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The Originality of Peano's 1886 Existence Theorem on Scalar Differential Equations

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Dedicated to the memory of Jean Jacques Moreau.

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The paper presents a revisited proof of Peano's existence theorem, emphasizing the historical background. The argument relies on methods and results due to Peano, such as comparison theorems by differential inequalities, Grönwall-type inequalities, sub- and super-solutions, maximum and minimum solutions, a characterization of continuously differentiable functions and Weierstrass maximum theorem for functions of several variables.

Keywords: Peano, ordinary differential equations, Peano's existence theorem, differential inequalities, Grönwall-type inequalities, strict sub-solutions, strict super-solutions, sub-solution method, super-solution method, maximum solution, minimum solution, continuously differentiable function, Weierstrass maximum theorem

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1. Introduction

In 1886 Giuseppe Peano [17] publishes the first proof that the continuity of a function $f\colon (a \vdash\!\dashv b)\times\mathbb{R} \to \mathbb{R}$ is a sufficient condition for an initial value problem([1])

$$y' = f(x,y), \qquad y(a) = c. \tag{ODE}$$

to have one or more local solutions. The concepts and instruments applied by Peano in his paper have been hardly ever recognized for the importance they deserve. Some authors (see for example, Kennedy [8, 9], Flett [2]) assert that

[1] According to Peano's notation in *Formulario Mathematico* [25], we shall denote the intervals of the real line in the following way: $a \relbar b := \{x \in \mathbb{R} : a < x < b\}$, $a \vdash b := \{x \in \mathbb{R} : a \leq x < b\}$, $a \dashv b := \{x \in \mathbb{R} : a < x \leq b\}$, $a \vdash\!\dashv b := \{x \in \mathbb{R} : a \leq x \leq b\}$.

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the proof is neither rigorous nor correct. Furthermore, ideas and techniques, introduced by Peano in [17] had been forgotten for several years before the rediscovery by Perron [26] (see, e.g., Müller [12] for a historical discussion of this topic). In particular, the method of super- and sub-solutions, ideated by Peano in [17], was erroneously attributed to Perron (see for example, Walter [33], Výborný [32]), the existence of the maximum and minimum solutions of an initial value problem (ODE) proved by Peano [17] was attributed to Osgood [13] by Simpson [30] and moreover, differential inequalities, credited to Grönwall [6], were used by Peano in [17] twenty-three years before [6].

Even if one may criticize some shortcuts of Peano's proof, its conceptual simplicity is appreciable. It involves neither polygonal approximations, nor the notion of integral, nor that of convergence of functions (neither pointwise nor uniform). It uses neither the Ascoli-Arzelà theorem nor the axiom of choice, but it is essentially based on the order relation on real numbers.

The aim of this paper is to give a complete statement and a revisited proof of the Peano's existence theorem, emphasizing its historical background. The proof relies on original methods and results due to Peano, such as comparison theorems by differential inequalities (see Lemma 2.1), sub- and super-solutions, maximum and minimum solutions, a characterization of continuously differentiable functions, Grönwall-type inequalities and Weierstrass maximum theorem for functions of several variables.

For a critical account of Peano's original statement and proof (an account, which fills gaps) see our paper [5]. A fine investigation on the logical ground of Peano's existence theorem in the setting of the second order arithmetic has been carried out by Simpson [30]. Recently López Pouso [10, 11] proposed very interesting, simple and readable revisited proofs of Peano's theorem, based on a conjoined use of Ascoli-Arzelà and Weierstrass maximum/minimum theorems.

2. Peano's 1886 existence theorem (a complete statement and preparatory lemmas)

Given a real function γ of a real variable, we shall denote the left (resp. right) derivative of γ at a point x by $D_-\gamma(x)$ (resp. $D_+\gamma(x)$), if it exists. Moreover, by an inequality of the type $D_\pm\gamma(x) > t$ we mean that both inequalities $D_+\gamma(x) > t$ and $D_-\gamma(x) > t$ hold (2).

Let $d \in a \dashv b$; a function $\gamma\colon a \vdash\!\dashv d \to \mathbb{R}$ is said to be a *strict sub-solution* (resp. *strict super-solution*) of the initial value problem (ODE) on interval $a \vdash\!\dashv d$, if γ has finite left and right derivatives, $\gamma[a] = c$ and

$$D_\pm\gamma(x) < f(x,\gamma(x))\big(\text{resp. } D_\pm\gamma(x) > f(x,\gamma(x))\big) \quad \text{for every } x \in a \vdash\!\dashv d. \qquad (1)$$

[2] Observe that it is implicitly assumed that the inequality "$D_\pm\gamma(x) > t$" should be understood as "$D_+\gamma(x) > t$" (resp. "$D_-\gamma(x) > t$"), whenever x is the lower extremum (resp. upper extremum) of the domain of γ. A similar convention holds for the other inequalities $D_\pm\gamma(x) \geq t$, $D_\pm\gamma(x) < t$, $D_\pm\gamma(x) \leq t$.

In the sequel, the set of such strict sub-solutions (resp. strict super-solutions) will be denoted by $\mathrm{Sol}_<(c;a,d)$ (resp. $\mathrm{Sol}_>(c;a,d)$). Moreover, $\mathrm{Sol}(c;a,d)$ will denote the set of all solutions of (ODE) on the interval $a \vdash\!\dashv d$. A greatest (resp. least) element of $\mathrm{Sol}(c;a,d)$ with respect the pointwise order will be called a *maximum solution* (resp. *minimum solution*) on $a \vdash\!\dashv d$, if they exist ([3]). In [17] Peano stated the following theorem.

Peano's 1886 Existence Theorem. *If f is a continuous function, then there exist a real number $\hat{a} \in a \,\bar{\dashv}\, b$ and two functions $\varphi_1, \varphi_2 : a \vdash\!\dashv \hat{a} \to \mathbb{R}$ such that*

(*i*) *there are infinitely many strict sub- and super-solutions of* (ODE) *on $a \vdash\!\dashv \hat{a}$, i.e.,*

$$\mathrm{Sol}_<(c;a,\hat{a}) \quad \textit{and} \quad \mathrm{Sol}_>(c;a,\hat{a}) \quad \textit{are infinite sets;}$$

(*ii*) *φ_1 is both a minimum solution of* (ODE) *and the pointwise supremum of the strict sub-solutions of* (ODE) *on $a \vdash\!\dashv \hat{a}$, i.e.,*

$$\varphi_1 \in \mathrm{Sol}(c;a,\hat{a}) \quad \textit{and} \quad \varphi_1 = \sup \mathrm{Sol}_<(c;a,\hat{a}) = \min \mathrm{Sol}(c;a,\hat{a});$$

(*iii*) *φ_2 is both a maximum solution of* (ODE) *and the pointwise infimum of the strict super-solutions of* (ODE) *on $a \vdash\!\dashv \hat{a}$, i.e.,*

$$\varphi_2 \in \mathrm{Sol}(c;a,\hat{a}) \quad \textit{and} \quad \varphi_2 = \inf \mathrm{Sol}_>(c;a,\hat{a}) = \max \mathrm{Sol}(c;a,\hat{a});$$

(*iv*) (ODE) *has a unique solution on $a \vdash\!\dashv \hat{a}$, if the partial derivative $\frac{\partial f}{\partial y}$ is bounded above on $(a \vdash\!\dashv b) \times \mathbb{R}$.*

Elements of originality in this Peano's theorem are the existence of a *minimum* and a *maximum solution* (φ_1 and φ_2, respectively) and the construction of these solutions as the supremum of the strict sub-solutions and infimum of the strict super-solutions, respectively ([4]). This particular type of construction, first introduced by Peano, was probably linked to his method for defining the *upper integral* and the *lower integral* [14], looking at the construction of solutions of differential equations as a generalized kind of integration, in the spirit of a former paper by Volterra [31]. Notice that Peano's paper [17] was written to give an affirmative answer to the question, first raised by Volterra [31], whether continuity of f is sufficient for the existence of solutions of (ODE).

In 1890, by means of a different technique, Peano [20] again proved the existence of maximum and minimum solutions without using sub- and super-solution method; more precisely, in [20] Peano defined every value of the maximum solution (resp. minimum solution) of (ODE), as the maximum (resp. minimum) of the corresponding section of Peano's 0-*approximated funnel*; see our paper [5] for further details.

[3] The maximum (resp. minimum) solution is usually and misleadingly referred to as maximal (resp. minimal) solution.

[4] This method was rediscovered later by Perron in [26], who first introduced the term *Oberfunktion* [over-functions] and *Unterfunktion* [under-functions].

Before proving Peano's 1886 Theorem, let us introduce some preparatory lemmas. The following comparison lemma is due to Peano [17, 20].

Lemma 2.1 (Peano, 1886). *Let $\psi_1\colon x_1 \vdash\!\dashv x_2 \to \mathbb{R}$ and $\psi_2\colon x_1 \vdash\!\dashv x_2 \to \mathbb{R}$ be two functions whith finite left and right derivatives such that $\psi_1(x_1) \le \psi_2(x_1)$ and either*

$$D_\pm\psi_1(x) < f(x,\psi_1(x)) \quad \textit{and} \quad f(x,\psi_2(x)) \le D_\pm\psi_2(x) \;\; \textit{for every } x \in x_1 \vdash\!\dashv x_2$$

or

$$D_\pm\psi_1(x) \le f(x,\psi_1(x)) \quad \textit{and} \quad f(x,\psi_2(x)) < D_\pm\psi_2(x) \;\; \textit{for every } x \in x_1 \vdash\!\dashv x_2\,.$$

Then $\psi_1(x) < \psi_2(x)$ for all $x \in x_1 \dashv x_2$.

Proof. Following Peano [17], *per absurdum* let us assume that there exists $\tilde{x} \in x_1 \dashv x_2$ such that $\psi_2(\tilde{x}) \le \psi_1(\tilde{x})$. Then, let $\bar{x} := \inf\{x \in x_1 \dashv x_2 : \psi_2(x) \le \psi_1(x)\}$. 1^{st} *case*: "$\bar{x} > x_1$". Since ψ_1 and ψ_2 are continuous, we have that $\psi_1(\bar{x}) = \psi_2(\bar{x})$; hence, $D_-(\psi_2 - \psi_1)(\bar{x}) = [D_-\psi_2(\bar{x}) - f(\bar{x},\psi_2(\bar{x}))] + [f(\bar{x},\psi_1(\bar{x})) - D_-\psi_1(\bar{x})] > 0$; therefore, there exists a $\delta > 0$ such that $(\psi_2 - \psi_1)(x) < 0$ for all $x \in (\bar{x}-\delta) \vdash \bar{x}$; hence, we have a contradiction. 2^{nd} *case*: "$\bar{x} = x_1$". Since $\psi_1(x_1) \le \psi_2(x_1)$ by hypothesis, we have that $\psi_1(\bar{x}) = \psi_2(\bar{x})$; hence $D_+(\psi_2 - \psi_1)(\bar{x}) > 0$. Therefore, there exists a $\delta > 0$ such that $\psi_2(x) > \psi_1(x)$ for all $x \in x_1 \dashv (x_1+\delta)$; thus $\inf\{x \in x_1 \dashv x_2 : \psi_2(x) \le \psi_1(x)\} \ge x_1 + \delta > x_1$; hence, we obtain a contradiction. □

For proving his theorem of existence of solutions of (ODE), Perron [26] used the following Lemma 2.2 which is proved here by means of the following inequality, due to Peano [19, 23, 24]:

(2) (Peano 1888). *Let $\mu\colon a \vdash\!\dashv b \to \mathbb{R}^n$ be a function with finite left and right derivatives. Then $\|\mu\|$ has finite left and right derivatives, and*

$$D_\pm\|\mu\|(x) \le \|D_\pm\mu(x)\| \;\; \text{for every } x \in a \vdash\!\dashv b.$$

Lemma 2.2 (Perron, 1915). $\inf\{\mu_1,\mu_2\} \in \mathrm{Sol}_>(c;a,b)$ *for every* $\mu_1,\mu_2 \in \mathrm{Sol}_>(c;a,b)$. *Analogously,* $\sup\{\mu_1,\mu_2\} \in \mathrm{Sol}_<(c;a,b)$ *for every* $\mu_1,\mu_2 \in \mathrm{Sol}_<(c;a,b)$.

Proof. Let ψ denote $\inf\{\mu_1,\mu_2\}$, where $\mu_1,\mu_2 \in \mathrm{Sol}_>(c;a,b)$. Clearly, for every $x \in a \vdash\!\dashv b$ such that $\mu_1(x) \ne \mu_2(x)$, by the continuity of μ_1 and μ_2 there exists a neighborhood of x on which ψ coincides with μ_1 or μ_2; consequently, the inequality $D_\pm\psi(x) > f(x,\psi(x))$ holds. In the case where $x \in a \vdash\!\dashv b$ is such that $\mu_1(x) = \mu_2(x)$, one has $\psi(x) = \mu_1(x) = \mu_2(x)$ and, by (2) the following relations hold:

$$\begin{aligned}
D_\pm\psi(x) &= \frac{1}{2} D_\pm(\mu_1 + \mu_2 - |\mu_1 - \mu_2|)(x) \\
&= \frac{1}{2}\left(D_\pm\mu_1(x) + D_\pm\mu_2(x) - D_\pm|\mu_1 - \mu_2|(x)\right) \\
&\ge \frac{1}{2}\left(D_\pm\mu_1(x) + D_\pm\mu_2(x) - |D_\pm\mu_1(x) - D_\pm\mu_2(x)|\right) \\
&= \inf\{D_\pm\mu_1(x), D_\pm\mu_2(x)\} > f(x,\psi(x)).
\end{aligned}$$

Hence, ψ is a strict super-solution. By duality, one can prove an analogous property for strict sub-solutions. $\square$

By analyzing the proof of mean value theorem given by Jordan in *Cours d'analyze* [7], in 1884 Peano stresses a difference between differentiable functions and continuously differentiable functions (see [15, 16], [18, p. 191]), [21, p. 30] [22]). The continuity of the derivative was expressed by Peano both in terms of uniform convergence of difference quotient and in the following way:

(3) (Peano, 1887). *A function* $\psi\colon a \vdash\!\dashv b \to \mathbb{R}$ *is continuously differentiable on* $a \vdash\!\dashv b$ *and a function* $h\colon a \vdash\!\dashv b \to \mathbb{R}$ *is its derivative if and only if for every* $x \in a \vdash\!\dashv b$

$$h(x) = \lim_{\substack{x_1, x_2 \to x \\ x_1 \neq x_2}} \frac{\psi(x_2) - \psi(x_1)}{x_2 - x_1}. \quad (^5)$$

To prove uniqueness of solutions of a differential equation and the continuous dependence of solutions with respect to an initial value (when Lipschitz condition holds), Peano [17, 23, 24] used an argument that amounts to the following Grönwall-type differential inequality:

(4) (Peano, 1886). *Let* $\psi\colon a \vdash\!\dashv b \to \mathbb{R}$ *be a differentiable function and* $c \in \mathbb{R}$. *Then*

$$\psi(b) \leq \psi(a) e^{c(b-a)} \ \text{ if } \psi'(x) \leq c\,\psi(x) \ \text{ for every } x \in a \vdash\!\dashv b. \ (^6)$$

Lemma 2.3 (Peano, 1886). *Let* $\varphi_1, \varphi_2 : a \vdash\!\dashv \hat{a} \to \mathbb{R}$ *be solutions of* (ODE) *with* $\varphi_1 \leq \varphi_2$. *Then* $\varphi_1 = \varphi_2$, *if the partial derivative* $\frac{\partial f}{\partial y}$ *is bounded above on* $(a \vdash\!\dashv \hat{a}) \times \mathbb{R}$.

Proof. (Peano's proof). Let $\partial f/\partial y$ be bounded above on $(a \vdash\!\dashv \hat{a}) \times \mathbb{R}$ by a real number M. Since φ_1 and φ_2 are solution of (ODE) on the interval $a \vdash\!\dashv \hat{a}$, we have $(\varphi_2 - \varphi_1)'(x) = f(x, \varphi_2(x)) - f(x, \varphi_1(x))$. Further, $\partial f/\partial y$ being bounded above by M and $\varphi_2 - \varphi_1 \geq 0$, we have:

$$(\varphi_2 - \varphi_1)'(x) \leq M(\varphi_2 - \varphi_1)(x) \ \text{ for every } x \in a \vdash\!\dashv \hat{a}. \tag{5}$$

Therefore, by (4) and $\varphi_2(a) = \varphi_1(a)$, we have $\varphi_2(x) - \varphi_1(x) \leq 0$ for every $x \in a \vdash\!\dashv \hat{a}$. Hence, by the inequality $\varphi_1 \leq \varphi_2$, one has $\varphi_1 = \varphi_2$, as required. $\square$

[5] Strict differentiability (i.e., the existence of the limit in (3)) plays a central role in Peano's differentiation of measure [18]. Peano's strict derivative (i.e., the function h in (3)), provided in [18] a solid base for the concept of "infinitesimal ratio" between two magnitudes, successfully used since Kepler (see Greco, Mazzucchi and Pagani [4] for details).

[6] In [20] Peano proved uniqueness of solutions of a differential equation by the following Grönwall-type differential inequality: "$\psi(c) < \frac{a}{b}(e^{bc} - 1)$, if $\psi(0) = 0$ and $\psi'(x) \leq a + b\psi(x)$ for every $x \in 0 \vdash\!\dashv c$."

3. Peano's 1886 existence theorem (a revisited proof)

The following proof retains many ideas of the original Peano's arguments. In addition to the previous lemmas, we will use the Weierstrass maximum theorem on the existence of maxima and minima of continuous functions of several variables on bounded closed sets. As Cantor [1, footnote p. 82 and pp. 85–86] noticed (see also Pincherle [27, p. 249]), Weierstrass proved maximum theorem for functions of one variable, i.e.,

> Eine in einen Intervalle (a . . . b) (die Grenzen inkl.) der reellen Veränderlichen x gegebene, stetige Funktion $\varphi(x)$ *erreicht* das Maximum.

As already noticed in *Encyclopädie der Mathematischen Wissenschaften* by Pringsheim [28, p. 48] and in *Enciclopédie des sciences mathématiques pures et appliquées* by Pringsheim and Molk [29, p. 101, 103], maximum/minimum theorem for functions of several variables was first proved by Peano in [3, p. 131] (see also [18, p. 166]):

> (Peano, 1884). *A continuous real-valued function on a closed bounded bounded set of* $\mathbb{R}^n$ *attains its minimum and maximum.* (6)

Proof of Peano's 1886 existence theorem. Clearly, from Lemma 2.3, (*ii*) and (*iii*) it follows immediately that (*iv*) holds. Moreover, by the duality $y \mapsto -y$ between the initial value problems "$y' = f(x,y)$ and $y(a) = c$" and "$y' = -f(x,-y)$ and $y(a) = -c$", the property (*ii*) amounts to (*iii*). Hence, it is enough to prove (*i*) and (*iii*). The proof of (*i*) is deferred to third part.

Preliminary claim: "there exist $\hat{a} \in a \mathop{\dashv} b$ and two functions $\gamma_1, \gamma_2 \colon a \mathop{\vdash\!\dashv} \hat{a} \to \mathbb{R}$ such that $\gamma_1 \in \mathrm{Sol}_<(c; a, \hat{a})$ and $\gamma_2 \in \mathrm{Sol}_>(c; a, \hat{a})$". To prove this, let us choose $m_1, m_2 \in \mathbb{R}$, with $m_1 < f(a,c) < m_2$ and consider the linear functions $\gamma_1, \gamma_2 \colon \mathbb{R} \to \mathbb{R}$ defined by $\gamma_1(x) := c + m_1(x-a)$ and $\gamma_2(x) := c + m_2(x-a)$. The continuous maps $x \mapsto \gamma_1'(x) - f(x, \gamma_1(x))$ and $x \mapsto \gamma_2'(x) - f(x, \gamma_1(x))$ are strictly negative and strictly positive at $x := a$, respectively. Hence, there exists a real number $\hat{a} \in a \mathop{\dashv} b$ such that $\gamma_1'(x) < f(x, \gamma_1(x))$ and $\gamma_2'(x) > f(x, \gamma_1(x))$ for all $x \in a \mathop{\vdash\!\dashv} \hat{a}$. Therefore, γ_1 (resp. γ_2) is a strict sub-solution (resp. a strict super-solution) of the initial value problem (ODE) on $a \mathop{\vdash\!\dashv} \hat{a}$. Hence, $\gamma_1 \in \mathrm{Sol}_<(c; a, \hat{a})$ and $\gamma_2 \in \mathrm{Sol}_>(c; a, \hat{a})$, as required.

Definition of φ_1 *and* φ_2. Let $\varphi_1, \varphi_2 \colon a \mathop{\vdash\!\dashv} \hat{a} \to \mathbb{R}$ be defined by

$$\varphi_1(x) := \sup\{\gamma(x) \colon \gamma \in \mathrm{Sol}_<(c; a, \hat{a})\},$$
$$\varphi_2(x) := \inf\{\gamma(x) \colon \gamma \in \mathrm{Sol}_>(c; a, \hat{a})\}.$$

These functions φ_1 and φ_2 are well defined and $\gamma_1 \le \varphi_1 \le \varphi_2 \le \gamma_2$, since $\mathrm{Sol}_>(c; a, \hat{a})$ and $\mathrm{Sol}_<(c; a, \hat{a})$ are non-empty, and since by Lemma 2.1 a strict sub-solution of the initial value problem (ODE) is less than any of its strict super-solution. Moreover, by Lemma 2.1 a strict sub-solution (resp. a strict super-solution) of (ODE) is less than (resp. greater than) every its solution, hence

$$\varphi_1 \le \varphi \le \varphi_2 \ \text{ for every } \varphi \in \mathrm{Sol}(c; a, \hat{a})\}. \tag{7}$$

Consequently, in order to prove (*iii*) it is enough to show that $\varphi_2 \in \mathrm{Sol}(c; a, \hat{a})$. For the sake of simplicity, let Φ denote φ_2.

First part: "$\Phi \in \mathrm{Sol}(c; a, \hat{a})$ for f continuous and bounded". Denote by M a real number greater than $\sup\{|f(x, y)| : x \in a \vdash\!\dashv \hat{a}$ and $y \in \mathbb{R}\}$.

1^{st} *claim*: "Φ is a Lipschitz function" ([7]). To prove this, let $x_1, x_2 \in a \vdash\!\dashv \hat{a}$ with $x_1 < x_2$. Choose an arbitrary strict super-solution $\gamma \in \mathrm{Sol}_{>}(c; a, \hat{a})$ and define a related function $\psi \colon x_1 \vdash\!\dashv x_2 \to \mathbb{R}$ by $\psi(x) := \gamma(x_1) - M(x - x_1)$. Since $\psi(x_1) = \gamma(x_1)$ and $D_{\pm}\psi(x) = -M < f(x, \psi(x))$ for every $x \in x_1 \vdash\!\dashv x_2$, Lemma 2.1 implies that $\gamma(x_2) \ge \psi(x_2)$, that is

$$\gamma(x_2) \ge \gamma(x_1) - M(x_2 - x_1). \tag{8}$$

By taking the infimum of both sides of this inequality when γ varies in $\mathrm{Sol}_{>}(c; a, \hat{a})$, one obtains:

$$\Phi(x_2) \ge \Phi(x_1) - M(x_2 - x_1). \tag{9}$$

Now, fix a real number $\varepsilon > 0$, choose an arbitrary strict super-solution $\gamma_\varepsilon \in \mathrm{Sol}_{>}(c; a, \hat{a})$ such that $\gamma_\varepsilon(x_1) - \Phi(x_1) < \varepsilon$, and define a function $\psi \colon a \vdash\!\dashv \hat{a} \to \mathbb{R}$ by $\psi(x) := \gamma_\varepsilon(x)$ for $x \in a \vdash\!\dashv x_1$, and $\psi(x) := \gamma_\varepsilon(x_1) + (M + \varepsilon)(x - x_1)$ for $x \in x_1 \vdash\!\dashv \hat{a}$. Clearly $\psi \in \mathrm{Sol}_{>}(c; a, \hat{a})$. Therefore, from the definition of Φ it follows that $\Phi(x_2) \le \psi(x_2)$. On the other hand, $\psi(x_2) - \gamma_\varepsilon(x_1) = (M + \varepsilon)(x_2 - x_1)$. Hence $\Phi(x_2) \le \psi(x_2) = \gamma_\varepsilon(x_1) + (\psi(x_2) - \gamma_\varepsilon(x_1)) \le \varepsilon + \Phi(x_1) + (M + \varepsilon)(x_2 - x_1)$. Since ε is an arbitrary positive real number, it follows that

$$\Phi(x_2) \le \Phi(x_1) + M(x_2 - x_1). \tag{10}$$

By combining (9) and (10), we have $|\Phi(x_2) - \Phi(x_1)| \le M|x_2 - x_1|$ for all $x_1, x_2 \in a \vdash\!\dashv \hat{a}$, i.e., Φ is a M-Lipschitzian function.

2^{nd} *claim*: "Let $x_0 \in a \vdash\!\dashv \hat{a}$ and $\varepsilon \in 0 - 1$. There exists $\delta > 0$ such that

$$\frac{\Phi(x_2) - \Phi(x_1)}{x_2 - x_1} \le f(x_0, \Phi(x_0)) + \varepsilon \tag{11}$$

for every $x_1, x_2 \in a \vdash\!\dashv \hat{a}$ with $x_1 < x_2$, $|x_1 - x_0| < \delta$ and $|x_2 - x_0| < \delta$". To prove this, define a function $H \colon (0 \vdash\!\dashv 1) \times (a \vdash\!\dashv \hat{a}) \times (a \vdash\!\dashv \hat{a}) \to \mathbb{R}$ by

$$H(\alpha, x, z) := f(x_0, \Phi(x_0)) + \varepsilon - f\big(x, \Phi(z) + \alpha + (f(x_0, \Phi(x_0)) + \varepsilon)(x - z)\big).$$

This function H is continuous, since Φ (by 1^{st} claim) and f (by hypothesis) are continuous. Since $H(0, x_0, x_0) = \varepsilon > 0$, from the continuity of Φ at $(0, x_0, x_0)$ it follows that there is a real number $\delta > 0$ such that

$$H(\alpha, x, z) > 0 \quad \text{for } \ \alpha \in 0 \vdash \delta,\ x, z \in a \vdash\!\dashv \hat{a},\ |x - x_0| \le \delta,\ |z - x_0| \le \delta. \tag{12}$$

[7] This claim which is absent in Peano [17], is suggested us by Perron's paper [26]. To prove a Lipschitz inequality for Φ, Perron used the notion of integral.

Now, fix $x_1, x_2 \in a \mathrel{\vdash\!\!\dashv} \hat{a}$ such that $x_1 < x_2$, $|x_1 - x_0| < \delta$ and $|x_2 - x_0| < \delta$. For every natural number $n \geq 1$ choose a strict super-solution $\varphi_n \in \mathrm{Sol}_>(c; a, \hat{a})$ such that $\varphi_n(x_1) - \Phi(x_1) < \frac{\delta}{n}$. Let $d := \min\{x_0 + \delta, \hat{a}\}$ and let the functions $\psi_n \colon a \mathrel{\vdash\!\!\dashv} \hat{a} \to \mathbb{R}$ be defined by

$$\psi_n(x) := \begin{cases} \varphi_n(x), & \text{for } x \in a \mathrel{\vdash\!\!\dashv} x_1 \\ \varphi_n(x_1) + (f(x_0, \Phi(x_0)) + \varepsilon)(x - x_1), & \text{for } x \in x_1 \mathrel{-\!\!\dashv} d \\ \psi_n(d) + M(x - d), & \text{for } x \in d \mathrel{-\!\!\dashv} \hat{a}. \end{cases} \tag{13}$$

Since $0 \leq \varphi_n(x_1) - \Phi(x_1) < \delta$, by (12) the inequality $H(\varphi_n(x_1) - \Phi(x_1), x, x_1) > 0$ holds for every x such that $x \in x_1 \mathrel{-\!\!\dashv} d$; hence, one can easily verify that $\psi_n \in \mathrm{Sol}_>(c; a, \hat{a})$. Consequently, from the definition of Φ it follows that $\Phi(x) \leq \psi_n(x)$ for every $x \in a \mathrel{\vdash\!\!\dashv} \hat{a}$ and every n. Then $\Phi(x_2) \leq \psi_n(x_2)$ for every n. Therefore, by (13) we have that

$$\Phi(x_2) \leq \Phi(x_1) + \frac{\delta}{n} + (f(x_0, \Phi(x_0)) + \varepsilon)(x_2 - x_1) \tag{14}$$

since $x_2 \in x_1 \mathrel{-\!\!\dashv} d$ and $\varphi_n(x_1) = \Phi(x_1) + (\varphi_n(x) - \Phi(x_1)) \leq \Phi(x_1) + \frac{\delta}{n}$. Now, by letting $n \to \infty$ in (14) we have:

$$\Phi(x_2) \leq \Phi(x_1) + (f(x_0, \Phi(x_0)) + \varepsilon)(x_2 - x_1).$$

Hence (11) holds.

3^{rd} *claim*: "Let $x_0 \in a \mathrel{\vdash\!\!\dashv} \hat{a}$ and $\varepsilon \in 0 \mathrel{-\!\!-} 1$. There exists $\delta > 0$ such that

$$\frac{\Phi(x_2) - \Phi(x_1)}{x_2 - x_1} \geq f(x_0, \Phi(x_0)) - \varepsilon \tag{15}$$

for every $x_1, x_2 \in a \mathrel{\vdash\!\!\dashv} \hat{a}$ with $x_1 < x_2$, $|x_1 - x_0| < \delta$ and $|x_2 - x_0| < \delta$". To prove this, let $H \colon (a \mathrel{\vdash\!\!\dashv} \hat{a}) \times (a \mathrel{\vdash\!\!\dashv} \hat{a}) \to \mathbb{R}$ be the function defined by

$$H(x, z) := f(x_0, \Phi(x_0)) - \varepsilon - f\big(x, \Phi(z) + (f(x_0, \Phi(x_0)) - \varepsilon)(x - z)\big).$$

Since $H(x_0, x_0) = -\varepsilon < 0$, from the continuity of Φ at (x_0, x_0) it follows that there is a real number $\delta > 0$ such that

$$H(x, z) < 0 \quad \text{for } \ x, z \in a \mathrel{\vdash\!\!\dashv} \hat{a}, \ |x - x_0| \leq \delta, \ |z - x_0| \leq \delta. \tag{16}$$

Now, fix $x_1, x_2 \in a \mathrel{\vdash\!\!\dashv} \hat{a}$ such that $x_1 < x_2$, $|x_1 - x_0| < \delta$ and $|x_2 - x_0| < \delta$. Let $d := \min\{x_0 + \delta, \hat{a}\}$; define the function $\psi \colon x_1 \mathrel{\vdash\!\!\dashv} d \to \mathbb{R}$ by

$$\psi(x) := \Phi(x_1) + (f(x_0, \Phi(x_0)) - \varepsilon)(x - x_1).$$

By (16) the inequality $H(x, x_1) < 0$ holds for every $x \in x_1 \mathrel{\vdash\!\!\dashv} d$; hence, one can easily verify that the inequality $D_\pm \psi(x) < f(x, \psi(x))$ holds for every $x \in x_1 \mathrel{\vdash\!\!\dashv} d$. Hence, by Lemma 2.1 we have $\psi \leq \varphi$ on $x_1 \mathrel{\vdash\!\!\dashv} d$ for every strict super-solution

$\varphi \in \mathrm{Sol}_{>}(c; a, \hat{a})$; consequently, $\Phi(x) \geq \psi(x)$ for every $x \in x_1 \mathrel{\vdash\!\dashv} x_2$; therefore $\Phi(x_2) \geq \psi(x_2)$, that is:

$$\Phi(x_2) \geq \Phi(x_1) + (f(x_0, \Phi(x_0)) - \varepsilon)(x_2 - x_1).$$

Hence (15) holds.

Conclusion of first part: "Φ is a solution of the initial problem (ODE)". By (3), from 2^{nd} and 3^{rd} claims it follows that Φ is differentiable and $\Phi'(x_0) = f(x_0, \Phi(x_0))$ for every $x_0 \in a \mathrel{\vdash\!\dashv} \hat{a}$. On the other hand $\Phi[a] = c$, since $\gamma_1(a) \leq \Phi(a) \leq \gamma_2(a)$ and $\gamma_1(a) = \gamma_2(a) = c$.

Second part: "$\Phi \in \mathrm{Sol}(c; a, \hat{a})$ for f continuous, not necessarily bounded". Define a function $g\colon (a \mathrel{\vdash\!\dashv} \hat{a}) \times \mathbb{R} \to \mathbb{R}$ by

$$g(x, y) := \begin{cases} f(x, y) & \text{if } \gamma_1(x) \leq y \leq \gamma_2(x) \\ f(x, \gamma_1(x)) & \text{if } y < \gamma_1(x) \\ f(x, \gamma_2(x)) & \text{if } y > \gamma_2(x). \end{cases} \tag{17}$$

Clearly, the function γ_1 (resp. γ_2) is a strict sub-solution (resp. a strict super-solution) of the following initial value problem

$$y' = g(x, y), \qquad y(a) = b. \tag{18}$$

By Weierstrass maximum theorem (6) the continuous function f is bounded on the compact set $K := \{(x, y) : x \in a \mathrel{\vdash\!\dashv} \hat{a} \text{ and } \gamma_1(x) \leq y \leq \gamma_2(x)\}$; therefore,

$$g \text{ is continuous and bounded.} \tag{19}$$

From Lemma 2.2 it follows that

$$\Phi = \inf\{\gamma : \gamma \in \mathrm{Sol}_{>}(c; a, \hat{a}) \text{ and } \gamma_1 \leq \gamma \leq \gamma_2\}. \tag{20}$$

On the other hand, a function $\gamma\colon a \mathrel{\vdash\!\dashv} \hat{a} \to \mathbb{R}$, verifying $\gamma_1 \leq \gamma \leq \gamma_2$, is a strict super-solution for (ODE) on $a \mathrel{\vdash\!\dashv} \hat{a}$ if and only if γ is a strict super-solution of (18) on $a \mathrel{\vdash\!\dashv} \hat{a}$. Therefore, by applying *First Part* to the initial value problem (18), we have that Φ is a solution of (18) and, consequently, a solution of (ODE), as required.

Third part: "$\mathrm{Sol}_{<}(c; a, \hat{a})$ and $\mathrm{Sol}_{>}(c; a, \hat{a})$ are infinite sets". By duality it is enough to prove that $\mathrm{Sol}_{>}(c; a, \hat{a})$ is an infinite set. A solution of (ODE) cannot be a strict super-solution. Hence, by Lemma 2.2 the solution Φ (which is the infimum of the strict super-solutions) cannot be an infimum of finitely many strict super-solutions. Therefore, $\mathrm{Sol}_{>}(c; a, \hat{a})$ must contain infinitely many elements. Alternatively, we may argue as in the *preliminary claim*. Let $m_2 > f(a, c)$ and let a function $h : (0 \mathrel{\vdash\!\dashv} 1) \times (a \mathrel{\vdash\!\dashv} b) \to \mathbb{R}$ defined by $h(\alpha, x) := m_2 + \alpha - f\big(x, c + (m_2 + \alpha)(x - a)\big)$. Since h is continuous and $h(0, a) = m_2 - f(a, c) > 0$, there exists a real number $\delta \in 0 \mathrel{-} 1$ and $\hat{a} \in a \mathrel{\vdash\!\dashv} b$ such that

$$h(\alpha, x) > 0 \ \text{ for every } x \in a \mathrel{\vdash\!\dashv} \hat{a}, \ \alpha \in 0 \mathrel{\vdash\!\dashv} \delta. \tag{21}$$

Therefore, for every $\alpha \in 0 \vdash\!\dashv \delta$ the functions $\gamma_\alpha : a \vdash\!\dashv \hat{a} \to \mathbb{R}$ defined by $\gamma_\alpha(x) := c + (m_2 + \alpha)(x - a)$ are strict super-solutions, that is $\gamma_\alpha \in \mathrm{Sol}_>(c; a, \hat{a})$. Therefore $\mathrm{Sol}_>(c; a, \hat{a})$ is an infinite set. $\square$

From the proof of Peano's theorem we infer global existence conditions:

Corollary 3.1. *Let f be continuous and $\hat{a} \in a \dashv b$. If $\mathrm{Sol}_<(c; a, \hat{a})$ and $\mathrm{Sol}_>(c; a, \hat{a})$ are non-empty (in particular, if f is bounded on $(a \vdash\!\dashv \hat{a}) \times \mathbb{R}$), then the initial value problem* (ODE) *has solutions on the interval $a \vdash\!\dashv \hat{a}$.*

References

[1] G. Cantor: Gesammelte Abhandlungen, Springer, Berlin (1932).

[2] T. M. Flett: Differential Analysis, Cambridge University Press, Cambridge (1980).

[3] A. Genocchi: Calcolo Differenziale e Principii di Calcolo Integrale, Fratelli Bocca, Torino (1884).

[4] G. H. Greco, S. Mazzucchi, E. Pagani: Peano on derivative of measures: strict derivative of distributive set functions, Atti Accad. Naz. Lincei, Rend. Lincei Mat. Appl. 21 (2010) 305–339.

[5] G. H. Greco, S. Mazzucchi: Peano's 1886 paper on existence theorem for scalar differential equations: a review, Boll. Unione Mat. Ital., to appear.

[6] T. H. Grönwall: Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Ann. Math. 20 (1919) 292–296.

[7] C. Jordan: Cours d'Analyse de l'École Polytechnique, Vol. I, Gauthier-Villars, Paris (1882).

[8] H. C. Kennedy: Is there an elementary proof of Peano's existence theorem for first order differential equations?, Amer. Math. Monthly 76 (1969) 1043–1045.

[9] H. C. Kennedy: Life and Works of Giuseppe Peano, 2nd Ed., Peremptory Edition (2006).

[10] R. López Pouso: Peano's existence theorem revisited, arXiv:1202.1152 [math.CA] (2012).

[11] R. López Pouso: Greatest solutions and differential inequalities: a journey in two directions, arXiv:1304.3576 [math.CA] (2013).

[12] M. Müller: Neuere Untersuchungen über den Fundamentalsatz in der Theorie der gewöhnlichen Differentialgleichungen, Jahresber. Dtsch. Math.-Ver. 37 (1928) 33–48.

[13] W. F. Osgood: Beweis der Existenz einer Lösung der Differentialgleichung $\frac{dy}{dx} = f(x, y)$ ohne Hinzunahme der Cauchy-Lipschitzchen Bedingung, Monatsh. Math. Phys. 9 (1898) 331–345.

[14] G. Peano: Sull'integrabilità delle funzioni, Atti. R. Accad. Sci. Torino (5) 18 (1882) 439–446.

[15] G. Peano: Extrait d'une lettre, Nouv. Ann. Math. 3 (1884) 45–47.

[16] G. Peano: Réponse à Ph. Gilbert, Nouv. Ann. Math. 3 (1884) 252–256.

[17] G. Peano: Sull'integrabilità delle equazioni differenziali del primo ordine, Atti R. Accad. Sci. Torino 21 (1886) 437–445.

[18] G. Peano: Applicazioni Geometriche del Calcolo Infinitesimale, Fratelli Bocca, Torino (1887).

[19] G. Peano: Intégration par séries des équations différentielles linéaires, Math. Ann. 32 (1888) 450–456.

[20] G. Peano: Démonstration de l'intégrabilité des équations différentielles ordinaires, Math. Ann. 37 (1890) 182–228.

[21] G. Peano: Gli Elementi di Calcolo Geometrico, Candeletti, Torino (1891).

[22] G. Peano: Sur la définition de la dérivée, Mathesis 2 (1892) 12–14.

[23] G. Peano: Sur un théorème général relatif à l'existence des intégrales des équations différentielles ordinaires, Nouv. Ann. Math. 11 (1892) 79–82.

[24] G. Peano: Generalità sulle equazioni differenziali ordinarie, Atti R. Accad. Sci. Torino 33 (1897) 9–18.

[25] G. Peano: Formulario Mathematico, Fratelli Bocca, Torino (1908).

[26] O. Perron: Ein neuer Existenzbeweis für die Integrale der Differentialgleichung $y' = f(x, y)$, Math. Ann. 76 (1915) 471–484.

[27] S. Pincherle: Saggio di una introduzione della teoria delle funzioni analitiche secondo i principi del prof. C. Weierstrass, Giorn. Mat. 18 (1880) 178–252, 317–357.

[28] A. Pringsheim: Grundlagen der allgemeinen Funktionenlehre, in: Encyclopädie der Mathematischen Wissenschaften II.1.1, H. Burkhardt et al. (ed.), Teubner, Leipzig (1899) 1–53.

[29] A. Pringsheim, J. Molk: Principes fondamentaux de la théorie des fonctions, in: Enciclopédie des Sciences Mathématiques Pures et Appliquées II.1, J. Molk et al. (ed.), Gauthiers-Villars, Paris (1909) 1–112.

[30] S. G. Simpson: Which set existence axioms are needed to prove the Cauchy/Peano theorem for ordianry differential equations?, J. Symbolic Logic 49 (1984) 783–802.

[31] V. Volterra: Sui principii del calcolo integrale, Giorn. Mat. 19 (1881) 333–372.

[32] R. Vyborny: A remark on Perron's method in the proof of the Peano theorem, Acta Math. Sci. 5 (1985) 349–352.

[33] W. Walter: Ordinary Differential Equations, Springer, New York (1998).