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Polarities and Generalized Extremal Convolutions

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Dedicated to the memory of Jean Jacques Moreau.

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A reflection on the inceptive role of Moreau in various studies is presented. Gustave Choquet (1915–2006), Jean Jacques Moreau (1923–2014) and Ennio De Giorgi (1928–1996) inspired developments of variational analysis. I had the fortune to know them all and to follow their tracks.

Jean Jacques Moreau presented in [18] and [19, p. 122] an abstract duality scheme of hulls with respect to arbitrary couplings, rather than the bilinear coupling used by Fenchel [12]. Moreau's scheme inspired Stanisław Kurcyusz and myself in [8, 9] in our study of generalized Lagrangeans. Investigations of stability of duality schemes lead Gabriele Greco and me to convergence theory [4, 5], founded by Gustave Choquet in [2], and later to the theory of variational convergences of Ennio De Giorgi and his School (see [13] of De Giorgi and Franzoni), theorized by Greco ([14, 16, 15]).

The inceptive role of Moreau was consequential in my subsequent mathematical activity. Moreau conjugates were a starting point to my paper with Kurcyusz [9] and his extremal convolutions to that with Guillerme, Lignola and Malivert [7] on stability of *generalized extremal convolutions*[1]

$$E_{\varphi}^{\alpha}(f,g) := \mathrm{ext}_X^{\alpha}\,\varphi(f,g), \tag{1}$$

of $f, g \in \overline{\mathbb{R}}^{X\times Y}$, of which Moreau conjugates and inf-convolutions[2] constitute

[1] Here $\varphi : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, $\alpha \in \{-1,+1\}$, $h : X \times Y \to \overline{\mathbb{R}}$ and $\mathrm{ext}_X^{-1} h := \inf_{x\in X} h(x,y)$ and $\mathrm{ext}_X^{+1} h := \sup_{x\in X} h(x,y)$.

[2] The *inf-convolution* of $f, h \in \overline{\mathbb{R}}^X$ where X is a vector space, defined by

$$(f\nabla h)(y) := \inf_{x\in X}\left(f(x) \dot{+} h(y-x)\right)$$

becomes (1) if $g(x,y) = h(y-x)$, $\alpha = -1$ and $\varphi(r,s) := r \dot{+} s$. Here $\dot{+}$ is the (commutative) *Moreau upper extension* of the addition such that $(+\infty) \dot{+} (-\infty) = +\infty$.

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special cases. The latter paper has been largely unnoticed, though it remains actual, probably partly because of its quite abstract appearance. Let me recall some of its main ideas.

1. Polarities

Both conjugates and extremal convolutions can be reformulated in terms of polarities. If $\Omega \subset V \times W$, that is, Ω is a binary relation, and A is a subset of V then we define

$$\Omega A := \bigcup\nolimits_{v\in A} \{w \in W : (v,w) \in \Omega\}$$

is the *image* of A, and

$$\Omega_* A := \bigcap\nolimits_{v\in A} \{w \in W : (v,w) \in \Omega\}.$$

is the Ω-*polar* of A. Of course, $B \subset \Omega_* A$ if and only if $A \times B \subset \Omega$. The *inverse binary relation* $\Omega^- \subset W \times V$ of $\Omega \subset V \times W$, defined by $v \in \Omega^- \{w\}$ whenever $w \in \Omega \{v\}$, determines the polars $\Omega_*^- B := (\Omega^-)_* B$ of subsets of W. By definition, the Ω-*bipolar* of $A \subset V$ is

$$\Omega_{**} A := \Omega_*^- (\Omega_* A).$$

Recall that $A \subset \Omega_{**}A, \Omega_{***}A = \Omega_* A$ and

$$\Omega_{**} A = \bigcap_{\{w : A \subset \Omega^- \{w\}\}} \Omega^- \{w\}. \tag{2}$$

2. Galois connections

Polarities are a special case of Galois connections. Let X, Y be preordered sets. A couple (f, g) of maps

$$f : X \to Y \quad \text{and} \quad g : Y \to X \tag{3}$$

is called a *connection*. The maps f and g are *branches* of the connection (3). A connection is a $(+,+)$-*connection* if f and g are *antitone* (that is, order-inverting) and $g \circ f$ and $f \circ g$ are *expansive*, that is,

$$x \le g \circ f(x), \qquad f \circ g(y) \ge y \tag{4}$$

for every $x \in X$ and $y \in Y$. If V, W are sets and $\Omega \subset V \times W$ is a binary relation, then $f : 2^V \to 2^W$ and $g : 2^W \to 2^V$ defined by

$$f(A) := \Omega_* A \quad \text{and} \quad g(B) := \Omega_*^-$$

constitute a $(+,+)$-connection. A map $f : X \to X$ on an ordered set is a *coprojector* if it is isotone, idempotent and

$$x \le f(x) \tag{expansive}$$

for every $x \in X$.

Proposition 2.1. *A couple* (f,g) *is a* $(+,+)$*-connection if and only if*

$$f(x) \geq y \Longleftrightarrow x \leq g(y). \tag{5}$$

Proof. If (3) is a $(+,+)$-connection, and if $f(x) \geq y$, then by the antitonicity of g and by (4), $x \leq g \circ f(x) \leq g(y)$. For the same reason $x \leq g(y)$ implies $f(x) \geq y$.

On setting $y := f(x)$ in (5), we get $x \leq g \circ f(x)$, that is, $g \circ f$ is expansive; $f \circ g$ is expansive by symmetry. To show that f is antitone, let $x_0 \leq x_1$, hence $g \circ f(x_1) \geq x_1 \geq x_0$, that is, $g\left(f(x_1)\right) \geq x_0$, thus by (5), $f\left(x_0\right) \geq f(x_1)$. By symmetry g is also antitone. □

Theorem 2.2. *If* (f,g) *is a* $(+,+)$*-connection* (3)*, then* $g \circ f$ *is a coprojector onto* $g(Y)$ *and* $f \circ g$ *is a coprojector onto* $f(X)$*. Moreover the restrictions of* (3)

$$f : g(Y) \to f(X) \quad \textit{and} \quad g : f(X) \to g(Y) \tag{6}$$

are dual ([3]) *order-isomorphisms, one inverse of the other.*

Proof. By definition, the map $g \circ f$ is expansive; it is order-preserving as the composition of two order-inverting maps. In order to prove that $g \circ f$ is idempotent, it is enough to show that for every $x \in X$,

$$f \circ g \circ f(x) = f(x). \tag{7}$$

On applying f to the first inequality of (4), we get $f(x) \geq f \circ g \circ f(x)$; on putting $y := f(x)$ in the second equation of (4), we get $f(x) \leq f \circ g \circ f(x)$, so that (7) holds. The coprojective class associated with $g \circ f$ is $g(f(X)) = \mathrm{fix}(g \circ f) = \{x : g \circ f(x) \leq x\}$ (The image of $g \circ f$ coincides with the set of its fixed points, because $g \circ f$ is idempotent). We conclude that

$$\mathrm{fix}(g \circ f) = g(Y), \tag{8}$$

because if $x \in g(Y)$, then there exists y such that $x = g(y)$, hence $g \circ f(x) = g \circ f \circ g(y) = g(y) = x$.

By definition, f and g are order-inverting. Therefore to see that (6) is an order isomorphism, it is enough to notice that the restrictions of f to $g\left(Y\right)$ and of g to $f\left(X\right)$ are injective, because we have already proved that they are surjective. So let $x_0, x_1 \in g(Y)$ and $f(x_0) = f(x_1)$. This and (8) imply that $x_0 = g \circ f(x_0) = g \circ f(x_1) = x_1$. A symmetric argument completes the proof. □

[3] That is, with the inverse order on one of the sides.

3. Conjugates and polarities

If $g \in \overline{\mathbb{R}}^{X\times Y}$ and $f \in \overline{\mathbb{R}}^{X}$, then the *Moreau conjugate* of f with respect to g

$$f^g(y) := \sup_{x\in X}\left(g(x,y) \dot{-} f(x)\right)$$

is a special case of (1) when $\alpha = +1$ and $\varphi(r,s) := r \dot{-} s$ ([4]). Moreau observed [19, p. 122] that the *biconjugate* f^{gg} of f is equal to $\mathrm{conv}_g f$, the upper bound of the functions of the form $g(\cdot,y) \dot{+} r$, with $y \in Y$ and $r \in \overline{\mathbb{R}}$, that minorize f, proving that the following *Fenchel-Moreau formula*

$$f^{gg} = \mathrm{conv}_g f \tag{9}$$

is true for an arbitrary coupling g, and not only for the bilinear couplings of Fenchel [12]. Incidentally, he called Γ, the class of upper envelopes of the elements of

$$\{g(\cdot,y) \dot{+} r : (y,r) \in Y \times \overline{\mathbb{R}}\},$$

the same character used later by De Giorgi and his School to design their variational convergences.

The conjugates can be reduced to polarities (see [3]) and thus to Galois connections. Together with the epigraphs of functions

$$\mathrm{epi}\, f := \{(x,r) \in X \times \mathbb{R} : r \geq f(x)\},$$

let us consider the diepigraphs. The *diepigraph* of $g \in \overline{\mathbb{R}}^{X\times Y}$ is is a binary relation between $(X \times \mathbb{R})$ and $(Y \times \mathbb{R})$, defined by

$$\mathrm{diepi}\, g := \{(x,r,y,s) : r + s \geq g(x,y)\}.$$

Theorem 3.1.

$$\mathrm{epi}\,(f^g) = (\mathrm{diepi}\, g)_* \,\mathrm{epi}\, f.$$

Proof. By definition, $(y,s) \in (\mathrm{diepi}\, g)_* \,\mathrm{epi}\, f$ whenever $s \geq g(x,y) - r$ for each $(x,r) \in \mathrm{epi}\, f$, that is,

$$s \geq \sup_{(x,r)\in \mathrm{epi}\, f}\left(g(x,y) - r\right) = \sup_{x\in X}\left(g(x,y) \dot{-} f(x)\right) = f^g(y).$$

It is now clear that (9) is a consequence of (2), which in turn, is a corollary of Theorem 2.2. □

[4]Where $\dot{+}$ is the (commutative) *Moreau lower extension* of the addition such that $(+\infty) \dot{+} (-\infty) = -\infty$ and a $\dot{-}\, b = a \dot{+} (-b)$.

4. Polar convergences

A convergence ξ is a subset of $\mathbb{F}X \times X$ ([5]) such that $(\mathcal{F}, x) \in \xi$ and $\mathcal{F} \leq \mathcal{G}$ implies that $(\mathcal{G}, x) \in \xi$ and $(\mathcal{N}_\iota(x), x) \in \xi$, where $\mathcal{N}_\iota(x) := \{F \subset X : x \in F\}$. To enhance readability, it is often preferable to write

$$x \in \lim_\xi \mathcal{F}$$

rather than $(\mathcal{F}, x) \in \xi$. A filter can converge to several points, that is, $\lim_\xi \mathcal{F}$ is a set.

One of the most important cases of non-topological convergences are polar convergences. Pseudotopologies, a pivotal concept in convergence theory, were introduced in [2] by Gustave Choquet in a study of hyperconvergences, that is, polar convergences on spaces of subsets. Choquet proved that upper convergence is not topological unless the underlying (Hausdorff) topology ([6]) is locally compact, but it is always pseudotopological, that is, the convergence of a filter $\mathcal{F}$ is determined by the convergence of the ultrafilters finer than $\mathcal{F}$.

If ξ is a topology on X, $\mathcal{H}$ is a filter on the set of ξ-closed subsets, then we define $A_0 \in \lim_{[\xi,\$]} \mathcal{H}$ whenever ([7])

$$\bigcap_{H \in \mathcal{H}} \mathrm{cl}_\xi\Big(\bigcup_{A \in H} A\Big) \subset A_0. \tag{10}$$

The *upper convergence* $[\xi, \$]$ is usually called *upper Kuratowski convergence*, although it was currently used by Giuseppe Peano in [21, (1903)], [22, (1908)] ([8]).

The *natural convergence* $[\xi, \sigma]$ (called also *polar, power* or *continuous convergence*) of a convergence ξ on X with respect to a convergence σ on Z is the coarsest convergence θ on $C(\xi, \sigma)$ ([9]), for which the coupling map defined by $\langle x, f\rangle := f(x)$ is (jointly) continuous from the product convergence $\xi \times \theta$ ([10]) to σ ([11]).

If $\langle \cdot, \cdot \rangle : X \times C(\xi, \sigma) \to Z, F \subset X$ and $H \subset C(\xi, \sigma)$, then

$$\langle F, H\rangle := \{\langle x, h\rangle : x \in F, h \in H\},$$

[5] Where $\mathbb{F}X$ stands for the set of filters on X.

[6] The necessary and sufficient condition without any separation condition is core-compactness, see [11] for example.

[7] The symbol $[\xi, \$]$ will be explained in a moment.

[8] When Kazimierz Kuratowski (1896–1980) was still a child.

[9] The set of maps, continuous from ξ to σ.

[10] By definition, if ξ is a convergence on X and θ is a convergence on Y, then $(x, y) \in \lim_{\xi\times\theta} \mathcal{H}$ whenever $x \in \lim_\xi p_X[\mathcal{H}]$ and $x \in \lim_\theta p_Y[\mathcal{H}]$, where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are the canonical projections.

[11] The upper convergence $[\xi, \$]$ is a special case with \$ being the *Sierpiński topology* on $\{0, 1\}$, the closed sets of which are $\varnothing, \{1\}$ and $\{0, 1\}$. Accordingly, a subset A of X is ξ-closed whenever $\chi_A \in C(\xi, \$)$. This constitutes a bijection between ξ-closed sets and elements of $C(\xi, \$)$.

and $\langle \mathcal{F}, \mathcal{H} \rangle$ denotes the filter generated by $\{\langle F, H \rangle : F \in \mathcal{F}, H \in \mathcal{H}\}$. By definition, $h_0 \in \lim_{[\xi,\sigma]} \mathcal{H}$ if

$$\langle x, h_0 \rangle \in \lim_\sigma \langle \mathcal{F}, \mathcal{H} \rangle$$

for every x and $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$.

The polar convergence $[\xi, \sigma]$ turns out to be the polar of ξ with respect to the initial convergence of σ by the coupling $\langle \cdot, \cdot \rangle$. If $h : W \to Z$ and σ is a convergence on Z, then there exists the coarsest convergence $h^{-}\sigma$ on X for which h is continuous. It is called the *initial convergence* of σ by h. It is now straightforward that

Theorem 4.1. *$[\xi, \sigma]$ is the polar of ξ with respect to $\langle \cdot, \cdot \rangle^{-}\sigma$.*

5. Variational convergences

Variational convergences appeared in the study of preservation of extrema and of the corresponding arguments, because pointwise convergence fails to preserve them ([12]).

In optimization only unilateral convergence and continuity are relevant, in particular, in minimization only lower semicontinuity and some compactness is needed to assure the existence of minima. Lower semicontinuity is equivalent to continuity with respect to the *lower topology* ν_- in the range in $\mathbb{R}$, that is, the topology which open sets are of the form $(r, +\infty)$ with $r \in \overline{\mathbb{R}}$. *Pointwise convergence* with respect to ν_- of a filter $\mathcal{F}$ on $\overline{\mathbb{R}}^X$ to a function f_∞ (that is, *pointwise lower convergence* with respect to the usual topology ν on $\mathbb{R}$) can be described by

$$f_\infty(x) \leq \sup_{F \in \mathcal{F}} \inf_{f \in F} f(x) \tag{11}$$

for each $x \in X$.

Consider a convergence θ of functions that implies (11) and preserves infima. The first condition means that there exists the greatest element of $\lim_\theta \mathcal{F}$ that we denote by $\overline{\lim_\theta}\mathcal{F} \in \overline{\mathbb{R}}^X$ such that

$$\left(\overline{\lim_\theta}\mathcal{F}\right)(x) \leq \sup_{F \in \mathcal{F}} \inf_{f \in F} f(x).$$

Moreover if we look for the coarsest convergence θ with these properties, then that it should coincide with pointwise lower convergence for constant functions. A convergence on $\overline{\mathbb{R}}^X$ designed to preserve infima, should preserve order of $\overline{\mathbb{R}}$, commute with order isomorphisms and should not carry out of the least complete lattice including the values of considered functions. The listed conditions are precisely the axioms of a *limitoid* operator introduced and studied by Greco (see e.g., [14]). More precisely, if ξ is a topology on X and $\mathcal{F}_\mathcal{G}$ is filter on $\overline{\mathbb{R}}^X$

[12] For instance, a classical example of the sequence of the characteristic functions f_n of $\left\{\frac{1}{n}\right\}$.

generated by a filter $\mathcal{G}$ on an index set I to the effect that if $F \in \mathcal{F}$ then there is $G \in \mathcal{G}$ so that $f_i \in F$ for each $i \in G$, and $x \in X$, then the functional

$$\left(\overline{\lim}_{\theta(\xi)}\mathcal{F}_{\mathcal{G}}\right)(x) : \overline{\mathbb{R}}^{I\times X} \to \overline{\mathbb{R}} \tag{12}$$

should be a limitoid, that is, as we have said, preserving order, commuting with order isomorphisms and remaining within the least complete lattice including the values of functions. Recall that for a family $\mathcal{A}$ of subsets of a set W, the *grill* $\mathcal{A}^{\#}$ of $\mathcal{A}$ is defined [1] by

$$\mathcal{A}^{\#} := \left\{ H \subset W : \underset{A\in\mathcal{A}}{\forall} H \cap A \neq \varnothing \right\}.$$

We say that $\mathcal{A}$ and $\mathcal{H}$ *mesh* (in symbols, $\mathcal{A}\#\mathcal{H}$) whenever $\mathcal{H} \subset \mathcal{A}^{\#}$, which is equivalent to $\mathcal{A} \subset \mathcal{H}^{\#}$.

Greco proved in [14] that each limitoid $T : \overline{\mathbb{R}}^{W} \to \overline{\mathbb{R}}$ admits a representation

$$T(f) = \sup\nolimits_{A\in\mathcal{A}} \inf\nolimits_{w\in A} f(w) = \inf\nolimits_{H\in\mathcal{A}^{\#}} \sup\nolimits_{w\in H} f(w),$$

where $\mathcal{A}$ is a family of subsets of W, which is stable for supersets. In this case, $\mathcal{A}$ is uniquely determined by T and is called the *support* of T. Mind that $\mathcal{A}$ is not a filter in general. In the case of (12), the support is a family of subsets of $I \times X$ constructed from $\mathcal{G}$ on I and the neighborhood filter $\mathcal{N}_{\xi}(x)$ on X.

In order to make preservation conditions on θ more explicit, consider a filter $\mathcal{F}$ on $\overline{\mathbb{R}}^{X}$ and let

$$\inf\nolimits_X \mathcal{F} := \{\{\inf\nolimits_X f : f \in F\} : F \in \mathcal{F}\}$$

denote a filter on $\overline{\mathbb{R}}$. In the same vein, $\operatorname{epi} f \subset X \times \mathbb{R}$ is a relation and $\operatorname{epi}[\mathcal{F}]$ denotes the following filter

$$\operatorname{epi}[\mathcal{F}] := \left\{ \bigcup\nolimits_{f\in F} \operatorname{epi} f : F \in \mathcal{F} \right\}.$$

on the set $2^{X\times\mathbb{R}}$ of relations. Accordingly, if $s \in \mathbb{R}$, since $(\operatorname{epi} f)^{-}\{s\} = \{x : f(x) \leq s\}$,

$$(\operatorname{epi}[\mathcal{F}])^{-}\{s\} := \left\{ \bigcup\nolimits_{f\in F} \{x : f(x) \leq s\} : F \in \mathcal{F} \right\}.$$

As $\inf_X f \leq f(x)$ for each f and each $x \in X$, an order-preserving convergence θ that coincides with the lower pointwise limit on constant functions, satisfies

$$\sup\nolimits_{F\in\mathcal{F}} \inf\nolimits_{f\in F} \inf\nolimits_X f = \overline{\lim}_{\theta} \inf\nolimits_X \mathcal{F} \leq \inf\nolimits_X \overline{\lim}_{\theta}\mathcal{F}.$$

Therefore, the quest is when the inequality can be reversed, that is,

$$\inf\nolimits_X \overline{\lim}_{\theta}\mathcal{F} \leq \sup\nolimits_{F\in\mathcal{F}} \inf\nolimits_{f\in F} \inf\nolimits_X f. \tag{13}$$

It is straightforward that

Lemma 5.1. *The condition* (13) *is fulfilled if and only if for each* $r \in \mathbb{R}$ *such that* $(\operatorname{epi}[\mathcal{F}])^{-}\{r\}$ *is a non-degenerate filter,* $(\operatorname{epi}(\overline{\lim}_{\theta}\mathcal{F}))^{-}\{r\} \neq \varnothing$.

This very general non-voidness condition can be realized as a compactness-like condition if θ is based on a convergence structure on X.

6. Γ-convergence

If $\mathcal{A}$ is a family of subsets of a set X, $f \in \overline{\mathbb{R}}^{X}$ and $\alpha \in \{-1, +1\}$, then

$$\Gamma(\mathcal{A}^{\alpha}) f := \operatorname{ext}^{-\alpha}_{A \in \mathcal{A}} \operatorname{ext}^{\alpha}_{x \in A} f(x).$$

If in particular, $\mathcal{A} = \{X\}$,then

$$\Gamma(\{X\}^{\alpha}) f = \operatorname{ext}^{\alpha}_{X} f$$

and if $\mathcal{A} = \{\{x\}\}$ or $\mathcal{N}_{\iota}(x) = \{A \subset X : x \in A\}$, then

$$\Gamma(\mathcal{N}_{\iota}(x)^{\alpha}) f = f(x).$$

Hence, the functionals of extremization over the whole space, are special cases of Γ-functionals.

If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are families of subsets of $X_1, \ldots, X_n$ respectively, $f \in \overline{\mathbb{R}}^{X_1 \times \ldots \times X_n}$ and $\alpha_1, \ldots, \alpha_n \in \{-1, +1\}$, then

$$\begin{aligned} & \Gamma\left(\mathcal{A}_1^{\alpha_1}, \ldots, \mathcal{A}_{n-1}^{\alpha_{n-1}}, \mathcal{A}_n^{\alpha_n}\right) f \\ := \; & \operatorname{ext}^{-\alpha}_{A \in \mathcal{A}_n} \Gamma\left(\mathcal{A}_1^{\alpha_1}, \ldots, \mathcal{A}_{n-1}^{\alpha_{n-1}}\right) \operatorname{ext}^{\alpha}_{x_n \in A} f(x_1, \ldots, x_n) \end{aligned} \tag{14}$$

defines a functional on $\overline{\mathbb{R}}^{X_1 \times \ldots \times X_n}$. Greco showed that each Γ-functional is a limitoid and gave an algorithm for calculation of its support [16]. If $\xi_1, \ldots, \xi_n$ are convergences on $X_1, \ldots, X_n$ respectively, then for each fixed $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$,

$$\begin{aligned} & \Gamma(\xi_1^{\alpha_1}, \ldots, \xi_n^{\alpha_n}) f(x_1, \ldots, x_n) \\ := \; & \operatorname{ext}^{\alpha_1}_{\mathcal{F}_1 \in \xi_1^{-}(x_1)} \ldots \operatorname{ext}^{\alpha_n}_{\mathcal{F}_n \in \xi_n^{-}(x_n)} \Gamma(\mathcal{F}_1^{\alpha_1}, \ldots, \mathcal{F}_n^{\alpha_n}) f \end{aligned} \tag{15}$$

is a functional obtained by successive extremizations of limitoids.

The resulting operators determine a larger (than convergences) category of structures, called *semi-convergences,* via the inequalities

$$\Gamma(\xi_1^{\alpha_1}, \ldots, \xi_n^{\alpha_n}) f \leq f \quad \text{or} \quad \Gamma(\xi_1^{\alpha_1}, \ldots, \xi_n^{\alpha_n}) f \geq f. \tag{16}$$

Applied to indicator functions of subsets of $X_1 \times \ldots \times X_n$, (16) become finite sequences of quantifiers that determine semi-convergences on $X_1 \times \ldots \times X_n$, where converging objects are families of subsets stable for supersets that Greco called *semi-filters.* They are actually supports of $\Gamma(\mathcal{F}_1^{\alpha_1}, \ldots, \mathcal{F}_n^{\alpha_n})$ from (15).

Semi-convergences can be seen as a sophistication of finite convergence products, which are their special cases when $\alpha_1 = \ldots = \alpha_n$.

In particular, if ξ is a convergence on X and $\mathcal{F}$ is a filter on $\overline{\mathbb{R}}^X$, then

$$\begin{aligned}\Gamma\left(\mathcal{F}^\alpha, \xi^\beta\right)(x_0) :=&\ \operatorname{ext}^\alpha_{\mathcal{G}\in\xi^-(x_0)} \Gamma\left(\mathcal{F}^\alpha, \mathcal{G}^\beta\right) \\ =&\ \operatorname{ext}^\alpha_{\mathcal{G}\in\xi^-(x_0)} \operatorname{ext}^{-\beta}_{G\in\mathcal{G}} \operatorname{ext}^\alpha_{F\in\mathcal{F}} \operatorname{ext}^{-\alpha}_{f\in F} \operatorname{ext}^{-\beta}_{x\in G} f(x),\end{aligned}$$

hence, on abridging, $- := -1$ and $+ := +1$,

$$\Gamma\left(\mathcal{F}^-, \xi^-\right)(x_0) := \inf_{\mathcal{G}\in\xi^-(x_0)} \sup_{G\in\mathcal{G}} \sup_{F\in\mathcal{F}} \inf_{f\in F} \inf_{x\in G} f(x),$$
$$\Gamma\left(\mathcal{F}^+, \xi^-\right)(x_0) := \inf_{\mathcal{G}\in\xi^-(x_0)} \sup_{G\in\mathcal{G}} \inf_{F\in\mathcal{F}} \sup_{f\in F} \inf_{x\in G} f(x).$$

Two basic variational convergences, defined by

$$f_\infty \in \lim_{\Gamma\left(-,\xi^-\right)} \mathcal{F} \Longleftrightarrow \Gamma\left(\mathcal{F}^-, \xi^-\right) \geq f_\infty,$$
$$f_\infty \in \lim_{\Gamma\left(+,\xi^-\right)} \mathcal{F} \Longleftrightarrow \Gamma\left(\mathcal{F}^+, \xi^-\right) \leq f_\infty,$$

are often called *lower* and *upper epi-convergences*, because they correspond respectively to the closedness (weak upper semicontinuity) and lower semicontinuity of inverse images of the epigraph relations. This terminology is rather arbitrary, as the same convergences can be described via the inverse images of the hypograph relations ([13]).

Here ι stands for the *discrete* topology and o for the *chaotic* topology. It is obvious that, in general, $\xi \leq \zeta$ implies

$$\begin{aligned}&\Gamma\left(\ldots, o^-, \ldots\right) \leq \Gamma\left(\ldots, \xi^-, \ldots\right) \leq \Gamma\left(\ldots, \zeta^-, \ldots\right) \leq \Gamma\left(\ldots, \iota^-, \ldots\right) \\ &= \Gamma\left(\ldots, \iota^+, \ldots\right) \leq \Gamma\left(\ldots, \zeta^+, \ldots\right) \leq \Gamma\left(\ldots, \xi^+, \ldots\right) \leq \Gamma\left(\ldots, o^+, \ldots\right),\end{aligned}$$

as $o \leq \xi \leq \iota$ for each convergence ξ. In particular, the *lower epi-convergence* implies the lower pointwise convergence and the upper pointwise convergence implies the *upper epi-convergence*. It follows that

$$\Gamma\left(\mathcal{F}^+, \xi^-\right) \leq f_\infty \Longrightarrow \Gamma\left(\mathcal{F}^+, o^-\right) \leq \inf_X f_\infty,$$

that is, $\Gamma\left(\mathcal{F}^+, \xi^-\right) \leq f_\infty$ implies that

$$\inf_{F\in\mathcal{F}} \sup_{f\in F} \inf_{x\in X} f(x) \leq \inf_{x\in X} f_\infty(x),$$

that is, the upper convergence of infima.

Lower convergence of infima was at the origin of variational convergences. Lower epi-convergence, however, is not sufficient to guarantee it. As anticipated in Lemma 5.1, non-voidness condition should be added. Recall that if ξ is a convergence, then by definition, $x \in \operatorname{adh}_\xi \mathcal{H}$ if there is a filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{H}$ and $x \in \lim_\xi \mathcal{G}$. By Lemma 5.1,

[13]The lower epiconvergence applied to indicator functions ψ_A, ψ_{A_∞}, yields the upper Peano-Kuratowski convergence: if $x \notin A_\infty$ then for each filter $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{G}$ there is $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $\bigcup_{\psi_A\in F} A \cap G = \varnothing$.

Proposition 6.1. *If* $\Gamma\left(\mathcal{F}^{-},\xi^{-}\right) \geq f_{\infty}$ *and*

$$\operatorname{adh}_{\xi} \operatorname{epi}^{-}[\mathcal{F}]\{r\} \neq \varnothing \tag{17}$$

for each $r > \inf_X f_\infty$, *then*

$$\sup_{F\in\mathcal{F}} \inf_{f\in F} \inf_X f \geq \inf_X f_\infty. \tag{18}$$

The non-voidness condition (17) can be realized via compactness-like assumptions. Recall [6] that a filter $\mathcal{H}$ is called ξ-*compactoid* if $\operatorname{adh}_{\xi}\mathcal{G} \neq \varnothing$ for each filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{H}$. Proposition 6.1 implies

Proposition 6.2. *If* $\Gamma\left(\mathcal{F}^{-},\xi^{-}\right) \geq f_{\infty}$ *and* $\operatorname{epi}^{-}[\mathcal{F}]\{r\}$ *is* ξ-*compactoid for each* $r > \inf_X f_\infty$, *then* (18) *holds.*

This fact is also a consequence of [7, Thorem 5.6].

7. Γ-inequalities

In [10, Lemma 3.1] M.-P. Moschen and myself proved that if $\mathcal{A},\mathcal{B},\mathcal{C}$ are families of subsets of X (stable for supersets) such that if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $A \cap B \in \mathcal{C}$, $f,g \in \overline{\mathbb{R}}^X$, $\varphi : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a (separately) increasing lower semicontinuous map, then

$$\varphi\left(\Gamma\left(\mathcal{A}^{-}\right)f, \Gamma\left(\mathcal{B}^{-}\right)g\right) \leq \Gamma\left(\mathcal{C}^{-}\right)\varphi\left(f,g\right).$$

Applied to Γ-functionals, this implies that [7, Theorem 3.3]

Theorem 7.1. *If*

$$\alpha_j + \beta_j \leq \gamma_j - 1, \tag{19}$$

$\mathcal{F}_j$ *is a filter on* X_j *for each* $1 \leq j \leq n$ *and* φ *is a (separately) increasing lower semicontinuous map, then*

$$\varphi\left(\Gamma(\ldots,\mathcal{F}_j^{\alpha_j},\ldots)f, \Gamma(\ldots,\mathcal{F}_j^{\beta_j},\ldots)g\right) \leq \Gamma(\ldots,\mathcal{F}_j^{\gamma_j},\ldots)\varphi\left(f,g\right). \tag{20}$$

Conditions dual of (19), that is,

$$\gamma_j + 1 \leq \alpha_j + \beta_j, \tag{21}$$

coupled with (separate) increasing upper semicontinuity imply inequalities that are dual of (20), that is,

$$\Gamma(\ldots,\mathcal{F}_j^{\gamma_j},\ldots)\varphi\left(f,g\right) \leq \varphi\left(\Gamma(\ldots,\mathcal{F}_j^{\alpha_j},\ldots)f, \Gamma(\ldots,\mathcal{F}_j^{\beta_j},\ldots)g\right). \tag{22}$$

Condition (21) excludes the case $\alpha_j = -1 = \beta_j$; if $\gamma_j = +1$ then $\alpha_j = +1$ and $\beta_j = +1$; if $\gamma_j = -1$ then either $\alpha_j = +1$ or $\beta_j = +1$.

Multiple applications of Theorem 7.1 were presented in [7]. For example, for the generalized infimal convolution E_{φ}^{-1},

Corollary 7.2. *If* (19) *holds, $\mathcal{F}_j$ is a filter on X_j for each $1 \leq j \leq 2$ and φ is a (separately) increasing upper semicontinuous map and* (21),

$$\Gamma(\mathcal{F}_1^{\alpha_1}, \mathcal{F}_2^{\alpha_2})f \leq f \quad \textit{and} \quad \Gamma(\mathcal{F}_1^{\beta_1}, \mathcal{F}_2^{\beta_2})g \leq g, \tag{23}$$

then

$$\Gamma(\mathcal{F}_1^{\gamma_1}, \mathcal{F}_2^{\gamma_2})E_\varphi^{-1}(f,g) \leq E_\varphi^{-1}(f,g)\,.$$

Proof. Indeed, by inverting the order on $\overline{\mathbb{R}}$ in Theorem 7.1 and on using (23), we get $\Gamma(\mathcal{F}_1^{\gamma_1}, \mathcal{F}_2^{\gamma_2})\varphi(f,g) \leq \varphi(f,g)$, hence

$$\Gamma(\mathcal{F}_1^{\gamma_1}, \mathcal{F}_2^{\gamma_2}) \inf_{X_1} \varphi(f,g) \leq \inf_{X1} \Gamma(\mathcal{F}_1^{\gamma_1}, \mathcal{F}_2^{\gamma_2})\varphi(f,g) \leq \inf_{X_1} \varphi(f,g)\,. \qquad \square$$

Observe that the convergence of conjugates is not topological in general.

Suppose that $g : V \times X \times Y \to \overline{\mathbb{R}}$ is a coupling function (between X and Y) depending on a parameter $v \in V$. Consider convergences θ on V, τ on X and σ on Y.

Proposition 7.3. *If*

$$\Gamma(\theta^-, \tau^-, \sigma^-)g \geq g \quad \textit{and} \quad \Gamma(\theta^+, \tau^-)f \leq f,$$

then

$$\Gamma(\theta^-, \sigma^-)f^g \geq f^g.$$

Corollary 7.4. *If* $\Gamma(\tau^-, \sigma^-)g \geq g$ *and* $\Gamma(\mathcal{F}^+, \tau^-)f \leq f$, *then*

$$\Gamma(\mathcal{F}^-, \sigma^-)f^g \geq f^g.$$

In other words, for an arbitrary lower semicontinuous coupling between two convergence spaces, the upper epi-convergence of functions implies the lower epiconvergence of the corresponding conjugates.

The inequality (20) can hold without (19) under assumptions that enable one to commute universal and existential quantifiers, like certain compactness-like conditions.

I have recalled here the simplest examples of applications of Γ-inequalities. Many others can be found in [7], like generalizations of the Moreau-Yosida inequality and of the theorems of Umberto Mosco [20] and Jean-Luc Joly [17] on continuity of bilinear polarities.

References

[1] G. Choquet: Sur les notions de filtre et de grille, C. R. Acad. Sci., Paris 224 (1947) 171–173.

[2] G. Choquet: Convergences, Ann. Univ. Grenoble, Sect. Sci. Math. Phys., II. Ser. 23 (1948) 55–112.

[3] S. Dolecki: Continuity of bilinear and non bilinear polarities, in: Optimization and Related Fields (Erice, 1984), R. Conti et al. (ed.), Springer, Berlin (1986).

[4] S. Dolecki, G. H. Greco: Cyrtologies of convergences, I, Math. Nachr. 126 (1986) 327–348.

[5] S. Dolecki, G. H. Greco: Cyrtologies of convergences, II: Sequential convergences, Math. Nachr. 127 (1986) 317–334.

[6] S. Dolecki, G. H. Greco, A. Lechicki: Compactoid and compact filters, Pacific J. Math. 117 (1985) 69–98.

[7] S. Dolecki, J. Guillerme, M. B. Lignola, C. Malivert: Γ-inequalities and stability of generalized extremal convolutions, Math. Nachr. 152 (191) 299–323.

[8] S. Dolecki, S. Kurcyusz: Convexité généralisée et optimisation, C. R. Acad. Sci., Paris, Sér. A 283 (1976) 91–94.

[9] S. Dolecki, S. Kurcyusz: On Φ-convexity in extremal problems, SIAM J. Control Optim. 16(2) (1978) 277–300.

[10] S. Dolecki, M.-P. Moschen: Inequalities involving Γ-functionals and semi-complete lattice homomorphisms, Rend. Semin. Mat. Univ. Padova 137 (1984) 61–82.

[11] S. Dolecki, F. Mynard: Hyperconvergences, Appl. Gen. Topol. 4(2) (2003) 391–419.

[12] W. Fenchel: Convex Cones, Sets and Functions, Princeton University, Princeton (1953).

[13] E. De Giorgi, T. Franzoni: Su un tipo di convergenza variazionale, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. 58 (1975) 842–850.

[14] G. H. Greco: Limitoidi e reticoli completi, Ann. Univ. Ferrara, N. Ser., Sez. VII 29 (1983) 153–164.

[15] G. H. Greco: Decomposizioni di semifiltri e Γ-limiti sequenziali in reticoli completamente distributivi, Ann. Mat. Pura Appl., IV. Ser. 137 (1984) 61–82.

[16] G. H. Greco: Limites et fonctions densemble, Rend. Sem. Mat. Univ. Padova 72 (1984) 89–97.

[17] J.-L. Joly: Une famille de topologies sur lensemble des fonctions convexes pour lesquelles la polarité est bicontinue, J. Math. Pures Appl., IX. Sér. 52 (1973) 421–441.

[18] J. J. Moreau: Fonctionnelles convexes, Séminaire Jean Leray 2 (1966–67) 1–108.

[19] J.-J. Moreau: Inf-convolutions, sous-additivité, convexité des fonctions numériques, J. Math. Pures Appl., IX. Sér. 49 (1970) 109–154.

[20] U. Mosco: On the continuity of Young-Fenchel transformations, J. Math. Anal. Appl. 35 (1971) 518–535.

[21] G. Peano: Formulaire Mathématique, 4th Ed., Bocca, Torino (1903).

[22] G. Peano: Formulario Mathematico, Bocca, Torino (1908).