Journal of Convex Analysis
Volume 23 (2016), No. 3, 719–732

Weak Convexity of Functions and the Infimal Convolution

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Dedicated to the memory of Jean Jacques Moreau.

Received: February 2, 2015
Revised manuscript received: 14, 2015
Accepted: March 7, 2015

In terms of sup-inf convolutions we characterize the class $\mathcal{CWC}(\gamma)$ of convolutionally weakly convex functions, that is a special class of Φ-convex functions, generated by the function γ. Motivated by Moreau–Yosida and Lasry–Lions regularizations, we consider a subclass of $\mathcal{CWC}(\gamma)$, namely, the class $\mathcal{RCWC}(\gamma)$ of regularly convolutionally weakly convex functions. We show that under reasonable assumptions on γ the class $\mathcal{RCWC}(\gamma)$ is the same as the class $\mathcal{WC}(\gamma)$, introduced from other considerations.

Keywords: Weak convexity, infimal convolution, Moreau–Yosida and Lasry–Lions regularizations

2010 Mathematics Subject Classification: 41A50, 41A65, 52A21

1. Convolutionally weakly convex functions

Let E be a real Banach space.

Recall that the *effective domain* of a function $f : E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is

$$\operatorname{dom} f = \{x \in E \mid f(x) \in \mathbb{R}\}.$$

Jean Jacques Moreau (see [8]–[12]) introduced and studied the following operations with functions.

The *infimal convolution* or the *episum* of the functions $f : E \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $g : E \to \overline{\mathbb{R}}$ is the function $f \boxplus g$, defined by

$$(f \boxplus g)(x) = \inf_{x_1,\, x_2 \in E:\ x_1 + x_2 = x} \Big(f(x_1) \dot{+} g(x_2) \Big), \qquad x \in E. \tag{1}$$

*Supported by the Russian Foundation for Basic Research, grant 16-01-00259.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

The *sup-convolution* or the *deconvolution* or the *epidifference* of f and g is defined by

$$(f \boxminus g)(x) = \sup_{x_1, x_2 \in E:\ x_1 - x_2 = x} \Big(f(x_1) \underset{\cdot}{+} (-g(x_2)) \Big), \quad x \in E. \tag{2}$$

Above $\dot{+}$ and $\underset{\cdot}{+}$ are the extensions of the sum operator to $\overline{\mathbb{R}}$:

$$(-\infty) \dot{+} (+\infty) = (+\infty) \dot{+} (-\infty) = (+\infty),$$
$$(-\infty) \underset{\cdot}{+} (+\infty) = (+\infty) \underset{\cdot}{+} (-\infty) = (-\infty),$$

otherwise (if $x, y \in \overline{\mathbb{R}}$ are not infinities with different signs) $x \dot{+} y = x \underset{\cdot}{+} y = x + y$ is defined as usual.

Recall the notion of Φ-convexity [1], [6] (or the abstract convexity in the terminology proposed in [14]). For a family Φ of functions $\varphi : E \to \mathbb{R}$, a function $f : E \to \overline{\mathbb{R}}$ is said to be Φ-*convex*, if there exists a subset $\Phi' \subset \Phi$ such that

$$f(x) = \sup_{\varphi \in \Phi'} \varphi(x), \quad x \in E. \tag{3}$$

For a function $\gamma : E \to \mathbb{R}$, we consider the family

$$\Phi(\gamma) = \{\varphi : E \to \mathbb{R} \mid \exists u \in E,\ c \in \mathbb{R} \ :\ \varphi(x) = c - \gamma(u - x),\ \forall x \in E\}. \tag{4}$$

For any function $f : E \to \overline{\mathbb{R}}$ we denote

$$f_-(x) = -f(-x), \quad x \in E. \tag{5}$$

Note that the graph of any function $\varphi \in \Phi(\gamma)$ is a shifted graph of the function γ_-.

A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be *convolutionally weakly convex* with respect to (w.r.t.) $\gamma : E \to \mathbb{R}$ if it is Φ-convex with $\Phi = \Phi(\gamma)$. We denote by $\mathcal{CWC}(\gamma)$ the class of functions $f : E \to \mathbb{R} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ which are convolutionally weakly convex w.r.t. γ. The term "convolutionally" is due to the following useful characterization of such functions by means of sup-inf convolutions.

Lemma 1.1. *For any functions $\gamma : E \to \mathbb{R}$ and $f : E \to \mathbb{R} \cup \{+\infty\}$ with* $\operatorname{dom} f \neq \emptyset$ *the following assertions are equivalent:*

(i) $f \in \mathcal{CWC}(\gamma)$;
(ii) *there exists a function $g : E \to \overline{\mathbb{R}}$ such that $f = g \boxminus \gamma$;*
(iii) $f = f \boxplus \gamma \boxminus \gamma$.

Lemma 1.1 implies that the class $\mathcal{CWC}(\gamma)$ is the same as the class of exactly γ-regular functions in the sense of [7].

We define $\mathcal{CWC}_-(\gamma)$ as the class of functions $f : E \to \mathbb{R} \cup \{-\infty\}$, which are *convolutionally weakly concave* w.r.t. γ, i.e. $f \boxminus \gamma \boxplus \gamma = f$.

By (1), (2), (5) for any functions $f, \gamma : E \to \overline{\mathbb{R}}$ we have

$$(f \boxplus \gamma)_- = f_- \boxminus \gamma. \tag{6}$$

Consequently, the inclusion $f \in \mathcal{CWC}_-(\gamma)$ is equivalent to $f_- \in \mathcal{CWC}(\gamma)$.

Let us denote

$$\beta_t(x) = \frac{1}{2t}\|x\|^2, \quad t > 0,\ x \in E. \tag{7}$$

Lemma 1.2. *Let $E = H$ be a Hilbert space and $t > 0$. Then $f \in \mathcal{CWC}(\beta_t)$ if and only if the function $f + \beta_t$ is lower semicontinuous and convex* (here $(f + \beta_t)(x) = f(x) + \beta_t(x)$ is the conventional sum of functions, rather than the infimal convolution).

So, if $E = \mathbb{R}^n$, then the class $\mathcal{CWC}(\beta_t)$ is the same as the class of weakly convex functions (or ϱ-convex functions with $\varrho = -\frac{1}{2t}$) in the terminology of Vial [15]. The local version of the weak convexity is the lower-C^2 property due to Rockafellar [13].

Lemma 1.3. *If $E = H$ is a Hilbert space, $f \in \mathcal{CWC}(\beta_t)$, $0 < s < t$, then $f \boxplus \beta_s \in \mathcal{CWC}(\beta_{t-s})$.*

In view of Lemma 1.2 the main result of [2] can be formulated as follows.

Proposition 1.4. *Let $E = H$ be a Hilbert space, $t > 0$. Then a function $f : E \to \mathbb{R}$ satisfies the inclusion $f \in \mathcal{CWC}(\beta_t) \cap \mathcal{CWC}_-(\beta_t)$ if and only if the Fréchet derivative f' exists and is Lipschitz continuous with constant $\frac{1}{t}$ on E.*

2. Regularization

Let H be a real Hilbert space. For $t > 0$ and function $f : H \to \mathbb{R} \cup \{+\infty\}$, the *Moreau–Yosida regularization* (or the *Moreau envelope*, or the *proximal envelope*) is defined (see [9], [16]) as

$$f^t(x) = \inf_{u \in H}\left(f(x) + \frac{1}{2t}\|x - u\|^2\right), \quad x \in H.$$

This can be rewritten in terms of the infimal convolution as $f^t = f \boxplus \beta_t$.

If f is lower semicontinuous at $x_0 \in \operatorname{dom} f$, then $f^t(x_0) \to f(x_0)$ as $t \to +0$. If $f \in \mathcal{CWC}(\beta_T)$ for a certain $T > 0$ (in particular, if f is convex), then for any $t \in (0, T)$ the function f^t is smooth, i.e. it possesses a Lipschitz continuous derivative. Moreover, the Moreau–Yosida regularization preserves the minimizers of the function. These well known facts give a lot of applications of the Moreau–Yosida regularization in Hamilton–Jacobi equations and optimization. If $f \notin \mathcal{CWC}(\beta_T)$ for all $T > 0$, instead of the Moreau–Yosida regularization one can use the Lasry–Lions [5] regularization $f^{t,s} = f \boxminus \beta_t \boxplus \beta_s$, which is also smooth and converges to f as $0 < s < t \to +0$.

Let us show that Lemma 1.3 and Proposition 1.4 induce smoothness of the Moreau–Yosida and the Lasry–Lions regularizations. If $f \in \mathcal{CWC}(\beta_T)$, $t \in (0,T)$, then by Lemma 1.3 we get $f \boxplus \beta_t \in \mathcal{CWC}(\beta_{T-t})$. On the other hand, Lemma 1.1 implies that $f \boxplus \beta_t \in \mathcal{CWC}_-(\beta_t)$. Hence, by Proposition 1.4 we deduce that the derivative of the Moreau–Yosida regularization $f^t = f \boxplus \beta_t$ does exist and is Lipschitz continuous with constant $\max\left\{\frac{1}{T-t}, \frac{1}{t}\right\}$ on E. Since $f \boxminus \beta_t \in \mathcal{CWC}(\beta_t)$, by the same reasons we obtain smoothness of the Lasry–Lions regularization $f^{t,s}$.

This reasoning motivates an attempt to extend Lemma 1.3 to Banach spaces and to more general functions than (7).

Given a function $\gamma : E \to \overline{\mathbb{R}}$ and a number $t > 0$, we denote

$$\gamma_t(x) = t \cdot \gamma\left(\frac{x}{t}\right), \quad x \in E. \tag{8}$$

A function $f \in \mathcal{CWC}(\gamma)$ is called *regularly convolutionally weakly convex* w.r.t. γ if

$$\forall \varepsilon \in (0,1)\ \exists \delta \in (0,1) :\ \forall t \in (0,\delta) \quad f \boxplus \gamma_t \in \mathcal{CWC}(\gamma_{1-\varepsilon}). \tag{9}$$

We denote by $\mathcal{RCWC}(\gamma)$ the class of functions, which are regularly convolutionally weakly convex w.r.t. γ. Lemma 1.3 implies that the classes $\mathcal{RCWC}(\beta_t)$ and $\mathcal{CWC}(\beta_t)$ coincide provided that the space is Hilbert and the function β_t is defined by (7).

Hereinafter we consider Example 4.5 with $E = \mathbb{R}^2$ and $\gamma(x_1, x_2) = (x_1^2 + x_2^2)^2$, which shows that $\mathcal{RCWC}(\gamma)$ may be a proper subset of $\mathcal{CWC}(\gamma)$. We prove that for a wide class of functions γ the class $\mathcal{RCWC}(\gamma)$ is the same as the class $\mathcal{WC}(\gamma)$ introduced in [3]. In the case $\gamma(x) = \|x\|^s$ the properties of the function $f \boxplus \gamma_t$ (called as Moreau s-envelope) are studied in [4].

3. Weakly convex functions

The γ-*predifferential* of a function $f : E \to \overline{\mathbb{R}}$ at a point $x \in \operatorname{dom} f$ is defined by

$$\pi_\gamma f(x) = \{u \in \operatorname{dom}\gamma \mid \exists t > 0 :\ (f \boxplus \gamma_t)(x + tu) = f(x) + \gamma_t(tu)\}. \tag{10}$$

A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be *weakly convex* with respect to (w.r.t.) $\gamma : E \to \mathbb{R}$ if

$$(f \boxplus \gamma)(x + u) = f(x) + \gamma(u), \quad \forall x \in \operatorname{dom} f,\ \forall u \in \pi_\gamma f(x). \tag{11}$$

We denote by $\mathcal{WC}(\gamma)$ the class of weakly convex (w.r.t. γ) lower semicontinuous functions $f : E \to \mathbb{R} \cup \{+\infty\}$ such that $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$. We use $\mathcal{WC}_-(\gamma)$ to denote the class of functions f such that $f_- \in \mathcal{WC}(\gamma)$ (see (5)).

Proposition 3.1 ([3, Theorem 3.4]). *Let $E = H$ be a Hilbert space and the function β_t be defined by* (7)*. Then $f \in \mathcal{WC}(\beta_t)$ if and only if the function $f + \beta_t$ is lower semicontinuous and convex.*

Lemma 1.2 and Proposition 3.1 imply that

$$\mathcal{WC}(\beta_t) = \mathcal{CWC}(\beta_t) \tag{12}$$

provided that the space is Hilbert and the function β_t is defined by (7).

A function γ is called *uniformly convex* on a convex set $X \subset E$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_1, x_2 \in X$ the inequality $\|x_1 - x_2\| \geq \varepsilon$ implies that $2\gamma\left(\frac{x_1+x_2}{2}\right) \leq \gamma(x_1) + \gamma(x_2) - \delta$.

A function $\gamma : E \to \overline{\mathbb{R}}$ is *coercive* if $\lim_{\|x\|\to\infty} \frac{\gamma(x)}{\|x\|} = +\infty$.

We say that a function $\gamma : E \to \mathbb{R}$ satisfies assumptions (A1) if

(A1) γ is coercive, bounded on any bounded set, and uniformly convex on any convex bounded set.

Note that function (7) in a Hilbert space (or, more generally, in a uniformly convex Banach space) satisfies (A1).

Proposition 3.2 ([3, Theorem 3.6]). *Let a function $\gamma : E \to \mathbb{R}$ satisfy* (A1)*, $f \in \mathcal{WC}(\gamma)$, $t \in (0, 1)$. Then*

(*i*) *the function $f \boxplus \gamma_t$ is real-valued and continuous on E;*

(*ii*) *for any $x_0 \in E$ the infimal convolution problem*

$$\min_{u \in E} \left(f(u) + \gamma_t(x_0 - u)\right) \tag{13}$$

has a unique solution $u_{\min}(x_0)$ and the function $u_{\min} : E \to E$ is continuous.

4. Main results

Theorem 4.1. *If a function $\gamma : E \to \mathbb{R}$ satisfies* (A1)*, $f \in \mathcal{WC}(\gamma_t)$, $0 < s < t$, then $f \boxplus \gamma_s \in \mathcal{WC}(\gamma_{t-s})$.*

Theorem 4.2. *If a function $\gamma : E \to \mathbb{R}$ satisfies* (A1)*, then $\mathcal{WC}(\gamma) \subset \mathcal{CWC}(\gamma)$.*

Example 4.5 shows that for a function γ satisfying (A1), the classes $\mathcal{WC}(\gamma)$ and $\mathcal{CWC}(\gamma)$ may be different.

Theorem 4.3. *Let a function* $\gamma : E \to \mathbb{R}$ *satisfy* (A1). *Then for all* $s > 0$ *and* $t > 0$ *we have*

$$\mathcal{CWC}(\gamma_s) \cap \mathcal{CWC}_-(\gamma_t) = \mathcal{WC}(\gamma_s) \cap \mathcal{WC}_-(\gamma_t).$$

Theorem 4.4. *If a function* $\gamma : E \to \mathbb{R}$ *satisfies* (A1), *then*

$$\mathcal{RCWC}(\gamma) = \mathcal{WC}(\gamma).$$

Example 4.5. Note that the function $\gamma(x_1, x_2) = (x_1^2+x_2^2)^2$ in $\mathbb{R}^2$ satisfies (A1). According to (3), (4) the function

$$f(x_1, x_2) = \max\{-\gamma(x_1 + 1, x_2), -\gamma(x_1 - 1, x_2)\} = -\left((|x_1| - 1)^2 + x_2^2\right)^2$$

satisfies the inclusion $f \in \mathcal{CWC}(\gamma)$. We claim that $f \notin \mathcal{WC}(\gamma_t)$ for all $t > 0$. Assume the contrary: $f \in \mathcal{WC}(\gamma_{2t})$ for some $t > 0$. By Proposition 3.2 the problem

$$\min_{u_1,u_2\in\mathbb{R}} (f(u_1, u_2) + \gamma_t(-u_1, -u_2)) \tag{14}$$

has a unique solution $(\widehat{u}_1, \widehat{u}_2)$. Since the function $F(u_1, u_2) = f(u_1, u_2) + \gamma_t(-u_1, -u_2)$ is even (i.e. $F(-u_1, -u_2) = F(u_1, u_2)$), it follows that $(-\widehat{u}_1, -\widehat{u}_2)$ is also a solution of problem (14). Due to the uniqueness of the solution we have $(\widehat{u}_1, \widehat{u}_2) = (0, 0)$. Consequently, $F(0, u) \geq F(0, 0)$ for all $u \in \mathbb{R}$, i.e.

$$-(1 + u^2)^2 + \frac{1}{t^3}u^4 \geq -1, \quad \forall u \in \mathbb{R}. \tag{15}$$

However, $-(1+u^2)^2+\frac{1}{t^3}u^4 = -1-2u^2+o(u^2)$ as $u \to 0$. This contradicts (15). So, $f \notin \mathcal{WC}(\gamma_t)$ for all $t > 0$. By Theorem 4.4 we deduce that $\mathcal{RCWC}(\gamma) = \mathcal{WC}(\gamma)$ is a proper subset of $\mathcal{CWC}(\gamma)$.

5. Proofs

By the definitions it immediately follows that

$$f \boxplus g \boxminus g \leq f \leq f \boxminus g \boxplus g. \tag{16}$$

Proof of Lemma 1.1. According to (4) any set $\Phi' \subset \Phi(\gamma)$ is characterized by a subset $\Omega \subset E \times \mathbb{R}$ such that

$$\Phi' = \{\varphi : E \to \mathbb{R} \mid \exists (u, c) \in \Omega : \ \varphi(x) = c - \gamma(u - x), \ \forall x \in E\}.$$

Using (3), we deduce that any function $f \in \mathcal{CWC}(\gamma)$ is determined by a subset $\Omega \subset E \times \mathbb{R}$:

$$f(x) = \sup_{(u,c)\in\Omega} (c - \gamma(u - x)), \quad x \in E. \tag{17}$$

Now we prove the equivalence of (i)–(iii) by the following scheme: $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

$(i) \Rightarrow (ii)$. Assume that $f \in \mathcal{CWC}(\gamma)$. Thus, there exists $\Omega \subset E \times \mathbb{R}$ such that (17) holds true. Then for the function

$$g(u) = \sup\{c \in \mathbb{R} \mid (u, c) \in \Omega\}, \quad u \in E$$

we get

$$f(x) = \sup_{u \in E} \left(g(u) - \gamma(u - x)\right), \quad x \in E, \tag{18}$$

that is $f = g \boxminus \gamma$. As usual, we believe $\sup \emptyset = -\infty$.

$(ii) \Rightarrow (i)$. Now we suppose that $f = g \boxminus \gamma$ for some function $g : E \to \overline{\mathbb{R}}$. One can easily see that the set $\Omega = \{(u, c) \in E \times \mathbb{R} \mid c \le g(u)\}$ satisfies (17).

$(ii) \Rightarrow (iii)$. Suppose again that $f = g \boxminus \gamma$. Using the second inequality in (16), we get $g \le g \boxminus \gamma \boxplus \gamma = f \boxplus \gamma$. Consequently, $f = g \boxminus \gamma \le f \boxplus \gamma \boxminus \gamma$. By the first inequality in (16) we obtain that $f = f \boxplus \gamma \boxminus \gamma$.

$(iii) \Rightarrow (ii)$. It suffices to take $g = f \boxplus \gamma$. $\square$

Proof of Lemma 1.2. Using the definitions of sup-inf convolutions (1), (2), one can easily obtain the equation

$$f \boxplus \beta_t \boxminus \beta_t + \beta_t = (f + \beta_t)^{**}.$$

Thus, the Fenchel–Moreau theorem yields the required statement. $\square$

Lemma 1.3 will be proved after the proof of Theorem 4.1.

Note that if the function $\gamma : E \to \overline{\mathbb{R}}$ is convex, then by (1) and (8) we have

$$\gamma_p \boxplus \gamma_s = \gamma_{p+s}, \quad \forall p > 0, \ s > 0. \tag{19}$$

Proof of Theorem 4.1. Let $x \in E$, $u \in \pi_{\gamma_{t-s}}(f \boxplus \gamma_s)(x)$. Then by (10) there exists $r \in (0, 1)$ such that $(f \boxplus \gamma_s \boxplus \gamma_p)(x + ru) = (f \boxplus \gamma_s)(x) + \gamma_p(ru)$, where $p = r(t - s)$. By Proposition 3.2 there exists $v \in E$ such that $(f \boxplus \gamma_s)(x) = f(x - sv) + \gamma_s(sv)$.

Putting $x_0 = x - sv$ and using (19), we have

$$\begin{aligned} &(f \boxplus \gamma_{s+p})(x_0 + ru + sv) = (f \boxplus \gamma_s \boxplus \gamma_p)(x + ru) \\ &= (f \boxplus \gamma_s)(x) + \gamma_p(ru) = f(x_0) + \gamma_s(sv) + \gamma_p(ru) \\ &= f(x_0) + s\gamma(v) + p\gamma\left(\frac{u}{t - s}\right). \end{aligned}$$

Suppose that $v \ne \frac{u}{t-s}$. Due to the uniform convexity of γ on any convex bounded set we have $s\gamma(v) + p\gamma\left(\frac{u}{t-s}\right) > (s + p)\gamma\left(\frac{w}{s+p}\right)$, where $w = sv + \frac{pu}{t-s} = sv + ru$.

Consequently, $(f \boxplus \gamma_{s+p})(x_0 + w) > f(x_0) + \gamma_{s+p}(w)$. This contradicts the definition of the infimal convolution (1). Therefore, $v = \frac{u}{t-s}$. Since $(f \boxplus \gamma_s)(x_0 + sv) = f(x_0) + \gamma_s(sv)$, it follows that $v \in \pi_\gamma f(x_0)$ and, consequently, $tv \in \pi_{\gamma_t} f(x_0)$. So, taking into account that $f \in \mathcal{WC}(\gamma_t)$, by (11) we get

$$(f \boxplus \gamma_t)(x_0 + tv) = f(x_0) + \gamma_t(tv) = f(x_0) + t\gamma(v).$$

Again, using the definition of the infimal convolution (1), we obtain

$$\begin{aligned}
&(f \boxplus \gamma_s)(x) + \gamma_{t-s}((t-s)v) \le f(x_0) + \gamma_s(sv) + \gamma_{t-s}((t-s)v) \\
&= f(x_0) + t\gamma(v) = (f \boxplus \gamma_t)(x_0 + tv) = (f \boxplus \gamma_s \boxplus \gamma_{t-s})(x_0 + tv) \\
&\le (f \boxplus \gamma_s)(x) + \gamma_{t-s}((t-s)v).
\end{aligned}$$

Thus,

$$\begin{aligned}
&(f \boxplus \gamma_s)(x) + \gamma_{t-s}(u) = (f \boxplus \gamma_t)(x_0 + tv) \\
&= (f \boxplus \gamma_t)(x + u) = (f \boxplus \gamma_s \boxplus \gamma_{t-s})(x + u).
\end{aligned}$$

So, by (11) the function $f \boxplus \gamma_s$ is weakly convex w.r.t. γ_{t-s}. By Proposition 3.2(i) the function $f \boxplus \gamma_s$ is continuous. Finally, $\operatorname{dom}(f \boxplus \gamma_s \boxplus \gamma_{t-s}) = \operatorname{dom}(f \boxplus \gamma_t) \neq \emptyset$. Thus, $f \boxplus \gamma_s \in \mathcal{WC}(\gamma_{t-s})$. □

Since the function $\beta(x) = \beta_1(x) = \frac{1}{2}\|x\|^2$ in a Hilbert space satisfies (A1), Lemma 1.3 follows from (12) and Theorem 4.1. Note that we haven't used Lemma 1.3 in the proof of Theorem 4.1, so the proof is correct.

Lemma 5.1. *Let a function $\gamma : E \to \mathbb{R}$ satisfy* (A1), $f \in \mathcal{WC}(\gamma)$. *Then*

$$(f \boxplus \gamma \boxminus \gamma)(x) \ge (f \boxplus \gamma_r)(x) - r\gamma(0), \quad \forall r \in (0,1),\ \forall x \in E.$$

Proof. Fix any $r \in (0,1)$ and $x \in E$. According to Proposition 3.2 there exists $x_0 \in E$ such that

$$(f \boxplus \gamma_r)(x) = f(x_0) + \gamma_r(x - x_0).$$

Putting $u = \frac{x - x_0}{r}$, we get $(f \boxplus \gamma_r)(x_0 + ru) = f(x_0) + \gamma_r(ru)$. By (10) we deduce that $u \in \pi_\gamma f(x_0)$. Using (11), we have

$$(f \boxplus \gamma)(x_0 + u) = f(x_0) + \gamma(u).$$

By (1) it follows that $(f \boxplus \gamma_r)(x) \le f(x_0) + \gamma_r(x - x_0) = f(x_0) + r\gamma(u)$. Consequently, using (2), we get

$$\begin{aligned}
&(f \boxplus \gamma \boxminus \gamma)(x) \ge (f \boxplus \gamma)(x_0 + u) - \gamma(x_0 + u - x) \\
&= f(x_0) + \gamma(u) - \gamma((1-r)u) \ge (f \boxplus \gamma_r)(x) + (1-r)\gamma(u) - \gamma((1-r)u) \\
&\ge (f \boxplus \gamma_r)(x) - r\gamma(0).
\end{aligned}$$

The last inequality is due to the convexity of γ. □

For $R > 0$ and $c \in E$ we denote by $\mathfrak{B}_R(c)$ the closed ball with center c and radius R.

Lemma 5.2. *Let a function $f : E \to \overline{\mathbb{R}}$ be lower semicontinuous at $x_0 \in E$. Let $\gamma : E \to \mathbb{R}$ be a convex, continuous and coercive function,* $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$. *Then*

$$\lim_{r \to +0} (f \boxplus \gamma_r)(x_0) = f(x_0).$$

Proof. First let us prove that for any $\varepsilon > 0$

$$\lim_{r \to +0} \inf_{x \in E \setminus \mathfrak{B}_\varepsilon(x_0)} (f(x) + \gamma_r(x_0 - x)) = +\infty. \tag{20}$$

Since the function γ is coercive, it follows that the function

$$\psi(t) = \inf_{x \in E \setminus \mathfrak{B}_t(0)} \frac{\gamma(x)}{\|x\|} \tag{21}$$

satisfies the following condition

$$\lim_{t \to +\infty} \psi(t) = +\infty. \tag{22}$$

Fix any $x_1 \in \operatorname{dom}(f \boxplus \gamma)$ and put $C_1 = (f \boxplus \gamma)(x_1)$. Fix also any $\varepsilon > 0$. Due to the continuity of γ at $x_1 - x_0$ there exist $\delta \in (0, \varepsilon)$ and $C_\gamma > 0$ such that

$$\gamma(x) \leq C_\gamma, \quad \forall x \in \mathfrak{B}_\delta(x_1 - x_0). \tag{23}$$

Let $r \in \left(0, \frac{1}{2}\right)$ be a sufficiently small number, namely $4r\|x_1 - x_0\| < \delta$. We claim that

$$\inf_{x \in E \setminus \mathfrak{B}_\varepsilon(x_0)} (f(x) + \gamma_r(x_0 - x)) \geq C_1 - C_\gamma + \frac{\delta}{4}\psi\left(\frac{\varepsilon}{r}\right). \tag{24}$$

Indeed, fix any $x \in E \setminus \mathfrak{B}_\varepsilon(x_0)$. Since $C_1 = (f \boxplus \gamma)(x_1)$, by (1) it follows that

$$f(x) \geq C_1 - \gamma(x_1 - x). \tag{25}$$

Let us put $t = \frac{r\delta}{4\|x_0 - x\|}$. The relations $x \in E \setminus \mathfrak{B}_\varepsilon(x_0)$ and $\delta \in (0, \varepsilon)$ imply that $t \in \left(0, \frac{r}{4}\right)$. Due to the convexity of γ we have

$$(r - t)\gamma\left(\frac{x_0 - x}{r}\right) - \gamma(x_1 - x) \geq -(1 - r + t)\gamma\left(\frac{x_1 - x_0 + \frac{t}{r}(x_0 - x)}{1 - r + t}\right). \tag{26}$$

Since

$$\left\|\frac{x_1 - x_0 + \frac{t}{r}(x_0 - x)}{1 - r + t} - (x_1 - x_0)\right\|$$
$$\leq \frac{(r - t)\|x_1 - x_0\|}{1 - r + t} + \frac{t\|x_0 - x\|}{r(1 - r + t)} < 2r\|x_1 - x_0\| + \frac{2t}{r}\|x_0 - x\| < \delta$$

by (23) and (26) it follows that

$$(r-t)\gamma\left(\frac{x_0-x}{r}\right)-\gamma(x_1-x)\geq-(1-r+t)C_\gamma\geq-C_\gamma.$$

This and (25) imply that

$$\begin{aligned}f(x)+\gamma_r(x_0-x)&=f(x)+r\gamma\left(\frac{x_0-x}{r}\right)\\ \geq C_1-C_\gamma+t\gamma\left(\frac{x_0-x}{r}\right)&=C_1-C_\gamma+\frac{r\delta}{4\|x_0-x\|}\gamma\left(\frac{x_0-x}{r}\right).\end{aligned}$$

Using (21), we get (24). The relations (22), (24) imply (20).

Since the function γ is convex, continuous and coercive, it follows that $\inf_{x\in E}\gamma(x)$ $=C_\gamma^{\min}\in\mathbb{R}$. Consequently,

$$\inf_{x\in\mathfrak{B}_\varepsilon(x_0)}(f(x)+\gamma_r(x_0-x))\geq\inf_{x\in\mathfrak{B}_\varepsilon(x_0)}f(x)+rC_\gamma^{\min}.$$

This and (20) yield

$$\liminf_{r\to+0}(f\boxplus\gamma_r)(x_0)\geq\inf_{x\in\mathfrak{B}_\varepsilon(x_0)}f(x),\quad\forall\varepsilon>0.$$

Since f is lower semicontinuous at x_0, passing to the limit as $\varepsilon\to+0$, we obtain

$$\liminf_{r\to+0}(f\boxplus\gamma_r)(x_0)\geq f(x_0).$$

Taking into account that

$$\limsup_{r\to+0}(f\boxplus\gamma_r)(x_0)\leq f(x_0)+\limsup_{r\to+0}\gamma_r(0)=f(x_0),$$

we complete the proof. □

Proof of Theorem 4.2. Let a function $\gamma:E\to\mathbb{R}$ satisfy (A1) and $f\in\mathcal{WC}(\gamma)$. Using Lemmas 5.1, 5.2, we get $f\boxplus\gamma\boxminus\gamma\geq f$. Taking into account the first inequality in (16), we obtain $f\boxplus\gamma\boxminus\gamma=f$. According to Lemma 1.1 we deduce that $f\in\mathcal{CWC}(\gamma)$. □

Lemma 5.3. *For any functions $f,g,h:E\to\mathbb{R}\cup\{+\infty\}$ and any vectors $x,y\in E$ the following inequality holds:*

$$(f\boxplus h)(x+y)\leq(f\boxminus g)(x)+(h\boxplus g)(y).$$

Proof. For any $u\in\operatorname{dom}g$ we have

$$\begin{aligned}(f\boxplus h)(x+y)&\leq f(x+u)+h(y-u)=f(x+u)-g(u)+h(y-u)+g(u)\\&\leq(f\boxminus g)(x)+h(y-u)+g(u).\end{aligned}$$

Consequently,

$$(f\boxplus h)(x+y)\leq(f\boxminus g)(x)+\inf_{u\in E}(h(y-u)+g(u))=(f\boxminus g)(x)+(h\boxplus g)(y).\quad\square$$

Lemma 5.4. *Let $\gamma : E \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function. Suppose that for some $u_0 \in \operatorname{dom}\gamma$ and $\delta > 0$ the function γ is uniformly convex on $\mathfrak{B}_\delta(u_0)$. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Suppose that for a vector $x_0 \in \operatorname{dom} f$ we have $u_0 \in \pi_\gamma f(x_0)$ and $(f \boxminus \gamma \boxplus \gamma)(x_0) = f(x_0)$. Then $(f \boxminus \gamma)(x_0 - u_0) = f(x_0) - \gamma(u_0)$.*

Proof. Since $(f \boxminus \gamma \boxplus \gamma)(x_0) = f(x_0) < f(x_0) + \frac{1}{k}$ by (1) it follows that for any $k \in \mathbb{N}$ there exists $u_k \in E$ such that

$$(f \boxminus \gamma)(x_0 - u_k) + \gamma(u_k) < f(x_0) + \frac{1}{k}. \tag{27}$$

Due to (10) and $u_0 \in \pi_\gamma f(x_0)$ there exists $r > 0$ such that

$$(f \boxplus \gamma_r)(x_0 + ru_0) = f(x_0) + \gamma_r(ru_0). \tag{28}$$

On the other hand, Lemma 5.3 gives $(f \boxplus \gamma_r)(x_0 + ru_0) \leq (f \boxminus \gamma)(x_0 - u_k) + (\gamma_r \boxplus \gamma)(ru_0 + u_k)$.

Thus, by (27), (28) we have

$$\gamma_r(ru_0) + \gamma(u_k) < (\gamma_r \boxplus \gamma)(ru_0 + u_k) + \frac{1}{k},$$

that is

$$r\gamma(u_0) + \gamma(u_k) < (1 + r)\gamma\left(\frac{ru_0 + u_k}{1 + r}\right) + \frac{1}{k}.$$

Taking into account that the function γ is convex and uniformly convex on $\mathfrak{B}_\delta(u_0)$, we deduce that $\lim_{k\to\infty} \|u_k - u_0\| = 0$. Since f is lower semicontinuous, it follows that $f \boxminus \gamma$ is lower semicontinuous as well. So, passing to the limit in (27), we obtain the inequality

$$(f \boxminus \gamma)(x_0 - u_0) + \gamma(u_0) \leq f(x_0).$$

The inverse inequality follows immediately from (2). $\square$

Proof of Theorem 4.3. Suppose that $f \in \mathcal{CWC}(\gamma_s) \cap \mathcal{CWC}_-(\gamma_t)$, $x \in E$, $u \in \pi_{\gamma_s} f(x)$. According to (10) there exists $r > 0$ such that $(f \boxplus \gamma_{rs})(x + ru) = f(x) + \gamma_{rs}(ru)$. Thus, for $r' = \frac{rs}{t}$ and $u' = \frac{t}{s}u$ we have $(f \boxplus \gamma_{r't})(x + r'u') = f(x) + \gamma_{r't}(r'u')$. Consequently,

$$u' \in \pi_{\gamma_t} f(x). \tag{29}$$

Since $f \in \mathcal{CWC}_-(\gamma_t)$ we have $f \boxminus \gamma_t \boxplus \gamma_t = f$. Using (29) and Lemma 5.4, we get $(f \boxminus \gamma_t)(x - u') = f(x) - \gamma_t(u')$. By (6) it means that $(f_- \boxplus \gamma_t)(-x + u') =$

$f_-(-x)+\gamma_t(u')$. Therefore, $(f_-\boxplus\gamma_{s\lambda})(-x+\lambda u) = f_-(-x)+\gamma_{s\lambda}(\lambda u)$ with $\lambda = \frac{t}{s}$. Consequently,

$$u \in \pi_{\gamma_s} f_-(-x). \tag{30}$$

Since $f \in \mathcal{CWC}(\gamma_s)$ by (6) and Lemma 1.1 it follows that $f_- \boxminus \gamma_s \boxplus \gamma_s = f_-$. Using (30) and Lemma 5.4, we get $(f_- \boxminus \gamma_s)(-x-u) = f_-(-x) - \gamma_s(u)$. By (6) it means that $(f \boxplus \gamma_s)(x+u) = f(x) + \gamma_s(u)$. Since $x \in E$, $u \in \pi_{\gamma_s} f(x)$ are arbitrary, we have proved that $f \in \mathcal{WC}(\gamma_s)$. The inclusion $f \in \mathcal{WC}_-(\gamma_t)$ can be proved analogously. □

Lemma 5.5. *Let a function $\gamma : E \to \mathbb{R}$ be convex. Suppose that for a function $f : E \to \mathbb{R} \cup \{+\infty\}$, vectors $x \in \operatorname{dom} f$, $u \in E$ and a number $T > 0$ we have $(f \boxplus \gamma_T)(x+Tu) = f(x) + \gamma_T(Tu)$. Then for any $t \in (0, T)$*

$$(f \boxplus \gamma_t)(x+tu) = f(x) + \gamma_t(tu), \quad u \in \pi_\gamma (f \boxplus \gamma_t)(x+tu).$$

Proof. By (1) and (19) we get

$$\begin{aligned} &f(x) + \gamma_T(Tu) = (f \boxplus \gamma_T)(x+Tu) \\ &= (f \boxplus \gamma_t \boxplus \gamma_{T-t})(x+Tu) \le (f \boxplus \gamma_t)(x+tu) + \gamma_{T-t}((T-t)u) \\ &\le f(x) + \gamma_t(tu) + \gamma_{T-t}((T-t)u) = f(x) + \gamma_T(Tu). \end{aligned}$$

Hence all inequalities in this chain became equalities. Using (10), we complete the proof. □

Lemma 5.6. *Let a function $\gamma : E \to \mathbb{R}$ be convex, $0 < t < T$. Then $\mathcal{CWC}(\gamma_T) \subset \mathcal{CWC}(\gamma_t)$.*

Proof. Assume that $f \in \mathcal{CWC}(\gamma_T)$. According to Lemma 1.1 there exists a function $g : E \to \overline{\mathbb{R}}$ such that $f = g \boxminus \gamma_T$. Using (19) we get $f = g \boxminus (\gamma_{T-t} \boxplus \gamma_t) = (g \boxminus \gamma_{T-t}) \boxminus \gamma_t$. Again by Lemma 1.1 we complete the proof. □

Lemma 5.7. *Let a function $\gamma : E \to \mathbb{R}$ satisfy* (A1)*. For a function $f : E \to \mathbb{R} \cup \{+\infty\}$ the following assertions are equivalent:*

(*i*) $f \in \mathcal{WC}(\gamma)$;
(*ii*) $f \in \mathcal{CWC}(\gamma)$ *and* $f \boxplus \gamma_t \in \mathcal{CWC}(\gamma_{1-t})$ *for all* $t \in (0, 1)$;
(*iii*) $f \in \mathcal{RCWC}(\gamma)$.

Proof. (*i*) ⇒ (*ii*). Assume that $f \in \mathcal{WC}(\gamma)$. By Theorems 4.1 and 4.2 for all $t \in (0, 1)$ we obtain $f \boxplus \gamma_t \in \mathcal{WC}(\gamma_{1-t}) \subset \mathcal{CWC}(\gamma_{1-t})$.

(*ii*) ⇒ (*iii*). According to Lemma 5.6 the assertion (*ii*) implies that for all $\varepsilon \in (0, 1)$ and $t \in (0, \varepsilon)$ we have $f \boxplus \gamma_t \in \mathcal{CWC}(\gamma_{1-t}) \subset \mathcal{CWC}(\gamma_{1-\varepsilon})$. Taking $\delta = \varepsilon$, we get (9).

(*iii*) ⇒ (*i*). Suppose that $f \in \mathcal{RCWC}(\gamma)$. Fix any $x \in \operatorname{dom} f$, $u \in \pi_\gamma f(x)$ and $\varepsilon \in (0, 1)$. According to (9) there exists $\delta \in (0, 1)$ such that for all $t \in (0, \delta)$ we have $f \boxplus \gamma_t \in \mathcal{CWC}(\gamma_{1-\varepsilon})$. By (6) and Lemma 1.1 we get $f \boxplus \gamma_t \in \mathcal{CWC}_-(\gamma_t)$. Using Theorem 4.3, we deduce that $f \boxplus \gamma_t \in \mathcal{CWC}(\gamma_{1-\varepsilon}) \cap \mathcal{CWC}_-(\gamma_t) = \mathcal{WC}(\gamma_{1-\varepsilon}) \cap \mathcal{WC}_-(\gamma_t)$. So,

$$f \boxplus \gamma_t \in \mathcal{WC}(\gamma_{1-\varepsilon}), \quad \forall t \in (0, \delta). \tag{31}$$

Since $u \in \pi_\gamma f(x)$, by (10) there exists $r > 0$ such that

$$(f \boxplus \gamma_r)(x + ru) = f(x) + \gamma_r(ru).$$

Take any t such that $0 < t < \min\{r, \delta\}$. According to Lemma 5.5 we have $(f \boxplus \gamma_t)(x + tu) = f(x) + \gamma_t(tu)$ and $u \in \pi_\gamma(f \boxplus \gamma_t)(x + tu)$. Hence, $(1 - \varepsilon)u \in \pi_{\gamma_{1-\varepsilon}}(f \boxplus \gamma_t)(x + tu)$. By (11) and (31) we obtain

$$\begin{aligned} &(f \boxplus \gamma_t \boxplus \gamma_{1-\varepsilon})(x + (t + 1 - \varepsilon)u) \\ &= (f \boxplus \gamma_t)(x + tu) + \gamma_{1-\varepsilon}((1 - \varepsilon)u) = f(x) + \gamma_t(tu) + \gamma_{1-\varepsilon}((1 - \varepsilon)u). \end{aligned}$$

So, $(f \boxplus \gamma_{t+1-\varepsilon})(x + (t + 1 - \varepsilon)u) = f(x) + \gamma_{t+1-\varepsilon}((t + 1 - \varepsilon)u)$. Using Lemma 5.5 once more, we deduce that

$$(f \boxplus \gamma_{1-\varepsilon})(x + (1 - \varepsilon)u) = f(x) + \gamma_{1-\varepsilon}((1 - \varepsilon)u), \quad \forall \varepsilon \in (0, 1).$$

Consequently,

$$\begin{aligned} &f(x) + \gamma_{1-\varepsilon}((1 - \varepsilon)u) \\ &\le f(x_1) + \gamma_{1-\varepsilon}(x + (1 - \varepsilon)u - x_1), \quad \forall x_1 \in E, \ \forall \varepsilon \in (0, 1), \end{aligned}$$

that is

$$\begin{aligned} &f(x) + (1 - \varepsilon)\gamma(u) \\ &\le f(x_1) + (1 - \varepsilon)\gamma\left(\frac{x - x_1}{1 - \varepsilon} + u\right), \quad \forall x_1 \in E, \ \forall \varepsilon \in (0, 1). \end{aligned}$$

Passing to the limit as $\varepsilon \to +0$ and using the continuity of γ, we get

$$f(x) + \gamma(u) \le f(x_1) + \gamma(x - x_1 + u), \quad \forall x_1 \in E.$$

So, $(f \boxplus \gamma)(x + u) = f(x) + \gamma(u)$ for all $x \in \operatorname{dom} f$, $u \in \pi_\gamma f(x)$. By (11) the function f is weakly convex w.r.t. γ.

Since $f \in \mathcal{CWC}(\gamma)$, it follows that $\operatorname{dom} f \neq \emptyset$ and by Lemma 1.1 we have $f \boxplus \gamma \boxminus \gamma = f$. Hence, $\operatorname{dom}(f \boxplus \gamma) \neq \emptyset$. Finally, the function $f \in \mathcal{CWC}(\gamma)$ is lower semicontinuous as the supremum of continuous functions $\varphi(x) = c - \gamma(u - x)$ (see (3), (4)). Thus, $f \in \mathcal{WC}(\gamma)$. □

Theorem 4.4 follows directly from Lemma 5.7.

Acknowledgements. The author thanks the referee for useful comments.

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