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Henig Approximate Proper Efficiency and Optimization Problems with Difference of Vector Mappings*

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Dedicated to the memory of Jean Jacques Moreau.

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This work focuses on approximate proper solutions of vector optimization problems. A concept of Henig approximate proper efficiency is introduced and analyzed from several points of view. First, its main properties are stated and the limit behavior in multiobjective problems of the whole Henig approximate proper efficient set is deduced. These results show that the introduced concept is suitable to approximate the efficient solution set of the problem. After that, the Henig approximate proper efficient solutions are characterized by linear scalarizations under convexity assumptions, and by ε-subgradients in optimization problems dealing with difference of vector mappings. For this last objective, a notion of ε-subdifferential is introduced and studied, obtaining, in particular, a Moreau-Rockafellar type theorem.

Keywords: Vector optimization, proper ε-efficiency, optimization of difference of vector mappings, ε-subdifferential, nearly cone-subconvexlikeness, linear scalarization

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1. Introduction

The concepts of proper efficient solution (see, for instance, [2, 3, 4, 7, 10, 11, 13, 20, 23, 24, 31, 33, 37]) are highly regarded by practitioners and researches in vector optimization, since by them, one obtains solutions of vector optimization problems that satisfy more suitable properties than the nondomination. For example, through notions of proper efficiency, one can search for nondominated solutions that, additionally, have bounded trade-offs [7], fulfil stability properties [4] or are solutions of associated scalar optimization problems [33].

Moreover, under certain conditions, the proper efficient solution set is dense in the set of nondominated solutions (see [6, 1, 12, 11]). From this point of view, to approximate the whole set of proper efficient solutions of a vector optimization problem is equivalent to approximate the set of nondominated solutions of the problem.

Recently, several authors have been interested in introducing and studying approximate proper efficiency notions. To the best of our knowledge, the first concepts of approximate proper efficient solutions of vector optimization problems were introduced by the end of the 1990's by Rong [36], Li and Wang [26] and Liu [27]. However, is just now, twenty years later, when this topic is taking up an important place in the theory and practice of vector optimization.

Let us underline that from the contributions that focus on scalarization techniques (see [8, 9, 16, 26, 27, 35, 36]) one can deal with some fundamental concepts and topics in optimization related to approximate solutions, as for example the ε-subdifferential, or the optimization problems with difference of vector mappings (see [14, 15, 28, 29, 30, 39]).

Since the set of proper efficient solutions of a vector optimization problem is included into the set of nondominated solutions, one hopes that approximate proper efficient solutions with a sufficient precision are close to nondominated solutions. To be precise, under the conditions that guarantee the density of the proper efficient solutions in the set of nondominated solutions, one hopes that the set of approximate proper solutions is a suitable surrogate of the whole set of nondominated solutions of the problem. However, some approximate proper efficiency concepts of the literature do not satisfy this limit behavior (see [14]).

In this work we define and study a notion of approximate proper efficiency in the Henig's sense (see [20, 28, 29]), that satisfies a suitable limit behavior when the error tends to zero. Moreover, some related topics are analyzed. To be precise, in Section 2, the framework and the main notations and concepts used throughout the paper are fixed. In particular, several concepts of exact and approximate solution of a vector optimization problem are recalled.

In Section 3, the concept of Henig (C, ε)-proper efficient solution of a vector optimization problem is introduced, where C is a nonempty set of the final space and ε is a nonnegative scalar. Several equivalent formulations are shown, and its main properties are studied. Moreover, some results for multiobjective problems describing the limit behavior when the error tends to zero of these solutions are obtained.

In Section 4, we study nondominated solutions of vector optimization problems with respect to open conic extensions of a set C. In particular, linear scalarization results are stated and, as a consequence, several relations between different notions of approximate efficiency and approximate proper efficiency are given. By these results, Henig (C,ε)-proper efficient solutions are characterized through linear scalarizations. Moreover, a weak ε-subdifferential for vector mappings is introduced and its basic properties and a Moreau-Rockafellar calculus rule are established. Finally, in Section 5, we also characterize Henig (C,ε)-proper efficient solutions via weak ε-subgradients, for convex problems and problems with difference of vector mappings, respectively.

2. Preliminaries

Let X and Y be two real locally convex Hausdorff topological vector spaces. We consider in Y a partial order given by an ordering cone D as usual, i.e.,

$$y_1, y_2 \in Y,\ y_1 \leq_D y_2 \iff y_2 - y_1 \in D.$$

We assume that D is proper ($\{0\} \neq D \neq Y$), convex, closed and pointed ($D \cap (-D) = \{0\}$). The topological dual space of Y is denoted by Y^*, and the duality pairing in Y is denoted by $\langle y^*, y\rangle$, $y^* \in Y^*$, $y \in Y$. We consider an arbitrary locally convex topology $\mathcal{T}$ on Y^* compatible with the dual pair, i.e., $(Y^*, \mathcal{T})^* = Y$ (see [32, 33]).

The nonnegative orthant of $\mathbb{R}^p$ is denoted by $\mathbb{R}^p_+$ and $\mathbb{R}_+ := \mathbb{R}^1_+$.

We define the element $+\infty_Y$ as the supremum in Y with respect to the ordering $\leq_D$, and we denote $\overline{Y} = Y \cup \{+\infty_Y\}$. In other words, it holds that $y \leq_D +\infty_Y$, for all $y \in \overline{Y}$, and if there exists $y \in \overline{Y}$ such that $+\infty_Y \leq_D y$, then $y = +\infty_Y$. The algebraic operations in Y are extended as follows:

$$+\infty_Y + y = y + (+\infty_Y) = +\infty_Y, \quad \alpha \cdot (+\infty_Y) = +\infty_Y, \quad \forall\, y \in \overline{Y},\ \forall\, \alpha > 0.$$

In particular, if $Y = \mathbb{R}$ and $D = \mathbb{R}_+$ then $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, where $+\infty := +\infty_{\mathbb{R}}$.

Given a nonempty set $F \subset Y$, we denote by $\operatorname{int} F$, $\operatorname{cl} F$, $\operatorname{co} F$ and $\operatorname{cone} F$ the topological interior, the closure, the convex hull and the cone generated by F, respectively. We assume that $0 \in \operatorname{cone} F$, i.e., $\operatorname{cone} F := \bigcup_{\alpha \geq 0} \alpha F$. We say that F is solid if $\operatorname{int} F \neq \emptyset$, and F is coradiant if $\alpha F \subset F$, $\forall\, \alpha \geq 1$. The coradiant set generated by F is denoted by $\operatorname{shw} F$ ("shadow" of F, see [41]), i.e.,

$$\operatorname{shw} F := \bigcup_{\alpha \geq 1} \alpha F.$$

Let $D' \subset Y$ be an arbitrary proper solid convex cone. We denote the conic extension of F with respect to D (resp. the open conic extension of F with respect to D') by $\mathcal{E}^D(F)$ (resp. $\mathcal{E}_0^{D'}(F)$), i.e.,

$$\mathcal{E}^D(F) := F + D \quad (\text{resp. } \mathcal{E}_0^{D'}(F) := F + \operatorname{int} D').$$

Moreover, we denote by D^+ and D^{s+} the positive and the strict positive polar cone of D, respectively, i.e.,

$$D^+ := \{\mu \in Y^* : \langle \mu, d\rangle \geq 0, \forall\, d \in D\},$$
$$D^{s+} := \{\mu \in Y^* : \langle \mu, d\rangle > 0, \forall\, d \in D\backslash\{0\}\}.$$

The set of continuous linear mappings from X to Y will be denoted by $\mathcal{L}(X, Y)$.

Given a mapping $f : X \to \overline{Y}$, we denote the effective domain of f by $\operatorname{dom} f$, i.e., $\operatorname{dom} f := \{x \in X : f(x) \in Y\}$, and we say that f is proper if $\operatorname{dom} f \neq \emptyset$. For each $y^* \in D^+\backslash\{0\}$ we assume that $\langle y^*, f(x)\rangle = +\infty$, $\forall\, x \in X, x \notin \operatorname{dom} f$. Thus, $y^* \circ f : X \to \overline{\mathbb{R}}$ and $\operatorname{dom}(y^* \circ f) = \operatorname{dom} f$.

In this work, we study the following implicitly constrained vector optimization problem:

$$\operatorname{Min}_D\{f(x) : x \in S\}, \qquad (\mathcal{P}_\mathcal{S})$$

where the objective mapping $f : X \to \overline{Y}$ is proper and the feasible set $S \subset X$ satisfies that $S_0 := S \cap \operatorname{dom} f \neq \emptyset$, in order to avoid trivial problems.

Let us recall some classical notions of exact efficient solution of problem $(\mathcal{P}_\mathcal{S})$ (see, for instance, [2, 20, 23, 24, 37]). A point $x_0 \in S_0$ is an efficient solution of problem $(\mathcal{P}_\mathcal{S})$ if there is not $x \in S_0$ such that $f(x) \leq_D f(x_0)$, $f(x) \neq f(x_0)$. The set of efficient solutions of problem $(\mathcal{P}_\mathcal{S})$ will be denoted by $\mathrm{E}(f, S, D)$.

Moreover, for D being solid, a point $x_0 \in S_0$ is a weak efficient solution of problem $(\mathcal{P}_\mathcal{S})$ if there is not a point $x \in S_0$ such that $f(x) \leq_{\operatorname{int} D\cup\{0\}} f(x_0)$, $f(x) \neq f(x_0)$. We denote the set of weak efficient solutions of $(\mathcal{P}_\mathcal{S})$ by $\mathrm{WE}(f, S, D)$.

A point $x_0 \in S_0$ is said to be a Benson proper efficient solution of problem $(\mathcal{P}_\mathcal{S})$, and we denote it by $x_0 \in \mathrm{Be}(f, S, D)$, if

$$\operatorname{cl}\operatorname{cone}(f(S_0) + D - f(x_0)) \cap (-D) = \{0\}.$$

On the other hand, a point $x_0 \in S_0$ is a proper efficient solution of $(\mathcal{P}_\mathcal{S})$ in the sense of Henig, and we denote it by $x_0 \in \mathrm{He}(f, S, D)$, if there exists a proper, solid and convex cone $D' \subset Y$ such that $D\backslash\{0\} \subset \operatorname{int} D'$ and $x_0 \in \mathrm{E}(f, S, D')$.

Remark 2.1. (a) It is easy to check that the notion of Henig proper efficient solution can be reformulated in the following equivalent way: a point $x_0 \in S_0$ is a proper efficient solution of $(\mathcal{P}_\mathcal{S})$ in the sense of Henig if there exists a proper, solid and convex cone $D' \subset Y$ such that $D\backslash\{0\} \subset \operatorname{int} D'$ and $x_0 \in \mathrm{WE}(f, S, D')$.

(b) It follows that $\mathrm{He}(f, S, D) \subset \mathrm{Be}(f, S, D) \subset \mathrm{E}(f, S, D)$ (see, for instance, [24, Proposition 2.4.6(i)]).

Concerning the approximate case, in the following definition we consider a general notion of approximate efficiency, which extends and unifies the most important concepts of approximate efficient solution of vector optimization problems introduced in the literature (see [17, 18] and the references therein).

For each nonempty set $C \subset Y\backslash\{0\}$, we define the set-valued mapping $C : \mathbb{R}_+ \to 2^Y$ as follows:

$$C(\varepsilon) := \begin{cases} \varepsilon C & \text{if } \varepsilon > 0 \\ (\operatorname{cone} C)\backslash\{0\} & \text{if } \varepsilon = 0 \end{cases}$$

and we consider the collections $\mathcal{H}, \overline{\mathcal{H}}$, $\mathcal{G}$, $\mathcal{O}$, $\mathcal{G}(C)$, $\mathcal{O}(C) \subset 2^Y$ given by

$$\begin{aligned} \mathcal{H} &:= \{\emptyset \neq C \subset Y\backslash\{0\} : C \cap (-D) = \emptyset\}, \\ \overline{\mathcal{H}} &:= \{\emptyset \neq C \subset Y\backslash\{0\} : \operatorname{cl}\operatorname{cone} C \cap (-D\backslash\{0\}) = \emptyset\}, \\ \mathcal{G} &:= \{D' \subset Y : D' \text{ is a proper solid convex cone}, D\backslash\{0\} \subset \operatorname{int} D'\}, \\ \mathcal{O} &:= \{D' \in \mathcal{G} : \operatorname{int} D' = D'\backslash\{0\}\}, \\ \mathcal{G}(C) &:= \{D' \in \mathcal{G} : C \cap (-\operatorname{int} D') = \emptyset\}, \\ \mathcal{O}(C) &:= \{D' \in \mathcal{G}(C) : \operatorname{int} D' = D'\backslash\{0\}\}. \end{aligned}$$

It is clear that $\overline{\mathcal{H}} \subset \mathcal{H}$, $D\backslash\{0\} \in \overline{\mathcal{H}}$ and $\mathcal{G}(D\backslash\{0\}) = \mathcal{G}$. Moreover, for each $D' \in \mathcal{G}(C)$ it is clear that $\mathcal{E}_0^{D'}(C) + q \in \mathcal{H}$, for all $q \in D$, $D' \in \mathcal{G}(\mathcal{E}^D(C))$ and $\operatorname{int} D' \cup \{0\} \in \mathcal{O}(C)$. On the other hand, $\operatorname{int} D' \cup \{0\} = D'$, for all $D' \in \mathcal{O}(C)$.

Lemma 2.2. *Consider a nonempty set $C \subset Y\backslash\{0\}$, $\varepsilon \geq 0$ and $D' \in \mathcal{G}$. Then,*

(a) $\mathcal{E}_0^{D'}(D\backslash\{0\}) = \operatorname{int} D'$ *and* $\mathcal{E}^D(\operatorname{int} D') = \operatorname{int} D'$.
(b) *If $D' \in \mathcal{G}(C)$, then $\mathcal{E}_0^{D'}(C)(\varepsilon) = \mathcal{E}_0^{D'}(C(\varepsilon))$.*
(c) *If $C \in \mathcal{H}$, then $\mathcal{E}^D(C)(\varepsilon) = \mathcal{E}^D(C(\varepsilon))$.*

Proof. (a) We have that

$$D\backslash\{0\} + \operatorname{int} D' \subset \operatorname{int} D' + \operatorname{int} D' = \operatorname{int} D'.$$

Reciprocally, let $d \in \operatorname{int} D'$ and consider an arbitrary $q \in D\backslash\{0\}$. There exists $\alpha > 0$ such that $d - \alpha q \in \operatorname{int} D'$, and so $d \in \alpha q + \operatorname{int} D' \subset D\backslash\{0\} + \operatorname{int} D'$. Thus, $\operatorname{int} D' \subset D\backslash\{0\} + \operatorname{int} D'$ and the first statement of part (a) is proved.

For the second one, observe that

$$\begin{aligned} \operatorname{int} D' \subset D + \operatorname{int} D' &= (D\backslash\{0\} \cup \{0\}) + \operatorname{int} D' \\ &\subset (\operatorname{int} D' \cup \{0\}) + \operatorname{int} D' \\ &= \operatorname{int} D'. \end{aligned}$$

(b) If $\varepsilon > 0$ then it is obvious that

$$(C + \operatorname{int} D')(\varepsilon) = \varepsilon(C + \operatorname{int} D') = \varepsilon C + \operatorname{int} D' = C(\varepsilon) + \operatorname{int} D',$$

since $\operatorname{int} D' \cup \{0\}$ is a cone. If $\varepsilon = 0$, as $C \cap (-\operatorname{int} D') = \emptyset$ we have $0 \notin C + \operatorname{int} D'$ and so

$$(C + \operatorname{int} D')(0) = \bigcup_{\alpha>0} \alpha(C + \operatorname{int} D') = \left(\bigcup_{\alpha>0} C(\alpha)\right) + \operatorname{int} D' = C(0) + \operatorname{int} D'.$$

The proof of part (c) is omitted, since it is similar to the previous one. □

Definition 2.3. Let $C \in \mathcal{H}$ and $\varepsilon \geq 0$. We say that $x_0 \in S_0$ is a (C, ε)-efficient solution of problem $(\mathcal{P}_S)$, denoted by $x_0 \in \mathrm{AE}(f, S, C, \varepsilon)$, if

$$(f(S_0) - f(x_0)) \cap (-C(\varepsilon)) = \emptyset. \tag{1}$$

Obviously, if $\operatorname{cone} C = D$ then $\mathrm{AE}(f, S, C, 0) = \mathrm{E}(f, S, D)$.

Remark 2.4. In Definition 2.3, we assume that $C \in \mathcal{H}$ in order to deal with a coherent set of approximate efficient solutions of problem $(\mathcal{P}_S)$. Indeed, suppose that $C \cap (-D) \neq \emptyset$, which is equivalent to $C(\varepsilon) \cap (-D) \neq \emptyset$, for all $\varepsilon \geq 0$. If $x_0 \in S_0$ is a (C, ε)-efficient solution of $(\mathcal{P}_S)$, by (1) we have in particular that

$$(f(S_0) - f(x_0)) \cap ((-C(\varepsilon)) \cap D) = \emptyset,$$

and we remove as possible approximate efficient solutions of $(\mathcal{P}_S)$ the feasible points $x_0 \in S_0$ for which there exists at least one point $\bar{x} \in S_0$ satisfying that $f(\bar{x}) - f(x_0) \in (-C(\varepsilon)) \cap D \subset D \backslash \{0\}$.

This last inclusion implies in particular that $f(x_0) \leq_D f(\bar{x})$, so by choosing a set $C \subset Y \backslash \{0\}$ with $C \cap (-D) \neq \emptyset$, we are following a pointless procedure for dealing with a suitable approximate efficient set. This situation is illustrated in the following example.

Example 2.5. Consider $X = Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$ and $C \subset \mathbb{R}^2 \backslash \{(0,0)\}$ such that $C \cap (-\mathbb{R}^2_+) \neq \emptyset$. Denote by $\mathbb{B}(0,1)$ the Euclidean unit closed ball in $\mathbb{R}^2$, and let $S = \mathbb{B}(0,1) + \mathbb{R}^2_+$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity mapping. It is easy to see that

$$\mathrm{E}(f, S, D) = \mathbb{S}(0,1) \cap (-\mathbb{R}^2_+),$$

where $\mathbb{S}(0,1)$ denotes the Euclidean unit sphere in $\mathbb{R}^2$. Let $\bar{d} \in C \cap (-\mathbb{R}^2_+)$. If $\varepsilon > 0$ and $x_0 \in \mathrm{AE}(f, S, C, \varepsilon)$, by Remark 2.4 we know that $f(x_0) - \varepsilon \bar{d} \notin \mathbb{B}(0,1) + \mathbb{R}^2_+$, which is a contradiction. As $\mathrm{AE}(f, S, C, 0) \subset \mathrm{AE}(f, S, C, \varepsilon)$ for all $\varepsilon > 0$ it follows that

$$\mathrm{AE}(f, S, C, \varepsilon) = \emptyset, \quad \forall \varepsilon \geq 0,$$

which shows that, in this case, the (C, ε)-efficiency concept is worthless.

With respect to the approximate proper efficiency, several concepts have been introduced in the literature. Next we recall some of them. The first one was introduced by Rong [36] and it is based on the Benson proper efficiency concept.

The second one was given by El Maghri [28] and it is motivated by the proper efficiency in the sense of Henig. Both notions can be considered as proper versions of an ε-efficiency notion due to Kutateladze [25].

In the third definition, we recall the concept of Benson (C,ε)-proper efficiency introduced by Gutiérrez, Huerga and Novo in [16], which can be considered as a proper version in the sense of Benson of the (C,ε)-efficiency notion.

Definition 2.6. Let $q \in D\backslash\{0\}$ and $\varepsilon \geq 0$. A point $x_0 \in S_0$ is a Benson ε-proper solution of problem $(\mathcal{P}_\mathcal{S})$ with respect to q, denoted by $x_0 \in \mathrm{Be}(f,S,q,\varepsilon)$, if

$$\operatorname{cl}\operatorname{cone}(f(S_0)+\varepsilon q + D - f(x_0)) \cap (-D) = \{0\}.$$

Definition 2.7. Consider $q \in Y\backslash\{0\}$ and $\varepsilon \geq 0$. A point $x_0 \in S_0$ is a Henig ε-proper solution of problem $(\mathcal{P}_\mathcal{S})$ with respect to q, denoted by $x_0 \in \mathrm{He}(f,S,q,\varepsilon)$, if there exists $D' \in \mathcal{G}$ such that

$$(f(S_0)+\varepsilon q - f(x_0)) \cap (-D'\backslash\{0\}) = \emptyset. \tag{2}$$

Definition 2.8. Let $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. A point $x_0 \in S_0$ is a Benson (C,ε)-proper solution of $(\mathcal{P}_\mathcal{S})$, denoted by $x_0 \in \mathrm{Be}(f,S,C,\varepsilon)$, if

$$\operatorname{cl}\operatorname{cone}(f(S_0)+C(\varepsilon) - f(x_0)) \cap (-D) = \{0\}.$$

Remark 2.9. (a) As in Remark 2.1(a), it is easy tho check that Definition 2.7 does not change if we consider the condition

$$(f(S_0)+\varepsilon q - f(x_0)) \cap (-\operatorname{int} D') = \emptyset$$

instead of (2).

(b) It follows that

$$\mathrm{He}(f,S,q,\varepsilon) \subset \mathrm{Be}(f,S,q,\varepsilon), \quad \forall\, q \in D\backslash\{0\}, \forall\, \varepsilon \geq 0.$$

(c) Observe that condition $C \in \overline{\mathcal{H}}$ in Definition 2.8 is needed in order to obtain a nonempty set of Benson (C,ε)-proper solutions of problem $(\mathcal{P}_\mathcal{S})$.

(d) In particular, if $\operatorname{cl}\operatorname{cone} C = D$, then $\mathrm{Be}(f,S,C,0) = \mathrm{Be}(f,S,D)$ (see Remark 2.4(a) in [14]), and if $C = \mathcal{E}^D(q)$, $q \in D\backslash\{0\}$, it follows that $\mathrm{Be}(f,S,C,\varepsilon) = \mathrm{Be}(f,S,q,\varepsilon)$, for all $\varepsilon \geq 0$.

(e) In [14, Section 3], the authors proved that for suitable sets C, the set of Benson (C,ε)-proper solutions has a good limit behavior when the precision ε tends to zero, in the sense that all its elements are close to the efficient set. This fundamental property, does not usually hold when one considers Benson ε-proper solutions of problem $(\mathcal{P}_\mathcal{S})$ with respect to a vector q (see [14, Example 3.10] and Remark 4.9(b)), due essentially to the fact that in this latter concept the error is quantified by a unique vector. Because of this, the notion of Benson (C,ε)-proper efficiency gains importance. For a more complete study of this type of solutions see [14, 15, 16].

Given a proper mapping $h : X \to \overline{\mathbb{R}}$, a nonempty set $M \subset X$, with $M_0 := M \cap \operatorname{dom} h \neq \emptyset$, and $\varepsilon \geq 0$, we denote

$$\varepsilon\text{-}\operatorname{argmin}_M h = \{x \in M : h(x) \leq h(z) + \varepsilon, \forall\, z \in M\}.$$

The elements of ε-$\operatorname{argmin}_M h$ are the suboptimal solutions with error ε (exact solutions if $\varepsilon = 0$) of the following scalar optimization problem:

$$\operatorname{Min}\{h(x) : x \in M\}.$$

On the other hand, let us recall that the mapping f is said to be D-convex on a set $M \subset X$ if M is convex and

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_D \lambda f(x_1) + (1-\lambda) f(x_2), \quad \forall\, x_1, x_2 \in M,\ \forall\, \lambda \in (0,1),$$

and f is said to be D-convexlike on M if $f(M_0) + D$ is convex. Notice that D-convexity of f on M implies D-convexlikeness of f on M.

The following generalized convexity notion introduced in [16] (see also [14]) is an "approximate" version of the well-known nearly subconvexlikeness concept (see [40]).

Definition 2.10. Consider $\varepsilon \geq 0$ and two nonempty sets $M \subset X$ and $C \subset Y\backslash\{0\}$. The mapping f is said to be nearly (C,ε)-subconvexlike on M if $\operatorname{cl}\operatorname{cone}(f(M_0) + C(\varepsilon))$ is convex.

In [14, Theorem 2.7] one can find sufficient conditions that guarantee the nearly (C,ε)-subconvexlikeness of the mapping f. In the following result we recall one of them for the convenience of the reader.

Theorem 2.11. *Let $\varepsilon \geq 0$ and two nonempty sets $M \subset X$ and $C \subset Y\backslash\{0\}$. If C is convex, $\mathcal{E}^D(C) = C$ and f is D-convexlike on M, then f is nearly (C,ε)-subconvexlike on M.*

Also, for a nonempty set $C \subset Y\backslash\{0\}$, we define the mapping $\tau_C : Y^* \to \mathbb{R} \cup \{-\infty\}$ as follows:

$$\tau_C(y^*) = \inf_{y \in C}\{\langle y^*, y\rangle\}, \quad \forall\, y^* \in Y^*.$$

Notice that $\tau_C = -\sigma_{-C}$, where $\sigma_C : Y^* \to \mathbb{R} \cup \{+\infty\}$ denotes the so-called support function (see, for instance, [42]). Moreover, we denote $C^+ := (\operatorname{cone} C)^+$.

For any nonempty sets $C_1, C_2 \subset Y\backslash\{0\}$ and reals $\alpha_1, \alpha_2 \in \mathbb{R}_+$, it is clear that

$$\tau_{C_1(\alpha_1)+C_2(\alpha_2)}(y^*) = \alpha_1 \tau_{C_1}(y^*) + \alpha_2 \tau_{C_2}(y^*), \quad \forall\, y^* \in C_1^+ \cap C_2^+.$$

Also, let us define the sets $\mathcal{F}_{D^{s+}}, \mathcal{F}_{D^+} \subset 2^Y$ as

$$\mathcal{F}_{D^{s+}} := \{\emptyset \neq C \subset Y\backslash\{0\} : D^{s+} \cap C^+ \neq \emptyset\},$$
$$\mathcal{F}_{D^+} := \{\emptyset \neq C \subset Y\backslash\{0\} : (D^+ \cap C^+)\backslash\{0\} \neq \emptyset\}.$$

It is clear that $\mathcal{F}_{D^{s+}} \subset \overline{\mathcal{H}} \cap \mathcal{F}_{D^+}$. Furthermore, if C is convex and $\operatorname{int} D^+ \neq \emptyset$, by [3, Proposition 2] we deduce that $C \in \mathcal{F}_{D^{s+}}$ if and only if $C \in \overline{\mathcal{H}}$.

3. Henig approximate proper efficiency

In this section, we study the following new notion of approximate proper efficiency in the sense of Henig.

Definition 3.1. Let $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. A point $x_0 \in S_0$ is a (C, ε)-proper efficient solution of problem $(\mathcal{P}_\mathcal{S})$ in the sense of Henig, denoted by $x_0 \in \mathrm{He}(f, S, C, \varepsilon)$, if there exists $D' \in \mathcal{G}(C)$ such that $x_0 \in \mathrm{AE}(f, S, \mathcal{E}_0^{D'}(C), \varepsilon)$.

Next, we clarify this notion.

Remark 3.2. (a) By Lemma 2.2(a) it is clear that

$$\mathrm{WE}(f, S, D') = \mathrm{AE}(f, S, \mathcal{E}_0^{D'}(D\backslash\{0\}), 1), \quad \forall\, D' \in \mathcal{G}.$$

Then, by Remark 2.1(a) we have that

$$\mathrm{He}(f, S, D) = \mathrm{He}(f, S, D\backslash\{0\}, 1)$$

and so, the notion of Henig approximate proper efficient solution introduced in Definition 3.1 reduces to the well-known exact concept by considering $C = D\backslash\{0\}$ and $\varepsilon = 1$.

(b) For each $q \in D\backslash\{0\}$ we have that $\{q\} \in \overline{\mathcal{H}}$, $\mathcal{G}(\{q\}) = \mathcal{G}$ and Definitions 3.1 and 2.7 coincide, i.e., $\mathrm{He}(f, S, \{q\}, \varepsilon) = \mathrm{He}(f, S, q, \varepsilon)$.

(c) Suppose that $D' \in \mathcal{G}$. Then condition $0 \notin \mathcal{E}_0^{D'}(C)$ is equivalent to $D' \in \mathcal{G}(C)$, and by Lemma 2.2(a) it follows that $\mathcal{E}^D(\mathrm{int}\, D') = \mathrm{int}\, D'$ and then

$$\mathcal{E}_0^{D'}(C) \in \mathcal{H} \iff D' \in \mathcal{G}(C).$$

Thus, condition $D' \in \mathcal{G}(C)$ is given in order to obtain a suitable set of approximate efficient solutions of problem $(\mathcal{P}_\mathcal{S})$ (see Remark 2.4 and Example 2.5). In particular, observe that

$$D' \in \mathcal{G}(C) \Rightarrow \mathrm{cl\, cone}\, C \cap (-D\backslash\{0\}) = \emptyset.$$

Because of this, we assume $C \in \overline{\mathcal{H}}$ in Definition 3.1.

(d) We have that

$$\begin{aligned} \mathrm{He}(f, S, C, \varepsilon) &= \bigcup_{D' \in \mathcal{G}(C)} \mathrm{AE}(f, S, \mathcal{E}_0^{D'}(C), \varepsilon) \\ &\subset \mathrm{AE}(f, S, C + D\backslash\{0\}, \varepsilon), \quad \forall\, C \in \overline{\mathcal{H}}. \end{aligned}$$

In the next result, we state four equivalent formulations for Henig (C, ε)-proper efficient solutions of problem $(\mathcal{P}_\mathcal{S})$.

Theorem 3.3. *Let $\varepsilon \geq 0$, $C \in \overline{\mathcal{H}}$ and $x_0 \in S_0$. The following statements are equivalent.*

(a) $x_0 \in \mathrm{He}(f, S, C, \varepsilon)$.

(b) *There exists* $D' \in \mathcal{O}(C)$ *such that* $x_0 \in AE(f, S, \mathcal{E}_0^{D'}(C), \varepsilon)$.

(c) *There exists* $D' \in \mathcal{O}(C)$ *such that*

$$\operatorname{cl}\operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-\operatorname{int} D') = \emptyset.$$

(d) *There exists* $D' \in \mathcal{O}(C)$ *such that*

$$(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-\operatorname{int} D') = \emptyset. \tag{3}$$

(e) $\mathcal{E}^D(C) \in \overline{\mathcal{H}}$ *and* $x_0 \in \mathrm{He}(f, S, \mathcal{E}^D(C), \varepsilon)$.

Proof. $(a) \Rightarrow (b)$. Consider a point $x_0 \in \mathrm{He}(f, S, C, \varepsilon)$. Then there exists $\bar{D} \in \mathcal{G}(C)$ such that $x_0 \in \mathrm{AE}(f, S, \mathcal{E}_0^{\bar{D}}(C), \varepsilon)$. Let us check that $D' := \operatorname{int} \bar{D} \cup \{0\}$ satisfies statement (b). Indeed, it is clear that $\operatorname{int} D' = \operatorname{int} \bar{D}$ and since $\bar{D} \in \mathcal{G}(C)$ we have that $D' \in \mathcal{O}(C)$. Therefore, $x_0 \in \mathrm{AE}(f, S, \mathcal{E}_0^{D'}(C), \varepsilon)$ and part (b) follows.

$(b) \Rightarrow (d)$. Let $D' \in \mathcal{O}(C)$ be such that $x_0 \in \mathrm{AE}(f, S, \mathcal{E}_0^{D'}(C), \varepsilon)$. Then,

$$(f(S_0) - f(x_0)) \cap (-\mathcal{E}_0^{D'}(C)(\varepsilon)) = \emptyset$$

and by Lemma 2.2(b) we deduce that

$$(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-\operatorname{int} D') = \emptyset.$$

Thus, statement (d) is satisfied.

$(d) \Leftrightarrow (c)$. Let $D' \in \mathcal{O}(C)$. As $\operatorname{int} D' \cup \{0\}$ is a cone we see that (3) is equivalent to

$$\operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-\operatorname{int} D') = \emptyset$$

and so

$$\operatorname{cl}\operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-\operatorname{int} D') = \emptyset,$$

which proves that parts (c) and (d) are equivalent.

$(d) \Rightarrow (e)$. By (3) we obtain

$$(f(S_0) - f(x_0)) \cap (-\mathcal{E}_0^{D'}(C(\varepsilon))) = \emptyset. \tag{4}$$

As $D' \in \mathcal{O}(C)$ we see that $C \cap (-D') = \emptyset$ and so $\mathcal{E}^D(C) \cap (-D') = \emptyset$, since $D \subset D'$. Thus,

$$\operatorname{cl}\left(\mathcal{E}^D(C)(0)\right) \cap (-D \backslash \{0\}) \subset \operatorname{cl}\left(\mathcal{E}^D(C)(0)\right) \cap (-\operatorname{int} D') = \emptyset$$

and we obtain that $\mathcal{E}^D(C) \in \overline{\mathcal{H}}$.

On the other hand, by parts (a)–(c) of Lemma 2.2 we have

$$\mathcal{E}_0^{D'}(C(\varepsilon)) = \mathcal{E}_0^{D'}(C)(\varepsilon) = \mathcal{E}_0^{D'}(\mathcal{E}^D(C))(\varepsilon)$$

and then (4) implies that $x_0 \in \mathrm{AE}(f, S, \mathcal{E}_0^{D'}(\mathcal{E}^D(C)), \varepsilon)$. Therefore, $x_0 \in \mathrm{He}(f, S, \mathcal{E}^D(C), \varepsilon)$ and statement (e) is verified.

Finally, $(e) \Rightarrow (a)$ follows directly by the definition and Lemma 2.2, and the proof finishes. □

Remark 3.4. (a) Observe that by Theorem 3.3(e), for each $C \in \overline{\mathcal{H}}$ and $\varepsilon \geq 0$ we have $\mathcal{E}^D(C) \in \overline{\mathcal{H}}$ whenever $\mathrm{He}(f, S, C, \varepsilon) \neq \emptyset$ and, in this case,

$$\mathrm{He}(f, S, C, \varepsilon) = \mathrm{He}(f, S, \mathcal{E}^D(C), \varepsilon).$$

Then, in Definition 3.1 one can assume without loss of generality that C coincides with its conic extension with respect to D. The sets $C \subset Y \backslash \{0\}$ satisfying this latter property are called improvement sets (see, for instance, [19]).

(b) Theorem 3.3(c) let us relate Benson (C, ε)-proper efficient solutions to Henig (C, ε)-proper efficient solutions of problem $(\mathcal{P}_{\mathcal{S}})$ (see Theorem 4.7 and Corollary 4.8).

By applying Theorem 3.3 in the case when $C = D\backslash\{0\}$ and $\varepsilon = 1$ we obtain the following equivalent formulations for the exact Henig proper efficiency of problem $(\mathcal{P}_{\mathcal{S}})$ (see Remark 3.2(a)).

Corollary 3.5. *Let $x_0 \in S_0$. The next statements are equivalent:*

(a) $x_0 \in \mathrm{He}(f, S, D)$.

(b) *There exists $D' \in \mathcal{O}$ such that $x_0 \in E(f, S, D')$.*

(c) *There exists $D' \in \mathcal{O}$ such that*

$$\mathrm{cl}\,\mathrm{cone}(f(S_0) + D - f(x_0)) \cap (-\,\mathrm{int}\, D') = \emptyset.$$

(d) *There exists $D' \in \mathcal{O}$ such that*

$$(f(S_0) + D\backslash\{0\} - f(x_0)) \cap (-\,\mathrm{int}\, D') = \emptyset.$$

Next, we show several properties of the set of Henig (C, ε)-proper efficient solutions of problem $(\mathcal{P}_{\mathcal{S}})$ when the error ε is fixed.

Theorem 3.6. *Let $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. The following properties are satisfied:*

(a) *Suppose that $0 \notin \mathrm{cl}\, C$ and consider a nonempty set $C' \subset D\backslash\{0\}$ such that $\mathrm{cone}\, C + \mathrm{cone}\, C'$ is closed. Then $\mathrm{cl}\, C \in \overline{\mathcal{H}}$, $C + \mathrm{cone}\, C' \in \overline{\mathcal{H}}$, $\mathcal{G}(C) = \mathcal{G}(\mathrm{cl}\, C) = \mathcal{G}(C + \mathrm{cone}\, C')$ and*

$$\mathrm{He}(f, S, C, \varepsilon) = \mathrm{He}(f, S, \mathrm{cl}\, C, \varepsilon) = \mathrm{He}(f, S, C + \mathrm{cone}\, C', \varepsilon). \tag{5}$$

(b) $\mathrm{He}(f, S, C', \delta) \subset \mathrm{He}(f, S, C, \varepsilon)$, *for all $C' \in \overline{\mathcal{H}}$ and $\delta \geq 0$ such that $C(\varepsilon) \subset \mathrm{cl}(C'(\delta))$.*

(c) *If $\mathcal{E}^D(C)$ is coradiant, then* $\mathrm{He}(f, S, C, \delta) \subset \mathrm{He}(f, S, C, \varepsilon)$, *for all $0 < \delta \leq \varepsilon$.*

(d) $\mathrm{He}(f, S, C + C', \varepsilon) = \mathrm{He}(f, S, \mathrm{cl}\, C + C', \varepsilon)$, *for all nonempty set $C' \subset Y$ such that $\mathrm{cl}\, C + C' \in \overline{\mathcal{H}}$.*

(e) *If C is convex and coradiant, then* $\mathrm{He}(f, S, C, \varepsilon) = \mathrm{He}(f, S, C + C(0), \varepsilon)$.

(f) *If $C \subset D\backslash\{0\}$, then* $\mathrm{He}(f, S, D) \subset \mathrm{He}(f, S, C, \varepsilon)$.

(g) *If $\mathrm{cl}\,\mathrm{cone}\, C = D$, then* $\mathrm{He}(f, S, C, 0) = \mathrm{He}(f, S, D)$.

Proof. First, observe that given two nonempty sets $F, C' \subset Y$ and $\delta \geq 0$ such that $C(\varepsilon) \subset \mathrm{cl}(C'(\delta))$, it follows that

$$\begin{aligned} \mathrm{cl}\,\mathrm{cone}(F + \mathrm{cl}\, C(\varepsilon)) &= \mathrm{cl}\,\mathrm{cone}(F + C(\varepsilon)) \subset \mathrm{cl}\,\mathrm{cone}(F + \mathrm{cl}(C'(\delta))) &(6)\\ &= \mathrm{cl}\,\mathrm{cone}(F + C'(\delta)) &(7)\end{aligned}$$

and

$$\mathrm{cl}\,\mathrm{cone}(F + (C + C')(\varepsilon)) = \mathrm{cl}\,\mathrm{cone}(F + (\mathrm{cl}\, C + C')(\varepsilon)). \tag{8}$$

(a) Let us prove (5), since the other statements of this part follow easily by the definitions. In view of Theorem 3.3(c) and statement (6) we deduce that

$$\mathrm{He}(f, S, C, \varepsilon) = \mathrm{He}(f, S, \mathrm{cl}\, C, \varepsilon).$$

On the other hand, as $\mathcal{E}_0^{D'}(\mathrm{cone}\, C') = \mathrm{int}\, D'$ for all $D' \in \mathcal{G}$, we have $\mathcal{E}_0^{D'}(C) = \mathcal{E}_0^{D'}(C + \mathrm{cone}\, C')$ for all $D' \in \mathcal{G}(C)$. Then,

$$\mathrm{AE}(f, S, \mathcal{E}_0^{D'}(C), \varepsilon) = \mathrm{AE}(f, S, \mathcal{E}_0^{D'}(C + \mathrm{cone}\, C'), \varepsilon), \quad \forall\, D' \in \mathcal{G}(C)$$

and $\mathrm{He}(f, S, C, \varepsilon) = \mathrm{He}(f, S, C + \mathrm{cone}\, C', \varepsilon)$.

(b) It is not hard to check that $\mathcal{O}(C') \subset \mathcal{O}(C)$. Then the result is a direct consequence of statements (6), (7) and Theorem 3.3(c).

(c) Let $0 < \delta \leq \varepsilon$. As $\mathcal{E}^D(C)$ is coradiant we have $(\varepsilon/\delta)\mathcal{E}^D(C) \subset \mathcal{E}^D(C)$ and so

$$C(\varepsilon) \subset \mathcal{E}^D(C)(\varepsilon) \subset \mathcal{E}^D(C)(\delta).$$

Suppose that $\mathrm{He}(f, S, C, \delta) \neq \emptyset$, since the result is obvious otherwise. Thus, by applying part (b) with $C' = \mathcal{E}^D(C)$ and Remark 3.4(a) we see that $\mathcal{E}^D(C) \in \overline{\mathcal{H}}$ and

$$\mathrm{He}(f, S, C, \delta) = \mathrm{He}(f, S, \mathcal{E}^D(C), \delta) \subset \mathrm{He}(f, S, C, \varepsilon).$$

(d) It is easy to check that $\mathcal{O}(\mathrm{cl}\, C + C') = \mathcal{O}(C + C')$. Then this part follows by statement (8) and Theorem 3.3(c).

(e) By [17, Lemma 3.1(v)] we have that

$$C + C(0) \subset C \subset \mathrm{cl}(C + C(0)).$$

By these inclusions it follows easily that $C + C(0) \in \overline{\mathcal{H}}$ and $\mathcal{O}(C) = \mathcal{O}(C + C(0))$. Then the result is a direct consequence of (6) and Theorem 3.3(c).

(f) It is deduced easily from Remark 3.2(a) by applying part (b) with $C' = D\backslash\{0\}$ and $\delta = 1$.

(g) Part (f) with $\varepsilon = 0$ implies that

$$\mathrm{He}(f, S, D) \subset \mathrm{He}(f, S, C, 0).$$

Reciprocally, by Remark 3.2(a) and part (b) with $D\backslash\{0\}$ instead of C, $C' = C$, $\delta = 0$ and $\varepsilon = 1$ we see that

$$\mathrm{He}(f, S, C, 0) \subset \mathrm{He}(f, S, D\backslash\{0\}, 1) = \mathrm{He}(f, S, D)$$

and the proof is complete. $\square$

In the rest of this section we assume that $(\mathcal{P}_\mathcal{S})$ is a multiobjective optimization problem (i.e., $Y = \mathbb{R}^p$), and we state some results in order to illustrate the limit behavior of Henig (C,ε)-proper efficient solutions when ε tends to zero and the set C is suitable.

The following lemma is needed. It collects some properties of the so-called Henig dilating cones. Given $\xi \in D^{s+}$, we denote

$$B_\xi := \{d \in D : \langle \xi, d\rangle = 1\},$$
$$D_\xi^\delta := \operatorname{cone}(B_\xi + \delta\mathbb{B}), \quad \forall\, \delta \in (0, r_\xi),$$

where $\mathbb{B}$ denotes the closed unit ball of $\mathbb{R}^p$, and $r_\xi := \inf\{\|b\| : b \in B_\xi\}$. It is easy to check that B_ξ is compact and $D_\xi^\delta \in \mathcal{G}$, for all $\delta \in (0, r_\xi)$ (see [11, Lemma 3.2.51]). Moreover, the Painlevé-Kuratowski upper limit of a set-valued mapping $F : \mathbb{R}_+ \rightrightarrows \mathbb{R}^p$ at $\varepsilon = 0$ is denoted by $\operatorname{Limsup}_{\varepsilon\to 0} F(\varepsilon)$, i.e.,

$$\operatorname*{Limsup}_{\varepsilon\to 0} F(\varepsilon) := \{y_0 \in \mathbb{R}^p : \text{there exist sequences } (y_n) \subset \mathbb{R}^p, (\varepsilon_n) \subset \mathbb{R}_+\backslash\{0\},$$
$$\varepsilon_n \to 0, \text{ such that } y_n \in F(\varepsilon_n), y_n \to y_0\}.$$

Lemma 3.7. *Suppose that $Y = \mathbb{R}^p$ and consider $\xi \in D^{s+}$.*

(*a*) *For each $K \in \mathcal{G}$, there exists $\delta \in (0, r_\xi)$ such that $D_\xi^\delta\backslash\{0\} \subset \operatorname{int} K$.*

(*b*) *Consider an arbitrary mapping $\delta : \mathbb{R}_+\backslash\{0\} \to (0, r_\xi)$. Then,*

$$\operatorname*{Limsup}_{\varepsilon\to 0}(D \cap (Y\backslash(\mathcal{E}_0^{D_\xi^{\delta(\varepsilon)}}(B_\xi)(\varepsilon)))) = \{0\}.$$

Proof. (*a*) If $K \in \mathcal{G}$, then $B_\xi \subset D\backslash\{0\} \subset \operatorname{int} K$. Suppose that $B_\xi + \delta\mathbb{B} \not\subset \operatorname{int} K$, for all $\delta \in (0, r_\xi)$. Then, for each $n \in \mathbb{N}$ there exist $d_n \in B_\xi$ and $v_n \in \mathbb{B}$ such that $d_n + (r_\xi/n)v_n \in Y\backslash \operatorname{int} K$. As B_ξ is compact, we can suppose without loss of generality that $d_n \to d \in B_\xi$. Therefore, $d \in \operatorname{int} K$. On the other hand, $Y\backslash \operatorname{int} K \ni d_n + (r_\xi/n)v_n \to d$, and so $d \notin \operatorname{int} K$, which is a contradiction. Then there exists $\delta \in (0, r_\xi)$ such that $B_\xi + \delta\mathbb{B} \subset \operatorname{int} K$ and $D_\xi^\delta\backslash\{0\} \subset \operatorname{int} K$.

(*b*) Consider a point $y \in \mathbb{R}^p$ and sequences $(y_n) \subset \mathbb{R}^p$ and $(\varepsilon_n) \subset \mathbb{R}_+\backslash\{0\}$, $\varepsilon_n \to 0$, such that $y_n \to y$, $y_n \in D \cap (Y\backslash(\mathcal{E}_0^{D_\xi^{\delta_n}}(B_\xi)(\varepsilon_n)))$, for all n, where $\delta_n := \delta(\varepsilon_n)$, for all n. It follows that $\langle \xi, y_n\rangle \le \varepsilon_n$, for all n. Indeed, if there exists n such that $\langle \xi, y_n\rangle > \varepsilon_n$, then by considering $z_n := (\varepsilon_n/\langle \xi, y_n\rangle)y_n \in B_\xi(\varepsilon_n)$ we have that

$$y_n = z_n + \left(1 - \frac{\varepsilon_n}{\langle \xi, y_n\rangle}\right) y_n \in B_\xi(\varepsilon_n) + D\backslash\{0\} \subset \mathcal{E}_0^{D_\xi^{\delta_n}}(B_\xi)(\varepsilon_n),$$

that is a contradiction. Then,

$$0 \le \langle \xi, y_n\rangle \le \varepsilon_n \to 0$$

and we obtain that $\langle \xi, y\rangle = 0$. As $y \in D$ we conclude that $y = 0$ and the proof is complete. $\square$

Theorem 3.8. *Suppose that $Y = \mathbb{R}^p$ and consider $\xi \in D^{s+}$. Let $x_0 \in S_0$ and $(x_n) \subset S_0$. Suppose that $x_n \in \mathrm{He}(f, S, B_\xi, \varepsilon_n)$, for all n, where $(\varepsilon_n) \subset \mathbb{R}_+\backslash\{0\}$, $\varepsilon_n \to 0$, $f(x_n) \to f(x_0)$ and $f(x_{n+1}) \leq_D f(x_n)$, $\forall\, n$. Then, $x_0 \in \mathrm{E}(f, S, D)$.*

Proof. Suppose by reasoning to the contrary that there exists $x \in S$ such that

$$f(x) - f(x_0) \in -D\backslash\{0\}.$$

As $x_n \in \mathrm{He}(f, S, B_\xi, \varepsilon_n)$ there exists a sequence $(D_n) \subset \mathcal{G}(B_\xi)$ such that $x_n \in \mathrm{AE}(f, S, \mathcal{E}_0^{D_n}(B_\xi), \varepsilon_n)$, for all n, and so

$$f(x) - f(x_n) \in Y\backslash(-\mathcal{E}_0^{D_n}(B_\xi)(\varepsilon_n)), \quad \forall\, n.$$

By Lemma 3.7(a), there exists a sequence $(\delta_n) \subset (0, r_\xi)$ such that $D_\xi^{\delta_n} \subset D_n$, for all n, and then

$$f(x) - f(x_n) \in Y\backslash(-\mathcal{E}_0^{D_\xi^{\delta_n}}(B_\xi)(\varepsilon_n)), \quad \forall\, n. \tag{9}$$

On the other hand, since the sequence $(f(x_n))$ is nonincreasing, D is closed and $f(x_n) \to f(x_0)$, it follows that

$$\begin{aligned} f(x) - f(x_n) &= (f(x) - f(x_0)) + (f(x_0) - f(x_n)) \\ &= f(x) - f(x_0) + \lim_{m\to\infty} (f(x_{n+m}) - f(x_n)) \\ &\in -D\backslash\{0\} - D = -D\backslash\{0\}. \end{aligned} \tag{10}$$

Then, by taking into account statements (9) and (10) and by applying Lemma 3.7(b) to an arbitrary mapping $\delta : \mathbb{R}_+\backslash\{0\} \to (0, r_\xi)$ such that $\delta(\varepsilon_n) = \delta_n$ we deduce that

$$\begin{aligned} f(x) - f(x_0) &= \lim_{n\to\infty} (f(x) - f(x_n)) \\ &\quad \in -\operatorname*{Limsup}_{\varepsilon\to 0} (D \cap (Y\backslash(\mathcal{E}_0^{D_\xi^{\delta(\varepsilon)}}(B_\xi)(\varepsilon)))) = \{0\}, \end{aligned}$$

that is a contradiction. Thus, $x_0 \in \mathrm{E}(f, S, D)$ and the proof finishes. □

Theorem 3.9. *Let $Y = \mathbb{R}^p$, $\xi \in D^{s+}$ and $(\varepsilon_n) \subset \mathbb{R}_+\backslash\{0\}$, $\varepsilon_n \to 0$. Then, there exists $n_0 \in \mathbb{N}$ such that*

$$\mathrm{He}(f, S, D) \subset \bigcap_{n\geq n_0} \mathrm{He}(f, S, B_\xi, \varepsilon_n) \subset \mathrm{E}(f, S, D).$$

Proof. Let $x_0 \in \mathrm{He}(f, S, D)$. Then, there exists $D' \in \mathcal{G}$ such that

$$(f(S_0) - f(x_0)) \cap (-\operatorname{int} D') = \emptyset.$$

By Lemma 3.7(a) there exists $\delta \in (0, r_\xi)$ such that $D_\xi^\delta \backslash \{0\} \subset \operatorname{int} D'$. On the other hand, since $\varepsilon_n \to 0$, there exists $n_0 \in \mathbb{N}$ such that $\varepsilon_n \leq \delta$, for all $n \geq n_0$. Thus, $D_\xi^{\varepsilon_n} \subset D_\xi^\delta$, for all $n \geq n_0$. Moreover, observe that

$$\varepsilon_n B_\xi = \varepsilon_n (B_\xi + 0) \subset \operatorname{cone}(B_\xi + \varepsilon_n \mathbb{B}) = D_\xi^{\varepsilon_n}, \quad \forall n \geq n_0.$$

Thus,

$$\varepsilon_n B_\xi + D_\xi^{\varepsilon_n} \subset D_\xi^{\varepsilon_n} \subset D_\xi^\delta \subset \operatorname{int} D' \cup \{0\}, \quad \forall n \geq n_0,$$

and we have in particular that

$$(f(S_0) - f(x_0)) \cap (-\mathcal{E}_0^{D_\xi^{\varepsilon_n}}(B_\xi)(\varepsilon_n)) = \emptyset, \quad \forall n \geq n_0,$$

which implies that $x_0 \in \operatorname{He}(f, S, B_\xi, \varepsilon_n)$, for all $n \geq n_0$, and the first inclusion is proved.

For the second inclusion, let $x_0 \in \operatorname{He}(f, S, B_\xi, \varepsilon_n)$, for all $n \geq n_0$, and consider the sequence $(x_n) \subset S$, $x_n := x_0$ for all n. It is clear that the assumptions of Theorem 3.8 are satisfied. Then $x_0 \in \operatorname{E}(f, S, D)$ and the proof is complete. □

In the following simple example we illustrate Theorems 3.8 and 3.9.

Example 3.10. Consider problem $(\mathcal{P}_\mathcal{S})$ with the data $X = Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $f(y_1, y_2) = (y_1, y_2)$, for all $(y_1, y_2) \in \mathbb{R}^2$ and

$$S = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0, y_2 \geq 0, \max\{y_1, y_2\} \geq 1\}.$$

We have that

$$\operatorname{E}(f, S, D) = \operatorname{He}(f, S, D) = \{(1, 0), (0, 1)\}.$$

Let $q \in \operatorname{int} \mathbb{R}^2_+$ and $\varepsilon \in (0, 1)$. It is not hard to check that

$$\operatorname{He}(f, S, q, \varepsilon) = \left(\operatorname{bd} S + \bigcup_{0 \leq \delta < \varepsilon} \{\delta q\} \right) \cup \{(1, 0) + \varepsilon q, (0, 1) + \varepsilon q\}$$

and so, for each sequence $(\varepsilon_n) \subset \mathbb{R}_+ \backslash \{0\}$, $\varepsilon_n \to 0$, it follows that

$$\operatorname*{Limsup}_{\varepsilon \to 0} \operatorname{He}(f, S, q, \varepsilon) = \bigcap_n \operatorname{He}(f, S, q, \varepsilon_n) = \operatorname{bd} S.$$

Analogously, if $q = (0, 1)$,

$$\begin{aligned} \operatorname{He}(f, S, q, \varepsilon) = {} & \{(y_1, y_2) \in \mathbb{R}^2 : 0 \leq y_1 < 1, 1 \leq y_2 < 1 + \varepsilon\} \cup \{(0, 1 + \varepsilon)\} \\ & \cup \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 1, 0 \leq y_2 < \varepsilon\} \cup \{(1, \varepsilon)\} \end{aligned}$$

and, for each sequence $(\varepsilon_n) \subset \mathbb{R}_+ \backslash \{0\}$, $\varepsilon_n \to 0$, we obtain that

$$\begin{aligned} \operatorname*{Limsup}_{\varepsilon \to 0} \operatorname{He}(f, S, q, \varepsilon) &= \left(\bigcap_n \operatorname{He}(f, S, q, \varepsilon_n) \right) \cup \{(1, 1)\} \\ &= \{(t, 1) \in \mathbb{R}^2 : 0 \leq t \leq 1\} \cup \{(t, 0) \in \mathbb{R}^2 : t \geq 1\}. \end{aligned}$$

A similar behavior happens by taking $q = (1,0)$. Thus, in order to obtain a suitable limit behavior of Henig (C,ε)-proper efficient solutions of the problem, we see that it is not sufficient to consider a singleton set $C = \{q\}$.

However, for $\xi = (1,1)$ and $\varepsilon \in (0,1)$, it is not hard to check that

$$\mathrm{He}(f,S,B_\xi,\varepsilon) = S \cap \{(y_1,y_2) \in \mathbb{R}^2_+ : y_1 + y_2 \le 1+\varepsilon\}.$$

Therefore, for each sequence $(\varepsilon_n) \subset \mathbb{R}_+\backslash\{0\}$, $\varepsilon_n \downarrow 0$, we have

$$\operatorname*{Limsup}_{\varepsilon\to 0} \mathrm{He}(f,S,B_\xi,\varepsilon) = \bigcap_n \mathrm{He}(f,S,B_\xi,\varepsilon_n) = \mathrm{E}(f,S,D)$$

and the set $\mathrm{He}(f,S,B_\xi,\varepsilon)$ is a suitable surrogate of $\mathrm{E}(f,S,D)$, for ε small enough.

4. Approximate efficiency with respect to open conic extensions

It is clear that Henig (C,ε)-proper efficient solutions of problem $(\mathcal{P}_S)$ are approximate efficient solutions of the same problem for open conic extensions defined by ordering cones belonging to $\mathcal{G}$, instead of the original ordering cone D. To be precise, we have that

$$\mathrm{He}(f,S,C,\varepsilon) = \bigcup_{D'\in\mathcal{G}(C)} \mathrm{AE}(f,S,\mathcal{E}_0^{D'}(C),\varepsilon), \tag{11}$$

as it was shown in Remark 3.2(d). In this section we obtain some properties of the sets $\mathrm{AE}(f,S,\mathcal{E}_0^{D'}(C),\varepsilon)$, that will be applied to characterize Henig (C,ε)-proper efficient solutions of problem $(\mathcal{P}_S)$ through linear scalarizations and via ε-subgradients of vector mappings.

Let $K \subset Y$ be a nonempty proper solid convex cone, $\emptyset \neq C \subset Y\backslash\{0\}$ be such that $C \cap (-\operatorname{int} K) = \emptyset$ and $\varepsilon \ge 0$. Next we characterize the elements of $\mathrm{AE}(f,S,\mathcal{E}_0^K(C),\varepsilon)$ by linear scalarizations. Let us observe that

$$0 \notin \mathcal{E}_0^K(C) \iff C \cap (-\operatorname{int} K) = \emptyset \iff \mathcal{E}_0^K(C) \cap (-K) = \emptyset.$$

Theorem 4.1. *It follows that*

$$\bigcup_{\mu\in(K^+\cap C^+)\backslash\{0\}} \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu\circ f) \subset \mathrm{AE}(f,S,\mathcal{E}_0^K(C),\varepsilon).$$

Proof. Let $\mu \in K^+ \cap C^+$, $\mu \neq 0$, and $x_0 \in \varepsilon\tau_C(\mu)\text{-}\}_S(\mu\circ f)$. Suppose by contradiction that $x_0 \notin \mathrm{AE}(f,S,\mathcal{E}_0^K(C),\varepsilon)$. Then, by Lemma 2.2($b$) we have that

$$(f(S_0) - f(x_0)) \cap (-C(\varepsilon) - \operatorname{int} K) \neq \emptyset,$$

and so there exist $\bar{x} \in S_0$, $c \in C(\varepsilon)$ and $d' \in \operatorname{int} K$ such that $f(\bar{x}) - f(x_0) = -c - d'$. Thus,

$$(\mu \circ f)(x_0) = (\mu \circ f)(\bar{x}) + \langle \mu, c\rangle + \langle \mu, d'\rangle > (\mu \circ f)(\bar{x}) + \varepsilon\tau_C(\mu),$$

which contradicts that $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f)$, and the proof is complete. □

Theorem 4.2. *Consider $x_0 \in S_0$ and suppose that $f - f(x_0)$ is nearly (C, ε)-subconvexlike on S. If $x_0 \in \mathrm{AE}(f, S, \mathcal{E}_0^K(C), \varepsilon)$, then there exists $\mu \in (K^+ \cap C^+)\backslash\{0\}$ such that $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f)$.*

Proof. Since $x_0 \in \mathrm{AE}(f, S, \mathcal{E}_0^K(C), \varepsilon)$, we have that

$$(f(S_0) - f(x_0)) \cap (-\mathcal{E}_0^K(C)(\varepsilon)) = \emptyset.$$

By Lemma 2.2(b) it follows that

$$(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-\operatorname{int} K) = \emptyset,$$

that is equivalent to

$$\operatorname{cl}\operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-\operatorname{int} K) = \emptyset,$$

since $\operatorname{int} K$ is open and $0 \notin \operatorname{int} K$. As $f - f(x_0)$ is nearly (C, ε)-subconvexlike on S, the cone $\operatorname{cl}\operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0))$ is convex. By Eidelheit's separation theorem (see [23, Theorem 3.16]), there exists $\mu \in Y^*\backslash\{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\begin{gathered}\langle \mu, y\rangle \geq \alpha \geq \langle \mu, -z\rangle, \quad \forall\, y \in \operatorname{cl}\operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)), \forall\, z \in K,\\ \langle \mu, z\rangle > \alpha, \quad \forall\, z \in \operatorname{int} K.\end{gathered}$$

From the first statement it is easy to see that $\alpha = 0$ and $\langle \mu, c\rangle \geq 0$, for all $c \in C$, which means that $\mu \in C^+$. From the second statement we deduce that $\mu \in K^+$. Thus, we have that $\mu \in (K^+ \cap C^+)\backslash\{0\}$. Finally, by the first inequality we have in particular that

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x) + \langle \mu, c\rangle, \quad \forall\, x \in S, \forall\, c \in C(\varepsilon),$$

which is equivalent to

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x) + \varepsilon\tau_C(\mu), \quad \forall\, x \in S,$$

as we want to prove. □

From Theorems 4.1 and 4.2 we deduce the following corollary.

Corollary 4.3. *Suppose that $f - f(x)$ is nearly (C, ε)-subconvexlike on S, for all $x \in S_0$. Then,*

$$\mathrm{AE}(f, S, \mathcal{E}_0^K(C), \varepsilon) = \bigcup_{\mu \in (K^+ \cap C^+)\backslash\{0\}} \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f).$$

In the following results we give necessary and sufficient conditions for Henig (C,ε)-proper efficient solutions through approximate solutions of related scalar optimization problems. They are direct consequences of the scalarization results derived in Theorems 4.1 and 4.2.

Theorem 4.4. *Let $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. We have that*

$$\bigcup_{\mu \in D^{s+} \cap C^{+}} \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f) \subset \mathrm{He}(f,S,C,\varepsilon).$$

Proof. Let $\mu \in D^{s+} \cap C^{+}$ and $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f)$. Consider the cone $D' := \{y \in Y : \langle \mu, y\rangle > 0\} \cup \{0\}$. Clearly, D' is proper, solid and convex with $D\backslash\{0\} \subset \operatorname{int} D'$ and $\mu \in D'^{+}$. Moreover, since $\mu \in C^{+}$ it follows that $C \cap (-\operatorname{int} D') = \emptyset$, and so $D' \in \mathcal{G}(C)$. Then, by applying Theorem 4.1 with $K := D'$ and by (11) we have

$$x_0 \in \mathrm{AE}(f,S,\mathcal{E}_0^{D'}(C),\varepsilon) \subset \mathrm{He}(f,S,C,\varepsilon),$$

and the proof is complete. □

Theorem 4.5. *Let $\varepsilon \geq 0$, $C \in \overline{\mathcal{H}}$ and $x_0 \in S_0$. Suppose that $f - f(x_0)$ is nearly (C,ε)-subconvexlike on S. If $x_0 \in \mathrm{He}(f,S,C,\varepsilon)$, then $C \in \mathcal{F}_{D^{s+}}$ and there exists $\mu \in D^{s+} \cap C^{+}$ such that $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f)$.*

Proof. Since $x_0 \in \mathrm{He}(f,S,C,\varepsilon)$, there exists $D' \in \mathcal{G}(C)$ such that $x_0 \in \mathrm{AE}(f,S,\mathcal{E}_0^{D'}(C),\varepsilon)$. By applying Theorem 4.2 with $K = D'$ we deduce that there exists $\mu \in (D'^{+} \cap C^{+})\backslash\{0\}$ such that $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f)$. Finally, as $D\backslash\{0\} \subset \operatorname{int} D'$ it follows that $\mu \in D'^{+} \subset D^{s+}$ and the proof finishes. □

The set of Henig (C,ε)-proper efficient solutions of problem $(\mathcal{P}_S)$ satisfies additional properties under generalized convexity assumptions. Some of these properties are stated in the following proposition.

Proposition 4.6. *Let $x_0 \in S_0$, $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. Suppose that $f - f(x_0)$ is nearly (C,ε)-subconvexlike on S. Then,*

$$x_0 \in \mathrm{He}(f,S,C,\varepsilon) \Longleftrightarrow x_0 \in \mathrm{He}(f,S,\operatorname{shw} C,\varepsilon).$$

If additionally $\operatorname{co} C \in \overline{\mathcal{H}}$, then

$$x_0 \in \mathrm{He}(f,S,C,\varepsilon) \Longleftrightarrow x_0 \in \mathrm{He}(f,S,\operatorname{co} C,\varepsilon).$$

Proof. Let us prove the first equivalence, since the proof of the second one is similar.

It is obvious that $\operatorname{shw} C \in \overline{\mathcal{H}}$. By Theorem 3.6($b$) we have that

$$x_0 \in \mathrm{He}(f,S,\operatorname{shw} C,\varepsilon) \Rightarrow x_0 \in \mathrm{He}(f,S,C,\varepsilon).$$

For the reciprocal implication, suppose that $x_0 \in \mathrm{He}(f, S, C, \varepsilon)$. Then, by Theorem 4.5 there exists $\mu \in D^{s+} \cap C^+$ such that $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f)$. It is easy to check (see, for instance, [14, Proposition 3.3(*b*)]) that $\tau_C(\mu) = \tau_{\operatorname{shw} C}(\mu)$. Therefore $\mu \in (\operatorname{shw} C)^+$ and by Theorem 4.4 we have that $x_0 \in \mathrm{He}(f, S, \operatorname{shw} C, \varepsilon)$, which finishes the proof. $\square$

In the next theorem we prove that the set of Henig (C, ε)-proper efficient solutions is included in the set of Benson (C, ε)-proper efficient solutions of problem $(\mathcal{P}_S)$.

Theorem 4.7. *Let $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. It follows that*

$$\mathrm{He}(f, S, C, \varepsilon) \subset \mathrm{Be}(f, S, C, \varepsilon).$$

Proof. Let $x_0 \in \mathrm{He}(f, S, C, \varepsilon)$. By Theorem 3.3(*c*) we have in particular that

$$\operatorname{cl} \operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-D) = \{0\}.$$

Thus $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$ and the proof is complete. $\square$

The next corollary follows directly from [14, Corollary 2.13] and Theorems 4.4, 4.5 and 4.7.

Corollary 4.8. *Let $\varepsilon \geq 0$ and $C \in \mathcal{F}_{D^{s+}}$.*

(*a*) *It follows that*

$$\bigcup_{\mu \in D^{s+} \cap C^+} \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f) \subset \mathrm{He}(f, S, C, \varepsilon) \subset \mathrm{Be}(f, S, C, \varepsilon). \quad (12)$$

Moreover, if $f - f(x)$ is nearly (C, ε)-subconvexlike on S, for all $x \in S_0$, then

$$\mathrm{He}(f, S, C, \varepsilon) = \bigcup_{\mu \in D^{s+} \cap C^+} \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f).$$

(*b*) *If additionally* $\operatorname{int}(D^+) \neq \emptyset$, *then*

$$\mathrm{He}(f, S, C, \varepsilon) = \mathrm{Be}(f, S, C, \varepsilon) = \bigcup_{\mu \in D^{s+} \cap C^+} \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f).$$

Remark 4.9. (a) For each $\varepsilon \geq 0$ and $C \in \mathcal{F}_{D^{s+}}$, statement (12) is stronger than [14, Theorem 2.11], where we proved that

$$\bigcup_{\mu \in D^{s+} \cap C^+} \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f) \subset \mathrm{Be}(f, S, C, \varepsilon).$$

In other words, by statement (12) we show that the set of Henig (C, ε)-proper efficient solutions of problem $(\mathcal{P}_S)$ is intermediate between the union set of approximate solutions of the scalar optimization problems

$$\operatorname{Min}\{(\mu \circ f)(x) : x \in S\}, \quad \forall\, \mu \in D^{s+},$$

and the set of Benson (C,ε)-proper efficient solutions of problem $(\mathcal{P}_{\mathcal{S}})$.

(b) Suppose that $D^{s+} \neq \emptyset$ and consider $C = q + D$, $q \in D\backslash\{0\}$. It is clear that $C \in \mathcal{F}_{D^{s+}}$ and by Theorem 2.11 we see that Corollary 4.8(a) reduces to [28, Theorem 3.1, $\sigma = p$]. If additionally $\operatorname{int} D^+ \neq \emptyset$, then by Lemma 2.2$(a)$ and Corollary 4.8(b) we obtain that

$$\mathrm{He}(f,S,q,\varepsilon) = \mathrm{He}(f,S,C,\varepsilon) = \mathrm{Be}(f,S,C,\varepsilon) = \mathrm{Be}(f,S,q,\varepsilon).$$

In the sequel, we introduce and study an ε-subdifferential for vector mappings in the sense of (C,ε)-efficiency, which will be an essential tool in the next section for obtaining a characterization of Henig (C,ε)-proper efficient solutions of optimization problems with difference of vector mappings. Some previous concepts are needed. The first one is the well-known Brøndsted-Rockafellar ε-subdifferential (see [5]), that we recall in the following definition.

Definition 4.10. Let $h : X \to \overline{\mathbb{R}}$ be a proper mapping and $\varepsilon \geq 0$. The ε-subdifferential of h at a point $x_0 \in \operatorname{dom} h$ is the set

$$\partial_\varepsilon h(x_0) = \{x^* \in X^* : h(x) \geq h(x_0) - \varepsilon + \langle x^*, x - x_0\rangle, \forall\, x \in X\}.$$

The elements of $\partial_\varepsilon h(x_0)$ are called ε-subgradients of h at x_0.

The next strong ε-subdifferential, which was introduced by Kutateladze [25], will be needed too. Let us observe that it reduces to the well-known strong subdifferential due to Raffin [34] by taking $\varepsilon = 0$, and also to the Brøndsted-Rockafellar ε-subdifferential whenever $Y = \mathbb{R}_+$, $D = \mathbb{R}_+$ and $q = \{1\}$.

Definition 4.11. Let $\varepsilon \geq 0$, $q \in D$ and $x_0 \in \operatorname{dom} f$. The εq-strong subdifferential of f at x_0 is the set

$$\hat{\partial}^D_{q,\varepsilon} f(x_0) := \{T \in \mathcal{L}(X,Y) : (f-T)(x_0) - \varepsilon q \leq_D (f-T)(x),\ \forall x \in X\}.$$

Moreover, in this paper we also use the following regularity conditions, which were introduced by El Maghri (see [28, Definition 3.1(ii)]).

Definition 4.12. Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$.

(a) It is said that f is p-regular ε-subdifferentiable (with respect to D) at x_0 if

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{q\in D\\ \langle \mu,q\rangle=\varepsilon}} \mu \circ \hat{\partial}^D_{q,1} f(x_0), \quad \forall\, \mu \in D^{s+}.$$

(b) It is said that f is w-regular ε-subdifferentiable (with respect to D) at x_0 if

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{q\in D_\varepsilon\\ \langle \mu,q\rangle=\varepsilon}} \mu \circ \hat{\partial}^D_{q,1} f(x_0), \quad \forall\, \mu \in D^{+}\backslash\{0\},$$

where $D_\varepsilon = D$ if $\varepsilon > 0$ and $D_0 = \{0\}$.

Finally, we say that the mapping f is star (resp., strict star) D-continuous at a point $x_0 \in \operatorname{dom} f$ if $\mu \circ f$ is continuous at x_0, for all $\mu \in D^+\backslash\{0\}$ (resp., $\mu \in D^{s+}$).

Definition 4.13. Let $K \subset Y$ be a nonempty proper solid convex cone, $\emptyset \neq C \subset Y\backslash\{0\}$ be such that $C \cap (-\operatorname{int} K) = \emptyset$, $\varepsilon \geq 0$ and $x_0 \in \operatorname{dom} f$. The weak (C, ε)-subdifferential of f (with respect to K) at x_0 is the set

$$\partial^K_{C,\varepsilon} f(x_0) = \{T \in \mathcal{L}(X,Y) : x_0 \in \operatorname{AE}(f - T, X, \mathcal{E}^K_0(C), \varepsilon)\}.$$

By taking $C = K\backslash\{0\}$ and $\varepsilon = 1$, this concept encompasses the weak version of the well-known exact subdifferential introduced by Sawaragi and Tanino (see [37, 38]). As a consequence, we denote $\partial^K f(x_0) := \partial^K_{K\backslash\{0\},1} f(x_0)$.

Analogously, if $q \in Y\backslash\{-\operatorname{int} K \cup \{0\}\}$, $C = \{q\}$ and $\varepsilon = 1$, then it reduces to the ε-w-subdifferential introduced and studied in deep by El Maghri (see [28, 29, 30]). Moreover, if $Y = \mathbb{R}$, it reduces to the Brøndsted-Rockafellar ε-subdifferential. To be precise, $\partial^{\mathbb{R}_+}_{1,\varepsilon} f(x_0) = \partial_\varepsilon f(x_0)$.

In the next result, we collect the basic properties of the weak (C, ε)-subdifferential. The proof is similar to the proof of Theorem 3.6 and it is omitted.

Proposition 4.14. *The weak (C, ε)-subdifferential of f (with respect to K) at x_0 satisfies the following properties:*

(a) *Assume that $0 \notin \operatorname{cl} C$ and consider a nonempty set $C' \subset K\backslash\{0\}$ such that $0 \notin C + \operatorname{cone} C'$ and $\operatorname{cone} C + \operatorname{cone} C'$ is closed. Then*

$$\partial^K_{C,\varepsilon} f(x_0) = \partial^K_{\operatorname{cl} C,\varepsilon} f(x_0) = \partial^K_{C+\operatorname{cone} C',\varepsilon} f(x_0).$$

(b) *Let $\emptyset \neq C' \subset Y\backslash\{0\}$ be such that $C' \cap (-\operatorname{int} K) = \emptyset$ and $C(\varepsilon) \subset \operatorname{cl}(C'(\delta))$. Then $\partial^K_{C',\delta} f(x_0) \subset \partial^K_{C,\varepsilon} f(x_0)$.*

(c) *If $\mathcal{E}^D(C)$ is coradiant, then $\partial^K_{C,\delta} f(x_0) \subset \partial^K_{C,\varepsilon} f(x_0)$, for all $0 < \delta \leq \varepsilon$.*

(d) *Consider a nonempty set $C' \subset Y$ such that $0 \notin \operatorname{cl} C + C'$ and $(\operatorname{cl} C + C') \cap (-\operatorname{int} K) = \emptyset$. Then $\partial^K_{\operatorname{cl} C+C',\varepsilon} f(x_0) = \partial^K_{C+C',\varepsilon} f(x_0)$.*

(e) *If C is convex and coradiant, then $\partial^K_{C+C(0),\varepsilon} f(x_0) = \partial^K_{C,\varepsilon} f(x_0)$.*

(f) *If $C \subset K\backslash\{0\}$, then $\partial^K_{K\backslash\{0\},1} f(x_0) \subset \partial^K_{C,\varepsilon} f(x_0)$.*

(g) *If $\operatorname{cl}\operatorname{cone} C = K$, then $\partial^K_{C,0} f(x_0) = \partial^K_{K\backslash\{0\},1} f(x_0)$.*

It is possible to state more properties by combining some of the previous ones. For example, by part (b) we deduce that for all $q_1, q_2 \in Y\backslash(-\operatorname{int} K \cup \{0\})$, $q_1 \leq_K q_2$,

$$\partial^K_{q_1,1} f(x_0) \subset \partial^K_{q_2,1} f(x_0).$$

This property was proved in [29, Proposition 2.1(f)].

The following scalarization result is a direct consequence of Theorems 4.1 and 4.2. For each $T \in \mathcal{L}(X,Y)$, we denote $f_T : X \to \overline{Y}$, $f_T(x) = f(x) - T(x)$, for all $x \in X$.

Theorem 4.15. *Let $x_0 \in \operatorname{dom} f$ and suppose that $f_T - f_T(x_0)$ is nearly (C, ε)-subconvexlike on S, for all $T \in \mathcal{L}(X, Y)$. Then,*

$$\partial^K_{C,\varepsilon} f(x_0) = \bigcup_{\mu \in (K^+ \cap C^+)\backslash\{0\}} \{T \in \mathcal{L}(X, Y) : \mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0)\}.$$

By applying Theorem 2.11 to $M = S$ and $C = q + \operatorname{int} K$, $q \notin -\operatorname{int} K$, we see that $f_T - f_T(x_0)$ is nearly (C, ε)-subconvexlike on S whenever f is K-convex on S. Thus, Theorem 4.15 reduces to the second part of [28, Theorem 3.2].

In the next result we establish a Moreau-Rockafellar type theorem for the weak $(C(\varepsilon) + q, 1)$-subdifferential of the sum of two proper mappings $f_1, f_2 : X \to \overline{Y}$.

For a set $F \subset Y$, we use the following notation:

$$F^0 := F\backslash\{0\}, \quad \text{if } 0 \in F, \quad F^0 := F, \text{ otherwise.}$$

Theorem 4.16. *Let $K \subset Y$ be a proper solid convex cone, $q \in \operatorname{int} K \cup \{0\}$, $\emptyset \neq C \subset Y\backslash\{0\}$ be convex such that $C \cap (-\operatorname{int} K) = \emptyset$, $\varepsilon \geq 0$, and $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Suppose that $N \subset Y$ is a convex cone such that $\mathcal{E}^K_0(N) = \operatorname{int} K$, and f_1, f_2 are N-convex on X, f_2 is w-regular ε'-subdifferentiable at x_0 (with respect to K), for all $\varepsilon' \geq 0$ and the following qualification condition holds*

(QC1) *There exists $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ such that f_1 or f_2 is star K-continuous at $\bar{x}$.*

Then,

$$\partial^K_{C(\varepsilon)+q,1}(f_1 + f_2)(x_0) = \bigcup_{\substack{(C_1,\bar{q}) \in \mathcal{F}_{K^+} \times K \\ C_1 = (C(\varepsilon) + \operatorname{int} K - \bar{q} + q)^0}} \left\{\partial^K_{C_1,1} f_1(x_0) + \hat{\partial}^K_{\bar{q},1} f_2(x_0)\right\}.$$

Proof. Inclusion "$\subset$". Consider an arbitrary $T \in \partial^K_{C(\varepsilon)+q,1}(f_1 + f_2)(x_0)$. Then $x_0 \in \operatorname{AE}(f_1 + f_2 - T, X, \mathcal{E}^K_0(C(\varepsilon) + q), 1)$. By [14, Theorem 2.7(a)] we see that $f_1 + f_2 - T - (f_1 + f_2 - T)(x_0)$ is $(\mathcal{E}^K_0(C(\varepsilon) + q), 1)$-nearly subconvexlike on X. Thus, by applying Theorem 4.2 we see that there exists $\mu \in (K^+ \cap \mathcal{E}^K_0(C(\varepsilon) + q)^+)\backslash\{0\}$ such that $x_0 \in \tau_{\mathcal{E}^K_0(C(\varepsilon)+q)}(\mu)\text{-}\operatorname{argmin}_X(\mu \circ (f_1 + f_2 - T))$. As $\mu \in K^+$, it is clear that $\mu \in (K^+ \cap (C(\varepsilon) + q)^+)\backslash\{0\}$, $\tau_{\mathcal{E}^K_0(C(\varepsilon)+q)}(\mu) = \tau_{C(\varepsilon)+q}(\mu)$, and so

$$\mu \circ T \in \partial_{\tau_{C(\varepsilon)+q}(\mu)}(\mu \circ f_1 + \mu \circ f_2)(x_0).$$

By applying the calculus rule for the ε-subdifferential of a sum of two convex mappings (see, for instance, [42, Theorem 2.8.7]) we see that there exist $\varepsilon_1, \varepsilon_2 \geq 0$ and $x^*_1, x^*_2 \in X^*$ such that $\varepsilon_1 + \varepsilon_2 = \tau_{C(\varepsilon)+q}(\mu)$, $x^*_1 \in \partial_{\varepsilon_1}(\mu \circ f_1)(x_0)$ and $x^*_2 \in \partial_{\varepsilon_2}(\mu \circ f_2)(x_0)$ and $x^*_1 + x^*_2 = \mu \circ T$.

The w-regular ε_2-subdifferentiability at x_0 (with respect to K) of f_2 implies that there exist $\bar{q} \in K_{\varepsilon_2}$ and $B \in \hat{\partial}^K_{\bar{q},1} f_2(x_0)$ such that $\langle \mu, \bar{q} \rangle = \varepsilon_2$ and $x_2^* = \mu \circ B$. Let $C_1 := (C(\varepsilon) + \operatorname{int} K + q - \bar{q})^0$. Since C is convex, it is easy to see that

$$\tau_{C_1}(\mu) = \tau_{C(\varepsilon)+q}(\mu) - \varepsilon_2 = \varepsilon_1 \geq 0.$$

Therefore, $\mu \in (K^+ \cap C_1^+) \backslash \{0\}$. Moreover,

$$\mu \circ (T - B) = x_1^* \in \partial_{\varepsilon_1}(\mu \circ f_1)(x_0)$$

and then $x_0 \in \tau_{C_1}(\mu)\text{-}\operatorname{argmin}_X \mu \circ (f_1 - (T - B))$. By Theorem 4.1 it follows that $x_0 \in \operatorname{AE}(f_1 - (T - B), X, \mathcal{E}^K_0(C_1), 1)$ and so $T - B \in \partial^K_{C_1,1} f_1(x_0)$. Therefore

$$T = (T - B) + B \in \partial^K_{C_1,1} f_1(x_0) + \hat{\partial}^K_{\bar{q},1} f_2(x_0)$$

and the proof of inclusion "$\subset$" finishes.

Inclusion "$\supset$". Let $(C_1, \bar{q}) \in \mathcal{F}_{K^+} \times K$, with $C_1 = (C(\varepsilon) + \operatorname{int} K - \bar{q} + q)^0$, $T_1 \in \partial^K_{C_1,1} f_1(x_0)$, $T_2 \in \hat{\partial}^K_{\bar{q},1} f_2(x_0)$, and suppose by contradiction that $T_1 + T_2 \notin \partial^K_{C(\varepsilon)+q,1}(f_1 + f_2)(x_0)$. Then, it follows that

$$\begin{aligned} &((f_1 + f_2 - T_1 - T_2)(\operatorname{dom} f_1 \cap \operatorname{dom} f_2) - (f_1 + f_2 - T_1 - T_2)(x_0)) \\ &\cap (-\mathcal{E}^K_0(C(\varepsilon) + q)) \neq \emptyset, \end{aligned}$$

and so there exists $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, such that

$$(f_1 + f_2 - T_1 - T_2)(\bar{x}) - (f_1 + f_2 - T_1 - T_2)(x_0) \in -\mathcal{E}^K_0(C(\varepsilon) + q). \tag{13}$$

On the other hand, since $T_2 \in \hat{\partial}^K_{\bar{q},1} f_2(x_0)$, we have in particular that

$$(f_2 - T_2)(\bar{x}) - (f_2 - T_2)(x_0) + \bar{q} =: \hat{d} \in K. \tag{14}$$

From (13) and (14) we deduce that

$$\begin{aligned} (f_1 - T_1)(\bar{x}) - (f_1 - T_1)(x_0) &\in -\mathcal{E}^K_0(C(\varepsilon) + q) - \hat{d} + \bar{q} \\ &\subset -\mathcal{E}^K_0(C(\varepsilon) + q) - K + \bar{q} \\ &= -\mathcal{E}^K_0(C(\varepsilon) + \operatorname{int} K + q - \bar{q}) \\ &= -\mathcal{E}^K_0((C(\varepsilon) + \operatorname{int} K + q - \bar{q})^0) \qquad (15) \\ &= -\mathcal{E}^K_0(C_1). \end{aligned}$$

Let us prove equality (15). If $0 \notin C(\varepsilon) + \operatorname{int} K + q - \bar{q}$, the equality is obvious. Suppose that $0 \in C(\varepsilon) + \operatorname{int} K + q - \bar{q}$ and let $k \in \operatorname{int} K$ and $\hat{q} \in (C(\varepsilon) + \operatorname{int} K + q - \bar{q}) \backslash \{0\}$. For $0 < \alpha < 1$ small enough, we have that $k - \alpha\hat{q} \in \operatorname{int} K$, so

$$k \in \alpha\hat{q} + \operatorname{int} K \subset (C(\varepsilon) + \operatorname{int} K + q - \bar{q}) \backslash \{0\} + \operatorname{int} K,$$

since $C(\varepsilon) + \operatorname{int} K + q - \bar{q}$ is convex. Thus $\operatorname{int} K \subset (C(\varepsilon) + \operatorname{int} K + q - \bar{q}) \backslash \{0\} + \operatorname{int} K$ and from that, equality (15) is easily deduced.

Then, we obtain that $(f_1 - T_1)(\bar{x}) - (f_1 - T_1)(x_0) \in -\mathcal{E}^K_0(C_1)$, which is a contradiction, since $T_1 \in \partial^K_{C_1,1} f_1(x_0)$, concluding the proof. □

Theorem 4.16 reduces to [28, Theorem 4.1] by considering $\varepsilon = 1$, $q = 0$, $C = \{e\}$ ($e \notin -\operatorname{int} K \cup \{0\}$) and $N = K$.

Following an analogous reasoning as in the proof of Theorem 4.16, in the next result we obtain an alternative Moreau-Rockafellar type theorem for the weak $(C(\varepsilon) + q, 1)$-subdifferential with respect to a cone $D' \in \mathcal{G}(C)$ of the sum of two mappings.

Theorem 4.17. *Let $\varepsilon \geq 0$, $\emptyset \neq C \subset Y\backslash\{0\}$ be convex, $D' \in \mathcal{G}(C)$, $q \in \operatorname{int} D' \cup \{0\}$ and $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Suppose that f_1, f_2 are D-convex on X, f_2 is p-regular ε'-subdifferentiable at x_0 (with respect to D), for all $\varepsilon' \geq 0$ and the following qualification condition holds*

(QC2) *There exists $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ such that f_1 or f_2 is strict star D-continuous at $\bar{x}$.*

Then,

$$\partial^{D'}_{C(\varepsilon)+q,1}(f_1 + f_2)(x_0) = \bigcup_{\substack{(C_1,\bar{q}) \in \mathcal{F}_{D^{s+}} \times D \\ C_1 = (C(\varepsilon) + \operatorname{int} D' - \bar{q} + q)^0}} \left\{ \partial^{D'}_{C_1,1} f_1(x_0) + \hat{\partial}^{D}_{\bar{q},1} f_2(x_0) \right\}.$$

5. Optimization problems with difference of vector mappings

In this section, we are going to study the following vector optimization problem

$$\operatorname{Min}_D\{f(x) - g(x) : x \in S\}, \qquad (\mathcal{DP}_S)$$

where the objective mapping is given by the difference of two proper mappings $f : X \to \overline{Y}$ and $g : X \to Y$. Observe that $\operatorname{dom}(f - g) = \operatorname{dom} f$. When the problem is unconstrained, i.e., $S = X$, we will denote it by $(\mathcal{DP}_X)$.

Next we characterize the Henig (C, ε)-proper efficient solutions of problem $(\mathcal{DP}_X)$. A previous lemma is needed, where the well-known transportation formula for ε-subgradients of extended real valued mappings (see [22, Proposition 4.2.2]) is generalized to strong εq-subgradients of extended vector valued mappings. Its proof is a direct consequence of the definitions and it is omitted.

Lemma 5.1. *Consider $\varepsilon \geq 0$, $q \in D$ and $\bar{x}, x_0 \in \operatorname{dom} f$. If $T \in \hat{\partial}^D_{q,\varepsilon} f(\bar{x})$, then $T \in \hat{\partial}^D_{q',1} f(x_0)$, where*

$$q' := (f - T)(x_0) - (f - T)(\bar{x}) + \varepsilon q \in D.$$

Theorem 5.2. *Let $\varepsilon \geq 0$, $C \in \overline{\mathcal{H}}$ and $x_0 \in \operatorname{dom} f$. Suppose that the following statement holds:*

$$\text{There exists } \bar{q} \in D\backslash\{0\} \text{ such that } \hat{\partial}^D_{\bar{q},\delta} g(x) \neq \emptyset,\ \forall x \in \operatorname{dom} f, \forall \delta > 0. \qquad (16)$$

Then, $x_0 \in \operatorname{He}(f - g, X, C, \varepsilon)$ if and only if there exists $D' \in \mathcal{G}(C)$ such that

$$\hat{\partial}^D_{q,1} g(x_0) \subset \partial^{D'}_{C(\varepsilon)+q,1} f(x_0), \quad \forall\, q \in D. \qquad (17)$$

Proof. Consider $x_0 \in \mathrm{He}(f-g, X, C, \varepsilon)$ and suppose by contradiction that for each $D' \in \mathcal{G}(C)$ there exist $q_{D'} \in D$ and $T_{D'} \in \mathcal{L}(X,Y)$ such that

$$T_{D'} \in \hat{\partial}^{D}_{q_{D'},1} g(x_0) \quad \text{and} \quad T_{D'} \notin \partial^{D'}_{C(\varepsilon)+q_{D'},1} f(x_0).$$

Since $T_{D'} \in \hat{\partial}^{D}_{q_{D'},1} g(x_0)$, it follows that

$$(g - T_{D'})(x_0) - q_{D'} \leq_D (g - T_{D'})(x), \quad \forall x \in X. \tag{18}$$

As $T_{D'} \notin \partial^{D'}_{C(\varepsilon)+q_{D'},1} f(x_0)$, taking into account Lemma 2.2(b), we have that

$$((f - T_{D'})(\mathrm{dom}\, f) - (f - T_{D'})(x_0)) \cap (-C(\varepsilon) - \mathrm{int}\, D' - q_{D'}) \neq \emptyset.$$

Then there exist $x_{D'} \in \mathrm{dom}\, f$, $c_{D'} \in C(\varepsilon)$ and $d'_{D'} \in \mathrm{int}\, D'$ such that

$$f(x_{D'}) - T_{D'}(x_{D'}) - f(x_0) + T_{D'}(x_0) = -c_{D'} - d'_{D'} - q_{D'}. \tag{19}$$

On the other hand, from (18) applied to $x = x_{D'}$, we deduce that

$$g(x_{D'}) - T_{D'}(x_{D'}) - g(x_0) + T_{D'}(x_0) + q_{D'} =: d_{D'} \in D. \tag{20}$$

Then, by (19), (20) and Lemma 2.2(a),(b) we obtain that

$$\begin{aligned}(f-g)(x_{D'}) - (f-g)(x_0) &= -c_{D'} - d'_{D'} - d_{D'} \\ &\in -(C(\varepsilon) + \mathrm{int}\, D' + D) \\ &= -\mathcal{E}^{D'}_0(C)(\varepsilon)\end{aligned}$$

and so $x_0 \notin \mathrm{AE}(f-g, X, \mathcal{E}^{D'}_0(C), \varepsilon)$, for all $D' \in \mathcal{G}(C)$. Then $x_0 \notin \mathrm{He}(f - g, X, C, \varepsilon)$, which is a contradiction.

For the reciprocal implication, suppose that there exists $D' \in \mathcal{G}(C)$ such that condition (17) is satisfied. Fix $\bar{x} \in \mathrm{dom}\, f$ and $\delta > 0$. By assumption (16) there exists $T \in \hat{\partial}^{D}_{\bar{q},\delta} g(\bar{x})$, and by applying Lemma 5.1 we obtain that $T \in \hat{\partial}^{D}_{q',1} g(x_0)$, where

$$q' := (g - T)(x_0) - (g - T)(\bar{x}) + \delta\bar{q} \in D. \tag{21}$$

Hence, by hypothesis we deduce that $T \in \partial^{D'}_{C(\varepsilon)+q',1} f(x_0)$, and then

$$((f - T)(\mathrm{dom}\, f) - (f - T)(x_0)) \cap (-\mathcal{E}^{D'}_0(C)(\varepsilon) - q') = \emptyset.$$

Since $\bar{x} \in \mathrm{dom}\, f$, by (21) we have

$$(f - T)(\bar{x}) - (f - T)(x_0) \notin -\mathcal{E}^{D'}_0(C)(\varepsilon) - (g - T)(x_0) + (g - T)(\bar{x}) - \delta\bar{q},$$

which is equivalent to

$$(f - g)(\bar{x}) - (f - g)(x_0) \notin \bigcup_{\delta>0} (-\mathcal{E}^{D'}_0(C)(\varepsilon) - \delta\bar{q}), \tag{22}$$

since δ is arbitrary. It is clear that $\mathcal{E}_0^{D'}(C)(\varepsilon) = \mathcal{E}_0^{D'}(C)(\varepsilon) + \operatorname{int} D'$ and

$$\bigcup_{\delta>0}(\delta\bar{q} + \operatorname{int} D') = \operatorname{int} D'.$$

Then, by (22) we have that

$$(f-g)(\bar{x}) - (f-g)(x_0) \notin -\mathcal{E}_0^{D'}(C)(\varepsilon).$$

As $\bar{x}$ has been chosen arbitrarily in $\operatorname{dom}(f-g)$, we deduce easily that

$$((f-g)(\operatorname{dom}(f-g)) - (f-g)(x_0)) \cap (-\mathcal{E}_0^{D'}(C)(\varepsilon)) = \emptyset,$$

i.e., $x_0 \in \operatorname{AE}(f-g, X, \mathcal{E}_0^{D'}(C), \varepsilon)$, and so $x_0 \in \operatorname{He}(f-g, X, C, \varepsilon)$, and the proof finishes. □

By considering $C = D\backslash\{0\}$ and $\varepsilon = 1$ in Theorem 5.2 we obtain the following characterization of (exact) Henig proper efficient solutions of problem $(\mathcal{DP}_X)$.

Corollary 5.3. *Let $x_0 \in \operatorname{dom} f$. Suppose that* (16) *holds. Then, $x_0 \in \operatorname{He}(f - g, X, D)$ if and only if there exists $D' \in \mathcal{G}$ such that*

$$\hat{\partial}^D_{q,1} g(x_0) \subset \partial^{D'}_{q+\operatorname{int} D',1} f(x_0), \quad \forall\, q \in D. \tag{23}$$

Remark 5.4. (a) Notice that the necessary condition of Theorem 5.2 and Corollary 5.3 does not require any assumption.

(b) If $Y = \mathbb{R}$ and $D = \mathbb{R}_+$, then (23) reduces to

$$\partial_\varepsilon g(x_0) \subset \partial_\varepsilon f(x_0), \quad \forall\, \varepsilon \geq 0,$$

and (16) reduces to $\partial_\varepsilon g(x) \neq \emptyset$, for all $x \in \operatorname{dom} f$, for all $\varepsilon > 0$. If additionally g is convex and lower semicontinuous on X, then $\partial_\varepsilon g(x) \neq \emptyset$, for all $\varepsilon > 0$ and $x \in X$ (see, for instance, [42, Theorem 2.4.4(iii)]). Therefore, Corollary 5.3 generalizes the well-known characterization of solutions of DC scalar optimization problems (see [21]).

Next, we are going to study Henig (C, ε)-proper efficient solutions of problem $(\mathcal{DP}_S)$. Let $\varepsilon \geq 0$ and $C \in \overline{\mathcal{H}}$. It is easy to see that $x_0 \in \operatorname{He}(f - g, S, C, \varepsilon)$ if and only if $x_0 \in \operatorname{He}(F, X, C, \varepsilon)$, where $F : X \to \overline{Y}$,

$$F(x) = f(x) - g(x) + I_S(x) = (f + I_S)(x) - g(x), \quad \forall\, x \in X,$$

and I_S is the indicator mapping of S. Thus, problem $(\mathcal{DP}_S)$ is equivalent to an unconstrained vector optimization problem and we can apply Theorem 5.2 in order to characterize Henig (C, ε)-proper efficient solutions of problem $(\mathcal{DP}_S)$.

In the following theorem we state a characterization of Henig (C, ε)-proper efficient solutions of problem $(\mathcal{DP}_S)$.

Theorem 5.5. *Let $\varepsilon \geq 0$, $C \in \overline{\mathcal{H}}$ and $x_0 \in S_0$. Suppose that C and S are convex, f is D-convex on X, star D-continuous at a point $\bar{x} \in S_0$ and the following statement is satisfied.*

$$\text{There exists } \bar{q} \in D\backslash\{0\} \text{ such that } \hat{\partial}^D_{\bar{q},\delta} g(x) \neq \emptyset, \quad \forall\, x \in S_0, \forall\, \delta > 0. \tag{24}$$

Then, $x_0 \in \mathrm{He}(f - g, S, C, \varepsilon)$ if and only if there exists $D' \in \mathcal{G}(C)$ such that

$$\hat{\partial}^D_{q,1} g(x_0) \subset \bigcup_{\substack{(C_1,\bar{q}) \in \mathcal{F}_{D'^+} \times D' \\ C_1 = (C(\varepsilon) + \operatorname{int} D' - \bar{q} + q)^0}} \left\{ \partial^{D'}_{C_1,1} f(x_0) + W^{D'}_{S,\bar{q}}(x_0) \right\}, \quad \forall\, q \in D,$$

where $W^{D'}_{S,\bar{q}}(x_0) = \{T \in \mathcal{L}(X, Y) : T(S - x_0) \subset \bar{q} - D'\}$.

Proof. We know that $x_0 \in \mathrm{He}(f - g, S, C, \varepsilon)$ if and only if $x_0 \in \mathrm{He}((f + I_S) - g, X, C, \varepsilon)$. Thus, by Theorem 5.2 there exists $D' \in \mathcal{G}(C)$ such that

$$\hat{\partial}^D_{q,1} g(x_0) \subset \partial^{D'}_{C(\varepsilon)+q,1}(f + I_S)(x_0), \quad \forall\, q \in D.$$

Since S is convex, it follows that I_S is D-convex on X, and by [28, Lemma 5.3] we see that I_S is also w-regular ε'-subdifferentiable at x_0 (with respect to D'), for all $\varepsilon' \geq 0$. Thus, by applying Theorem 4.16 with $K = D'$ and $N = D$ we deduce that for each $q \in D$,

$$\partial^{D'}_{C(\varepsilon)+q,1}(f + I_S)(x_0) = \bigcup_{\substack{(C_1,\bar{q}) \in \mathcal{F}_{D'^+} \times D' \\ C_1 = (C(\varepsilon) + \operatorname{int} D' - \bar{q} + q)^0}} \left\{ \partial^{D'}_{C_1,1} f(x_0) + \hat{\partial}^{D'}_{\bar{q},1} I_S(x_0) \right\}. \tag{25}$$

On the other hand, observe that

$$T \in \hat{\partial}^{D'}_{\bar{q},1} I_S(x_0) \Leftrightarrow T(x_0 - x) + \bar{q} \in D', \quad \forall x \in S \Leftrightarrow T \in W^{D'}_{S,\bar{q}}(x_0). \tag{26}$$

Thus, the theorem is proved taking into account (25) and (26). $\square$

With a similar reasoning as in the result above, but applying Theorem 4.17 instead of Theorem 4.16 we obtain the following characterization.

Theorem 5.6. *Let $\varepsilon \geq 0$, $C \in \overline{\mathcal{H}}$ and $x_0 \in S_0$. Suppose that C and S are convex, f is D-convex on X, strict star D-continuous at a point $\bar{x} \in S_0$ and statement (24) holds. Then, $x_0 \in \mathrm{He}(f - g, S, C, \varepsilon)$ if and only if there exists $D' \in \mathcal{G}(C)$ such that*

$$\hat{\partial}^D_{q,1} g(x_0) \subset \bigcup_{\substack{(C_1,\bar{q}) \in \mathcal{F}_{D^{s+}} \times D \\ C_1 = (C(\varepsilon) + \operatorname{int} D' - \bar{q} + q)^0}} \left\{ \partial^{D'}_{C_1,1} f(x_0) + W^{D}_{S,\bar{q}}(x_0) \right\}, \quad \forall\, q \in D.$$

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