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Integration of Nonconvex Epi-Pointed Functions in Locally Convex Spaces*

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Dedicated to the memory of Jean Jacques Moreau.

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We extend the results of Correa, Garcia and Hantoute [6], dealing with the integration of nonconvex epi-pointed functions using the Fenchel subdifferential. In this line, we prove that the classical formula of Rockafellar in the convex setting is still valid in general locally convex spaces for an appropriate family of nonconvex epi-pointed functions, namely those we call SDPD. The current integration formulas use the Fenchel subdifferential of the involved functions to compare the corresponding closed convex envelopes. Some examples of SDPD functions are investigated. This analysis leads us to approach a useful family of locally convex spaces, referred to as the SDPD, having an RNP-like property.

1. Introduction

The problem with integration of lower semicontinuous convex functions in terms of their Fenchel subdifferential, was started and solved by Moreau [11] in Hilbert spaces. The problem was then studied by Rockafellar [14] in the 60's: He proved that every two proper lower semicontinuous convex functions f, g defined over a

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Banach space X with values in $\mathbb{R} \cup \{+\infty\}$ satisfying

$$\partial f(x) \subseteq \partial g(x), \quad \forall x \in X,$$

are in fact equal up to an additive constant. Motivated by this result, some generalizations to nonconvex cases have been made by, among others, Benoist, Correa, Daniilidis, Jofré, Poliquin, Thibault, Zagrodny and Zlateva involving the Fenchel subdifferential and other kinds of subdifferentials as well (see, for example [2, 7, 13, 17, 18, 19]). The literature on the subject is vast. In particular, in the article of R. Correa, Y. Garcia and A. Hantoute [6], they accomplish many integration results for a particular family of nonconvex functions introduced by Benoist and Hiriart-Urruty in [9]: the ones which are *epi-pointed*. A function f defined over a Banach space X with values in $\mathbb{R} \cup \{+\infty\}$ is epi-pointed if the effective domain of its Legendre-Fenchel conjugate f^* has nonempty interior.

The most interesting result in this line is the following: If X is a Banach space with the Radon-Nikodým property, then for each epi-pointed and lower-semicontinuous function $f : X \to \mathbb{R}\cup\{+\infty\}$ and any function $g : X \to \mathbb{R}\cup\{+\infty\}$ we have that

$$\partial f(x) \subseteq \partial g(x), \quad \forall x \in X \;\Rightarrow\; \exists c \in \mathbb{R}, \quad \overline{\mathrm{co}}\, f = \overline{\overline{\mathrm{co}}\, g \Box \sigma_{\mathrm{dom} f^*}} + c,$$

where $\overline{\mathrm{co}}\, f$ denotes the closed convex envelope of f, $\sigma_{\mathrm{dom} f^*}$ is the support function of the effective domain of f^*, $\mathrm{dom} f^*$, and $\Box$ is the classic Moreau inf-convolution. In this work, we generalize this result replacing the Banach space by a general (Hausdorff) locally convex space (X, τ_X). For this we introduce a new notion of epi-pointedness, depending of the topology used in the dual space X^*, and replace the RNP hypothesis with a special property of the function itself, namely to be *SDPD*.

In Section 2 we fix notation and recall some useful definitions and properties. Section 3 shows some preliminary results about properties of continuous convex functions. Section 4 is the core of the paper: We define epi-pointedness and integrability and prove a general integration theorem. Finally, in Section 5, we provide some examples of SDPD functions and describe a necessary condition for a function to be SDPD that can be translated into an RNP-like property of the space X itself.

2. Notation

In the following, X will be a real (Hausdorff) locally convex space endowed with a topology θ. The topological dual of X will be denoted by X^*. On X we consider two other topologies: The weak topology, $w(X, X^*)$, and the Mackey topology, $\tau(X, X^*)$. On X^* we consider three topologies given by the primal: The weak-star topology, denoted by $w^*(X^*, X)$, the Mackey topology due to the primal, denoted by $\tau(X^*, X)$, and the strong topology, denoted by $\beta(X^*, X)$. For the definitions of these topologies we refer the reader to [16, Chapter IV].

Also, X^{**} will be the bidual of X, defined as the dual space of $(X^*, \beta(X^*, X))$. The same above topologies considered for X and X^* are considered for X^* and X^{**}, where X^* plays the role of the primal, $\beta(X^*, X)$ the role of θ, and X^{**} the role of the dual. When there is no confusion, we will simplify the notation for the weak and strong topologies, using only w and β respectively. In the particular case of weak-star topology, we will write w^* for $w^*(X^*, X)$, w^{**} for $w^*(X^{**}, X^*)$.

Let us now recall some definitions and notations of convex analysis which will be needed. We will write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$. Also, τ_0 will denote the usual topology in $\mathbb{R}$ (and its natural extension to $\mathbb{R}_\infty$ and $\overline{\mathbb{R}}$). For a given function $f : X \to \overline{\mathbb{R}}$, we use $\operatorname{dom} f$ to denote the (effective) *domain* of f, namely $\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\}$. We say that f is *proper*, if $\operatorname{dom} f \neq \emptyset$ and $f > -\infty$. Following [10], we denote by $\Gamma_0(X, \theta)$ the set of all θ-lower semicontinuous (lsc, for short) convex proper functions defined on X; clearly

$$\Gamma_0(X, w(X, X^*)) = \Gamma_0(X, \theta),$$

and so, for any two (Hausdorff) locally convex topologies τ_1 and τ_2 over X such that $(X, \tau_1)^* = (X, \tau_2)^*$, we have that $\Gamma_0(X, \tau_1) = \Gamma_0(X, \tau_2)$.

The (Legendre-Fenchel) *conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ given by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\},$$

and the *subdifferential* of f at a point $x \in \operatorname{dom} f$ is the (possibly empty) subset of X^* given by

$$\partial f(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq f(y) - f(x), \ \forall y \in X\}.$$

Also, for $\varepsilon > 0$, the *ε-subdifferential* of f at x is the subset of X^* given by

$$\partial_\varepsilon f(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon, \ \forall y \in X\}.$$

Similarly, the *biconjugate* of f is the function $f^{**} : X^{**} \to \overline{\mathbb{R}}$ defined as the conjugate function of f^*. In particular, provided that f^* is proper, the restriction of f^{**} on X coincides with the *θ-closed convex envelope* of f, $\overline{\operatorname{co}} f : X \to \overline{\mathbb{R}}$ defined by

$$\overline{\operatorname{co}} f(x) = \sup\{g(x) \mid g \in \Gamma_0(X, \theta), \ g \leq f\}.$$

It is worth pointing out that the notation $\overline{\operatorname{co}} f$ is not ambiguous, because for any locally convex topology τ considered between $w(X, X^*)$ and $\tau(X, X^*)$, the τ-lsc convex envelope of f is $\overline{\operatorname{co}} f$ (this function is completely determined by the duality $\langle X, X^*\rangle$). The *θ-closed envelope* $\overline{f}^\theta$ of f is given by

$$\overline{f}^\theta(x) = \sup\{g(x) \mid g : X \to \overline{\mathbb{R}} \text{ is } \theta\text{-lsc, and } g \leq f\}.$$

Again, if there is no confusion, we simply write $\overline{f}$. Finally, $\mathrm{Cont}[f,\theta]$ denotes the set of points where f is finite and θ-continuous. When there is no ambiguity concerning the topology, we simply write $\mathrm{Cont}[f]$.

In the case of f^*, the subdifferential and ε-subdifferential mappings $\partial f^*, \partial_\varepsilon f^* : X^* \rightrightarrows X^{**}$ can be written as

$$\partial f^*(x^*) = \{x^{**} \in X^{**} \mid \langle x^{**}, x^* \rangle = f^*(x^*) + f^{**}(x^{**})\}$$
$$\partial_\varepsilon f^*(x^*) = \{x^{**} \in X^{**} \mid \langle x^{**}, x^* \rangle \geq f^*(x^*) + f^{**}(x^{**}) - \varepsilon\}.$$

The *indicator* and the *support* functions of a set A ($\subseteq X, X^*$) are, respectively,

$$I_A(x) := \begin{cases} 0 & x \in A \\ +\infty & x \notin A, \end{cases} \qquad \sigma_A := I_A^*.$$

The *(Moreau) inf-convolution* of two functions $f, g : X \to \mathbb{R}_\infty$ is $f \square g := \inf_{x \in X}\{f(x) + g(\cdot - x)\}$. With $f(x) \in \mathbb{R}$, the directional derivative of f at x in the direction $u \in X$ is $f'(x;u)$, where

$$f'(x;u) = \lim_{t \searrow 0} \frac{f(x+tu) - f(x)}{t}, \tag{1}$$

whenever the above limit exists. Also, for a set $U \subseteq X$, ρ_U is the Minkowski functional of U, given by $\rho_U(u) = \inf\{t > 0 \ : \ t^{-1}u \in U\}$. Recall that f is said to be Lipschitz θ-continuous near $x \in X$ if $f(x) \in \mathbb{R}$ and there exist $\gamma > 0$ and a θ-neighborhood U of zero such that f is finite on $x + U$ and

$$|f(y) - f(z)| \leq \gamma \rho_U(y - z), \quad \forall z, y \in x + U.$$

Note that the neighborhood U of zero given in the definition of Lipschitz continuity of f near x can always be assumed convex and symmetric (just take a convex neighborhood V of zero contained in U and replace U by $V \cap (-V)$). Also, it is known that a convex function f is Lipschitz θ-continuous near $x \in X$ if and only if $x \in \mathrm{Cont}[f,\theta]$ (see [20, Lemma 2.2.8]).

If $M : Y \rightrightarrows Z$ is a set-valued operator, for two sets Y, Z, we denote by $M^{-1}(z) := \{y \in Y \mid z \in My\}$, $\mathrm{Im}\, M := \bigcup_{y \in Y} M(y)$ and $\mathrm{dom}\, M := \{y \in Y \mid M(y) \neq \emptyset\}$. The graph $\mathrm{gph}\, M$ of M is the set

$$\mathrm{gph}\, M := \{(y,z) \in Y \times Z \mid z \in M(y)\}.$$

It will be convenient sometimes to identify $\mathrm{gph}\, M$ with M; that is, to write $(y,z) \in M$ instead of $z \in M(y)$.

For a set $A \subseteq X$, $\mathrm{co}(A)$, $\overline{\mathrm{co}}^\theta(A)$ and A^o denote the *convex hull*, the *θ-closed convex hull* and the *polar* set of A, respectively, where the polar set is the subset of X^* given by

$$A^o = \{x^* \in X^* \mid \langle x^*, x \rangle \leq 1, \forall x \in A\}.$$

The polar set depends on the duality considered (in this case, $\langle X, X^*\rangle$). We will use the same notation for each possible duality, and if there is any confusion we will specify which duality is used. When there is no ambiguity on the topology, we write simply $\overline{\text{co}}(A)$: in particular, if A is convex, then $\overline{\text{co}}^w(A) = \overline{\text{co}}^\theta(A)$, and so the notation $\overline{\text{co}}(A)$ is not ambiguous. Observe also that for any function $f : X \to \mathbb{R}_\infty$, we have that $\overline{\text{co}}\, f$ is the (unique) convex lsc function satisfying

$$\text{epi}(\overline{\text{co}}\, f) = \overline{\text{co}}(\text{epi}\, f).$$

Finally, we will write $\text{Int}[A, \theta]$ and $\overline{A}^\theta$ for the θ-interior and the θ-closure of A, respectively (if there is no confusion, we simply write $\text{Int}[A]$ and $\overline{A}$).

3. Preliminary results on continuous functions

For any function $f \in \Gamma_0(X, \theta)$, the θ-continuity points play a fundamental role. It is known that if $x \in \text{Cont}[f, \theta]$, then $\partial f(x)$ is nonempty and w^*-compact and

$$f'(x; u) = \max\{\langle x^*, u\rangle \ : \ x^* \in \partial f(x)\}, \quad \forall u \in X. \tag{2}$$

Also, if $\text{Cont}[f, \theta]$ is nonempty, then $\text{Cont}[f, \theta] = \text{Int}[\text{dom}\, f, \theta]$. Nevertheless, $\text{Cont}[f, \theta]$ can be empty even if $\text{Int}[\text{dom}\, f, \theta]$ is not: A counterexample is easily found in a nonreflexive dual Banach space endowed with the w^*-topology. Just consider the support function of the unit ball $\sigma_{\mathbb{B}_X}$ which is the conjugate of $I_{\mathbb{B}_X}$ and so is w^*-lower semicontinuous. Because of the nonreflexivity, there is a functional $x^* \in X^*$ that doesn't achieve its norm and so

$$X \cap \partial\sigma_{\mathbb{B}_X}(x^*) = \emptyset.$$

Then, $\sigma_{\mathbb{B}_X}$ cannot be w^*-continuous at x^* and because $\text{dom}\, \sigma_{\mathbb{B}_X} = X^*$ the counterexample is accomplished. In view of Sections 4 and 5, we will study here some results on θ-continuity that are known in the Banach space setting.

Lemma 3.1. *Let $f \in \Gamma_0(X, \theta)$ with $D = \text{Cont}[f, \theta] \neq \emptyset$. Then, for each $x \in \overline{\text{dom}\, f}^\theta$ we have that*

$$f(x) = \liminf_{D \ni y \to x} f(y).$$

Proof. In this proof, the topological notation will refer to the θ topology for X. We know that for $x \in \overline{\text{dom}\, f}$,

$$f(x) = \liminf_{y \to x} f(y) = \liminf_{\text{dom}\, f \ni y \to x} f(y).$$

On the other hand, since $\text{Cont}[f]$ is nonempty, the epigraph of f has nonempty interior in the $\theta \times \tau_0$ topology, and so

$$\text{epi}\, f = \overline{\text{Int}[\text{epi}\, f]}.$$

Let us consider $f\big|_D$. It is direct that $\mathrm{Int}[\mathrm{epi}\, f] \subseteq \mathrm{epi}\, f\big|_D \subseteq \mathrm{epi}\, f$, thus $\overline{\mathrm{epi}\, f\big|_D} = \mathrm{epi}\, f$. So, there exists a net $(x_i, r_i)_{i\in I} \subseteq \mathrm{epi}\, f\big|_D$ such that $(x_i, r_i) \to (x, f(x))$. Thus,

$$f(x) = \liminf_{\mathrm{dom}\, f \ni y \to x} f(y) \leq \liminf_{D \ni y \to x} f(y) \leq \liminf_{i\in I} f(x_i) \leq \liminf_{i\in I} r_i = f(x),$$

and so $f(x) = \liminf_{D\ni y\to x} f(y)$. □

The next lemma is related to maximal monotone operators. Recall that a set-valued operator $M : X \rightrightarrows X^*$ is *monotone* provided that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0,$$

for all $(x_i, x_i^*) \in M$, $i = 1, 2$. When in addition there is no different set-valued monotone operator whose graph contains $\mathrm{gph}\, M$, one says that M is *maximal monotone.*

Lemma 3.2. *Let $f \in \Gamma_0(X, \theta)$ with $\mathrm{Cont}[f, \theta] \neq \emptyset$. Then ∂f is a maximal monotone operator.*

Proof. We follow the proof in [12, Theorem 2.25], where f is assumed to be continuous in all of X and X is a Banach space. Let $(y, y^*) \in X \times X^*$ such that $y^* \notin \partial f(y)$. In order to prove that ∂f is maximal monotone, it is sufficient to show that there exists some $(x, x^*) \in \partial f$ such that

$$\langle x^* - y^*, x - y \rangle < 0.$$

Without loss of generality, we can suppose that $(y, y^*) = (0, 0)$. If not, we can replace f with the function $g \in \Gamma_0(X, \theta)$ given by $g(x) = f(x + y) - \langle y^*, y\rangle$ and it is easily verified that

$$\partial g(x) = \partial f(x + y) - y^*, \quad \forall x \in X.$$

Since $0 \notin \partial f(0)$, we know that 0 is not a global minimum of f. Then, there exists $x \in \mathrm{dom}\, f$ such that $f(x) < f(0)$. This and Lemma 3.1 guarantee that there exists $x_1 \in \mathrm{Cont}[f]$ such that $f(x_1) < f(0)$. Consider now the function $h : [0, 1] \subset \mathbb{R} \to \mathbb{R}$ given by $h(t) = f(tx_1)$. For $\lambda \in (0, 1)$ we have that

$$\begin{aligned} h'(\lambda; 1) &= \lim_{t\searrow 0} \frac{h(\lambda + t) - h(\lambda)}{t} \\ &= \lim_{t\searrow 0} \frac{f(\lambda x_1 + tx_1) - f(\lambda x_1)}{t} = f'(\lambda x_1; x_1). \end{aligned}$$

Then, because $h(1) = f(x_1) < f(0) = h(0)$, there exists some $t_1 \in (0, 1)$ such that $h'(t_1; 1) < 0$. Fixing $x = t_1 x_1$ and noting that $x \in \mathrm{Int}[\mathrm{dom}\, f] = \mathrm{Cont}[f]$, we have that

$$\max\{\langle x^*, x\rangle \mid x^* \in \partial f(x)\} = t_1 f'(x; x_1) < 0,$$

and hence there is an $x^* \in \partial f(x)$ such that $\langle x^*, x\rangle \leq t_1 f'(x; x_1) < 0$, which completes the proof. □

The next proposition is probably known but we still present a detailed demonstration because of its utility.

Proposition 3.3. *Let $f : X \to \mathbb{R}_\infty$ be a proper convex function and let $x \in \mathrm{Cont}[f, \theta]$. Then, ∂f is locally w^*-bounded near x, which means that there exists a θ-neighborhood U of zero such that $\partial f(x+U)$ is w^*-bounded. Even more, U can be chosen such that $\partial f(x+U)$ is contained in an absolutely convex (convex and balanced) w^*-compact set of X^*.*

Proof. In this proof, the topological notation will refer to the θ topology for X. Since f is continuous at x, there exists a convex symmetric neighborhood U of zero such that, f is Lipschitz continuous on $x + 2U$ with constant $\gamma > 0$. Then, for $z \in x + U$ and $z^* \in \partial f(z)$ we have that for all $h \in U$,

$$\begin{aligned} |\langle z^*, h\rangle| &= \max(\langle z^*, h\rangle, \langle z^*, -h\rangle) \\ &\le \max(|f(z+h) - f(z)|, |f(z-h) - f(z)|) \le \gamma \rho_{2U}(h). \end{aligned}$$

Then, because ρ_{2U} and $|\langle z^*, \cdot\rangle|$ are positively homogeneous,

$$z^* \in B = \{x^* \in X^* \mid |\langle x^*, h\rangle| \le \gamma \rho_{2U}(h),\ \forall h \in X\},$$

and so $\partial f(x+U) \subseteq B$. It is easy to verify that $B = (2\gamma^{-1}U)^o$ and so, by the Alaoglu-Bourbaki theorem, B is w^*-compact. From the construction, B is also absolutely convex, and so the proof is finished. □

The last interesting property of the subdifferential of a proper convex function f is its outer-semicontinuity at continuity points of f. Let us recall first the definition of outer-semicontinuity:

Definition 3.4 (Outer-semicontinuity). Let $(T, \tau), (S, \sigma)$ be two Hausdorff topological spaces. A set-valued mapping $\mathcal{R} : T \rightrightarrows S$ is τ-σ-outer semicontinuous at a point $t_0 \in T$ if for each net $(t_i, s_i) \in \mathcal{R}$ which is $\tau \times \sigma$-convergent in $T \times S$ with $t_i \to t_0$, we have that

$$\lim_i s_i \in \mathcal{R}(t_0).$$

Proposition 3.5. *Let $f : X \to \mathbb{R}_\infty$ be a proper convex function with $\mathrm{Cont}[f, \tau(X, X^*)] \neq \emptyset$. Then ∂f is $\tau(X, X^*)$-w^*-outer semicontinuous at each point of $\mathrm{Cont}[f, \tau(X, X^*)]$.*

Proof. In this proof, the topological notation will refer to the Mackey topology $\tau(X, X^*)$ for X and the w^*-topology for X^*. Fix $x \in \mathrm{Cont}[f]$ and let $(x_i, x_i^*)_{i\in I} \subseteq \partial f$ be a convergent net with limit $(x, x^*) \in X \times X^*$. Applying Proposition 3.3, there exists an open neighborhood V of x such that $\partial f(V)$ is contained in an absolutely convex w^*-compact set K of X^*. Thus, there exists an $i_0 \in I$ such that for all $i \ge i_0$, $x_i \in V$, and so $(x_i^*)_{i \ge i_0} \subseteq K$. Now, fix $y \in X$. We have that

$$\langle x_i^*, y - x_i\rangle = \langle x_i^*, y - x\rangle + \langle x_i^*, x - x_i\rangle \le \langle x_i^*, y - x\rangle + \sup_{z^* \in K} \langle z^*, x - x_i\rangle.$$

It is clear that $\langle x_i^*, y-x\rangle \to \langle x^*, y-x\rangle$. Also, since $(x_i - x)_{i\geq i_0}$ is convergent to zero, by definition of the Mackey topology in [16, Chapter IV.3, Corollary 1], it converges to zero uniformly over absolutely convex w^*-compact sets of X^*. In particular, we have that

$$\lim_{i\geq i_0} \sup_{z^*\in K} \langle z^*, x_i - x\rangle = 0.$$

Then, provided f is continuous at x, we have that

$$0 \leq f(y) - f(x_i) - \langle x_i^*, y - x_i\rangle \to f(y) - f(x) - \langle x^*, y - x\rangle,$$

and so $f(x) + \langle x^*, y-x\rangle \leq f(y)$. Since y is arbitrary, $x^* \in \partial f(x)$. This finishes the proof. □

Remark 3.6. The above Proposition replaces the well-known norm-w^*-upper semicontinuity property of ∂f, when X is a Banach space [12, Proposition 2.5]. In fact, combining Propositions 3.3 and 3.5 we can derive that ∂f is also $\tau(X, X^*)$-w^*-upper semicontinuous at each point of $\mathrm{Cont}[f, \tau(X, X^*)]$ (see, e.g., [1, Definition 6.2.4 and Proposition 6.3.2]). Even though we won't use it directly, this last result is worth mentioning, since upper-semicontinuity has been an extremely useful property of the subdifferential in the development of convex analysis. Here, the use of Mackey topology $\tau(X, X^*)$ is fundamental.

4. Main result: Integration theorem

The first concept we need is an appropriate extension of the notion of epi-pointedness:

Definition 4.1 (τ-epi-pointedness). Let τ be a locally convex topology on X^* finer than the w^*-topology. A function over X with values in $\mathbb{R}_\infty$ is said to be τ-epi-pointed if the set of τ-continuity points of its conjugate is nonempty.

Outside the RNP setting, epi-pointedness and lower semicontinuity are not enough to ensure the nonemptiness of the subdifferential, and therefore it is not possible to perform any type of integration as we want to, having only these hypotheses.

Proposition 4.2. *Let X be a Banach space which lacks the RNP. Then there exists an epi-pointed lower semicontinuous function $f : X \to \mathbb{R}_\infty$ such that $\partial f(x) = \emptyset$, for all $x \in X$.*

Proof. Since X is a space lacking the RNP, then, $X \times \mathbb{R}$ also lacks the RNP. Applying [4, Theorem 3.7.8], there exists an equivalent norm p over $X \times \mathbb{R}$ and a closed set $D \subseteq \mathbb{B}'_{(X\times\mathbb{R},p)}$ (where $\mathbb{B}'_{(X\times\mathbb{R},p)}$ denotes the open unit ball in $X \times \mathbb{R}$ given by p), such that

$$\mathbb{B}_{(X\times\mathbb{R},p)} = \overline{\mathrm{co}}(D).$$

Let us consider the set $E = \{(x, \lambda) \ : \ \exists \alpha \le \lambda, \ (x, \alpha) \in D\}$. Evidently, E is the epigraph of the function

$$\begin{aligned} f : X &\to \mathbb{R}_\infty \\ x &\mapsto \inf\{t \in \mathbb{R} \ : \ (x, t) \in E\}. \end{aligned}$$

We claim that f is the function that we are looking for. First, we show the epi-pointedness. Note that, since $\overline{\text{co}}(D) = \mathbb{B}_{(X\times\mathbb{R},p)}$, we have

$$\overline{\text{co}}(E) = \{(x, \lambda) \ : \ \exists \alpha \le \lambda, \ p(x, \alpha) \le 1\},$$

and so $\overline{\text{co}}\, f$ is given by

$$\overline{\text{co}}\, f(x) = \inf\{t \in \mathbb{R} \ : \ p(x, t) \le 1\}.$$

Then we have that

$$\begin{aligned} f^*(x^*) &= \sup_{x\in X} \{\langle x^*, x\rangle - \inf\{t \in \mathbb{R} \ : \ p(x, t) \le 1\}\} \\ &= \sup_{(x,t)\in X\times\mathbb{R}} \left\{\langle x^*, x\rangle - t + I_{\mathbb{B}_{(X\times\mathbb{R},p)}}(x, t)\right\} \\ &= p_*(x^*, -1), \end{aligned}$$

where p_* denotes the dual norm on $X^* \times \mathbb{R}$ induced by p. We see that f^* is continuous and real-valued, which proves that f is epi-pointed. Let us suppose now, by contradiction, that f is not lower semicontinuous. Then, there exists a sequence $(x_n) \in \operatorname{dom} f$ converging to some point $x \in X$ such that

$$\bar{t} = \liminf f(x_n) < f(x).$$

We may assume that $(f(x_n))$ is converging to $\bar{t}$. Since $(x_n) \subset \operatorname{dom} f$, we have that $(x_n, f(x_n)) \subset D$ and $(x, \bar{t}) = \lim(x_n, f(x_n)) \in D$. Then we have that

$$f(x) = \inf\{t \in \mathbb{R} \ : \ \exists \alpha \le t, \ (x, \alpha) \in D\} \le \bar{t} < f(x),$$

which is clearly a contradiction. It only remains to verify the emptyness of $\partial f(x)$, for $x \in \operatorname{dom} f$. Let $x \in \operatorname{dom} f$. Since $(x, f(x)) \in D$, $(x, \overline{\text{co}}\, f(x)) \in \mathbb{S}_{(X\times\mathbb{R},p)}$ (where $\mathbb{S}_{(X\times\mathbb{R},p)}$ denotes the unit sphere in $X\times\mathbb{R}$ given by p) and $D\cap\mathbb{S}_{(X\times\mathbb{R},p)} = \emptyset$, we have that $\overline{\text{co}}\, f(x) < f(x)$, and so $\partial f(x) = \emptyset$. $\square$

The next definition provides the sufficient conditions needed to perform integration of a function using the same techniques applied in [6], as it will be shown in Theorem 4.8.

Definition 4.3 (SDPD function). We say that a function $f : X \to \mathbb{R}_\infty$ is Subdifferential Dense Primal Determined (SDPD) if it is $\tau(X^*, X^{**})$-epi-pointed and the set of functionals $x^* \in \text{Cont}[f^*, \tau(X^*, X^{**})]$ which satisfy the equality

$$\partial f^*(x^*) = \overline{\text{co}}^{w^{**}} \left[(\partial f)^{-1}(x^*)\right] \tag{3}$$

is $\tau(X^*, X^{**})$-dense in $\text{Cont}[f^*, \tau(X^*, X^{**})]$.

The choice of the Mackey topology is crucial for three reasons: First, we have more epi-pointed functions; Second, in the Banach spaces setting, the Mackey topology $\tau(X^*, X^{**})$ coincides with the norm topology in X^*; And finally, the subdifferential of any conjugate function is $\tau(X^*, X^{**})$-w^{**}-outer semicontinuous at each point of $\tau(X^*, X^{**})$-continuity, according to Proposition 3.5. This final property will be a key to prove the integration theorem. The problem with this choice is that the density of the functionals which satisfy equality (3) is harder: We will need the equation to hold at more points.

Lemma 4.4. *Let $f, h \in \Gamma_0(X, \theta)$, both having at least one point of $\tau(X, X^*)$-continuity and satisfying the following two conditions:*

1. $D := \mathrm{Cont}[f, \tau(X, X^*)] = \mathrm{Cont}[h, \tau(X, X^*)]$.
2. $\partial f(x) \subseteq \partial h(x)$, *for all* $x \in D$.

Then f and h are equal up to an additive constant.

Proof. In this proof, the topological notation will refer to the Mackey topology $\tau(X, X^*)$ for X. Without loss of generality, we may suppose that $0 \in D$. Fix then $x \in D$ and define the functions $\varphi_f, \varphi_h : \mathbb{R} \to \mathbb{R}_\infty$ given by

$$\varphi_f(t) = f(tx) \quad \text{and} \quad \varphi_h(t) = h(tx).$$

It is clear that $\varphi_f, \varphi_h \in \Gamma_0(\mathbb{R})$ and that both are continuous in $[0, 1]$. Defining the linear continuous operator $A : \mathbb{R} \to X$ given by $A(t) = tx$, we have that $\varphi_f = f \circ A$ and $\varphi_h = h \circ A$. Since $0 \in D$, the subdifferential chain rule holds and then for all $t \in [0, 1]$,

$$\partial\varphi_f(t) = \langle \partial f(tx), x\rangle \subseteq \langle \partial h(tx), x\rangle = \partial\varphi_h(t).$$

Then, again because of the continuity of φ_f and φ_h, the subdifferential sum rule holds and then, for all $t \in \mathbb{R}$,

$$\begin{aligned} \partial\left(\varphi_f + I_{[0,1]}\right)(t) &= \partial\varphi_f(t) + \partial I_{[0,1]}(t) \\ &\subseteq \partial\varphi_h(t) + \partial I_{[0,1]}(t) \\ &= \partial\left(\varphi_h + I_{[0,1]}\right)(t). \end{aligned}$$

Thus, by the classical results of integration in real analysis, we conclude that φ_f and φ_h are equal in the interval $[0, 1]$ up to an additive constant. In particular,

$$f(x) - f(0) = \varphi_f(1) - \varphi_f(0) = \varphi_h(1) - \varphi_h(0) = h(x) - h(0).$$

Fixing $c = f(0) - h(0)$, we have $f(x) = h(x) + c$ for all $x \in D$. To finish the proof, consider $x \in \overline{D}$ $\left(= \overline{\mathrm{dom}\, f} = \overline{\mathrm{dom}\, h}\right)$. Applying Lemma 3.2 we obtain

$$\begin{aligned} f(x) &= \liminf_{y\to x} f(y) \\ &= \liminf_{D\ni y\to x} f(y) = \liminf_{D\ni y\to x} h(x) + c \\ &= \liminf_{y\to x} h(y) + c = h(x) + c. \end{aligned}$$

Noting that the equality is trivial at points $x \in X\setminus\overline{D}$, the proof is concluded. $\square$

Lemma 4.5. *Let $f, g \in \Gamma_0(X, \theta)$ and $V \subseteq \mathrm{Cont}[f, \tau(X, X^*)] \cap \mathrm{Cont}[g, \tau(X, X^*)]$ be a nonempty $\tau(X, X^*)$-open set. If there exists a $\tau(X, X^*)$-dense subset $D \subseteq V$ for which*

$$\partial f(x) \subseteq \partial g(x), \quad \text{for all } x \in D,$$

then $\partial f(x) \subseteq \partial g(x)$, for all $x \in V$.

Proof. In this proof, the topological notation will refer to the Mackey topology $\tau(X, X^*)$ for X. Let us suppose that there exist $x \in V$ and $x^* \in X^*$ such that $x^* \in \partial f(x) \setminus \partial g(x)$. Since $\partial g(x)$ is w^*-closed and nonempty, we can apply the separation theorem with the w^*-topology on X^* to obtain $z \in X \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\langle x^*, z\rangle < \alpha < \langle u^*, z\rangle, \quad \text{for all } u^* \in \partial g(x).$$

Consider then the w^*-open set $W = \{u^* \in X^* \; : \; \langle u^*, z\rangle > \alpha\}$, which contains $\partial g(x)$, and the sequence $(z_n) \subseteq X$ given by

$$z_n = x - \frac{1}{n} z.$$

Without loss of generality, we may assume that $(z_n) \subseteq V$. We will show first that there exists an $n_0 \in \mathbb{N}$ such that $\partial g(z_{n_0}) \subseteq W$. If not, we choose $z_n^* \in \partial g(z_n) \setminus W$ for each $n \in \mathbb{N}$. Since ∂g is locally w^*-bounded in V (see Proposition 3.3), it is direct that $\{z_n^* \; : n \in \mathbb{N}\}$ is a w^*-bounded set, and so (z_n^*) has a w^*-convergent subnet $(z^*_{\varphi(i)})_{i \in I}$. Since (z_n) converges to x, so does $(z_{\varphi(i)})_{i \in I}$. Thus, from the $\tau(X, X^*)$-w^*-outer semicontinuity of ∂g at x (see Proposition 3.5), we have that

$$w^*\text{-}\lim z^*_{\varphi(i)} \in \partial g(x) \subseteq W,$$

which is clearly a contradiction. Fix then $y = z_{n_0}$ such that $\partial g(y) \subseteq W$. Because of the density of D, there exists a net $(y_i) \subseteq D$ converging to y and a net $(y_i^*) \subseteq X^*$ with $y_i^* \in \partial f(y_i)$. By Proposition 3.3 again, we can assume that $(y_i^*)_{i \in I}$ is included in a w^*-compact set of X^*. In particular, $(y_i^*)_{i \in I}$ has a w^*-convergent subnet, that we will continue denoting by $(y_i^*)_{i \in I}$. We have then $(y_i, y_i^*) \in \partial f \cap \partial g$ and since both subdifferentials are $\tau(X, X^*)$-w^*-outer semicontinuous at y, we deduce

$$y^* = w^*\text{-}\lim y_i^* \in \partial f(y) \cap \partial g(y).$$

Then, $y^* \in W$ and

$$\langle x^* - y^*, x - y\rangle = \frac{1}{n_0}\langle x^* - y^*, z\rangle < 0,$$

which is a contradiction with the monotonicity of ∂f. $\square$

Lemma 4.6. *Let (T,τ) be a Hausdorff topological space, $f : T \to \mathbb{R}_\infty$ be a proper function and $V \subseteq \operatorname{dom} f$ be a τ-open set. If there exists a τ-dense subset $D \subseteq V$ for which*

$$\limsup_{D \ni s \to t} f(s) = f(t), \quad \forall t \in V,$$

then f is τ-upper semicontinuous in V.

Proof. Let us fix $\bar{t} \in V$ and a net $(t_i)_{i\in I} \subseteq T$ converging to $\bar{t}$. Without loss of generality, $(t_i)_{i\in I} \subseteq V$. Then, for each $i \in I$, there exists a net $(t_{i,j})_{j\in J(i)} \subseteq D$ converging to t_i with

$$\limsup_{j} f(t_{i,j}) = f(t_i).$$

Fix $\varepsilon > 0$ and consider the set of indexes Λ given by the tuples (i, j, W) such that

1. $j \in J(i)$ and W is an open set with $\bar{t} \in W \subseteq V$.
2. $t_{i,j} \in W$.
3. $f(t_{i,j}) \geq f(t_i) - \varepsilon$.

Consider the following preorder in Λ: $(i_1, j_1, W_1) \prec (i_2, j_2, W_2)$ if $i_1 \leq i_2$ and $W_1 \supseteq W_2$, in the case that $i_1 \neq i_2$, or if $j_1 \leq j_2$ and $W_1 \supseteq W_2$, in the case that $i_1 = i_2$.

We claim that the net $(t_\alpha)_{\alpha\in\Lambda}$, where $t_{(i,j,W)} := t_{i,j}$ for each $(i, j, W) \in \Lambda$, converges to $\bar{t}$. Let W be an open set with $\bar{t} \in W \subseteq V$. Since $t_i \to \bar{t}$, there exists $i_0 \in I$ such that, for each $i \geq i_0$, $t_i \in W$. In particular, $t_{i_0} \in W$ and, since W is an open set, we have that $W \in \mathcal{N}(t_{i_0})$. Then, provided

$$\limsup_{j\in J(i_0)} f(t_{i_0,j}) = f(t_{i_0}),$$

there exists $j_0 \in J(i_0)$ such that $t_{i_0,j_0} \in W$ and $f(t_{i_0,j_0}) \geq f(t_{i_0}) - \varepsilon$. Let $\alpha_0 = (i_0, j_0, W)$ which, by construction, belongs to Λ. For each $\alpha = (i', j', W') \in \Lambda$ such that $\alpha_0 \prec \alpha$, we get that $t_\alpha \in W' \subseteq W$, and so, the net $(t_\alpha)_{\alpha\in\Lambda}$ is eventually in W. Since W is an arbitrary open neighborhood of $\bar{t}$, we conclude that $(t_\alpha)_{\alpha\in\Lambda}$ converges to $\bar{t}$, as we claimed. In particular, we have that

$$\limsup_{\alpha} f(t_\alpha) \leq \limsup_{D\ni t\to\bar{t}} f(t) = f(\bar{t}).$$

Also, noting that $\limsup_\alpha f(t_\alpha) \geq \limsup_i f(t_i) - \varepsilon$ we deduce

$$\limsup_{i} f(t_i) - \varepsilon \leq f(\bar{t}).$$

Thus, because ε is arbitrary, we conclude that $\limsup_i f(t_i) \leq f(\bar{t})$, which finishes the proof. $\square$

Note that Lemma 4.6 remains true if we replace upper semicontinuity by lower semicontinuity (and lim sup by lim inf): Just apply the same proof to $-f$ instead of f. Thus, it also remains true if we replace upper semicontinuity by continuity (and lim sup by lim).

Proposition 4.7. *Let $f : X \to \mathbb{R}_\infty$ be an SDPD function and $g : X \to \mathbb{R}_\infty$ be any function satisfying*

$$\partial f(x) \subseteq \partial g(x), \quad \forall x \in X.$$

Then, $\operatorname{Cont}[f^*, \tau(X^*, X^{**})] \subseteq \operatorname{Cont}[g^*, \tau(X^*, X^{**})]$.

Proof. Endow X^* with the Mackey topology $\tau(X^*, X^{**})$. Since f is SDPD, there exists a dense subset $D \subseteq \operatorname{Cont}[f^*]$ such that the equality (3) holds for each $x^* \in D$. In particular, we have that for each $x^* \in D$

$$\partial f^*(x^*) \subseteq \overline{\operatorname{co}}^{w^{**}} \left[(\partial f)^{-1}(x^*)\right] \subseteq \overline{\operatorname{co}}^{w^{**}} \left[(\partial g)^{-1}(x^*)\right] \subseteq \partial g^*(x^*).$$

Fix $z^* \in \operatorname{Cont}[f^*]$ and let $(z^*_\alpha) \subseteq D$ be a net converging to z^*. Consider $(z_\alpha) \subseteq X$ such that $z_\alpha \in X \cap \partial f^*(z^*_\alpha)$. Since f^* is continuous at z^*, we can assume that (z_α) is bounded (see Proposition 3.3) and so there exist $z^{**} \in X^{**}$ and a subnet $(z_{\varphi(i)})_{i\in I}$ such that $z_{\varphi(i)} \rightharpoonup^{w^{**}} z^{**}$. Since $z_{\varphi(i)} \in X \cap \partial g^*(z^*_{\varphi(i)})$ we have that for all $y^* \in X^*$

$$\langle z_{\varphi(i)}, y^* - z^*_{\varphi(i)}\rangle + g^*(z^*_{\varphi(i)}) \leq g^*(y^*).$$

Define $K = \{-z_{\varphi(i)}, z_{\varphi(i)} \ : \ i \in I\}$, which is a bounded set. Since $(z_{\varphi(i)})$ w^{**}-converges to z^{**} and $(z^*_{\varphi(i)})$ converges to z^*, we have that

$$\langle z_{\varphi(i)}, y^* - z^*\rangle \to \langle z^{**}, y^* - z^*\rangle$$
$$|\langle z_{\varphi(i)}, z^* - z^*_{\varphi(i)}\rangle| \leq \sup_{z\in K}\langle z, z^* - z^*_\alpha\rangle \to 0.$$

Thus, we deduce

$$\langle z_{\varphi(i)}, y^* - z^*_{\varphi(i)}\rangle = \langle z_{\varphi(i)}, y^* - z^*\rangle + \langle z_{\varphi(i)}, z^* - z_{\varphi(i)}\rangle \to \langle z^{**}, y^* - z^*\rangle.$$

Finally, we have that

$$\begin{aligned}
\langle z^{**}, y^* - z^*\rangle + g^*(z^*) &\leq \langle z^{**}, y^* - z^*\rangle + \liminf_i g^*(z^*_{\varphi(i)}) \\
&= \liminf_i \langle z_{\varphi(i)}, y^* - z^*_{\varphi(i)}\rangle + \liminf_i g^*(z^*_{\varphi(i)}) \\
&\leq \liminf_i [\langle z_{\varphi(i)}, y^* - z^*_{\varphi(i)}\rangle + g^*(z^*_{\varphi(i)})] \\
&\leq g^*(y^*).
\end{aligned}$$

Then $z^{**} \in \partial g^*(z^*)$ and so, $z^* \in \operatorname{dom}\partial g^* \subseteq \operatorname{dom} g^*$. Furthermore, evaluating the last inequality in $y^* = z^*$, we deduce that $\liminf_i g^*(z^*_{\varphi(i)}) = g^*(z^*)$, and so there exists a second subnet $(z^*_{\psi(j)})_{j\in J}$ such that $g^*(z^*_{\psi(j)}) \to g^*(z^*)$. Since (z^*_α) was an arbitrary net in D converging to z^*, we have that

$$\lim_{D\ni y^*\to z^*} g^*(y^*) = g^*(z^*).$$

The conclusion follows directly from Lemma 4.6. $\square$

We can now state and prove our integration theorem.

Theorem 4.8. *Let $f : X \to \mathbb{R}_\infty$ be an SDPD function and $g : X \to \mathbb{R}_\infty$ be any function satisfying the following condition:*

$$\partial f(x) \subseteq \partial g(x), \quad \forall x \in X.$$

Then there exists a constant $c \in \mathbb{R}$ such that

$$\overline{\mathrm{co}}\, f = \overline{(\overline{\mathrm{co}}\, g)\Box\sigma_{\mathrm{dom}\, f^*}} + c.$$

If, in addition, $\mathrm{dom}\, g^* \subseteq \overline{\mathrm{dom}\, f^*}$, then $\overline{\mathrm{co}}\, f$ and $\overline{\mathrm{co}}\, g$ are equal up to an additive constant.

Proof. In this proof, the topological notation will refer to the Mackey topology $\tau(X^*, X^{**})$ for X^*. Without loss of generality, we may assume $g \in \Gamma_0(X,\theta)$ (due to the trivial inclusion $\partial g \subseteq \partial(\overline{\mathrm{co}}\, g)$). Because of the inclusion $\partial f(X) \subseteq \mathrm{dom}\, f^*$, it is clear that

$$\partial f(x) \subseteq \partial g(x) \cap \mathrm{dom}\, f^*, \quad \forall x \in X.$$

Applying [20, Corollary 2.4.7], we have that for all $x \in X$ with $\partial f(x) \neq \emptyset$,

$$\partial f(x) \subseteq \partial g(x) \cap \partial\sigma_{\mathrm{dom}\, f^*}(0) = \partial(g\Box\sigma_{\mathrm{dom}\, f^*})(x).$$

Defining $h = g\Box\sigma_{\mathrm{dom}\, f^*}$, the inclusion $\partial f(x) \subseteq \partial h(x)$ holds for all $x \in X$, and so $\mathrm{Cont}[f^*] \subseteq \mathrm{Cont}[h^*]$ (see Proposition 4.7). On the other hand, the equality $h^* = g^* + I_{\overline{\mathrm{dom}\, f^*}}$ provides the inclusion

$$\mathrm{Cont}[h^*] = \mathrm{Cont}[g^*] \cap \mathrm{Cont}[f^*] \subseteq \mathrm{Cont}[f^*],$$

which implies that $\mathrm{Cont}[f^*] = \mathrm{Cont}[h^*]$. Let D be the dense subset of $\mathrm{Cont}[f^*]$ given in the definition of SDPD functions. We have that for all $x^* \in D$,

$$\partial f^*(x^*) = \overline{\mathrm{co}}^{w^{**}} \left[(\partial f)^{-1}(x^*)\right] \subseteq \overline{\mathrm{co}}^{w^{**}} \left[(\partial h)^{-1}(x^*)\right] \subseteq \partial h^*(x^*),$$

and so, applying Lemma 4.5, we have that $\partial f^*(x^*) \subseteq \partial h^*(x^*)$, for all $x^* \in \mathrm{Cont}[f^*]$. Now, all the hypotheses of Lemma 4.4 hold and so there exists a constant $c \in \mathbb{R}$ such that $f^* = h^* - c$. Finally,

$$\overline{\mathrm{co}}\, f = f^{**}\big|_X = h^{**}\big|_X + c = \overline{g\Box\sigma_{\mathrm{dom}\, f^*}} + c,$$

which finishes the first part of the proof. For the second part, note that if $\mathrm{dom}\, g^* \subseteq \overline{\mathrm{dom}\, f^*}$, then

$$h^* = g^* + I_{\overline{\mathrm{dom}\, f^*}} = g^*,$$

and so $h^{**}\big|_X = g^{**}\big|_X = g$, according to the fact that g^* is proper. The conclusion is direct. $\square$

5. Some examples and SDPD spaces

This section gives some examples of certain classes of functions that are SDPD, and shows that the integrability property has a component that can be written as a property from the space where these functions are defined.

The first class involves the spaces with the Radon-Nikodým property (Radon-Nikodým spaces). Recall that a Banach space X is known to have the Radon-Nikodým property (RNP, for short) if and only if each norm-epi-pointed function $f : X \to \mathbb{R}_\infty$ satisfies that its conjugate $f^* : X^* \to \mathbb{R}_\infty$ is Fréchet-Differentiable in a dense set of $\mathrm{Cont}[f^*]$.

Proposition 5.1. *Let X be a Banach space with the RNP. Then every norm-epi-pointed and norm-lsc function on X is SDPD.*

Proof. Let $f : X \to \overline{\mathbb{R}}$ be a norm-epi-pointed and norm-lsc function. In [6, Proposition 6], it is proved that if f^* is Fréchet-differentiable at $x^* \in X^*$, then

$$\partial f^*(x^*) = (\partial f)^{-1}(x^*).$$

In particular, f satisfies the equation (3) at each functional $x^* \in X^*$ where f^* is Fréchet-differentiable. Thus f is SDPD, since the set of point of Fréchet-differentiability of f^* is a dense subset of $\mathrm{Int}[\mathrm{dom}\, f^*]$ and that the Mackey topology $\tau(X^*, X^{**})$ on X^* coincides with the norm topology on X^*. □

Recall that, for a nonempty set $K \subseteq X$, $\mathrm{ext}[K]$ denotes the set of extreme points of K, and for two points $x, y \in X$, $[x, y]$ denotes the convex segment between x and y, that is,

$$[x, y] = \{tx + (1 - t)y \ : \ t \in [0, 1]\}.$$

Recall also that a locally convex space X is semi-reflexive if $(X^*, \beta(X^*, X))^* = X$, or equivalently, if the topologies $\tau(X^*, X)$ and $\beta(X^*, X)$ coincide in X^*.

Proposition 5.2. *Let X be a semi-reflexive locally convex space. Then every $\tau(X^*, X)$-epi-pointed and w-lower semicontinuous function on X is SDPD.*

Proof. In this proof, the topological notation will refer to the w-topology for X and the Mackey topology due to the primal, $\tau(X^*, X)$, for X^*. Because X is semi-reflexive, $\tau(X^*, X)$ is the same as the Mackey topology over X^* due to the bidual. Also, at each functional $x^* \in \mathrm{Cont}[f^*]$, the subdifferential $\partial f^*(x^*)$ is a nonempty convex w-compact subset of X. Thus, by the Krein-Milman theorem (see [8, Theorem 3.65]), for each $x^* \in \mathrm{Cont}[f^*]$,

$$\partial f^*(x^*) = \overline{\mathrm{co}}\left[\mathrm{ext}[\partial f^*(x^*)]\right].$$

Therefore, it is sufficient to prove that for any functional $x^* \in \mathrm{Cont}[f^*]$,

$$\mathrm{ext}(\partial f^*(x^*)) \subseteq (\partial f)^{-1}(x^*). \tag{4}$$

Let us consider then a functional $x^* \in \mathrm{Cont}[f^*]$ and the function $\sigma_{\mathrm{epi}\, f}$ ($= (I_{\overline{\mathrm{co}}(\mathrm{epi}\, f)})^*$) on $X^* \times \mathbb{R}$ (endowed with the product topology $\tau(X^*, X) \times \tau_0$). For $\alpha > 0$ we have that

$$\begin{aligned}\sigma_{\mathrm{epi}\, f}(x^*, -\alpha) &= \sup\{\langle x^*, x\rangle - \alpha f(x) \mid x \in \mathrm{dom}\, f\}\\ &= \alpha \sup\left\{\langle \alpha^{-1} x^*, x\rangle - f(x) \mid x \in \mathrm{dom}\, f\right\}\\ &= \alpha f^*(\alpha^{-1} x^*).\end{aligned}$$

Then, because f^* is continuous at x^*, $\sigma_{\mathrm{epi}\, f}$ is continuous at $(x^*, -1)$. In particular, for each $\varepsilon > 0$, $\partial_\varepsilon \sigma_{\mathrm{epi}\, f}(x^*, -1)$ is a nonempty, convex and $w \times \tau_0$-compact set. Let us now fix an $\varepsilon > 0$ and define the set

$$C := \partial_\varepsilon \sigma_{\mathrm{epi}\, f}(x^*, -1).$$

Because $\partial \sigma_{\mathrm{epi}\, f}(x^*, -1) \subseteq C$ we have that $\partial \sigma_C(x^*, -1) \equiv \partial \sigma_{\mathrm{epi}\, f}(x^*, -1)$. Also,

$$\begin{aligned}C &= \{(x, \lambda) \in X \times \mathbb{R} \mid \langle x^*, x\rangle - \lambda \geq \sigma_{\mathrm{epi}\, f}(x^*, -1) + I_{\overline{\mathrm{co}}(\mathrm{epi}\, f)}(x, \lambda) - \varepsilon\}\\ &= \overline{\mathrm{co}}(\mathrm{epi}\, f) \cap \{(x, \lambda) \in X \times \mathbb{R} \mid \langle x^*, x\rangle - \lambda \geq \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon\}.\end{aligned}$$

Since $\mathrm{ext}[\overline{\mathrm{co}}(\mathrm{epi}\, f)] \cap C \subseteq C$, by [3, Lemma 2.7.1] (which still holds in every locally convex space due to its strictly algebraic proof), we have that

$$\mathrm{ext}\,[\partial \sigma_{\mathrm{epi}\, f}(x^*, -1)] \subseteq \mathrm{ext}[\overline{\mathrm{co}}(\mathrm{epi}\, f)] \cap C \subseteq \mathrm{ext}(C).$$

Consider now the set

$$H = \overline{\mathrm{co}}(\mathrm{epi}\, f) \cap \{(x, \lambda) \in X \times \mathbb{R} \mid \langle x^*, x\rangle - \lambda = \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon\}.$$

It is clear that $\overline{\mathrm{co}}(H \cup [C \cap \mathrm{epi}\, f]) \subseteq C$. For the reverse inclusion, let us consider $(x, \lambda) \in C$. Because $C \subseteq \overline{\mathrm{co}}(\mathrm{epi}\, f)$, there exists a net $(x_i, \lambda_i)_{i \in \Lambda}$ in $\mathrm{co}(\mathrm{epi}\, f)$ converging to (x, λ). We have two cases: First, $\langle x^*, x\rangle - \lambda = \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon$ which means that $(x, \lambda) \in H \subseteq \overline{\mathrm{co}}(H \cup [C \cap \mathrm{epi}\, f])$. Second, $\langle x^*, x\rangle - \lambda > \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon$. In such a case, there exists $i_0 \in \Lambda$, such that,

$$\langle x^*, x_i\rangle - \lambda_i > \sigma_{\mathrm{epi}\, f}(x^*, -1) - \varepsilon, \quad \forall i \geq i_0.$$

In particular, $(x_i, \lambda_i)_{i \geq i_0}$ is contained in $C \cap \mathrm{co}(\mathrm{epi}\, f)$. Note that, for each $i \geq i_0$, we can find $(x_i^1, \lambda_i^1), (x_i^2, \lambda_i^2) \in \mathrm{co}(\mathrm{epi}\, f)$ and $t_i \in [0, 1]$ such that

- $(x_i^1, \lambda_i^1) \in \mathrm{co}(C \cap \mathrm{epi}\, f)$, which implies that $\langle x^*, x_i^1\rangle - \lambda_i^1 \geq \sigma_{\mathrm{epi}\, f}(x_i^1, \lambda_i^1) - \varepsilon$.
- $(x_i^2, \lambda_i^2) \in \mathrm{co}(\mathrm{epi}\, f \setminus C)$, which implies that $\langle x^*, x_i^2\rangle - \lambda_i^2 \leq \sigma_{\mathrm{epi}\, f}(x_i^2, \lambda_i^2) - \varepsilon$.
- $(x_i, \lambda_i) = t_i(x_i^1, \lambda_i^1) + (1 - t_i)(x_i^2, \lambda_i^2)$.

Then, it is direct from the first two conditions that we can choose a point $(x_i^3, \lambda_i^3) \in [(x_i^1, \lambda_i^1), (x_i^2, \lambda_i^2)] \cap H$, and because $(x_i, \lambda_i) \in C$ for all $i \geq i_0$, we know that

$$(x_i, \lambda_i) \in [(x_i^1, \lambda_i^1), (x_i^3, \lambda_i^3)] \subseteq \overline{\mathrm{co}}(H \cup [C \cap \mathrm{epi}\, f]).$$

Finally, $(x,\lambda) \in \overline{\text{co}}(H \cup [C \cap \operatorname{epi} f])$, concluding that $\overline{\text{co}}(H \cup [C \cap \operatorname{epi} f]) = C$. Applying the Milman theorem (see [8, Theorem 3.66]), provided that H and $C \cap \operatorname{epi} f$ are $w \times \tau_0$-closed (and thus their union) and the trivial fact that $\partial\sigma_{\operatorname{epi} f}(x^*, -1) \cap H = \emptyset$, we have that

$$\operatorname{ext}[\partial\sigma_{\operatorname{epi} f}(x^*, -1)] \subseteq \operatorname{epi} f. \tag{5}$$

To conclude, let us recall three known facts:

(i) $(x,\lambda) \in \partial\sigma_{\operatorname{epi} f}(x^*, -1) \implies \lambda = \overline{\text{co}}\, f(x)$.

(ii) $x \in \partial f^*(x^*)$ if and only if $(x, \overline{\text{co}}\, f(x)) \in \partial\sigma_{\operatorname{epi} f}(x^*, -1)$.

(iii) $x \in (\partial f)^{-1}(x^*)$ if and only if $x \in \partial f^*(x^*)$ and $f(x) = \overline{\text{co}}\, f(x)$.

Take now $x \in \operatorname{ext}(\partial f^*(x^*))$. We have that $(x, \overline{\text{co}}\, f(x)) \in \operatorname{ext}[\partial\sigma_{\operatorname{epi} f}(x^*, -1)]$. If not, due to (i), there would be two distinct points $(x_1, \overline{\text{co}}\, f(x_1)), (x_2, \overline{\text{co}}\, f(x_2))$ in $\partial\sigma_{\operatorname{epi} f}(x^*, -1)$ and a real number $t \in (0,1)$ such that

$$(x, \overline{\text{co}}\, f(x)) = t(x_1, \overline{\text{co}}\, f(x_1)) + (1-t)(x_2, \overline{\text{co}}\, f(x_2)).$$

In particular, $x_1 \neq x_2$ and $x = tx_1 + (1-t)x_2$, and because of (ii), $x_1, x_2 \in \partial f^*(x^*)$, which is a contradiction. Then, applying the inclusion (5), we have that $(x, \overline{\text{co}}\, f(x)) \in \operatorname{epi} f$ and so $\overline{\text{co}}\, f(x) = f(x)$. Finally, because of (iii), $x \in (\partial f)^{-1}(x^*)$, which proves the inclusion (4), finishing the proof. $\square$

What do the semi-reflexive spaces and the normed spaces with the Radon-Nikodým property have in common that allow the previous functions to be SDPD? It is not easy to give an answer, but clearly it is related to the subdifferential of the conjugate functions. The following small characterization helps us to understand this situation better:

Proposition 5.3. *For a function $f : X \to \overline{\mathbb{R}}$, a functional $x^* \in X^*$ satisfies the equality* (3) *if and only if the following two conditions hold:*

(i) $X \cap \partial f^*(x^*) = \overline{\text{co}}\, [(\partial f)^{-1}(x^*)]$.

(ii) $\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}$.

In particular, a $\tau(X^*, X^{**})$-epi-pointed function is SDPD whenever there exists a $\tau(X^*, X^{**})$-dense subset D of $\operatorname{Cont}[f^*, \tau(X^*, X^{**})]$ *such that every $x^* \in D$ satisfies (i) and (ii).*

Proof. The second part of the proposition is direct, so we will show only the first equivalence:

$\Leftarrow$) Assuming (i) and (ii), it is direct that

$$\begin{aligned}\overline{\text{co}}^{w^{**}}[(\partial f)^{-1}(x^*)] &= \overline{\overline{\text{co}}[(\partial f)^{-1}(x^*)]}^{w^{**}}\\ &\overset{(i)}{=} \overline{X \cap \partial f^*(x^*)}^{w^{**}}\\ &\overset{(ii)}{=} \partial f^*(x^*).\end{aligned}$$

$\Rightarrow$) Suppose now that $x^* \in X^*$ satisfies (3). Then

$$\begin{aligned} X \cap \partial f^*(x^*) &= X \cap \overline{\mathrm{co}}^{w^{**}}[(\partial f)^{-1}(x^*)] \\ &= \overline{\mathrm{co}}^{w}[(\partial f)^{-1}(x^*)] \\ &= \overline{\mathrm{co}}[(\partial f)^{-1}(x^*)], \end{aligned}$$

so (i) holds. Finally, noting again that

$$\overline{\mathrm{co}}^{w^{**}}[(\partial f)^{-1}(x^*)] = \overline{\overline{\mathrm{co}}[(\partial f)^{-1}(x^*)]}^{w^{**}},$$

the statement (ii) is concluded, according to (3) and to (i). $\square$

So, the necessary condition that doesn't depend on the function f but entirely on the conjugate is that the equation

$$\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}} \tag{6}$$

must hold in a $\tau(X^*, X^{**})$-dense set of functionals. Motivated by this observation, we introduce the following definition:

Definition 5.4 (SDPD space). We say that a locally convex space X is an SDPD space if for every function $f^* \in \Gamma_0(X^*, w^*)$ with $\mathrm{Cont}[f^*, \tau(X^*, X^{**})]$ nonempty, there exists a $\tau(X^*, X^{**})$-dense subset D of $\mathrm{Cont}[f^*, \tau(X^*, X^{**})]$, such that for each $x^* \in D$, the equation (6) holds.

Note that in an SDPD space, each $\tau(X^*, X^{**})$-epi-pointed function in $\Gamma_0(X)$ is an SDPD function, since for each $f \in \Gamma_0(X)$ we have that

$$X \cap \partial f^*(x^*) = (\partial f)^{-1}(x^*), \quad \forall x^* \in X^*,$$

and so the condition (i) in Proposition 5.3 always holds. In fact, this is a characterization: The locally convex space X is an SDPD space if and only if each $\tau(X^*, X^{**})$-epi-pointed function in $\Gamma_0(X)$ is an SDPD function.

Now, we establish a second characterization of SDPD spaces as an RNP-like property:

Proposition 5.5. *Let X be a locally convex space, $f \in \Gamma_0(X)$ be a $\tau(X^*, X^{**})$-epi-pointed function and $x^* \in \mathrm{Cont}\,[f^*, \tau(X^*, X^{**})]$. We have that*

$$\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}} \iff (f^*)'(x^*, \cdot) \ \text{ is } w^*\text{-lsc.}$$

In particular, X is an SDPD space if and only if for each function $f \in \Gamma_0(X)$, the set

$$D = \{x^* \in \mathrm{Int}\,[\mathrm{dom}\, f^*, \tau(X^*, X^{**})] \ : \ (f^*)'(x^*, \cdot) \text{ is } w^*\text{-lsc.}\}$$

is $\tau(X^*, X^{**})$-dense in $\mathrm{dom}\, f^*$.

Proof. The second part of the proposition is direct from the first part, so we only need to prove that one. Assume then that $f \in \Gamma(X)$ and $x^* \in \mathrm{Cont}\,[f^*, \tau(X^*, X^{**})]$. On one hand, we know (see comments before equation (2)) that

$$(f^*)'(x^*, u^*) = \sup\{\langle x^{**}, u^*\rangle \ : \ x^{**} \in \partial f^*(x^*)\}, \quad \forall u^* \in X^*.$$

On the other hand, by [5, Corollary 2.4.15],

$$\overline{(f^*)'(x^*, \cdot)}^{w^*}(u^*) = \sup\{\langle x, u^*\rangle \ : \ x \in X \cap \partial f^*(x^*)\}, \quad \forall u^* \in X^*.$$

Therefore,

$$\begin{aligned}(f^*)'(x^*, \cdot) = \overline{(f^*)'(x^*, \cdot)}^{w^*} &\iff \sigma_{\partial f^*(x^*)} = \sigma_{X \cap \partial f^*(x^*)} \\ &\iff \overline{\mathrm{co}}^{w^{**}}[\partial f^*(x^*)] = \overline{\mathrm{co}}^{w^{**}}[X \cap \partial f^*(x^*)] \\ &\iff \partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}},\end{aligned}$$

which finishes the proof. □

From the definition of SDPD spaces follows this direct corollary:

Corollary 5.6. *Suppose that X is an SDPD space. Let $f : X \to \mathbb{R}_\infty$ be a $\tau(X^*, X^{**})$-epi-pointed function and define the sets*

$$D_1 = \left\{x \in \mathrm{Cont}[f^*, \tau(X^*, X^{**})] \mid \partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}\right\}.$$

$$D_2 = \left\{x \in \mathrm{Cont}[f^*, \tau(X^*, X^{**})] \mid X \cap \partial f^*(x^*) = \overline{\mathrm{co}}\left[(\partial f)^{-1}(x^*)\right]\right\}.$$

Then f is SDPD if and only if $D_2 \cap D_1$ is $\tau(X^*, X^{**})$-dense in D_1.

Our main goal for nearest future is to give sufficient conditions for the situation described in the latter corollary, in terms of known properties (for example, continuity-like conditions). The idea is to better understand which families of epi-pointed functions are SDPD when the space is SDPD.

Even if this work is written in general locally convex spaces, an important question related to SDPD spaces is in the Banach spaces setting: Namely, how far is the class of SDPD spaces with respect to the class of spaces with the RNP?. This question will be developed in a future paper.

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