{p-1}\mathrm{E}\int_0^t\sup_{u\in[0,s]}\|b(u,X_n(u))\|^p\,du + 2^{p-1}\kappa_p\,t^{\frac{p}{2}-1}\mathrm{E}\int_0^t\|\sigma(s,X_n(s))\|^p\,ds\\
&\leq 4^{p-1}T^p\mathfrak{c}^p + 4^{p-1}T^{p-1}\mathfrak{c}^p\mathrm{E}\int_0^t\sup_{u\in[0,s]}\|X_n(u)\|^{p/2}\,du\\
&\qquad + 4^{p-1}\kappa_pT^p\mathfrak{c}^p + 4^{p-1}\kappa_pT^{p-1}\mathfrak{c}^p\mathrm{E}\int_0^t\|X_n(s)\|^{p/2}\,ds\\
&\leq 4^{p-1}T^{p-1}\mathfrak{c}^p\,(1+\kappa_p)\left(T+\mathrm{E}\int_0^t\sup_{u\in[0,s]}\|X_n(u)\|^{p/2}\,du\right).
\end{aligned}$$

Then, by (25), we have

$$\begin{aligned}
&\mathrm{E}\left(\sup_{s\in[0,t]}\|Z_n(s)\|^p\right)\\
&\leq 4^{p-1}T^{p-1}\mathfrak{c}^p\,(1+\kappa_p)\Bigg(T+2^{\frac{p}{2}-1}T\mathrm{E}(A^{p/2})\\
&\qquad +2^{\frac{p}{2}-1}\left(\frac{1}{2r}+1\right)^{p/2}\mathrm{E}\int_0^t\sup_{u\in[0,s]}\|Z_n(u)\|^p\;ds\Bigg)\\
&:= \mathfrak{a}+\mathfrak{b}\mathrm{E}\int_0^t\sup_{u\in[0,s]}\|Z_n(u)\|^p\;ds.
\end{aligned}$$

By Gronwall's lemma, this yields

$$\mathrm{E}\left(\sup_{s\in[0,t]}\|Z_n(s)\|^p\right)\leq\mathfrak{a}e^{\mathfrak{b}t}.$$

By (26), $\mathfrak{b}$ is finite for $p = 4$, which shows that

$$\begin{cases} \sup_n \mathrm{E}\left(\sup_{t\in[0,T]} \|Z_n(t)\|^4\right) < +\infty, \\ \sup_n \mathrm{E}\left(\sup_{t\in[0,T]} \rho_n^2(t)\right) = \sup_n \mathrm{E}\left(\rho_n^2(T)\right) < +\infty, \\ \sup_n \mathrm{E}\left(\sup_{t\in[0,T]} \|X_n(t)\|^2\right) < +\infty. \end{cases} \tag{27}$$

Let us now prove that $(Z_n)$, considered as a sequence of $\mathrm{C}([0,T];\mathbb{R}^d)$-valued random variables, is tight (here and in the sequel, $\mathrm{C}([0,T];\mathbb{R}^d)$ is endowed with the topology of uniform convergence). Recall that a sequence $(U_n)$ of random variables with values in a topological space $\mathbb{X}$ is said to be tight if, for every $\epsilon > 0$, there exists a compact subset $\mathcal{K}$ of $\mathbb{X}$ such that $\mathrm{P}(U_n \notin \mathcal{K}) \le \epsilon$ for every $n$.

Let $\mathbb{T}$ denote the set of stopping times with values in $[0,T]$. For $\sigma, \tau \in \mathbb{T}$, we have, for any $n$,

$$\begin{aligned} &\mathrm{E}\,\|Z_n(\tau) - Z_n(\sigma)\|^2 \\ &\le 2\mathrm{E}\left(\int_{\sigma\wedge\tau}^{\sigma\vee\tau} \|b(s, X_n(s))\|\, ds\right)^2 + 2\mathrm{E}\int_{\sigma\wedge\tau}^{\sigma\vee\tau} \|\sigma(s, X_n(s))\|^2\, ds \\ &\le 8\mathfrak{c}^2 \mathrm{E}\int_{\sigma\wedge\tau}^{\sigma\vee\tau} (1 + \|X_n(s)\|)\, ds \\ &\le 8\mathfrak{c}^2\left(\mathrm{E}\,|\tau-\sigma| + \left(\mathrm{E}\int_0^T \|X_n(s)\|^2\, ds\right)^{1/2} \left(\mathrm{E}\,|\tau-\sigma|^2\right)^{1/2}\right). \end{aligned}$$

Therefore, for every $\delta > 0$,

$$\begin{aligned} &\sup_{n\ge 1} \sup_{\substack{\sigma,\tau\in\mathbb{T} \\ 0\le|\tau-\sigma|\le\delta}} \mathrm{E}\,\|Z_n(\tau) - Z_n(\sigma)\|^2 \\ &\le 8\mathfrak{c}^2\delta\left(1 + T\sup_n \mathrm{E}\left(\sup_{t\in[0,T]} \|X_n(t)\|\right)\right) \to 0 \quad \text{as } \delta \to 0. \end{aligned} \tag{28}$$

By a criterion of Aldous ([2], see also [16, Theorem VI.3.26 and Proposition VI.4.5]), we deduce from (28) and (27) that $(Z_n)_{n\ge 1}$ is tight in $\mathrm{C}([0,T];\mathbb{R}^d)$.

To show that $(X_n)$ is tight, we use the reasoning of Theorem 2.3. Let $\sigma$ and $\tau$ be stopping times, with $0 \le \sigma \le \tau \le T$. Let $m \ge 1$ be fixed. From Equation (5) we get, for any $n \ge 1$,

$$\int_\sigma^\tau \langle X_n - X_m, D(Y_n - Y_m)\rangle \le 0 \quad \text{a.e.} \tag{29}$$

We have also

$$\begin{aligned}
&\langle (Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma), (Y_n - Y_m)(\tau) - (Y_n - Y_m)(\sigma)\rangle \\
= &\int_\sigma^\tau \langle (Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma), D(Y_n - Y_m)(s)\rangle \\
= &\int_\sigma^\tau \langle (Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma) - (Z_n - Z_m)(s), D(Y_n - Y_m)(s)\rangle \\
&\quad + \int_\sigma^\tau \langle (Z_n - Z_m)(s), D(Y_n - Y_m)(s)\rangle
\end{aligned} \tag{30}$$

and

$$\|(Y_n - Y_m)(\tau) - (Y_n - Y_m)(\sigma)\|^2 = 2\int_\sigma^\tau \langle (Y_n - Y_m)(s), D(Y_n - Y_m)(s)\rangle\,. \tag{31}$$

Since $X_n - X_m = (Y_n - Y_m) + (Z_n - Z_m)$, we get, combining (29), (30) and (31),

$$\begin{aligned}
&\|(X_n - X_m)(\tau) - (X_n - X_m)(\sigma)\|^2 \\
= &\ \|(Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma)\|^2 \\
&\quad + 2\langle (Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma), (Y_n - Y_m)(\tau) - (Y_n - Y_m)(\sigma)\rangle \\
&\quad + \|(Y_n - Y_m)(\tau) - (Y_n - Y_m)(\sigma)\|^2 \\
= &\ \|(Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma)\|^2 \\
&\quad + 2\int_\sigma^\tau \langle (Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma) - (Z_n - Z_m)(s), D(Y_n - Y_m)(s)\rangle \\
&\quad + 2\int_\sigma^\tau \langle (Z_n - Z_m)(s), D(Y_n - Y_m)(s)\rangle \\
&\quad + 2\int_\sigma^\tau \langle (Y_n - Y_m)(s), D(Y_n - Y_m)(s)\rangle \\
= &\ \|(Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma)\|^2 \\
&\quad + 2\int_\sigma^\tau \langle (Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma) - (Z_n - Z_m)(s), D(Y_n - Y_m)(s)\rangle \\
&\quad + 2\int_\sigma^\tau \langle (X_n - X_m)(s), D(Y_n - Y_m)(s)\rangle \\
\leq &\ \|(Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma)\|^2 \\
&\quad + 2\int_\sigma^\tau \langle (Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma) - (Z_n - Z_m)(s), D(Y_n - Y_m)(s)\rangle\,.
\end{aligned}$$

Using (22), we deduce

$$\begin{aligned}
&\|(X_n - X_m)(\tau) - (X_n - X_m)(\sigma)\|^2 \\
\leq &\|(Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma)\|^2 \\
&\quad + 2\rho(T)\,\|(Z_n - Z_m)(\tau) - (Z_n - Z_m)(\sigma)\| \\
&\quad - 2\int_\sigma^\tau \langle (Z_n - Z_m)(s), D(Y_n - Y_m)(s)\rangle\,.
\end{aligned} \tag{32}$$

For $s \in [0,T]$, let us denote $\tilde{s}^n = (s-\frac{1}{n})_+$ and $\tilde{s}^m = (s-\frac{1}{m})_+$. By the integration by parts formula for Itô's integral, we have, since $Z_n - Z_m$ has finite variation,

$$\begin{aligned}
&\int_\sigma^\tau \langle (Z_n - Z_m)(s), D(Y_n - Y_m)(s) \rangle \\
= \; & \langle (Z_n - Z_m)(\tau), (Y_n - Y_m)(\tau) \rangle - \langle (Z_n - Z_m)(\sigma), (Y_n - Y_m)(\sigma) \rangle \\
& \quad - \int_\sigma^\tau \langle (Y_n - Y_m)(s), D(Z_n - Z_m)(s) \rangle \\
= \; & \langle (Z_n - Z_m)(\tau), (Y_n - Y_m)(\tau) \rangle - \langle (Z_n - Z_m)(\sigma), (Y_n - Y_m)(\sigma) \rangle \\
& \quad - \int_\sigma^\tau \langle (Y_n - Y_m)(s), (b(\tilde{s}^n, X_n(\tilde{s}^n)) - b(\tilde{s}^m, X_m(\tilde{s}^m))) \rangle \, ds \\
& \quad - \int_\sigma^\tau \langle (Y_n - Y_m)(s), \sigma(\tilde{s}^n, X_n(\tilde{s}^n)) \, dB(\tilde{s}^n) \rangle \\
& \quad + \int_\sigma^\tau \langle (Y_n - Y_m)(s), \sigma(\tilde{s}^m, X_m(\tilde{s}^m)) \, dB(\tilde{s}^m) \rangle .
\end{aligned}$$

Using (24), this leads to

$$\begin{aligned}
& \mathrm{E} \left| \int_\sigma^\tau \langle (Z_n - Z_m)(s), D(Y_n - Y_m)(s) \rangle \right| \\
\leq \; & 2\mathrm{E} \Big( (\|a\| + \rho_n(T)) \left( \|Z_n(\tau) - Z_n(\sigma)\| + \|Z_m(\tau) - Z_m(\sigma)\| \right) \Big) \\
& \quad + \mathrm{E} \int_\sigma^\tau |\langle (Y_n - Y_m)(s), (b(s, X_n(\tilde{s}^n)) - b(s, X_m(\tilde{s}^m))) \rangle| \, ds \\
& \quad + \mathrm{E} \left( \int_\sigma^\tau \langle (Y_n - Y_m)(s), \sigma(\tilde{s}^n, X_n(\tilde{s}^n)) \rangle^2 \, ds \right)^{1/2} \\
& \quad + \mathrm{E} \left( \int_\sigma^\tau \langle (Y_n - Y_m)(s), \sigma(\tilde{s}^m, X_n(\tilde{s}^m)) \rangle^2 \, ds \right)^{1/2} \\
\leq \; & 2\mathrm{E} \Big( (\|a\| + \rho_n(T)) \left( \|Z_n(\tau) - Z_n(\sigma)\| + \|Z_m(\tau) - Z_m(\sigma)\| \right) \Big) \\
& \quad + 2\mathfrak{c} \int_\sigma^\tau \mathrm{E} \left( \sup_{k\geq 1} \sup_{s\in[0,T]} \|Y_n(s)\| \sup_{k\geq 1} \sup_{s\in[0,T]} (1 + \|X_n(s)\|^{1/2}) \right) \, ds \\
& + 2\mathfrak{c} \left( \int_\sigma^\tau \mathrm{E} \left( \sup_{k\geq 1} \sup_{s\in[0,T]} \|Y_n(s)\|^2 \sup_{k\geq 1} \sup_{s\in[0,T]} (2 + \|X_n(s)\|) \right) \, ds \right)^{1/2} .
\end{aligned}$$

By (27), all terms in the preceding integrals are uniformly integrable. Thus, by (28), we have

$$\sup_{n\geq 1} \sup_{\sigma,\tau\in\mathbb{T}0\leq|\tau-\sigma|\leq\delta} \mathrm{E} \left| \int_\sigma^\tau \langle (Z_n - Z_m)(s), D(Y_n - Y_m)(s) \rangle \right| \to 0 \;\text{ as } \delta \to 0.$$

Using again (28), we deduce that

$$\sup_{n\geq 1} \sup_{\sigma,\tau\in\mathbb{T}0\leq|\tau-\sigma|\leq\delta} \mathrm{E} \left\| (X_n - X_m)(\tau) - (X_n - X_m)(\sigma) \right\|^2 \to 0 \;\text{ as } \delta \to 0. \tag{33}$$

By (27), we can apply Aldous' criterion ([2] or [16, Theorem VI.3.26 and Proposition VI.4.5]), which proves that $(X_n - X_m)_{n\geq 1}$ is tight in $\mathrm{C}([0,T];\mathbb{R}^d)$. As the $\mathrm{C}([0,T];\mathbb{R}^d)$-valued random variable $X_m$ (for fixed $m$) is necessarily tight, we deduce that $(X_n)_{n\geq 1}$ is tight in $\mathrm{C}([0,T];\mathbb{R}^d)$. Thus $(Y_n) = (X_n - Z_n)_{n\geq 1}$ is tight too.

Furthermore, by (23), $\rho_n$ is a continuous functional of $Z_n$ and $\gamma$, thus the sequence $(\rho_n)$ is tight in the space $\mathrm{C}([0,T];\mathbb{R})$ endowed with the topology of uniform convergence. This implies that any limit in distribution of $(Y_n)$ has P-a.e. finite variation. Indeed, if this was not the case, there would exist a subsequence of $(Y_n)$ converging in distribution to some $Y$ with $\mathrm{P}(\|DY\| = +\infty) \geq \delta > 0$. Let us denote for simplicity $(Y_n)$ this subsequence. Since the mapping $u \mapsto \|Du\|$, $\mathrm{C}([0,T];\mathbb{R}) \to \mathbb{R}$, is lower semicontinuous (this easy result is contained in Helly's theorem), we would have, for any number $M > 0$, $\mathrm{P}(\|DY_n\| \geq M) \geq \delta/2$ for $n$ large enough. But then, by (22), we would have $\mathrm{P}(\rho_n(T) \geq M) \geq \delta/2$ for $n$ large enough. This would contradict the fact that $(\rho_n)$, or a subsequence of $(\rho_n)$, converges in distribution in the space $\mathrm{C}([0,T];\mathbb{R})$.

*Second step: construction of a weak solution using Young measures.* Let us denote by $\mathbb{X}$ the space of trajectories of $X_n$, $Y_n$ and $Z_n$, i.e., $\mathbb{X} = \mathrm{C}([0,T];\mathbb{R}^d)$. We are going to construct a solution $(X, Y)$ to (16) on an extended stochastic basis

$$\underline{\mathfrak{B}} = (\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_{0\leq t\leq T}, \mu),$$

where $\mu$ is a Young measure on $\mathbb{X}$ based on P. The idea of using Young measures to construct weak solutions to stochastic equations goes back to Pellaumail [29], who reinvented Young measures under the name of "rules". We hope to convince the reader that this is a very natural way to construct weak solutions.

We refer the reader interested in Young measures theory to the nice introductions by Valadier and Balder [33, 3], or to the recent self-contained book by Florescu and Godet-Thobie [11], or the monograph [8].

Let us denote by $\mathcal{M}^1_+(\mathbb{X})$ the set of Borel probability measures on $\mathbb{X}$, endowed with the narrow topology, i.e., the coarsest topology such that the mapping $\nu \mapsto \nu(f)$ is continuous for every bounded continuous function $f : \mathbb{X} \to \mathbb{R}$. Recall that a Young measure $\mu$ on $\mathbb{X}$ based on P is a measurable mapping

$$\begin{cases} \Omega \to \mathcal{M}^1_+(\mathbb{X}) \\ \omega \mapsto \mu_\omega. \end{cases}$$

We denote by $\mathcal{Y} := \mathcal{Y}(\Omega, \mathcal{F}, \mathrm{P}; \mathcal{M}^1_+(\mathbb{X}))$ the space of Young measures on $\mathbb{X}$ based on P. This space is endowed with the stable topology, i.e., the coarsest topology such that the mapping

$$\begin{cases} \mathcal{Y} \to \mathbb{R} \\ \mu \mapsto \int_\Omega \int_{\mathbb{X}} f(\omega, u)\, d\mu_\omega(u)\, d\mathrm{P}(\omega) \end{cases}$$

is continuous for every bounded measurable mapping $f : \Omega \times \mathbb{X} \to \mathbb{R}$ such that $f(\omega, .)$ is continuous for every $\omega \in \Omega$. Each random variable $U : \Omega \to \mathbb{X}$ can be identified with the Young measure $\omega \mapsto \delta_{U(\omega)}$, where $\delta_x$ denotes the Dirac measure at $x$. A variant for Young measures of Prohorov's compactness criterion implies that, if a sequence $(U_n)$ of $\mathbb{X}$-valued random variables is tight, then it is sequentially relatively compact in $\mathcal{Y}$, i.e., from any subsequence of $(U_n)$ one can extract a further subsequence $(U'_n)$ which converges in $\mathcal{Y}$ to some Young measure $\mu$.

From the result of the first step, we can extract a subsequence of $(X_n)$ (still denoted by $(X_n)$ for simplicity) which stably converges to some Young measure $\mu \in \mathcal{Y}_{\mathbb{X}} := \mathcal{Y}(\Omega, \mathcal{F}, \mathrm{P}; \mathcal{M}^1_+(\mathbb{X}))$. In particular, $(X_n)$ converges in probability distribution to the marginal measure of $\mu$ on $\mathbb{X}$.

Let us now construct an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathrm{P})$.

Let $\mathcal{C}$ the Borel $\sigma$-algebra of $\mathbb{X}$, and, for each $t \in [0, T]$, let $\mathcal{C}_t$ be the sub-$\sigma$-algebra of $\mathcal{C}$ generated by the restrictions to $[0, t]$ of elements of $\mathbb{X}$. We define a stochastic basis

$$\underline{\mathfrak{B}} = (\underline{\Omega}, \underline{\mathcal{F}}, (\underline{\mathcal{F}}_t)_{0\le t\le T}, \mu)$$

by

$$\underline{\Omega} = \Omega \times \mathbb{X}, \qquad \underline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{C} \qquad \underline{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{C}_t.$$

We can identify $\mathcal{F}$ (respectively $\mathcal{F}_t$) with the sub-$\sigma$-algebra $\{A \times \mathbb{X};\ A \in \mathcal{F}\}$ (respectively with $\{A\times\mathbb{X};\ A \in \mathcal{F}_t\}$) of $\underline{\mathcal{F}}$ (respectively of $\underline{\mathcal{F}}_t$). Random processes defined on $[0, T] \times \Omega$ are extended to $[0, T] \times \underline{\Omega}$ in a natural way: For $t \in [0, T]$ and $(\omega, u) \in \underline{\Omega}$, we set

$$C(t, \omega, u) = C(t, \omega), \qquad B(t, \omega, u) = B(t, \omega),$$
$$X_n(t, \omega, u) = X_n(t, \omega) \quad (n \ge 1).$$

The processes $Y_n$ and $Z_n$ are extended similarly. In this way, $(\mathcal{F}_t)$-adapted processes remain $(\mathcal{F}_t)$-adapted, and thus are $(\underline{\mathcal{F}}_t)$-adapted.

We now define $X$ on $\underline{\Omega}$ as an $\mathbb{X}$-valued random element by

$$X(\omega, u) = u,$$

i.e., $X(t, \omega, u) = u(t)$, $t \in [0, T]$. The idea behind this construction is that the given random elements of the problem, i.e., the Brownian motion $B$ and the random moving set $C$, are not necessarily sufficient to "explain" the solution $X$, and that we have to add some randomness. This is done by defining, for each $\omega \in \Omega$, a new probability measure $\mu_\omega$, and then by drawing $X(\omega, .)$ accordingly.

Clearly, $X$ is $(\underline{\mathcal{F}}_t)$–adapted, and the distribution of $X$ is the projection of $\mu$ on $\mathbb{X}$, thus $(X_n)$ converges in distribution to $X$. Furthermore, it is easy to check that $B$ remains an $(\underline{\mathcal{F}}_t)$-Brownian motion under $\mu$ (see, e.g., [17]), and that the sequence $(X_n, B)$ converges in distribution to $(X, B)$ in $\mathrm{C}([0, T]; \mathbb{R}^d) \times \mathrm{C}([0, T]; \mathbb{R}^m)$. From the continuity of $b(t, .)$ and $\sigma(t, .)$, the sequence $((b(s, X_n), \sigma(s, X_n)))$ converges in distribution in $\mathrm{C}([0,T];\mathbb{R}^d\times\mathcal{M}_{d\times m})$ to $(b(s,X),\sigma(s,X))$. Using [18, Théorème 26], we deduce that

$$\left(\int_0^{\cdot} b(s,X_n(s))\,ds+\int_0^{\cdot}\sigma(s,X_n(s))\,dB(s)\right)$$

converges in distribution to $Z$ defined on $\underline{\mathfrak{B}}$ by

$$\int_0^{\cdot} b(s,X(s))\,ds+\int_0^{\cdot}\sigma(s,X(s))\,dB(s).$$

This sequence is slightly different from $(Z_n)$, but we have also

$$\begin{aligned}\mathrm{E}\left\|\int_{t-1/n}^{t}\sigma(s,X_n(s))\,dB(s)\right\|^2 &= \mathrm{E}\int_{t-1/n}^{t}\|\sigma(s,X_n(s))\|^2\,ds\\ &\le \mathfrak{c}^2\mathrm{E}\int_{t-1/n}^{t}\left(1+\|X_n(s)\|^{1/2}\right)^2 ds\\ &\le 2\mathfrak{c}^2\mathrm{E}\int_{t-1/n}^{t}\left(1+\|X_n(s)\|\right) ds\\ &\to 0 \quad \text{as } n\to 0\end{aligned}$$

because, by (27), the sequence $(1+\|X_n(s)\|)$ is uniformly integrable on $[0,T]\times\Omega$. A similar calculation shows that

$$\lim_{n\to\infty}\mathrm{E}\left\|\int_{t-1/n}^{t} b(s,X_n(s))\,ds\right\|^2=0.$$

We deduce that $(Z_n)$ converges in distribution to $Z$, and thus $(X_n,Y_n,Z_n)$ converges in distribution to $(X,Y,Z)$, where $Y:=X-Z$. This proves

$$X(t)=\int_0^t b(s,X(s))\,ds+\int_0^t\sigma(s,X(s))\,dB(s)+Y(t),\quad \forall t\in[0,T].$$

Extracting if necessary a further subsequence of $(X_n)$, we have also that the sequence $(X_n,C)$ converges in distribution to $(X,C)$ in $\mathrm{C}([0,T];\mathbb{R}^d)\times\mathrm{C}([0,T];\mathrm{cc}(\mathbb{R}^d))$, where $\mathrm{cc}(\mathbb{R}^d)$ is endowed with a topology which generates the Effros $\sigma$-algebra, e.g., the Wijsman topology. Let $(x_i)$ be a dense sequence in $\mathbb{R}^d$. For each $i$, the continuous processes $d(x_i,X_n)-d(x_i,C)\ge 0$ converge in distribution to $d(x,X)-d(x,C)$, which proves that $X(t)\subset C(t)$ for every $t\in[0,T]$, $\mu$-a.e.

We have seen that $Y$ has BV continuous trajectories $\mu$-a.e. To prove (16), there only remains to show that $\frac{dDY}{d|DY|}(t)\in -N_{C(t)}(X(t))$, $|DY|$-a.e. Let $\eta$ be an $(\underline{\mathcal{F}}_t)$-adapted (or $(\mathcal{F}_t)$-adapted, it doesn't matter here) continuous selection of $C$. As $Y_n=\Psi(X_n)$, we have, for each integer $n$, and all $s,t\in[0,T]$ such that $s<t$,

$$\int_s^t\langle\eta(u)-X_n(u),dY_n(u)\rangle\ge 0\ \ \text{P-a.e.}$$

Now, $(X_n, Y_n, \eta)$ (or an extracted sequence) converges in distribution to $(X, Y, \eta)$ in $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$. By Skorokhod's representation theorem, we can find a sequence $(\widehat{X}_n, \widehat{Y}_n, \widehat{\eta}_n)_{n\geq 1}$ of $\mathbb{X}\times\mathbb{X}\times\mathbb{X}$-valued random variables defined on some probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu})$, and an $\mathbb{X}\times\mathbb{X}\times\mathbb{X}$-valued random variable $(\widehat{X}, \widehat{Y}, \widehat{\eta})$, also defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu})$, such that, for each $n$, $(\widehat{X}_n, \widehat{Y}_n, \widehat{\eta}_n)$ has the same distribution as $(X, Y, \eta)$, and $(\widehat{X}_n, \widehat{Y}_n, \widehat{\eta}_n)$ converges $\widehat{\mu}$-a.e. to $(\widehat{X}, \widehat{Y}, \widehat{\eta})$. We thus have, for each integer $n$, and all $s, t \in [0, T]$ such that $s < t$,

$$\int_s^t \left\langle \widehat{\eta}_n(u) - \widehat{X}_n(u), d\widehat{Y}_n(u) \right\rangle \geq 0 \ \ \widehat{\mu}\text{-a.e.} \tag{34}$$

Recall that $(\rho_n)$ is tight, and thus $(\|Y_n\|_{\mathbb{X}})_{n\geq 1}$ is uniformly bounded, P-a.e. Using Helly's theorem on a subsequence (see, e.g., [24, Theorem 0.2.1]), we deduce that

$$\int_s^t \left\langle \widehat{\eta}(u) - \widehat{X}(u), d\widehat{Y}(u) \right\rangle \geq 0 \ \ \widehat{\mu}\text{-a.e.},$$

and thus

$$\begin{aligned} &\int_s^t \left\langle \eta(u) - X(u), \frac{dY}{d\,|DY|}(u) \right\rangle \, d\,|DY|\,(u) \\ = &\int_s^t \langle \eta(u) - X(u), dY(u) \rangle \geq 0 \ \ \mu\text{-a.e.} \end{aligned}$$

As $\eta$ is arbitrary, this proves that $\frac{dDY}{d|DY|}(t) \in -N_{C(t)}(X(t))$, $|DY|$-a.e., $\mu$-a.e.

*II. Proof of* (*a*) *in the general case.* We define as before the sequences $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$ and $(Z_n)_{n\geq 1}$. Condition $(C4)$ was used in the first part to prove that $(X_n)$ is tight. In this part, we use this partial result and Lemma 5.1 to show that $(X_n)$ is tight in the general case.

Let $(r_k)_{k\geq 1}$ be a decreasing sequence of positive numbers which converges to 0, and let $(\tau_k)$ be a sequence of stopping times associated with $(r_k)$ as in Condition $(C3)$ of Lemma 5.1. We can assume without loss of generality that $(\tau_k)$ is nondecreasing. Further, as noted in the remark following the proof of Lemma 5.1, we can assume that $\tau_k = T$ for $k$ large enough. Define, for every $t \in [0, T]$ and every $k$,

$$C_k(t) = C(t \wedge \tau_k) \quad \text{and} \quad \gamma_k(t) = \gamma(t \wedge \tau_n).$$

We have, P-a.e., for every $t \in [0, T]$ and every $k$,

$$\overline{B}(\gamma_k(t), r_k) \subset C_k(t), \quad \text{and} \quad \mathrm{E}\left(\sup_{t\in[0,T]} \|\gamma_k(t)\|^2\right) < +\infty.$$

For each $k$, let $\Phi_k$ and $\Psi_k$ be the mappings defined in Lemma 5.3 associated with $C_k$, and define as in Part *I* the corresponding sequences $(X_{n,k})_{n\geq 1}$, $(Y_{n,k})_{n\geq 1}$ and

$(Z_{n,k})_{n\geq 1}$. The processes $X_n$ and $X_{n,k}$ (respectively, $Y_n$ and $Y_{n,k}$, $Z_n$ and $Z_{n,k}$) coincide on the stochastic interval

$$[\![0,\tau_k]\!] := \{(t,\omega) \in [0,T]\times\Omega;\ 0\leq t\leq \tau_k(\omega)\}\,.$$

We deduce that $(X_n)$ is tight. Indeed, let $\epsilon > 0$. Let $k$ such that $\mathrm{P}\{\tau_k < T\} < \epsilon$. Let $\mathcal{K}$ be a compact subset of $\mathbb{X} = \mathrm{C}([0,T];\mathbb{R}^d)$ such that $\sup_{n\geq 1}\mathrm{P}\{X_{n,k}\notin\mathcal{K}\} < \epsilon$. We have, for every $n\geq 1$,

$$\begin{aligned}\mathrm{P}\{X_n\notin\mathcal{K}\} &= \mathrm{P}\left(\{X_n\notin\mathcal{K}\}\cap\{\tau_k = T\}\right) + \mathrm{P}\left(\{X_n\notin\mathcal{K}\}\cap\{\tau_k < T\}\right)\\ &\leq \mathrm{P}\left(\{X_{n,k}\notin\mathcal{K}\}\cap\{\tau_k = T\}\right) + \mathrm{P}\left(\{\tau_k < T\}\right) < 2\epsilon.\end{aligned}$$

The tightness of $(Y_n)$ and $(Z_n)$ is deduced similarly.

From Part $I$ of the proof, we can extract from each subsequence of $(X_{n,k})_{n\geq 1}$ a further subsequence which converges in $\mathcal{Y}$ to a Young measure $\mu^k$. We can thus construct nested convergent sequences, which we still denote by $(X_{n,k})_{n\geq 1}$, $k\geq 1$, in a such a way that, if $l\geq k$, the indexes of $(X_{n,l})_{n\geq 1}$ are contained in those of $(X_{n,k})_{n\geq 1}$. By a diagonal procedure, we get a subsequence of $(X_n)$ (still denoted by $(X_n)$) such that, for every $k\geq 1$, all but a finite number of indices of $(X_n)$ are contained in those of $(X_{n,k})$. Each element of $(X_n)$ coincides on $[\![0,\tau_k]\!]$ with the element of $(X_{n,k})$ which has the same index. Thus, if $\mu$ is a limit of $(X_n)$ in $\mathcal{Y}$, we have, for any bounded measurable mapping $f:\Omega\times\mathbb{X}\to\mathbb{R}$ such that $f(\omega,.)$ is continuous for every $\omega\in\Omega$,

$$\begin{aligned}\left|\mu(f)-\mu^k(f)\right| &= \left|\lim_{n\to\infty}\int_\Omega f(\omega,X_n(\omega))\,d\mathrm{P}(\omega) - \lim_{n\to\infty}\int_\Omega f(\omega,X_{n,k}(\omega))\,d\mathrm{P}(\omega)\right|\\ &\leq \lim_{n\to\infty}\int_\Omega \left|f(\omega,X_n(\omega)) - f(\omega,X_{n,k}(\omega))\right|\,d\mathrm{P}(\omega)\\ &\leq M\mathrm{P}\{\tau_k < T\}\to 0 \ \text{ as } k\to\infty,\end{aligned}$$

where $M$ is the supremum of $|f|$. We deduce that $(\mu^k)$ converges to $\mu$ in $\mathcal{Y}$, thus the limit of $(X_n)$ in $\mathcal{Y}$ is unique, i.e., $(X_n)$ converges to $\mu$ in $\mathcal{Y}$.

Note that, a similar calculation shows that, if $f(\omega,u)$ depends only on the restriction of $u$ to the interval $[0,\tau_k(\omega)]$, then $\mu(f)=\mu^k(f)$.

The rest of the proof of $(a)$ goes as in the first part.

*III. Proof of* $(b)$. If (17) is satisfied, then pathwise uniqueness holds for the problem (16), i.e., two solutions defined on the same stochastic basis with the same Brownian motion coincide. Indeed, let us reproduce the reasoning of Tanaka [31]: Let $(X,Y)$ and $(\widetilde{X},\widetilde{Y})$ be two solutions defined on the same stochastic basis with the same Brownian motion $B$. Let us denote

$$K(t)=\int_0^t b(s,X(s))\,ds, \qquad M(t)=\int_0^t \sigma(s,X(s))\,dB(s)$$
$$\widetilde{K}(t)=\int_0^t b(s,\widetilde{X}(s))\,ds, \qquad \widetilde{M}(t)=\int_0^t \sigma(s,\widetilde{X}(s))\,dB(s).$$

Using the fact that

$$\int_0^t \left\langle X(s)-\widetilde{X}(s), d(Y-\widetilde{Y})(s)\right\rangle \leq 0,$$

we get

$$\begin{aligned}
&\left\|X(t)-\widetilde{X}(t)\right\|^2\\
&= \left\|M(t)-\widetilde{M}(t)\right\|^2 + 2\left\langle M(t)-\widetilde{M}(t), K(t)-\widetilde{K}(t)+Y(t)-\widetilde{Y}(t)\right\rangle\\
&\qquad + \left\|K(t)-\widetilde{K}(t)+Y(t)-\widetilde{Y}(t)\right\|^2\\
&= \left\|M(t)-\widetilde{M}(t)\right\|^2 + 2\int_0^t \left\langle M(t)-\widetilde{M}(t), d(K-\widetilde{K}+Y-\widetilde{Y})(s)\right\rangle\\
&\qquad + 2\int_0^t \left\langle K(s)-\widetilde{K}(s)+Y(s)-\widetilde{Y}(s), d(K-\widetilde{K}+Y-\widetilde{Y})(s)\right\rangle\\
&= \left\|M(t)-\widetilde{M}(t)\right\|^2\\
&\qquad + 2\int_0^t \left\langle M(t)-\widetilde{M}(t)-M(s)+\widetilde{M}(s), d(K-\widetilde{K}+Y-\widetilde{Y})(s)\right\rangle\\
&\qquad + 2\int_0^t \left\langle M(s)-\widetilde{M}(s), d(K-\widetilde{K}+Y-\widetilde{Y})(s)\right\rangle\\
&\qquad + 2\int_0^t \left\langle K(s)-\widetilde{K}(s)+Y(s)-\widetilde{Y}(s), d(K-\widetilde{K}+Y-\widetilde{Y})(s)\right\rangle\\
&= \left\|M(t)-\widetilde{M}(t)\right\|^2\\
&\qquad + 2\int_0^t \left\langle M(t)-\widetilde{M}(t)-M(s)+\widetilde{M}(s), d(K-\widetilde{K}+Y-\widetilde{Y})(s)\right\rangle\\
&\qquad + 2\int_0^t \left\langle X(s)-\widetilde{X}(s), d(K-\widetilde{K}+Y-\widetilde{Y})(s)\right\rangle\\
&\leq \left\|M(t)-\widetilde{M}(t)\right\|^2\\
&\qquad + 2\int_0^t \left\langle M(t)-\widetilde{M}(t)-M(s)+\widetilde{M}(s), d(K-\widetilde{K}+Y-\widetilde{Y})(s)\right\rangle\\
&\qquad + 2\int_0^t \left\langle X(s)-\widetilde{X}(s), d(K-\widetilde{K})(s)\right\rangle.
\end{aligned}$$

We deduce

$$\begin{aligned}
&\mathrm{E}\left\|X(t)-\widetilde{X}(t)\right\|^2\\
&\le \mathrm{E}\left\|M(t)-\widetilde{M}(t)\right\|^2\\
&\qquad + 2\mathrm{E}\int_0^t \left\langle M(t)-\widetilde{M}(t)-M(s)+\widetilde{M}(s), d(K-\widetilde{K}+Y-\widetilde{Y})(s)\right\rangle\\
&\qquad + 2\mathrm{E}\int_0^t \left\langle X(s)-\widetilde{X}(s), d(K-\widetilde{K})(s)\right\rangle.
\end{aligned}$$

By Fubini's theorem, the second term in the right hand side is zero because, by reproducing the reasoning that led to (27), we get

$$\mathrm{E}\int_0^T \|\sigma(s,X(s))\|^2\,ds < +\infty \quad\text{and}\quad \mathrm{E}\int_0^T \left\|\sigma(s,\widetilde{X}(s))\right\|^2 ds < +\infty,$$

which entail that $M$ and $\widetilde{M}$ are martingales with zero expectation. Thus

$$\begin{aligned}
\mathrm{E}\left\|X(t)-\widetilde{X}(t)\right\|^2 &\le \mathrm{E}\left\|\int_0^t b(s,X(s))\,ds - \int_0^t b(s,\widetilde{X}(s))\,ds\right\|^2\\
&\qquad + 2\mathrm{E}\int_0^t \left\langle X(s)-\widetilde{X}(s), b(s,X(s))-b(s,\widetilde{X}(s))\right\rangle ds\\
&\le \mathfrak{d}^2\mathrm{E}\int_0^t \left\|X(s)-\widetilde{X}(s)\right\|^2 ds + 2\mathfrak{d}\mathrm{E}\int_0^t \left\|X(s)-\widetilde{X}(s)\right\|^2 ds.
\end{aligned}$$

We deduce by Gronwall's inequality that $X(t)=\widetilde{X}(t)$ a.e. for every $t\in[0,T]$, which proves pathwise uniqueness of the solution.

Now, the set of probability distributions of solutions is convex. Indeed, let $(X,Y)$ and $(\widetilde{X},\widetilde{Y})$ be two solutions, associated with Brownian motions $B$ and $\widetilde{B}$ respectively. As we are only interested in the distributions $\mathrm{law}(X,Y)$ and $\mathrm{law}(\widetilde{X},\widetilde{Y})$ of $(X,Y)$ and $(\widetilde{X},\widetilde{Y})$, we can assume without loss of generality that $(X,Y,B)$ and $(\widetilde{X},\widetilde{Y},\widetilde{B})$ are constructed on the same stochastic basis $\mathfrak{B}=(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},\mathrm{P})$, that they are independent, and that there exists an $\mathcal{F}_0$-measurable random variable $\eta$ which is independent of $(X,Y,B)$ and $(\widetilde{X},\widetilde{Y},\widetilde{B})$, with $\mathrm{P}(\eta=1)=\mathrm{P}(\eta=-1)=1/2$. Then

$$\underline{B} := \frac{1}{2}\mathbb{1}_{\{\eta=1\}}B + \frac{1}{2}\mathbb{1}_{\{\eta=1\}}\underline{B}$$

is an $(\mathcal{F}_t)$-Brownian motion, and the process $(\underline{X},\underline{Y})$, defined similarly to $\underline{B}$, is a solution driven by $\underline{B}$ with distribution

$$\mathrm{law}(\underline{X},\underline{Y}) = \frac{1}{2}\,\mathrm{law}(X,Y) + \frac{1}{2}\,\mathrm{law}(\widetilde{X},\widetilde{Y}).$$

By Kurtz's version of Yamada and Watanabe's theorem [20, Theorem 3.15], this implies that there exists a unique pair $(X,Y)$ defined on $\mathfrak{B}$ which is solution to (16). □

**Remark.** In the deterministic case (i.e., when $\sigma = 0$), Part $(b)$ of Theorem 5.2 is a variant of Theorem 4.1. Note also that, in this case, tightness amounts to compactness in $\mathrm{C}([0,T];\mathbb{R}^d)$.

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