

Coderivative Characterizations of Maximal Monotonicity for Set-Valued Mappings

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This paper concerns generalized differential characterizations of maximal monotone set-valued mappings. Using advanced tools of variational analysis, we establish coderivative criteria for maximal monotonicity of set-valued mappings, which seem to be the first infinitesimal characterizations of maximal monotonicity outside the single-valued case. We also present second-order necessary and sufficient conditions for lower- $\mathcal{C}^2$ functions to be convex and strongly convex. Examples are provided to illustrate the obtained results and the imposed assumptions.

Keywords: Maximal monotone mappings, convex lower- $\mathcal{C}^2$ functions, variational analysis, coderivatives, second-order subdifferentials

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1. Introduction

The notion of maximal monotone operators appeared in the early 1960s and since that has been well recognized as a fundamental tool in the study of various aspects of partial differential equations, optimization, equilibrium theory, etc.; particularly those concerning existence and uniqueness of solutions, stability issues, convergence of numerical algorithms, and related topics; see, e.g., [2, 4, 5, 6, 19, 23, 32, 36, 38, 39] and the references therein. We specially emphasize a crucial role of maximal monotonicity in the theory and applications of the *sweeping process* (“processus de raffle”) introduced and investigated by Jean Jacques Moreau [30]; see more details and recent developments in the survey paper [12].

Since a *single-valued* and *continuous* monotone mapping is automatically maximal monotone (see, e.g., [2, Corollary 20.25]), the maximality issue does not arise in this case. The classical criterion of monotonicity presented in [38, Proposition 12.3] tells us that a differentiable single-valued mapping is monotone if and only if its derivative is positive-semidefinite at every point. Infinitesimal characterizations of monotonicity for possibly nondifferentiable mappings have attracted much attention in the literature. In his landmark paper [23], Minty established a sufficient condition for monotonicity of nondifferentiable mappings by using directional derivatives. Jiang and Qi [22, Proposition 2.3] and Luc and Schaible [20, Proposition 2.1] independently proved that a locally Lipschitzian mapping defined on an open convex set of $\mathbb{R}^n$ is monotone if and only if its Clarke’s generalized Jacobian is pointwise positive-semidefinite at every point. Replacing generalized Jacobian matrices by their approximate Jacobian matrices, Jeyakumar et al. [21, Theorem 3.1] derived a sufficient condition for monotonicity of continuous single-valued mappings between finite-dimensional spaces. More recently [9], Chieu and Trang obtained necessary and sufficient conditions for monotonicity of continuous single-valued mappings in both finite-dimensional and infinite-dimensional settings via positive-semidefiniteness of the coderivative constructions by Mordukhovich [25] (see Sections 2), extending in this way the classical characterization of monotonicity for smooth mappings.

However, the major role in nonlinear and variational analysis and their applications is played not by single-valued but *intrinsically set-valued* maximal monotone operators that include, e.g., subdifferential mappings for lower semi-continuous proper convex functions and normal cone mappings associated with close convex sets. In particular, such set-valued mappings allow us to adequately describe variational inequalities and complementarity problems in Robinson’s framework of generalized equations [34], the aforementioned Moreau’s sweeping process, etc. It is very attractive and challenging therefore to establish verifiable infinitesimal conditions (better, complete characterizations) of maximal monotonicity and related properties for set-valued mappings in finite and infinite dimensions.

To the best of our knowledge, the first result in this direction was obtained by

Poliquin and Rockafellar [33, Theorem 2.1] who derived a *necessary* condition for the maximal monotonicity of set-valued mappings between finite-dimensional spaces in terms of the positive-semidefiniteness of the limiting coderivative; this condition was extended in [8, 26] to Hilbert spaces and reversed in [9] for single-valued mappings. Note that the motivation of [33] came from the application to *tilt stability* in optimization whose theory has been flowering during the recent years; see, e.g., [3, 15, 16, 17, 26, 27] and the references therein.

Another impact to the study of monotonicity properties for set-valued mappings has been recently done by Mordukhovich and Nghia [28] who established complete coderivative characterizations of *strong local* maximal monotonicity in finite and infinite dimensions with applications to *full stability* (in the Lipschitzian and Hölderian frameworks) of parametric variational systems. The approach of [28] used, along with advanced tools of variational analysis and generalized differentiation, *hypomonotonicity* properties of set-valued mappings, which will be exploited in what follows.

The main goal of this paper is to establish complete *coderivative characterizations* of *maximal monotonicity* of set-valued mappings in Hilbert spaces via pointwise *positive-semidefiniteness* conditions and appropriate properties of global and (semi)local hypomonotonicity. The results obtained seem to be the first infinitesimal characterizations of maximal monotonicity outside the single-valued setting even in the case of finite dimensions. As consequences of these characterizations, we derive second-order necessary and sufficient conditions for lower- $\mathcal{C}^2$ functions to be *convex* and *strongly convex*. These conditions are expressed in terms of *second-order subdifferentials/generalized Hessians* of extended-real-valued functions and extend the classical result of real analysis saying that a $\mathcal{C}^2$ function is convex if and only if its Hessian is positive-semidefinite at any point.

The paper is organized as follows. Section 2 recalls some basic notions and facts from variational analysis that are employed in the sequel. Section 3 is the main part of our analysis, which contains several coderivative characterizations of maximal monotonicity for set-valued mappings in Hilbert spaces. Section 4 is devoted to the study of convexity and strong convexity for lower- $\mathcal{C}^2$ functions via second-order subdifferentials. Finally, we present some concluding remarks and formulate open questions in Section 5.

Our notation is standard in variational analysis and generalized differentiation; cf. [25, 38]. Throughout the paper we assume that X is a *Hilbert* space being identified with its dual space. As usual, the symbol $\langle \cdot, \cdot \rangle$ signifies the canonical pairing in X with the norm $\|x\| = \sqrt{\langle x, x \rangle}$. We denote by $\mathbb{B}$ the closed unit ball in X and by $\mathbb{B}_r(\bar{x}) := \bar{x} + r\mathbb{B}$ the ball with radius $r > 0$ and center $\bar{x} \in X$. The notation $\xrightarrow{w}$ indicates for the weak convergence in X . Given a set-valued

mapping $F: X \rightrightarrows X$ and a point $\bar{u} \in X$, the symbol

$$\begin{aligned} \limsup_{u \rightarrow \bar{u}} F(u) := \{v \in X \mid \exists \text{ sequences } u_k \rightarrow \bar{u}, v_k \xrightarrow{w} v \text{ such that} \\ v_k \in F(u_k) \text{ for all } k \in \mathbb{N} := \{1, 2, \dots\}\} \end{aligned} \quad (1)$$

stands for the *sequential Painlevé-Kuratowski outer/upper limit* of $F(u)$ as $u \rightarrow \bar{u}$.

2. Preliminaries

Here we mainly follow the book [25] referring the reader also to [4, 38] for related and additional material. Given a proper (i.e., not identically equal to infinity) extended-real-valued function $f: X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and its domain point $\bar{u} \in \text{dom } f := \{u \in X \mid f(u) < \infty\}$, the *regular/Fréchet subdifferential* (known also as the presubdifferential and the viscosity subdifferential) of f at $\bar{u}$ is

$$\widehat{\partial}f(\bar{u}) := \left\{v \in X \mid \liminf_{u \rightarrow \bar{u}} \frac{f(u) - f(\bar{u}) - \langle v, u - \bar{u} \rangle}{\|u - \bar{u}\|} \geq 0\right\} \quad (2)$$

with $\widehat{\partial}f(\bar{u}) := \emptyset$ if $\bar{u} \notin \text{dom } f$. The *limiting/Mordukhovich subdifferential* (known also as the basic subdifferential) of f at $\bar{u} \in \text{dom } f$ is defined via (1) by

$$\partial f(\bar{u}) := \limsup_{u \xrightarrow{f} \bar{u}} \widehat{\partial}f(u), \quad (3)$$

where the notation $u \xrightarrow{f} \bar{u}$ means that $u \rightarrow \bar{u}$ with $f(u) \rightarrow f(\bar{u})$. It is well known that both regular and limiting subdifferentials reduce to the classical subdifferential of convex analysis when the function f is convex. On the other hand, the limiting subdifferential of nonconvex functions and related normal cone/coderivative constructions for sets and mappings enjoy *full calculus*, which is not the case for (2) and its set/mapping counterparts.

Given further a set $\Omega \subset X$ with its indicator function $\delta(u; \Omega)$ equal to 0 for $u \in \Omega$ and to ∞ otherwise, the regular and limiting *normal cones* to Ω at $\bar{u} \in \Omega$ are defined, respectively, via the corresponding subdifferentials (2) and (3) by

$$\widehat{N}(\bar{u}; \Omega) := \widehat{\partial}\delta(\bar{u}; \Omega) \quad \text{and} \quad N(\bar{u}; \Omega) := \partial\delta(\bar{u}; \Omega). \quad (4)$$

Now we consider a set-valued mapping $F: X \rightrightarrows X$ and associate with it the *domain* $\text{dom } F$ and the *graph* $\text{gph } F$ by

$$\text{dom } F := \{u \in X \mid F(u) \neq \emptyset\} \quad \text{and} \quad \text{gph } F := \{(u, v) \in X \times X \mid v \in F(u)\}.$$

The mapping F is said to be *proper* when $\text{dom } F \neq \emptyset$, which is always assumed. Define by

$$\widehat{D}^*F(\bar{u}, \bar{v})(w) := \left\{z \in X \mid (z, -w) \in \widehat{N}((\bar{u}, \bar{v}); \text{gph } F)\right\} \quad \text{for all } w \in X \quad (5)$$

the *regular coderivative* of F at $(\bar{u}, \bar{v}) \in \text{gph } F$ and by

$$D_M^* F(\bar{u}, \bar{v})(w) := \limsup_{\substack{(u,v) \rightarrow (\bar{u}, \bar{v}) \\ y \rightarrow w}} \widehat{D}^* F(u, v)(y) \quad \text{for all } w \in X \quad (6)$$

the *mixed limiting coderivative* of F at $(\bar{u}, \bar{v})$, where the convergence $y \rightarrow w$ is strong in X while the outer limit in (6) is taken by (1) in the weak topology of X ; see [25] for more discussions. We omit the subscript “ M ” in (6) when X is finite-dimensional and also drop indicating $\bar{v} = F(\bar{u})$ if F is single-valued. When F is single-valued and continuously differentiable around $\bar{u}$ (or strictly differentiable at this point), we get

$$\widehat{D}^* F(\bar{u})(w) = D_M^* F(\bar{u})(w) = \{\nabla F(\bar{u})^* w\} \quad \text{for all } w \in X$$

via the adjoint derivative operator $\nabla F(\bar{u})^*$; see, e.g., [25, Theorem 1.38].

Next we recall two second-order subdifferential/generalized Hessian constructions introduced by the scheme suggested in [24] as a coderivative of a first-order subdifferential mapping; see [25, 26, 29] for more details and discussions.

Definition 2.1 (second-order subdifferentials). Let $f : X \rightarrow \overline{\mathbb{R}}$ with $\bar{u} \in \text{dom } f$, and let $\bar{v} \in \partial f(\bar{u})$. Then we say that:

- (i) The *combined second-order subdifferential* of f at $\bar{u}$ relative to $\bar{v}$ is the set-valued mapping $\check{\partial}^2 f(\bar{u}, \bar{v}) : X \rightrightarrows X$ with the values

$$\check{\partial}^2 f(\bar{u}, \bar{v})(w) := \left(\widehat{D}^* \partial f \right) (\bar{u}, \bar{v})(w) \quad \text{for all } w \in X. \quad (7)$$

- (ii) The *mixed second-order subdifferential* of f at $\bar{u}$ relative to $\bar{v}$ is the set-valued mapping $\partial_M^2 f(\bar{u}, \bar{v}) : X \rightrightarrows X$ with the values

$$\partial_M^2 f(\bar{u}, \bar{v})(w) := (D_M^* \partial f) (\bar{u}, \bar{v})(w) \quad \text{for all } w \in X. \quad (8)$$

It is worth mentioning that if f is $\mathcal{C}^2$ around $\bar{u}$ with $\bar{v} = \nabla f(\bar{u})$, then both $\check{\partial}^2 f(\bar{u}, \bar{v})(w)$ and $\partial_M^2 f(\bar{u}, \bar{v})(w)$ reduce to the classical (symmetric) single-valued Hessian operator:

$$\check{\partial}^2 f(\bar{u}, \bar{v})(w) = \partial_M^2 f(\bar{u}, \bar{v})(w) = \{\nabla^2 f(\bar{u})^* w\} = \{\nabla^2 f(\bar{u}) w\} \quad \text{for all } w \in X.$$

One of the main fact of generalized differentiation largely employed in our paper is the following *mean value inequality* for Lipschitz continuous functions; [25, Corollary 3.50(ii)]. For the reader’s convenience we formulate it here.

Mean value inequality. Let $f : X \rightarrow \overline{\mathbb{R}}$ be a lower semicontinuous (l.s.c.) function with $a \in \text{dom } f$. Then for any $b \in X$ and $\varepsilon > 0$ we have the estimate

$$|f(b) - f(a)| \leq \|b - a\| \sup \left\{ \|v\| \mid v \in \widehat{\partial} f(c), c \in [a, b] + \varepsilon B \right\}, \quad (9)$$

where $[a, b] := \{\lambda a + (1 - \lambda)b \mid \lambda \in [0, 1]\}$.

3. Characterizations of Maximal Monotonicity

The following notions of (global) monotonicity for set-valued mappings are main objects of our study and applications in this paper.

Definition 3.1 (monotone set-valued operators). Given $T : X \rightrightarrows X$, we say that:

(i) T is *monotone* on X if

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq 0 \text{ for all } (u_1, v_1), (u_2, v_2) \in \text{gph } T. \quad (10)$$

T is said to be *maximal monotone* on X if in addition we have $\text{gph } T = \text{gph } S$ whenever S is monotone with $\text{gph } T \subset \text{gph } S$.

(ii) T is *hypomonotone* on X if there exists a number $r > 0$ such that $T + rI$ is monotone, where $I : X \rightarrow X$ is the identity mapping. This means that

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq -r\|u_1 - u_2\|^2 \text{ for all } (u_1, v_1), (u_2, v_2) \in \text{gph } T. \quad (11)$$

First we present a characterization of maximal monotonicity via the positive-semidefiniteness condition for the regular coderivative and global hypomonotonicity.

Theorem 3.2 (regular coderivative and global hypomonotonicity characterization of maximal monotonicity). Let $T : X \rightrightarrows X$ be a set-valued mapping with closed graph in the norm topology of $X \times X$. The following assertions are equivalent:

(i) T is maximal monotone on X .

(ii) T is hypomonotone on X and for any $(u, v) \in \text{gph } T$ we have

$$\langle z, w \rangle \geq 0 \text{ whenever } z \in \widehat{D}^*T(u, v)(w). \quad (12)$$

Proof. Since the monotonicity obviously yields hypomonotonicity, implication (i) $\implies$ (ii) follows from [8, Lemma 5.2] and also from [26, Lemma 6.2].

To verify the converse implication (ii) $\implies$ (i), suppose that T is hypomonotone and that condition (12) is satisfied. Then there is some number $r > 0$ such that $T + rI$ is monotone. Take any $s > r$ and define $F : X \rightrightarrows X$ by $\text{gph } F := \text{gph}(T + sI)^{-1}$. For any $(v_i, u_i) \in \text{gph } F$, $i = 1, 2$ we have $(u_i, v_i - su_i) \in \text{gph } T$ and thus deduce from (11) that

$$\langle v_1 - su_1 - v_2 + su_2, u_1 - u_2 \rangle \geq -r\|u_1 - u_2\|^2.$$

The latter implies in turn the inequalities

$$\|v_1 - v_2\| \cdot \|u_1 - u_2\| \geq \langle v_1 - v_2, u_1 - u_2 \rangle \geq (s - r)\|u_1 - u_2\|^2,$$

which allow us to arrive at the estimate

$$\|u_1 - u_2\| \leq \frac{1}{s - r}\|v_1 - v_2\|, \quad (13)$$

verifying that F is single-valued and Lipschitz continuous with modulus $(s-r)^{-1}$ on its domain. To proceed further, fix any $z \in X$ and define $f_z : X \rightarrow \overline{\mathbb{R}}$ by

$$f_z(v) := \begin{cases} \langle z, F(v) \rangle & \text{if } v \in \text{dom } F, \\ \infty & \text{otherwise.} \end{cases} \quad (14)$$

Since $\text{gph } T$ is closed, it is easy to check that $\text{gph } F$ is also closed in $X \times X$. Next we show that f_z is lower semicontinuous on X . Arguing by contradiction, suppose that there exist $\varepsilon > 0$ and a sequence v_k converging to some $v \in X$ such that $f_z(v_k) < f_z(v) - \varepsilon$. If $f_z(v) = \infty$, then $v \notin \text{dom } F$ and $v_k \in \text{dom } F$. It follows from (13) that $\|F(v_k) - F(v_j)\| \leq (s-r)^{-1}\|v_k - v_j\|$, and so $\{F(v_k)\}$ is a Cauchy sequence converging to some $u \in X$. Hence the sequence $(v_k, F(v_k)) \in \text{gph } F$ converges to $(v, u) \in \text{gph } F$ due to the closedness of $\text{gph } F$. This gives us $F(v) = u$ and contradicts $v \notin \text{dom } F$. In the remaining case of $f_z(v) < \infty$ we get from (13) and (14) the estimates

$$\begin{aligned} |f_z(v_k) - f_z(v)| &\leq \|z\| \cdot \|F(v_k) - F(v)\| \\ &\leq \|z\| \cdot \frac{1}{s-r} \|v_k - v\| \rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

which is also a contradiction due to the assumption $f_z(v_k) < f_z(v) - \varepsilon$. This justifies the lower semicontinuity of f_z on the space X for any fixed $z \in X$.

Now we claim that T is *monotone*. To proceed, pick two pairs $(u_i, v_i) \in \text{gph } T$ and get

$$(y_i, u_i) \in \text{gph } F \text{ with } y_i := v_i + su_i, \ i = 1, 2.$$

Applying the mean value inequality (9) to the l.s.c. function f_z tells us that

$$\begin{aligned} |\langle z, u_1 - u_2 \rangle| &= |f_z(y_1) - f_z(y_2)| \\ &\leq \|y_1 - y_2\| \sup \left\{ \|w\| \mid w \in \widehat{\partial} f_z(y), \ y \in [y_1, y_2] + \varepsilon \mathcal{B} \right\} \end{aligned} \quad (15)$$

with $[y_1, y_2] := \{\lambda y_1 + (1-\lambda)y_2 \mid \lambda \in [0, 1]\}$ and fixed $\varepsilon > 0$. Since $\widehat{\partial} f_z(y) = \emptyset$ if $y \notin \text{dom } f_z$, it suffices to consider the case of $y \in \text{dom } f_z \cap ([y_1, y_2] + \varepsilon \mathcal{B}) = \text{dom } F \cap ([y_1, y_2] + \varepsilon \mathcal{B})$ in (15). Take any y from the latter set and observe that

$$w \in \widehat{D}^* F(y)(z) \text{ whenever } w \in \widehat{\partial} f_z(y). \quad (16)$$

Indeed, it follows from the definition of the regular subgradient $w \in \widehat{\partial} f_z(y)$ that

$$\liminf_{v \rightarrow y} \frac{f_z(v) - f_z(y) - \langle w, v - y \rangle}{\|v - y\|} \geq 0,$$

which can be equivalently written by the construction of f_z in (14) as

$$\liminf_{v \xrightarrow{\text{dom } F} y} \frac{\langle z, F(v) \rangle - \langle z, F(y) \rangle - \langle w, v - y \rangle}{\|v - y\|} \geq 0.$$

The latter readily implies that

$$\liminf_{(v,u) \xrightarrow{\text{gph } F} (y,F(y))} \frac{\langle z, u - F(y) \rangle - \langle w, v - y \rangle}{\|v - y\| + \|u - F(y)\|} \geq 0.$$

Hence we get from (2), (4), and (5) that

$$(w, -z) \in \widehat{N}((y, F(y)); \text{gph } F) \iff w \in \widehat{D}^*F(y)(z) = \widehat{D}^*(T + sI)^{-1}(y)(z)$$

and therefore $-z \in \widehat{D}^*(T + sI)(F(y), y)(-w)$. It easily follows from the coderivative sum rule in [25, Theorem 1.62] that

$$-z + sw \in \widehat{D}^*T(F(y), y - sF(y))(-w). \quad (17)$$

Combining this with (12) tells us that $\langle -z + sw, -w \rangle \geq 0$, which yields

$$\|z\| \cdot \|w\| \geq \langle z, w \rangle \geq s\|w\|^2 \quad (18)$$

and together with the estimate (15) implies further that

$$|\langle z, u_1 - u_2 \rangle| \leq s^{-1}\|z\| \cdot \|y_1 - y_2\|.$$

Since this inequality holds for all $z \in X$, we get

$$\|u_1 - u_2\| \leq s^{-1}\|y_1 - y_2\| = s^{-1}\|v_1 + su_1 - v_2 - su_2\|$$

and then deduce by the elementary transformation that

$$\begin{aligned} s^2\|u_1 - u_2\|^2 &\leq \|(v_1 - v_2) + s(u_1 - u_2)\|^2 \\ &= \|v_1 - v_2\|^2 + 2s\langle v_1 - v_2, u_1 - u_2 \rangle + s^2\|u_1 - u_2\|^2. \end{aligned}$$

Therefore we arrive at the inequality

$$0 \leq \frac{1}{2s}\|v_1 - v_2\|^2 + \langle v_1 - v_2, u_1 - u_2 \rangle \quad \text{for any } s > r.$$

Letting there $s \rightarrow \infty$ shows that

$$0 \leq \langle v_1 - v_2, u_1 - u_2 \rangle \quad \text{for all } (u_1, v_1), (u_2, v_2) \in \text{gph } T$$

and thus verifies the monotonicity of T .

It remains to prove that T is *maximal monotone*. Since T is proper, there exists a pair $(u_0, v_0) \in \text{gph } T$ such that

$$u_0 = (T + sI)^{-1}(y_0) \quad \text{with } y_0 := v_0 + su_0.$$

Applying again the mean value inequality (9) to the function f_z from (14), we have

$$|f_z(y) - f_z(y_0)| \leq \|y - y_0\| \sup \left\{ \|w\| \mid w \in \widehat{\partial}f_z(x), x \in [y, y_0] + \varepsilon \mathcal{B} \right\} \quad (19)$$

for any $y \in X$. It follows similarly to (18) that $\|w\| \leq s^{-1}\|z\|$ for all $w \in \widehat{\partial}f_z(x)$ with $x \in \text{dom } F \cap ([y, y_0] + \varepsilon B)$. This together with (19) gives us the estimates

$$|f_z(y) - f_z(y_0)| \leq s^{-1}\|z\| \cdot \|y - y_0\|.$$

Hence $f_z(y) < \infty$ and so $F(y) \neq \emptyset$ for all $y \in X$, which means that $\text{dom } (T + sI)^{-1} = X$. Employing now the classical Minty theorem (see, e.g., [6, Theorem 4.4.7 and Remark 4.4.8]) and taking into account the monotonicity of T justified above, we conclude that T is maximal monotone and thus complete the proof of the theorem. $\square$

Our next goal is to obtain another version of the coderivative characterization in Theorem 3.2 with replacing the global hypomonotonicity of T in assertion (ii) therein by a certain local hypomonotonicity. Besides being interesting for its own sake, it is needed for the subsequent applications in Section 4 to characterize convexity and strong convexity of lower- $\mathcal{C}^2$ functions. In fact, for these purposes we need to modify the conventional notion of local monotonicity and hypomonotonicity, which are dealing with neighborhoods in the *product* space $X \times X$; see, e.g., [28, 31] with more references and discussions. The notions we use in what follows concern neighborhoods only in the *domain* space X . Such a local monotonicity has been considered in [38, Example 12.28]. We will name the domain versions as *semilocal* monotonicity and hypomonotonicity, which reflects their nature and distinguishes them from their fully localized product counterparts.

Definition 3.3 (semilocal monotonicity and hypomonotonicity). We say that the mapping $T: X \rightrightarrows X$ is *semilocally hypomonotone* (resp. *semilocally monotone*) at $\bar{u} \in \text{dom } T$ if there exist a neighborhood U of $\bar{u}$ and a number $r > 0$ (resp. $r = 0$) with

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq -r\|u_1 - u_2\|^2 \quad \text{for all } (u_1, v_1), (u_2, v_2) \in \text{gph } T \cap (U \times X). \quad (20)$$

Given a set $\Omega \subset X$, we say that T is *semilocally hypomonotone* (resp. *monotone*) *on* Ω if it is semilocally hypomonotone (resp. monotone) at every point $\bar{u} \in \Omega \cap \text{dom } T$.

Establishing the desired semilocal version of Theorem 3.2 requires an additional convexity assumption on the domain of T , which is shown below to be essential by providing a counterexample. To proceed in this direction, we first present the following lemma proved in [23, Theorem 5] by similar arguments for single-valued operators.

Lemma 3.4 (semilocal monotonicity of set-valued mappings with convex domains). *Let $T: X \rightrightarrows X$ be a semilocally monotone mapping on X , and let its domain $\text{dom } T$ be convex. Then T is (globally) monotone on X .*

Proof. Pick any $(u_1, v_1), (u_2, v_2) \in \text{gph } T$ and get $[u_1, u_2] \subset \text{dom } T$ by the convexity of $\text{dom } T$. Since T is semilocally monotone, for each $x \in [u_1, u_2]$ there is

$\gamma_x > 0$ such that

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \quad \text{whenever } (x_1, y_1), (x_2, y_2) \in \text{gph } T \cap (\text{int } \mathcal{B}_{\gamma_x}(x) \times X). \quad (21)$$

The compactness of $[u_1, u_2]$ allows us to select $x_i \in [u_1, u_2]$, $i = 1, \dots, n$, satisfying

$$[u_1, u_2] \subset \bigcup_{i=1}^n (\text{int } \mathcal{B}_{\gamma_{x_i}}(x_i)).$$

Thus we can find $0 = t_0 < t_1 < \dots < t_m = 1$ such that for each $j \in \{0, \dots, m-1\}$ it holds

$$[\hat{u}_j, \hat{u}_{j+1}] \subset \text{int } \mathcal{B}_{\gamma_{x_{i_j}}}(x_{i_j}) \quad \text{with some } i_j \in \{1, \dots, n\},$$

where $\hat{u}_j := u_1 + t_j(u_2 - u_1)$. Since $\hat{u}_j \in [u_1, u_2] \subset \text{dom } T$ for each $j \in \{0, \dots, m\}$, there exist $\hat{v}_j \in T(\hat{u}_j)$ with $\hat{v}_0 = v_1$ and $\hat{v}_m = v_2$. It follows from (21) that

$$(t_{j+1} - t_j) \langle \hat{v}_{j+1} - \hat{v}_j, u_2 - u_1 \rangle = \langle \hat{v}_{j+1} - \hat{v}_j, \hat{u}_{j+1} - \hat{u}_j \rangle \geq 0,$$

which implies that $\langle \hat{v}_{j+1} - \hat{v}_j, u_2 - u_1 \rangle \geq 0$ whenever $j \in \{0, \dots, m-1\}$. Hence we get

$$\langle v_2 - v_1, u_2 - u_1 \rangle = \sum_{j=0}^{m-1} \langle \hat{v}_{j+1} - \hat{v}_j, u_2 - u_1 \rangle \geq 0$$

and thus verify the global monotonicity of the operator T . $\square$

Now we are ready to obtain a semilocal counterpart of the coderivative characterization in Theorem 3.2 under the convexity assumption on $\text{dom } T$. Example 3.6 below demonstrates that the latter assumption cannot be dropped. Since the proof of the following theorem is similar in some places to that of Theorem 3.2, we omit the corresponding details.

Theorem 3.5 (regular coderivative and semilocal hypomonotonicity characterization of maximal monotonicity). *Let $T : X \rightrightarrows X$ be a set-valued mapping with closed graph and convex domain. The following assertions are equivalent:*

- (i) T is maximal monotone on X .
- (ii) T is semilocally hypomonotone on X and satisfies the regular coderivative condition (12) for any $(u, v) \in \text{gph } T$.

Proof. Implication (i) $\implies$ (ii) follows from Theorem 3.2. To verify the converse implication, suppose that condition (12) holds and that T is semilocally hypomonotone. The latter allows us to find, for each $\bar{u} \in \text{dom } T$, positive numbers δ and r such that

$$\begin{aligned} \langle v_1 - v_2, u_1 - u_2 \rangle &\geq -r \|u_1 - u_2\|^2 \\ &\quad \text{whenever } (u_1, v_1), (u_2, v_2) \in \text{gph } T \cap (\mathcal{B}_\delta(\bar{u}) \times X). \end{aligned} \quad (22)$$

Take any $s > r$ and define the mapping $F : X \rightrightarrows X$ by $\text{gph } F := \text{gph}(T + sI)^{-1} \cap (X \times \mathcal{B}_\delta(\bar{u}))$. Picking arbitrarily pairs $(v_i, u_i) \in \text{gph } F$, $i = 1, 2$ we have $(u_i, v_i - su_i) \in \text{gph } T \cap (\mathcal{B}_\delta(\bar{u}) \times X)$. It follows from (22) that

$$\langle v_1 - su_1 - v_2 + su_2, u_1 - u_2 \rangle \geq -r\|u_1 - u_2\|^2.$$

Similarly to (15) we deduce from the latter that

$$\|u_1 - u_2\| \leq \frac{1}{s-r}\|v_1 - v_2\| \quad \text{for all } (v_1, u_1), (v_2, u_2) \in \text{gph } F. \quad (23)$$

This implies that F is single-valued and Lipschitz continuous on $\text{dom } F$. For any fixed vector $z \in X$ we also define the function $f_z : X \rightarrow \overline{\mathcal{R}}$ as in (14) and prove similarly to Theorem 3.2 that f_z is lower semicontinuous on X .

Now pick arbitrary pairs $(u_1, v_1), (u_2, v_2) \in \text{gph } T \cap (\text{int } \mathcal{B}_\delta(\bar{u}) \times X)$ and fix $\bar{v} \in T(\bar{u})$. Then $F(y_i) = u_i \in \mathcal{B}_\delta(\bar{u})$ with $y_i := v_i + su_i$. Applying the mean value inequality (9) for any $\varepsilon \in (0, \sqrt{s})$ tells us that

$$\begin{aligned} |\langle z, u_1 - u_2 \rangle| &= |f_z(y_1) - f_z(y_2)| \\ &\leq \|y_1 - y_2\| \sup \left\{ \|w\| \mid w \in \widehat{\partial} f_z(y), y \in [y_1, y_2] + \varepsilon \mathcal{B} \right\}. \end{aligned} \quad (24)$$

Similar to (16) we get $\widehat{\partial} f_z(y) \subset \widehat{D}^* F(y)(z)$ for all $y \in \text{dom } F \cap ([y_1, y_2] + \varepsilon \mathcal{B})$ and then for any $y \in \text{dom } F \cap ([y_1, y_2] + \varepsilon \mathcal{B})$ find some $y_0 \in \varepsilon \mathcal{B}$ and $t \in [0, 1]$ satisfying $y = ty_1 + (1-t)y_2 + y_0$. Since $F(\bar{v} + s\bar{u}) = \bar{u}$, it follows from (23) that

$$\begin{aligned} \|F(y) - \bar{u}\| &= \|F(ty_1 + (1-t)y_2 + y_0) - F(\bar{v} + s\bar{u})\| \\ &\leq \frac{1}{s-r}\|ty_1 + (1-t)y_2 + y_0 - \bar{v} - s\bar{u}\| \\ &= \frac{1}{s-r}\|t(v_1 + su_1) + (1-t)(v_2 + su_2) + y_0 - \bar{v} - s\bar{u}\| \\ &= \frac{1}{s-r}\|t(v_1 - \bar{v}) + st(u_1 - \bar{u}) + (1-t)(v_2 - \bar{v}) + s(1-t)(u_2 - \bar{u}) + y_0\| \\ &\leq \frac{1}{s-r} [t\|v_1 - \bar{v}\| + (1-t)\|v_2 - \bar{v}\| + st\|u_1 - \bar{u}\| + s(1-t)\|u_2 - \bar{u}\| + \|y_0\|] \\ &\leq \frac{1}{s-r} [\max \{\|v_1 - \bar{v}\|, \|v_2 - \bar{v}\|\} + \varepsilon] + \frac{s}{s-r} \max \{\|u_1 - \bar{u}\|, \|u_2 - \bar{u}\|\} \\ &\leq \frac{1}{s-r} [\max \{\|v_1 - \bar{v}\|, \|v_2 - \bar{v}\|\} + \sqrt{s}] + \frac{s}{s-r} \max \{\|u_1 - \bar{u}\|, \|u_2 - \bar{u}\|\}. \end{aligned} \quad (25)$$

Since the choice of $(u_1, v_1), (u_2, v_2), (\bar{u}, \bar{v}) \in \text{gph } T \cap (\text{int } \mathcal{B}_\delta(\bar{u}) \times X)$ was independent of the parameter $s > r$ and by $\max \{\|u_1 - \bar{u}\|, \|u_2 - \bar{u}\|\} < \delta$, we can find M so large that

$$\begin{aligned} &\frac{1}{s-r} \max \{\|v_1 - \bar{v}\|, \|v_2 - \bar{v}\| + \sqrt{s}\} \\ &+ \frac{s}{s-r} \max \{\|u_1 - \bar{u}\|, \|u_2 - \bar{u}\|\} < \delta \quad \text{if } s > M. \end{aligned}$$

This together with (25) ensures that $F(y) \in \text{int } \mathcal{B}_\delta(\bar{x})$ and thus

$$\begin{aligned}\widehat{N}((y, F(y)); \text{gph } F) &= \widehat{N}((y, F(y)); \text{gph}(T + sI)^{-1} \cap (X \times \mathcal{B}_\delta(\bar{x}))) \\ &= \widehat{N}((y, F(y)); \text{gph}(T + sI)^{-1}),\end{aligned}$$

which clearly implies in turn the equality

$$\widehat{D}^*F(y)(z) = \widehat{D}^*(T + sI)^{-1}(y, F(y))(z).$$

Similarly to (17), for any $w \in \widehat{\partial}f_z(y) \subset \widehat{D}^*F(y)(z)$ we get from the latter that $-z + sw \in \widehat{D}^*T(F(y), y - sF(y))(-w)$. It follows from (12) that $\langle -z + sw, -w \rangle \geq 0$, which yields

$$\|z\| \cdot \|w\| \geq \langle z, w \rangle \geq s\|w\|^2, \quad \text{i.e., } \|z\| \geq s\|w\|.$$

This together with (24) tells us that

$$\langle z, u_1 - u_2 \rangle \leq s^{-1}\|y_1 - y_2\| \cdot \|z\|.$$

Since the latter holds for any $z \in X$, we have

$$\begin{aligned}\|u_1 - u_2\|^2 &\leq s^{-2}\|y_1 - y_2\|^2 = s^{-2}\|v_1 + su_1 - v_2 - su_2\|^2 \\ &= s^{-2}\|(v_1 - v_2) + s(u_1 - u_2)\|^2\end{aligned}$$

and hence arrive at the estimate

$$0 \leq \frac{1}{s}\|v_1 - v_2\|^2 + 2\langle v_1 - v_2, u_1 - u_2 \rangle \quad \text{when } s > M.$$

Letting there $s \rightarrow \infty$ shows that

$$0 \leq \langle v_1 - v_2, u_1 - u_2 \rangle \quad \text{for all } (u_1, v_1), (u_2, v_2) \in \text{gph } T \cap (\text{int } \mathcal{B}_\delta(\bar{u}) \times X),$$

which verifies the semilocal monotonicity of T at any $\bar{u} \in \text{dom } T$. Since $\text{dom } T$ is convex, Lemma 3.4 tells us that T is globally monotone. Now we are in a position to apply Theorem 3.2 and conclude therefore that T is maximal monotone on X . $\square$

It is well known in monotone operator theory that the maximal monotonicity of T always yields the convexity of the *closure* of the domain $\text{cl}(\text{dom } T)$; see, e.g., [2, Corollary 21.12]. This naturally gives a raise to the question: whether Theorem 3.5 is true when the condition on the convexity of $\text{dom } T$ is replaced by the convexity of $\text{cl}(\text{dom } T)$? The following simple example shows that it is not true and consequently that the convexity assumption on $\text{dom } T$ in Theorem 3.5 cannot be dropped.

Example 3.6 (semilocal monotonicity does not yield the convexity of the domain). Define the mapping $T : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$T(x) := \begin{cases} \left\{ -\frac{1}{x} \right\} & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ \emptyset & \text{if } x = 0. \end{cases}$$

Observe that $\text{gph } T$ is closed, T is semilocally monotone on $\mathbb{R}$, $\text{dom } T = \mathbb{R} \setminus \{0\}$ is nonconvex while $\text{cl}(\text{dom } T) = \mathbb{R}$ is convex. Moreover, it is obvious that assertion (ii) of Theorem 3.5 is valid, but T is not globally monotone on $\mathbb{R}$.

The next theorem provides other coderivative characterizations of maximal monotonicity, where the regular coderivative condition (12) is replaced by the positive-semidefiniteness conditions imposed on the *mixed limiting coderivative* (6). These characterizations are clearly equivalent to those presented in Theorems 3.2 and 3.5, but in this paper is more convenient for us to derive them by passing to the limit in (12). Note that the limiting coderivative characterizations have a strong advantage in comparison with (12) due to well-developed calculus rules for (6); see Remark 3.8 and Section 5 for more discussions.

Theorem 3.7 (limiting coderivative characterizations of maximal monotonicity). *Let $T : X \rightrightarrows X$ be a set-valued mapping with closed graph. The following are equivalent:*

- (i) T is maximal monotone on X .
- (ii) T is hypomonotone on X and for any $(u, v) \in \text{gph } T$ we have

$$\langle z, w \rangle \geq 0 \quad \text{whenever } z \in D_M^*T(u, v)(w), \quad w \in X. \quad (26)$$

If in addition the operator domain $\text{dom } T$ is convex, then the (global) hypomonotonicity in assertion (ii) can be equivalently replaced by the semilocal one.

Proof. Implication (ii) $\implies$ (i) is straightforward from Theorem 3.2 due to

$$\widehat{D}^*T(u, v)(w) \subset D_M^*T(u, v)(w) \quad \text{for all } (u, v) \in \text{gph } T, \quad w \in X.$$

Thus (12) follows from (26), and T is maximal monotone by Theorem 3.2.

To justify the reverse implication (i) $\implies$ (ii), suppose that (i) holds, and so (12) is valid due to Theorem 3.2. Picking any $(u, v) \in \text{gph } T$ and $z \in D_M^*T(u, v)(w)$ and using definition (6) of the mixed limiting coderivative, we find sequences $(u_k, v_k) \xrightarrow{\text{gph } T} (u, v)$ with $z_k \xrightarrow{w} z$ and $w_k \rightarrow w$ satisfying $z_k \in \widehat{D}^*T(u_k, v_k)(w_k)$ for all $k \in \mathbb{N}$. It follows from (12) that $\langle z_k, w_k \rangle \geq 0$. Letting $k \rightarrow \infty$ and taking into account that sequence $\{w_k\}$ converges *strongly* in X give us that $\langle z, w \rangle \geq 0$, which verifies (26).

If $\text{dom } T$ is convex, we proceed in the same way with replacing hypomonotonicity by semilocal hypomonotonicity and using Theorem 3.5 instead of Theorem 3.2. $\square$

Remark 3.8 (advantages of limiting coderivative characterizations).

Although the coderivative conditions (12) and (26) married with the corresponding hypomonotonicity give us the equivalent characterizations of maximal monotonicity, the limiting coderivative condition (26) has clear advantages in comparison with the regular coderivative one (12). This is due to the well-developed *full calculus* for the limiting coderivative (in contrast to its regular counterpart) presented in the first volume of [25]. The comprehensive calculus rules developed for (6) allow us to deal with various compositions of set-valued and single-valued mappings and to establish maximal monotonicity of structurally composed operators under the validity of the corresponding qualification conditions. We refer the reader to both volumes of [25] for numerous applications of the coderivative calculus to different issues of variational analysis, optimization, and control while not related to monotonicity.

Note also that a similar full calculus is available in [25] for the *normal* limiting coderivative, which is defined by scheme (6) with replacing the strong convergence $y \rightarrow w$ therein by the weak convergence in X . However, the corresponding positive-semidefiniteness condition in terms of the normal coderivative is only sufficient (together with the imposed hypomonotonicity) for the maximal monotonicity of T outside finite-dimensional spaces. The proof of the necessity part given in Theorem 3.7 does not hold true for the normal coderivative, since we cannot pass to the limit in the inequality $\langle z_k, w_k \rangle \geq 0$ when both sequences $\{z_k\}$ and $\{w_k\}$ converge only weakly in X as $k \rightarrow \infty$.

The following one-dimensional example shows that the hypomonotonicity conditions in (ii) in Theorem 3.2, Theorem 3.5, and Theorem 3.7 are essential for the obtained coderivative characterizations of maximal monotonicity.

Example 3.9 (hypomonotonicity is essential). Given $\eta \geq 0$, define the set-valued mapping $T: \mathbb{R} \rightrightarrows \mathbb{R}$ with full domain by:

$$T(x) := \eta x + [0, 1] \quad \text{for all } x \in \mathbb{R}.$$

It is easy to calculate directly by the definitions (or using elementary calculus) that

$$D^*T(u, v)(w) = \widehat{D}^*T(u, v)(w) = \begin{cases} \{0\} & \text{if } w = 0, v - \eta u \in (0, 1), \\ \{\eta w\} & \text{if } w \geq 0, v - \eta u = 0, \\ \{\eta w\} & \text{if } w \leq 0, v - \eta u = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus both coderivative conditions (12) and (26) are satisfied. However, T is not monotone. The reason is that this mapping is not semilocally hypomonotone.

As consequences of the obtained results, we derive in the next corollary the corresponding regular and limiting coderivative characterizations of strong maximal

monotonicity for set-valued mappings in Hilbert spaces. Recall that $T : X \rightrightarrows X$ is (globally) *strongly maximal monotone* on X with modulus $\kappa > 0$ if it is maximal monotone and the shifted mapping $T - \kappa I$ is monotone on X . It follows from the classical Minty theorem that T is strongly maximal monotone with modulus κ if and only if $T - \kappa I$ is maximal monotone.

Corollary 3.10 (coderivative characterizations of strong maximal monotonicity). *Let $T : X \rightrightarrows X$ be a set-valued mapping with closed graph. The following are equivalent:*

- (i) T is strongly maximal monotone on X with modulus $\kappa > 0$.
- (ii) T is hypomonotone on X and for any $(u, v) \in \text{gph } T$ we have

$$\langle z, w \rangle \geq \kappa \|w\|^2 \quad \text{whenever } z \in \widehat{D}^*T(u, v)(w), \quad w \in X.$$

- (iii) T is hypomonotone on X and for any $(u, v) \in \text{gph } T$ we have

$$\langle z, w \rangle \geq \kappa \|w\|^2 \quad \text{whenever } z \in D_M^*T(u, v)(w), \quad w \in X.$$

If in addition the operator domain $\text{dom } T$ is convex, then the (global) hypomonotonicity in assertions (ii) and (iii) can be equivalently replaced by the semilocal one.

Proof. Define $S := T - \kappa I$ and get from the corresponding coderivative sum rules in [25, Theorem 1.62] the equalities

$$\begin{aligned} \widehat{D}^*T(u, v)(w) &= \widehat{D}^*S(u, v - \kappa u)(w) + \kappa w \\ \text{and } D_M^*T(u, v)(w) &= D_M^*S(u, v - \kappa u)(w) + \kappa w \end{aligned}$$

for all $(u, v) \in \text{gph } T$ and $w \in X$. Thus the validity of (ii) (resp. (iii)) for T is equivalent to the fulfillment of all the conditions in Theorem 3.2(ii) (resp. in Theorem 3.7(ii)) for the operator S ; it is obvious for hypomonotonicity. Applying now Theorem 3.2 and Theorem 3.7, respectively, we get that either assertion (ii) or (iii) of this corollary is equivalent to the maximal monotonicity of S . Since the latter is equivalent to the strong maximal monotonicity of T with modulus κ , we complete the proof of the corollary. $\square$

Note that a certain localized regular coderivative characterization of the *local* strong maximal monotonicity for set-valued mappings with respect to *product* neighborhoods (see the discussion right before Definition 3.3) has been recently obtained in [28, Theorem 3.4] while being independent of the global characterizations in Corollary 3.10.

4. Second-Order Characterizations of Convexity

In this section we apply the obtained coderivative characterizations of maximal monotonicity and second-order subdifferential constructions to characterize *convexity* and *strong convexity* for the remarkable class of lower- $\mathcal{C}^2$ functions.

Recall [38, Definition 10.29] that a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is *lower- $\mathcal{C}^k$* with $k \in \mathbb{N} \cup \{\infty\}$ if for each $\bar{x} \in \mathbb{R}^n$ there is a neighborhood V of $\bar{x}$ on which f admits the representation

$$f(x) = \max_{t \in T} f_t(x), \quad x \in V,$$

where the functions f_t are of class $\mathcal{C}^k$ on V , where T is compact, and where $f_t(x)$ and all their partial derivatives in x through order k depend continuously on $(t, x) \in T \times V$. This class of functions introduced by Rockafellar [37] is among the favorable classes of functions in variational analysis and optimization. Many nice properties and equivalent descriptions of such functions can be found, e.g., in [1, 37, 38]. As shown in [38, Corollary 10.34], for each $k > 2$ the class of lower- $\mathcal{C}^k$ functions coincides with the class of lower- $\mathcal{C}^2$ functions. However, the latter is a proper subclass of lower- $\mathcal{C}^1$ functions. In fact, between the class of lower- $\mathcal{C}^1$ functions and the class of lower- $\mathcal{C}^2$ functions there are the classes of lower- $\mathcal{C}^{1,\alpha}$ functions, $0 < \alpha \leq 1$, which have been studied recently by Daniilidis and Malick [14].

The next theorem is the main result of this section, which provides complete second-order characterizations of convexity for lower- $\mathcal{C}^2$ functions in finite dimensions.

Theorem 4.1 (second-order subdifferential characterizations of convexity for the class of lower- $\mathcal{C}^2$ functions). *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a lower- $\mathcal{C}^2$ function. Then the following assertions are equivalent:*

- (i) f is convex on $\mathbb{R}^n$.
- (ii) For each $(u, v) \in \text{gph } \partial f$ we have the condition

$$\langle z, w \rangle \geq 0 \quad \text{whenever } z \in \partial^2 f(u, v)(w), \quad w \in \mathbb{R}^n. \quad (27)$$

- (iii) For each $(u, v) \in \text{gph } \partial f$ we have the condition

$$\langle z, w \rangle \geq 0 \quad \text{whenever } z \in \check{\partial}^2 f(u, v)(w), \quad w \in \mathbb{R}^n. \quad (28)$$

Proof. To verify implication (i) $\implies$ (ii), observe that the subdifferential operator $\partial\varphi$ is maximal monotone for any convex l.s.c. function by the classical result of convex analysis. Hence condition (27) follows from implication (i) $\implies$ (ii) in Theorem 3.7 by construction (8) of the second-order subdifferential in finite dimensions. The next implication (ii) $\implies$ (iii) of the theorem is obvious due to the inclusion

$$\check{\partial}^2 f(u, v)(w) \subset \partial^2 f(u, v)(w) \quad \text{for every } (u, v) \in \text{gph } \partial f, \quad w \in \mathbb{R}^n.$$

To prove finally implication (iii) $\implies$ (i), suppose that f is lower- $\mathcal{C}^2$ and that condition (28) holds. It follows from [38, Example 12.28] that the subdifferential mapping $T(x) := \partial\varphi(x)$ for the lower- $\mathcal{C}^2$ function φ is semilocally hypomonotone with $\text{dom } T = \mathbb{R}^n$. Then (iii) amounts to saying that for any $(u, v) \in \text{gph } T$ we have

$$\langle z, w \rangle \geq 0 \quad \text{for all } z \in D^*T(u, v)(w), \quad w \in \mathbb{R}^n.$$

Since the set $\partial\varphi$ is closed in this setting, we deduce from Theorem 3.5 that T is monotone, and thus φ is convex by the result of [13]; see also [25, Theorem 3.56]. $\square$

Remark 4.2 (discussion on second-order subdifferential characterizations of convexity.) The following comments are in order:

- (i) Consider the pointwise maximum of finitely many $\mathcal{C}^2$ functions

$$f(x) := \max \{f_1(x), \dots, f_m(x)\}, \quad x \in \mathbb{R}, \quad (29)$$

which surely belongs to the class of lower- $\mathcal{C}^2$ functions. Based in the recent precise calculation [18] of the second-order subdifferential $\partial^2 f$ of (29) via the generated functions f_i and the appropriate index subsets of $\{1, \dots, m\}$, we can express the second-order characterization (27) entirely in terms of the initial data of (29).

- (ii) Second-order subdifferential characterizations of convexity have been recently obtained in [7, 8, 9, 10, 11] for functions that may not be necessarily lower- $\mathcal{C}^2$, and thus they are generally independent of Theorem 4.1. Note also that it is not clear by now whether the characterizations of Theorem 4.1 hold if f is merely lower- $\mathcal{C}^1$ or lower- $\mathcal{C}^{1,\alpha}$ with $0 < \alpha \leq 1$.

Finally in this section, we present the second-order subdifferential characterizations of strong convexity for lower- $\mathcal{C}^2$ functions, which can be treated as consequences of Theorem 4.1. Recall that f is *strongly convex* on $\mathbb{R}^n$ with modulus $\kappa > 0$ if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \frac{\kappa}{2}\lambda(1 - \lambda)\|x - y\|^2$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$. It is well known that the strong convexity of f is equivalent to the convexity of the shifted function

$$g(x) := f(x) - \frac{\kappa}{2}\|x\|^2, \quad x \in \mathbb{R}^n. \quad (30)$$

Corollary 4.3 (second-order subdifferential characterizations of strong convexity for lower- $\mathcal{C}^2$ functions). *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be lower- $\mathcal{C}^2$. The following are equivalent.*

- (i) f is strongly convex on $\mathbb{R}^n$ with modulus $\kappa > 0$.
(ii) We have the second-order subdifferential condition

$$\langle z, w \rangle \geq \kappa\|w\|^2 \quad \text{for all } z \in \partial^2 f(u, v)(w), \quad (u, v) \in \text{gph } \partial f, \quad w \in \mathbb{R}^n.$$

- (iii) We have the combined second-order subdifferential condition

$$\langle z, w \rangle \geq \kappa\|w\|^2 \quad \text{for all } z \in \check{\partial}^2 f(u, v)(w), \quad (u, v) \in \text{gph } \partial f, \quad w \in \mathbb{R}^n.$$

Proof. It can be derived from Theorem 4.1 by applying it to the shifted function (30) and taking into account the obvious subdifferential relationship

$$\partial g(x) = \partial f(x) - \kappa x, \quad x \in \mathbb{R}^n.$$

On the other hand, we can justify the results by applying the characterizations of strong maximal monotonicity from Corollary 3.10 similarly to the proof of Theorem 4.1. $\square$

5. Concluding Remarks

The main results of this paper provide coderivative characterizations of maximal monotonicity for set-valued mappings under one of the following assumptions: (a) the mapping is globally hypomonotone without imposing the convexity of its domain, and (b) the mapping is semilocally hypomonotone with convex domain. Example 3.9 shows that removing hypomonotonicity may destroy these characterizations. However, it is proved in [9] that for single-valued mappings hypomonotonicity can be replaced by continuity. Thus the *first* open question is to clarify what is common for both hypomonotonicity of set-valued mappings and continuity of single-valued ones. We intend to develop general results in this direction, which unify these two requirements.

The *second* area of the promising future research is to find an umbrella, which covers all the second-order subdifferential characterizations of convexity for functions developed in the previous investigations [7, 8, 9, 10, 11] and those obtained in Theorem 4.1 for lower- $\mathcal{C}^2$ functions. So far we can deduce [7, Theorem 4.1] from Theorem 4.1 while the other major results of the aforementioned papers seem to be independent. It is also desired to obtain second-order subdifferential characterizations of convexity for the important classes of lower- $\mathcal{C}^1$ and lower- $\mathcal{C}^{1,\alpha}$ ($0 < \alpha \leq 1$) functions.

The *third* and probably most important direction of the future research is employing the limiting coderivative calculus to derive from the pointwise coderivative characterization (26) verifiable conditions for *preserving maximal monotonicity* under various combinations (including sums, compositions, etc.) of set-valued and single-valued maximal monotone operators. The classical result in this vein is Rockafellar's theorem [35] about maximal monotonicity of sums of maximal monotone operators under certain interiority or local boundedness assumptions. It seems that the coderivative characterizations of maximal monotonicity obtained in our paper open a new gate (in Hilbert spaces so far) to proceed in this direction via verifying qualification conditions that ensure the validity of the corresponding coderivative calculus rules from [25].

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