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Bi- and Multifocal Curves and Surfaces for Gauges

Thomas Jahn
Faculty of Mathematics, Technische Universität Chemnitz, 09107 Chemnitz, Germany
thomas.jahn@mathematik.tu-chemnitz.de

Horst Martini
Faculty of Mathematics, Technische Universität Chemnitz, 09107 Chemnitz, Germany
horst.martini@mathematik.tu-chemnitz.de

Christian Richter
Institute of Mathematics, Friedrich Schiller University, 07737 Jena, Germany
christian.richter@uni-jena.de

Dedicated to the memory of Jean Jacques Moreau.

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Combining methods from analysis and (convex) geometry, we derive geometric properties of generalizations of the following four classical types of bifocal curves: ellipses, hyperbolas, Cassini curves, and Apollonius circles. It is natural to consider them together, since the points of one such curve have (in the same sequence) constant sum, constant difference, constant product, and constant ratio of distances to the two given foci. Our generalization of this classical concept is two-fold: first we extend the used distance functions from the Euclidean and the (in the literature only partially investigated) normed case to gauges, where the unit ball no longer has to be symmetric with respect to the origin. And second we study, in the cases where it makes sense (ellipses and Cassini curves), even multifocal curves. Most of our investigations are even extended to higher dimensions. We also survey known results about these classes of curves which are close to our purpose (namely, referring to the normed case, to multifocality, and to higher dimensions), since no comparable survey exists and these results are widespread and not systematized in the literature.

Keywords: Apollonius circle, bifocal curve, Birkhoff orthogonality, (multifocal) Cassini curve, convex distance function, (multifocal) ellipse, gauge, (generalized) hyperbola, generalized Minkowski space, multifocal curve, n-lemniscate, sublevel set, Voronoi cell

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1. Introduction

In the classical theory of curves, there are several examples of curves which deserve to be called *bifocal curves.* Such curves can occur as loci of points in the plane whose sum, difference, product or ratio of distances from two fixed points – called *foci* – are constant.

Ellipses are sets of points which have a *constant sum of distances* from the two given foci. They appear in almost all fields of mathematics. For example, ellipses are the unit circles of two-dimensional inner product spaces, they occur as plane sections of cones in $\mathbb{R}^3$, they model planetary orbits in astronomy, and they can be constructed via the gardener's string construction (for these and many further properties see [3, Chapters 15–17]).

Hyperbolas are sets of points which have a *constant* (absolute value of the) *difference of distances* from the two fixed foci. They also appear as plane sections of cones in $\mathbb{R}^3$, they come along with interference phenomena in physics, they model interplanetary trajectories in manned spaceflight etc. (see again [3, Chapters 15–17] for related material).

Cassini curves are named after the Italian astronomer G. D. Cassini. They are defined as loci of points which have a *constant product of distances* from two fixed points [41, 23, 39, 78]. In differential and algebraic geometry, they can be found as plane sections of tori (see [8, p. 17]). The most prominent member of the family of Cassini curves is the lemniscate of Bernoulli, presenting also the intersection with many other interesting classes of curves.

The locus of all points in the plane having *constant ratio of distances* from two given points is a circle. For varying ratio of distances these *Apollonius circles* form so-called hyperbolic pencils of coaxial circles (cf. [12, § 6.6] and [63, § 8.3, Chapter III]).

The natural idea of considering these four classes of curves together (since their definitions correspond to the four basic arithmetic operations) was already proposed in [56]. In the present paper, we extend them to *bi- and* (in two cases) *multifocal curves in generalized Minkowski planes.*

That is, we consider sets of points whose sum and product of distances from a finite number of foci is constant (in the two other cases we naturally stay with two foci). In addition, we replace the Euclidean distance measurement by the Minkowski functional of a compact, convex set B in $\mathbb{R}^2$ having the origin 0 as interior point. These are precisely the non-negative, positively homogeneous, and convex functions which vanish only at the origin. For such distance functions also the synonyms *gauge* and *convex distance function* occur in the literature. Since gauges are a straightforward generalization of norms obtained by dropping the symmetry condition, it is a promising task to study finite-dimensional vector spaces endowed with gauges as direct generalizations of the concept of normed spaces. Since the latter are also called Minkowski spaces, it is natural to call finite-dimensional vector spaces endowed with a gauge *generalized Minkowski* *spaces* (and for two dimensions *generalized Minkowski planes*). Topological notions with respect to a gauge, like convergence or continuity, coincide with the usual Euclidean ones. For the sake of convenience, we adopt the vocabulary from normed spaces (like, e.g., the notion of unit ball). The reader should notice that we show gauges already in the figures in our introducing sections for every class of curves, i.e., before we come to our new results really referring to gauges (see, e.g., Figures 3.1 and 3.2 below).

Our results clearly demonstrate how complicated the geometry, and in some cases even the topology, of the investigated types of curves can be. Besides a bunch of results on the starshapedness of related sublevel sets we prove also several characterizations of special classes of gauges, e.g., characterizing inner product spaces or strictly convex gauges. We also give generalizations of classical geometric topics, such as Voronoi regions and orthogonality of confocal conics (using, in the latter case, Birkhoff orthogonality extended for gauges). In addition, we show how exotic the connectedness of Cassini sublevel sets can be.

In some cases, we assign to each of the foci its own gauge. In other words, we practically combine the geometry of several convex bodies with non-empty interior. As already mentioned, we also extend our results, whenever it is possible, to higher dimensions.

2. Preliminaries

Although we mainly work in the two-dimensional vector space $\mathbb{R}^2$, we introduce basic notions for the d-dimensional space $\mathbb{R}^d$. Thus, let $\mathbb{R}^d$ be equipped with the topology induced by the standard inner product $\langle x \,|\, y\rangle = \sum_{i=1}^d \xi_i\eta_i$ for $x = (\xi_i)_{i=1}^d$, $y = (\eta_i)_{i=1}^d$. For $\alpha \in \mathbb{R}$ and $A, B \subseteq \mathbb{R}^d$, set $\alpha A + B = \{\alpha a + b \,|\, a \in A, b \in B\}$. If $B = \{b\}$ is a singleton, we write $A + b$ rather than $A + \{b\}$. *Segments* or *intervals* shall be denoted by $[x, y]$, (x, y), $[x, y)$, or $(x, y]$ corresponding to which of the endpoints $x, y \in \mathbb{R}^d$ is contained. A set $B \subseteq \mathbb{R}^d$ is called *convex* iff $\lambda B + (1-\lambda)B \subseteq B$ for all $\lambda \in [0, 1]$. The standing assumption of our paper is that B is convex, compact, and contains the origin 0 as interior point. The straight line passing through two distinct points shall be abbreviated by $\langle x, y\rangle$, the ray starting at x and passing through y by $[x, y\rangle$. As usual, the abbreviations bd, int, co, and lin stand for *boundary*, *interior*, *convex hull*, and *linear hull*, respectively. If $f : \mathbb{R}^d \to \mathbb{R}$ is a function and $\alpha \in \mathbb{R}$ a number, we denote its *level set* by $f_{=\alpha} = \left\{x \in \mathbb{R}^d \,\middle|\, f(x) = \alpha\right\}$ and its *sublevel set* by $f_{\leq\alpha} = \left\{x \in \mathbb{R}^d \,\middle|\, f(x) \leq \alpha\right\}$.

A *gauge* is a function $\gamma : \mathbb{R}^d \to \mathbb{R}$ which vanishes only at 0, is positively homogeneous ($:\Leftrightarrow \gamma(\alpha x) = \alpha\gamma(x)$ for all $x \in \mathbb{R}^d$ and $\alpha > 0$), and convex ($:\Leftrightarrow \gamma(\alpha x + (1-\alpha)y) \leq \alpha\gamma(x) + (1-\alpha)\gamma(y)$ for all $x, y \in \mathbb{R}^d$ and $\alpha \in [0, 1]$). There is a one-to-one correspondence between gauges and sets B satisfying our standing assumption via the relations $\gamma(x) = \inf\{\lambda > 0 \,|\, x \in \lambda B\}$ and $B = \left\{x \in \mathbb{R}^d \,\middle|\, \gamma(x) \leq 1\right\}$. If we want to emphasize the dependence on B, we shall write γ_B instead of γ. Finite-dimensional real vector spaces endowed with a gauge shall be called *generalized Minkowski spaces* and are denoted by $\left(\mathbb{R}^d, \gamma\right)$. In the subcase of a (normed or) *Minkowski space* with norm $\|\cdot\|$ we simply write $\left(\mathbb{R}^d, \|\cdot\|\right)$. Adopting common notions from the theory of normed spaces encourages to get familiar with the general theory. A *ball* is a set of the form

$$B(x,\alpha) = x + \alpha B = \left\{ y \in \mathbb{R}^d \,\middle|\, \gamma_B(y-x) \leq \alpha \right\},$$

with $x \in X$ and $\alpha \geq 0$. The respective *sphere* is

$$S(x,\alpha) = x + \alpha \operatorname{bd}(B) = \left\{ y \in \mathbb{R}^d \,\middle|\, \gamma_B(y-x) = \alpha \right\}.$$

In particular, the set $B = \left\{ y \in \mathbb{R}^d \,\middle|\, \gamma_B(y) \leq 1 \right\}$ is called *unit ball*, and its boundary *unit sphere*, of our space. In order to avoid confusion when multiple gauges are involved, the notations $B_\gamma(x,\alpha)$ and $S_\gamma(x,\alpha)$ will be used to stress the respective gauge γ.

The gauge γ_B is said to be *strictly convex* iff $\gamma_B(x) = \gamma_B(y) = \gamma_B(\frac{1}{2}x + \frac{1}{2}y)$ implies $x = y$. Geometrically, this is equivalent to the strict convexity of the unit ball B, i.e., the non-existence of non-degenerate segments contained in $\operatorname{bd}(B)$. (Note that our notion of strictly convex gauges differs from the usual notion of strictly convex functions f for which $f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y)$ for all $x, y \in \mathbb{R}^d$, $x \neq y$, and $\alpha \in (0,1)$.) We say that a hyperplane H supports a set B iff one of the closed half-spaces bounded by B fully contains B, but none of the correspondingly open half-spaces does. The corresponding *supporting set* $H \cap B$ is called a $(d-1)$-*face* of B if the affine dimension of $H \cap B$ is $d-1$ (see [72, pp. 18, 62, 63] and [33, Lemma 3.6]).

3. Multifocal Ellipses

It is well-known that ellipses can be defined as planar curves whose points have constant sum of distances to two given points p_1, p_2 (called foci), this distance sum being larger than the distance of the foci to each other. This definition trivially yields the famous gardener's string construction of ellipses, and it is well known that they also occur as special conic sections. These and many further properties of ellipses are folklore and collected in [3, Chapters 15–17]. We survey now the less known properties of ellipses with an arbitrary number $n > 1$ of foci, called *multifocal ellipses* or, shortly, n-*ellipses*, and of such curves in normed planes. The literature on these curves is widespread, not systematized and presents therefore a typical example of *reinventing the wheel* again and again. Our Figure 3.1 shows pencils of bifocal ellipses and the respective unit circles for the Euclidean norm, a non-Euclidean normed plane and two gauges, these latter two cases referring to the new concepts introduced in the present paper and in [33].

The history of the famous Fermat–Torricelli problem started with de Fermat and Descartes who proposed to study 4-focal ellipses [4, p. 235]. The Saxonian

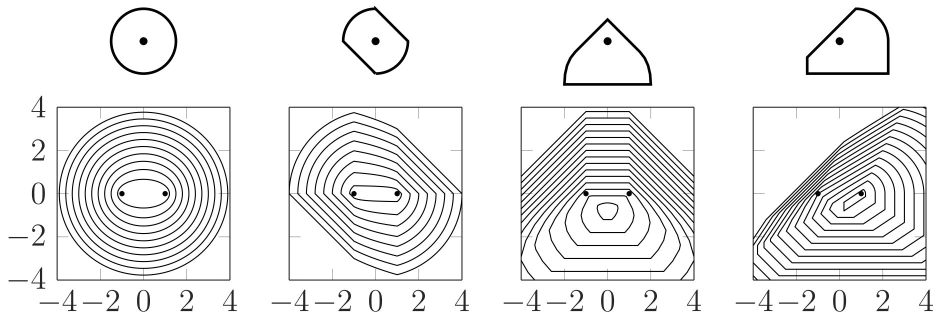


Figure 3.1: Four unit circles and corresponding bifocal ellipses with foci $(1, 0)$ and $(-1, 0)$.

polymath von Tschirnhaus was the first who extended the gardener's string construction to multifocal ellipses [90]. Constructions of their tangents and normals can be found already in books like [70, 62]. The possibly first thorough study of multifocal ellipses is the dissertation [80], and in classical books on algebraic curves (see [41, § V.6], [100, p. 84], and [21, § VII.21]) n-ellipses are classified as special generalized polyzomal curves, which go back to Cayley and are themselves defined by using ternary forms of equal degree no two of which have a quadratic quotient. As special polyzomal curves, the so-called curves of constant potential (see again the monographs above, but in particular [71] and also our Section 5) unify n-ellipses with curve classes like that of Cassini ovals. With this background, already in [71] multifocal ellipses were studied regarding normals, cusps, and their properties as level curves of convex functions. Further contributions to the geometry of these curves over the 19th and 20th century, mostly not citing each other, are [87, 88, 27, 52, 17, 73, 104] and Pick's mathematical appendix to the influential monograph [22] (this list is not complete). In these papers basic geometric properties of n-ellipses are derived, many of them repeatedly. In Figure 3.2 one sees pencils of 4-ellipses for the same norms and gauges as in Figure 3.1.

For example, also for $n > 2$ these curves are strictly convex in the Euclidean plane, but do no longer have direct optical interpretations (for $n = 2$ really justifying the name "foci"). Their normal vector at a regular curve point x is the sum s of the unit vectors from x to the n foci, and they have cusps exactly at foci p_i where the sum vector s_i of the unit vectors from the focus p_i to the other foci is halving the corresponding angle of the half-tangents at these cusps. The latter statements explain nice curvature properties of these curves, and also their gradient properties, useful for Weiszfeld's procedure in location science (see again [4, Chapter II]), are obvious. All these results have natural analogues in higher dimensions; see, e.g., [87, 88, 27, 104]. Furthermore, in [38] highly symmetric n-ellipses (with foci at vertices of regular n-gons) are investigated regarding limiting eccentricity.

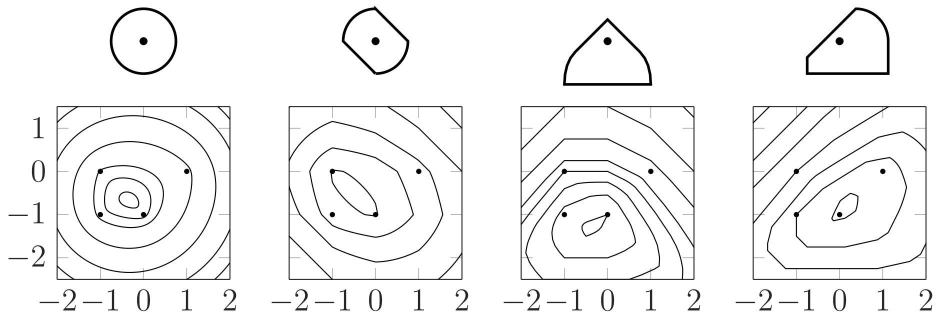


Figure 3.2: Four unit circles and corresponding 4-focal ellipses with foci $(1,0)$, $(-1,0)$, $(-1,-1)$, and $(0,-1)$.

The paper [19] makes the geometry of n-focal ellipses attractive for approximation theory: the authors show that multifocal ellipses are useful as approximation curves for arbitrary simple closed curves that contain at most one segment, but are everywhere else strictly convex (i.e., they do not contain any further segments).

These investigations were continued in [91], where also lower and upper bounds for the curvature of n-ellipses, involving explicit constants, are given. They depend on the number of foci, the rate of the level, and the global minimum of the function measuring the sum of distances. Having also approximation in mind, [64] presents various multifocal surfaces (defined by constant distance sums and products).

A deeper study of n-ellipses in the spirit of algebraic geometry is given in [58] (see also [59] and [82]; weighted foci and higher-dimensional analogues are studied, too). Their defining polynomial equation has degree 2^n if n is odd, and $2^n - \binom{n}{n/2}$ if n is even, and this polynomial equation is expressed as the determinant of a symmetric matrix of linear polynomials, yielding new geometric applications of semidefinite programming.

We now come to generalizations of ellipses in other directions. Since ellipses can also be presented as sets of points being equidistant to two non-concentric circles, the authors of [65] propose to take the notion of equidistant set as a natural starting point for generalizing conics, such that classical properties of conics can be retrieved from more general equidistant sets. In [94] a generalized conic is introduced as set of points with the same average distance from a (not necessarily finite) point set in $\mathbb{R}^d$, and this concept is used to construct (non-Euclidean) Minkowski functionals from generalized conics. Applications of so-defined generalized conics in geometric tomography are derived in [95].

The authors of [25, 26] go further, proposing the combination of multifocality with the concept of normed spaces; see also [31, 20, 49] for a deeper study of the bifocal subcases. In [33] the notion of multifocal ellipses is extended in a two-fold way: the given foci can be general convex sets, and the distance function used for the defining constant distance sums is determined by a gauge (also for higher dimensions). Thus, for completing the picture of our results below, the reader should also consult [33]. Continuing parts of this paper, we here present two new results on higher-dimensional analogues of multifocal ellipses with respect to gauges. Namely, in [87] it was shown that for any higher-dimensional analogue of a multifocal ellipse the following holds: the normal at any regular point meets the convex hull of the foci. This is the Euclidean version of the limit case of the result proved below for gauges.

Definition 3.1. The *bisector* of two distinct points $p_1, p_2 \in \mathbb{R}^d$ with respect to a gauge γ is the set

$$\operatorname{bis}(p_1, p_2) := \left\{ x \in \mathbb{R}^d \,\middle|\, \gamma(x - p_1) = \gamma(x - p_2) \right\}.$$

We write $\operatorname{bis}_\gamma(p_1, p_2)$ if it is not apparent from the context which gauge we use.

Theorem 3.2. *Let $p_1, \ldots, p_n$, $n \geq 1$, be (not necessarily different) points of a generalized Minkowski space $\left(\mathbb{R}^d, \gamma\right)$. If e_1, e_2 are points of the same level set of the function $f : \mathbb{R}^d \to \mathbb{R}, f(x) = \sum_{i=1}^n \gamma(x - p_i)$, then the bisector* $\operatorname{bis}_{\tilde{\gamma}}(e_1, e_2)$ *of e_1 and e_2 with respect to the opposite gauge $\tilde{\gamma} : \mathbb{R}^d \to \mathbb{R}, \tilde{\gamma} := \gamma(-x)$, has non-empty intersection with* $\operatorname{co}\{p_1, \ldots, p_n\}$.

Proof. Since e_1 and e_2 belong to the same level set, we have

$$\sum_{i=1}^n \gamma(e_1 - p_i) = \sum_{i=1}^n \gamma(e_2 - p_i).$$

Then there exist $i, j \in \{1, \ldots, n\}$ such that

$$\gamma(e_1 - p_i) \leq \gamma(e_2 - p_i) \quad \text{and} \quad \gamma(e_1 - p_j) \geq \gamma(e_2 - p_j).$$

The intermediate value theorem gives a point $x \in [p_i, p_j] \subseteq \operatorname{co}\{p_1, \ldots, p_n\}$ such that

$$\gamma(e_1 - x) = \gamma(e_2 - x).$$

The last equation says that $\tilde{\gamma}(x - e_1) = \tilde{\gamma}(x - e_2)$; i.e., $x \in \operatorname{bis}_{\tilde{\gamma}}(e_1, e_2)$. $\square$

Corollary 3.3. *Let $p_1, \ldots, p_n$, $n \geq 1$, be points of a Minkowski space $\left(\mathbb{R}^d, \|\cdot\|\right)$. If e_1, e_2 are points of the same level set of $f : \mathbb{R}^d \to \mathbb{R}, f(x) = \sum_{i=1}^n \|x - p_i\|$, then* $\operatorname{bis}(e_1, e_2) \cap \operatorname{co}\{p_1, \ldots, p_n\} \neq \emptyset$.

The last claim is not true for arbitrary gauges, as the following example will demonstrate.

Example 3.4. Let the plane $\mathbb{R}^2$ be equipped with the asymmetric gauge γ whose unit sphere is given by

$$S(0,1) = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 \,\middle|\, \xi_1^2 + (\xi_2 - 4)^2 = 5^2 \right\}.$$

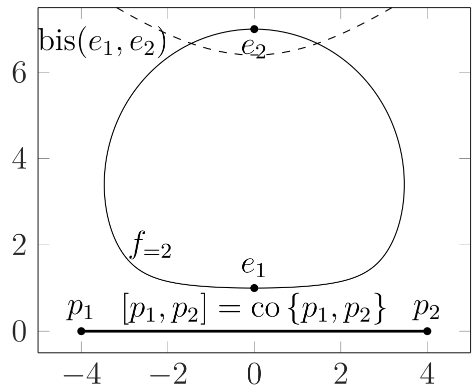


Figure 3.3: Illustration of Example 3.4.

Then $e_1 = (0, 1)$ and $e_2 = (0, 7)$ belong to the level set $f_{=2}$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \gamma(x - p_1) + \gamma(x - p_2)$ with $p_1 = (-4, 0)$ and $p_2 = (4, 0)$. On the other hand,

$$\mathrm{bis}(e_1, e_2) = \left\{ x \in \mathbb{R}^2 \,\middle|\, \gamma(x - e_1) = \gamma(x - e_2) \right\} = \bigcup_{\alpha>0} S(e_1, \alpha) \cap S(e_2, \alpha)$$

consists of points from intersections of pairs of distinct Euclidean circles of the same radius whose Euclidean centers are of the form $(0, \xi_2)$ with $\xi_2 \geq 1$. This shows that the second coordinates of all points from $\mathrm{bis}(e_1, e_2)$ are larger than 1. Thus

$$\mathrm{bis}(e_1, e_2) \cap \mathrm{co}\,\{p_1, p_2\} = \mathrm{bis}(e_1, e_2) \cap [(-4, 0), (4, 0)] = \emptyset,$$

see Figure 3.3.

Theorem 3.5. *The following statements are equivalent for every generalized Minkowski plane* $(\mathbb{R}^2, \gamma)$.

(a) *The gauge γ is not strictly convex.*

(b) *There exist points $p_1, \ldots, p_n \in \mathbb{R}^2$ such that some level set of the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \sum_{i=1}^{n} \gamma(x - p_i)$, contains a non-degenerate segment $[e_1, e_2]$ satisfying $\{p_1, \ldots, p_n\} \not\subseteq \langle e_1, e_2 \rangle$.*

(c) *Given finitely many points $p_1, \ldots, p_n \in \mathbb{R}^2$, there exists $\alpha_0 \in \mathbb{R}$ such that, for every $\alpha > \alpha_0$, the level set $f_{=\alpha}$ of the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \sum_{i=1}^{n} \gamma(x - p_i)$, contains a non-degenerate segment.*

Proof. (a)$\Longrightarrow$(c). If γ is not strictly convex, there is a non-degenerate segment $[s, t] \subseteq S(0, 1)$. The cone $\bigcap_{i=1}^{n}(p_i + C)$, where $C := \{\lambda u \mid u \in [s, t], \lambda \geq 0\}$, has non-empty interior, and f is an affine function on this cone.

(c)$\Longrightarrow$(b). Consider $n = 1$, $p_1 = 0$ and pick some level $\alpha > \alpha_0$. Then $f_{=\alpha}$ contains a non-degenerate segment $[e_1, e_2]$. Since $f_{=\alpha} = S(p_1, \alpha)$, $p_1 \notin \langle e_1, e_2 \rangle$.

$(b) \Longrightarrow (a)$. By (b), there are distinct points $e_1, e_2 \in \mathbb{R}^2$ and a number $\alpha > 0$ such that $[e_1, e_2] \subseteq f_{=\alpha}$ and $\{p_1, \ldots, p_n\} \not\subseteq \langle e_1, e_2 \rangle$. Suitably shortening the segment, we can assume that $e_1, e_2 \notin \{p_1, \ldots, p_n\}$. For each $\lambda \in [0, 1]$, we get $f(\lambda e_1 + (1-\lambda)e_2) = \alpha = \lambda f(e_1) + (1-\lambda) f(e_2)$. Due to the triangle inequality, we have $\gamma(\lambda e_1 + (1-\lambda)e_2 - p_i) = \lambda \gamma(e_1 - p_i) + (1-\lambda)\gamma(e_2 - p_i)$ for each $i \in \{1, \ldots, n\}$. Taking [33, Lemma 3.5] into account, we obtain $\left[\frac{e_1 - p_i}{\gamma(e_1 - p_i)}, \frac{e_2 - p_i}{\gamma(e_2 - p_i)}\right] \subseteq S(0, 1)$ for each $i \in \{1, \ldots, n\}$. If a segment $\left[\frac{e_1 - p_i}{\gamma(e_1 - p_i)}, \frac{e_2 - p_i}{\gamma(e_2 - p_i)}\right]$ is degenerate, then $e_1 - p_i$ and $e_2 - p_i$ are linearly dependent, and in turn $p_i \in \langle e_1, e_2 \rangle$. The assumption $\{p_1, \ldots, p_n\} \not\subseteq \langle e_1, e_2 \rangle$ ensures that one of the segments is non-degenerate. Hence, γ is not strictly convex. □

The above observation gives rise to a characterization of strictly convex gauges by a property of bifocal ellipsoids.

Corollary 3.6. *The gauge γ of a generalized Minkowski space $\left(\mathbb{R}^d, \gamma\right)$ is not strictly convex if and only if there exist two distinct points $p_1, p_2 \in \mathbb{R}^d$ such that some level set of the function $f : \mathbb{R}^d \to \mathbb{R}$, $f(x) = \gamma(x - p_1) + \gamma(x - p_2)$, contains a non-degenerate segment $[e_1, e_2]$ satisfying $\langle p_1, p_2 \rangle \neq \langle e_1, e_2 \rangle$.*

Proof. If γ is not strictly convex, then there exists a two-dimensional subspace $X \subseteq \mathbb{R}^d$ such that $\gamma|_X$ is not strictly convex. We apply $(a) \Longrightarrow (c)$ from Theorem 3.5. Then there exist two points $p_1, p_2 \in X$ such that the corresponding function f has several level sets that contain non-degenerate segments. Since the restriction $f|_{\langle p_1, p_2 \rangle}$ is convex, at most one level set of f contains a non-degenerate segment of $\langle p_1, p_2 \rangle$. Thus, there is another level set satisfying the claim of the corollary.

The proof of the converse implication follows the lines of the proof of $(b) \Longrightarrow (a)$ from Theorem 3.5. □

4. Hyperbolas

Hyperbolas again occur as special conic sections, e.g. defined via constancy of (the absolute value of) the difference of their distances to two given foci. Many properties and applications of them can also be found in [3, Chapters 15–17]. Although the combination of constancy of the *difference* of distances makes sense only for $n = 2$, one can consider constancy of the sum of $n > 1$ distances to n foci with a mixture of positive and negative weights. To some extent this general viewpoint was investigated, suggested, or at least mentioned by several authors [88, 52, 17], but there are no deeper results in this direction. However, in the two papers [25, 26] it was at least proposed to combine this with norms. Furthermore, [104] investigated generalized hyperbolas whose foci are convex sets. In [31] and [20] a purely geometric study of conics and, in particular, hyperbolas in normed planes is presented. Several definitions of hyperbolas, all of them equivalent in the Euclidean plane, are shown to yield partially different classes of curves in

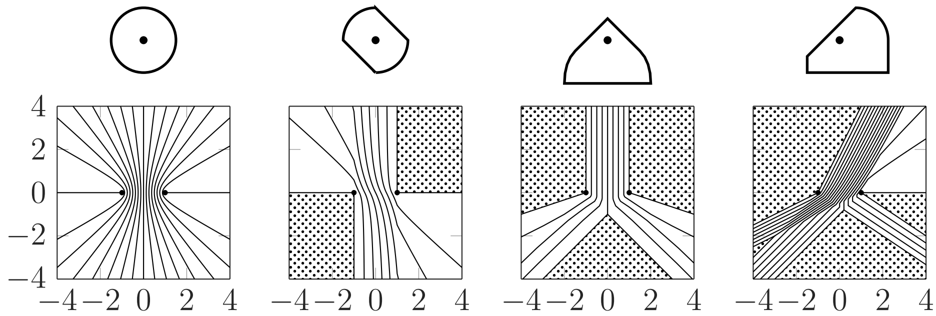


Figure 4.1: Four unit circles and corresponding hyperbolas with foci $(1, 0)$ and $(-1, 0)$.

normed planes, and in [20] it was proved that metrically defined hyperbolas for norms, which are not strictly convex, need not be curves, but may even have two-dimensional parts. Our Figure 4.1 shows pencils of hyperbolas for the same unit circles as in Figures 3.1 and 3.2. The shaded regions demonstrate that such "curves" may contain inner points.

Now we present new results on hyperbolas and their higher-dimensional analogues for gauges.

Lemma 4.1. *Let $\left(\mathbb{R}^d, \gamma\right)$ be a generalized Minkowski space and let $p_1, p_2 \in \mathbb{R}^d$, $p_1 \neq p_2$. Then the function $f : \mathbb{R}^d \to \mathbb{R}$, $f(x) = \gamma(x-p_1) - \gamma(x-p_2)$, is bounded from above by $\gamma(p_2 - p_1)$ and from below by $-\gamma(p_1 - p_2)$. These extremal values are attained, e.g., at p_2 and p_1, respectively.*

Proof. The estimates are obtained from the triangle inequalities

$$\begin{aligned} &\gamma(x - p_1) \leq \gamma(x - p_2) + \gamma(p_2 - p_1) \\ \text{and} \quad &\gamma(x - p_2) \leq \gamma(x - p_1) + \gamma(p_1 - p_2). \end{aligned} \qquad \square$$

Proposition 4.2. *The set of maximizers of the function f from Lemma* 4.1 *is the cone*

$$\begin{aligned} \arg\max f :=& \left\{ z \in \mathbb{R}^d \,\middle|\, f(z) = \max_{x \in \mathbb{R}^d} f(x) \right\} \\ =& \{p_2\} \cup \left\{ z \in \mathbb{R}^d \setminus \{p_2\} \,\middle|\, \left[\frac{z - p_2}{\gamma(z - p_2)}, \frac{p_2 - p_1}{\gamma(p_2 - p_1)} \right] \subseteq S(0, 1) \right\}. \end{aligned}$$

In particular, if γ is strictly convex, then $\arg\max f$ *is a ray.*

Proof. Let $z \in \arg\max f \setminus \{p_2\}$. Then $f(z) = \gamma(p_2 - p_1)$, which is equivalent to $\gamma(z-p_1) = \gamma(z-p_2)+\gamma(p_2-p_1)$. Due to [33, Lemma 3.5], this is equivalent to $\left[\frac{z-p_2}{\gamma(z-p_2)}, \frac{p_2-p_1}{\gamma(p_2-p_1)}\right] \subseteq S(0, 1)$. $\square$

For $p_1, p_2 \in \mathbb{R}^d$, $p_1 \neq p_2$, and $\alpha \in \mathbb{R}$, we define the *level set* and the *sublevel set for hyperbolas* by

$$H_{p_1,p_2,=\alpha} := \left\{ x \in \mathbb{R}^d \;\middle|\; \gamma(x-p_1) - \gamma(x-p_2) = \alpha \right\}$$

and

$$H_{p_1,p_2,\leq\alpha} := \left\{ x \in \mathbb{R}^d \;\middle|\; \gamma(x-p_1) - \gamma(x-p_2) \leq \alpha \right\},$$

respectively.

Lemma 4.3. *Let $(\mathbb{R}^2, \gamma)$ be a generalized Minkowski plane and let $h : \mathbb{R}^2 \to \mathbb{R}$ be a linear functional such that $h^{-1}(\{1\}) \cap S(0,1) = [s,t]$ is a non-degenerate segment (i.e., γ is not strictly convex). Then, for any two points $p_1, p_2 \in \mathbb{R}^2$, the level set $H_{p_1,p_2,=h(p_2-p_1)}$ contains a translate of the convex cone $C = \{\lambda u \mid u \in [s,t], \lambda \geq 0\}$; namely,*

$$(p_1 + C) \cap (p_2 + C) \subseteq H_{p_1,p_2,=h(p_2-p_1)}.$$

Proof. We have

$$\gamma(x) = h(x) \quad \text{for all } x \in C,$$

because $x = \lambda c \in \lambda[s,t] \subseteq S(0,\lambda)$ with $\lambda \geq 0$ and $c \in [s,t]$, and in turn $\gamma(x) = \lambda = \lambda h(c) = h(\lambda c) = h(x)$. Since every $y \in (p_1 + C) \cap (p_2 + C)$ satisfies $y - p_1, y - p_2 \in C$, we get

$$\gamma(y-p_1) - \gamma(y-p_2) = h(y-p_1) - h(y-p_2) = h(p_2-p_1).$$

This implies $(p_1 + C) \cap (p_2 + C) \subseteq H_{p_1,p_2,=h(p_2-p_1)}$. □

The following lemma (in a natural way called "Butterfly Lemma") is a direct extension of [47, Proposition 7] from norms to gauges.

Lemma 4.4 (Butterfly Lemma). *Suppose that $\{z, p_1, p_2, w\}$ are in successive order the vertices of a plane convex non-degenerate quadrilateral in a generalized Minkowski space $\left(\mathbb{R}^d, \gamma\right)$. Then*

$$\gamma(z-p_1) + \gamma(w-p_2) \leq \gamma(z-p_2) + \gamma(w-p_1),$$

with equality if and only if $\left[\frac{z-p_2}{\gamma(z-p_2)}, \frac{w-p_1}{\gamma(w-p_1)}\right] \subseteq S(0,1)$.

Proof. Let $\{s\} = [z, p_2] \cap [p_1, w]$, see Figure 4.2. Using the triangle inequality, we have

$$\begin{aligned}\gamma(z-p_1) + \gamma(w-p_2) &\leq \gamma(z-s) + \gamma(s-p_1) + \gamma(w-s) + \gamma(s-p_2) \\ &= \gamma(z-p_2) + \gamma(w-p_1).\end{aligned}$$

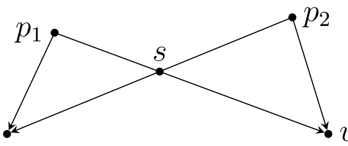


Figure 4.2: Illustration for Lemma 4.4.

Note that $\frac{z-s}{\gamma(z-s)} = \frac{s-p_2}{\gamma(s-p_2)} = \frac{z-p_2}{\gamma(z-p_2)}$ and $\frac{w-s}{\gamma(w-s)} = \frac{s-p_1}{\gamma(s-p_1)} = \frac{w-p_1}{\gamma(w-p_1)}$. Thus, if the segment $\left[\frac{z-p_2}{\gamma(z-p_2)}, \frac{w-p_1}{\gamma(w-p_1)}\right]$ is a subset of $S(0,1)$, then $\gamma(z-p_1) = \gamma(z-s)+\gamma(s-p_1)$ and $\gamma(w-p_2) = \gamma(w-s)+\gamma(s-p_2)$ by [33, Lemma 3.5]. It follows that

$$\begin{aligned}\gamma(z-p_1)+\gamma(w-p_2) &= \gamma(z-s)+\gamma(s-p_1)+\gamma(w-s)+\gamma(s-p_2)\\ &= \gamma(z-p_2)+\gamma(w-p_1).\end{aligned}$$

Conversely, if

$$\begin{aligned}\gamma(z-p_1)+\gamma(w-p_2) &= \gamma(z-p_2)+\gamma(w-p_1)\\ &= \gamma(z-s)+\gamma(s-p_1)+\gamma(w-s)+\gamma(s-p_2),\end{aligned}$$

then $\gamma(z-p_1) = \gamma(z-s)+\gamma(s-p_1)$ and $\gamma(w-p_2) = \gamma(w-s)+\gamma(s-p_2)$. (In both lines, the triangle inequality says "$\leq$". Since the sum of these two inequalities is in fact an equality, we have equality in the single inequalities.) Using [33, Lemma 3.5] again, we have $\left[\frac{z-p_2}{\gamma(z-p_2)}, \frac{w-p_1}{\gamma(w-p_1)}\right] \subseteq S(0,1)$. □

Lemma 4.5. *Let p_1, p_2 be two distinct points of a strictly convex generalized Minkowski space $\left(\mathbb{R}^d, \gamma\right)$. Then, for every $x \in \operatorname{bis}(p_1,p_2)$ and $\lambda > 0$,*

$$\gamma((x+\lambda(p_1-p_2))-p_1) < \gamma((x+\lambda(p_1-p_2))-p_2).$$

Proof. If $x \in \langle p_1, p_2\rangle$, then we have a one-dimensional problem on the straight line $\langle p_1, p_2\rangle$, and the claim is easily seen.

If $x \notin \langle p_1, p_2\rangle$, then the points $z := x+\lambda(p_1-p_2), p_1, p_2$ and $w := x$ are (in this cyclic order) the vertices of a convex quadrangle. Lemma 4.4 gives

$$\gamma((x+\lambda(p_1-p_2))-p_1)+\gamma(x-p_2) < \gamma((x+\lambda(p_1-p_2))-p_2)+\gamma(x-p_1)$$

with strict inequality, because $S(0,1)$ does not contain the segment

$$\left[\frac{(x+\lambda(p_1-p_2))-p_2}{\gamma((x+\lambda(p_1-p_2))-p_2)}, \frac{x-p_1}{\gamma(x-p_1)}\right]$$

by strict convexity of γ. Using $x \in \operatorname{bis}(p_1,p_2)$, i.e., $\gamma(x-p_1) = \gamma(x-p_2)$, we obtain our claim. □

Theorem 4.6. *The following statements are equivalent for every generalized Minkowski plane $(\mathbb{R}^2, \gamma)$.*

(*a*) *The gauge γ is not strictly convex.*

(*b*) *For all distinct points $p_1, p_2 \in \mathbb{R}^2$, there is a number $\alpha \in [-\gamma(p_1 - p_2), \gamma(p_2 - p_1)]$ and a straight line L parallel but not identical to $\langle p_1, p_2 \rangle$ such that the set $L \cap H_{p_1,p_2,=\alpha}$ contains at least two points.*

(*c*) *For all distinct points $p_1, p_2 \in \mathbb{R}^2$, there is a number $\alpha \in [-\gamma(p_1 - p_2), \gamma(p_2 - p_1)]$ such that $H_{p_1,p_2,=\alpha}$ has non-empty interior.*

(*d*) *There is a convex cone $C \subseteq \mathbb{R}^2$ with non-empty interior such that, for all distinct points $p_1, p_2 \in \mathbb{R}^2$, there exists a number $\alpha \in [-\gamma(p_1 - p_2), \gamma(p_2 - p_1)]$ such that $H_{p_1,p_2,=\alpha}$ contains a translate of C.*

(*e*) *There are distinct points $p_1, p_2 \in \mathbb{R}^2$, a number $\alpha \in [-\gamma(p_1 - p_2), \gamma(p_2 - p_1)]$ and a straight line L parallel but not identical to $\langle p_1, p_2 \rangle$ such that the set $L \cap H_{p_1,p_2,=\alpha}$ contains at least two points.*

(*f*) *There are distinct points $p_1, p_2 \in \mathbb{R}^2$ and a number $\alpha \in [-\gamma(p_1 - p_2), \gamma(p_2 - p_1)]$ such that $H_{p_1,p_2,=\alpha}$ has non-empty interior.*

(*g*) *There are distinct points $p_1, p_2 \in \mathbb{R}^2$ and a number $\alpha \in (-\gamma(p_1 - p_2), \gamma(p_2 - p_1)) \setminus \{0\}$ such that $H_{p_1,p_2,=\alpha}$ has non-empty interior.*

(*h*) *There are distinct points $p_1, p_2 \in \mathbb{R}^2$ such that $H_{p_1,p_2,=0} = \operatorname{bis}(p_1, p_2)$ has non-empty interior.*

Proof. Lemma 4.3 shows the implications from (*a*) to (*d*), (*g*) and (*h*). Indeed, (*a*)$\Longrightarrow$(*d*) is obvious. For (*a*)$\Longrightarrow$(*g*), we apply Lemma 4.3 to a maximal non-degenerate segment $[s, t] \subseteq S(0, 1)$ and two points p_1, p_2 such that $\langle p_1, p_2 \rangle$ is not parallel to $[s, t]$ (hence, $h(p_2 - p_1) \neq 0$) and $\frac{p_1 - p_2}{\gamma(p_1 - p_2)}, \frac{p_2 - p_1}{\gamma(p_2 - p_1)} \notin [s, t]$ (thus, $h(p_2 - p_1) \notin \{-\gamma(p_1 - p_2), \gamma(p_2 - p_1)\}$). Similarly, for (*a*)$\Longrightarrow$(*h*), we pick p_1, p_2 such that $\langle p_1, p_2 \rangle$ is parallel to $[s, t]$.

The implications (*d*)$\Longrightarrow$(*c*)$\Longrightarrow$(*b*)$\Longrightarrow$(*e*), (*g*)$\Longrightarrow$(*f*)$\Longrightarrow$(*e*) and (*h*)$\Longrightarrow$(*e*) are obvious. (Note that (*a*)$\Longleftrightarrow$(*h*) is already stated in [42, Corollary 2.1.1.2].)

It remains to show that (*e*)$\Longrightarrow$(*a*). By (*e*), there exist four distinct points p_1, p_2, w, z that are, in this cyclic order, the vertices of a convex quadrilateral with parallel edges $[p_1, p_2]$ and $[z, w]$ such that $z, w \in H_{p_1,p_2,=\alpha}$ for some real number α. The last inclusion yields

$$\gamma(z - p_1) + \gamma(w - p_2) = \gamma(z - p_2) + \gamma(w - p_1).$$

Now the Butterfly Lemma implies $\left[\frac{z - p_2}{\gamma(z - p_2)}, \frac{w - p_1}{\gamma(w - p_1)}\right] \subseteq S(0, 1)$. Thus, γ is not strictly convex. $\square$

Problem 4.7. *In every generalized Minkowski plane $(\mathbb{R}^2, \gamma)$, non-strict convexity of γ implies that there is a vector $v \in \mathbb{R}^2 \setminus \{0\}$ such that for all $p_1, p_2 \in \mathbb{R}^2$ there exists a number $\alpha \in [-\gamma(p_1 - p_2), \gamma(p_2 - p_1)]$ implying that $H_{p_1,p_2,=\alpha}$ contains a non-degenerate line segment parallel to v. This is a consequence of Theorem 4.6. Is the reverse implication also true?*

Closely related to the notion of bisector, Voronoi cells encode proximity to an element of a finite point set with respect to a gauge. Given a finite number of points $p_1, \dots, p_n \in \mathbb{R}^d$, the *Voronoi cell* of p_i is usually defined as

$$V_i := \left\{ x \in \mathbb{R}^d \,\middle|\, \gamma(x - p_i) \le \gamma(x - p_j)\, \forall\, j \in \{1, \dots, n\} \setminus \{i\} \right\}.$$

This notion is also important in computational geometry and location science; see [46, § 4] (for norms) and [57, § 3.2]. Generalizing a lemma from [10, p. 237], we show now that Voronoi cells (and certain relaxations) are star-shaped. Recall that a set $K \subseteq \mathbb{R}^d$ is called *star-shaped with respect to* $x \in \mathbb{R}^d$ if $[x, y] \subseteq K$ for all $y \in K$.

Lemma 4.8. *Let $p_1, \dots, p_n \in \mathbb{R}^d$ be arbitrary points, and $\gamma : \mathbb{R}^d \to \mathbb{R}$ be a gauge. Let us denote by $\star$ either addition or multiplication. If $\star$ is addition, let $\alpha \in \mathbb{R}$. Otherwise let $\alpha \in [0, 1]$. Then the set*

$$V_i^{\alpha,\star} := \left\{ x \in \mathbb{R}^d \,\middle|\, \gamma(x - p_i) \le \alpha \star \gamma(x - p_j)\, \forall\, j \in \{1, \dots, n\} \setminus \{i\} \right\}$$

is star-shaped with respect to p_i.

Proof. Let $x \in V_i^{\alpha,\star}$, $\mu \in [0, 1]$. We have

$$\begin{aligned} \gamma(x - p_i) &\le \alpha \star \gamma(x - p_j) \\ &= \alpha \star \gamma(\mu x + (1 - \mu)p_i - p_j + (1 - \mu)(x - p_i)) \\ &\le \alpha \star \gamma(\mu x + (1 - \mu)p_i - p_j) + (1 - \mu)\gamma(x - p_i), \end{aligned}$$

and thus

$$\begin{aligned} \alpha \star \gamma(\mu x + (1 - \mu)p_i - p_j) &\ge \gamma(x - p_i) - (1 - \mu)\gamma(x - p_i) \\ &= \gamma(\mu(x - p_i)) \\ &= \gamma(\mu x + (1 - \mu)p_i - p_i) \end{aligned}$$

for all $j \in \{1, \dots, n\} \setminus \{i\}$. Therefore, $\mu x + (1 - \mu)p_i \in V_i^{\alpha,\star}$. □

The set $V_i^{\alpha,\star}$ coincides with the Voronoi cell V_i if $\star$ denotes addition and $\alpha = 0$ or if $\star$ denotes multiplication and $\alpha = 1$. From this we get immediately the following

Theorem 4.9. *Let $p_1, \dots, p_n$ be points in a generalized Minkowski space $\left(\mathbb{R}^d, \gamma\right)$. Then the Voronoi cell V_i is star-shaped with respect to p_i. In addition, for any two points p_1, p_2 in a generalized Minkowski space $\left(\mathbb{R}^d, \gamma\right)$ and any $\alpha \in \mathbb{R}$, the sublevel set $H_{p_1,p_2,\le\alpha}$ is star-shaped with respect to p_1.*

Asking even for convexity properties, we are able to prove some characterization theorems for inner-product spaces.

Proposition 4.10. *Let $\gamma : \mathbb{R}^d \to \mathbb{R}$ be a gauge, $d \ge 2$. The following statements are equivalent.*

(*a*) *The gauge γ is a norm induced by an inner product.*

(b) *The bisector* $\operatorname{bis}(p_1, p_2)$ *is a hyperplane for all distinct points* $p_1, p_2 \in \mathbb{R}^d$.
(c) *The bisector* $\operatorname{bis}(p_1, p_2)$ *is convex for all distinct points* $p_1, p_2 \in \mathbb{R}^d$.

Proof. The implications $(a)\Longrightarrow(b)\Longrightarrow(c)$ are clear. The equivalence $(a)\Longleftrightarrow(c)$ can be found in [101]. $\square$

In Euclidean geometry the sublevel sets $H_{p_1,p_2,\leq\alpha}$ are convex as soon as $\alpha \leq 0$. This turns out to be a characteristic property and is closely related to Proposition 4.10.

Theorem 4.11. *The following are equivalent for every generalized Minkowski space* $\left(\mathbb{R}^d, \gamma\right)$, $d \geq 2$.

(a) *The gauge* γ *is a norm induced by an inner product.*
(b) *For any two distinct points* $p_1, p_2 \in \mathbb{R}^d$ *and any* $\alpha \in (-\gamma(p_1 - p_2), 0)$, *the sublevel set* $H_{p_1,p_2,\leq\alpha}$ *is convex.*
(c) *For any two distinct points* $p_1, p_2 \in \mathbb{R}^d$ *and any* $\alpha \in (-\infty, 0]$, *the sublevel set* $H_{p_1,p_2,\leq\alpha}$ *is convex.*
(d) *For any two distinct points* $p_1, p_2 \in \mathbb{R}^d$, *the sublevel set* $H_{p_1,p_2,\leq 0}$ *is convex.*

We prepare the proof by a lemma on generalized Minkowski planes.

Lemma 4.12. *Let* $(\mathbb{R}^2, \gamma)$ *be a generalized Minkowski plane that is not strictly convex. Then there exist distinct points* $p_1, p_2 \in \mathbb{R}^2$ *and* $\alpha \in (-\gamma(p_1 - p_2), 0)$ *such that* $H_{p_1,p_2,\leq\alpha}$ *is not convex.*

Proof. Choose p_1, p_2 and a maximal non-degenerate segment $[s, t] \in S(0, 1)$ such that $\langle p_1, p_2\rangle$ is not parallel to $[s, t]$, the condition $\left\{\frac{p_1-p_2}{\gamma(p_1-p_2)}, \frac{p_2-p_1}{\gamma(p_2-p_1)}\right\} \cap [s, t] = \emptyset$ is satisfied, and $\langle p_1, p_1 + t\rangle$ and $\langle p_2, p_2 + s\rangle$ are the supporting lines of the cone $(p_1 + C) \cap (p_2 + C)$, where $C := \{\lambda u \mid u \in [s, t], \lambda \geq 0\}$; see Figure 4.3. Like in Lemma 4.3, $x \mapsto \gamma(x - p_1) - \gamma(x - p_2)$ is constant on $(p_1 + C) \cap (p_2 + C)$. Denote this constant by α. By construction, $\alpha \notin \{-\gamma(p_1 - p_2), 0, \gamma(p_2 - p_1)\}$. If $\alpha > 0$, change the roles of p_1 and p_2, and those of s and t accordingly. Assume that, contrary to our claim, $H_{p_1,p_2,\leq\alpha}$ is convex. Due to Lemma 4.4, the function $x \mapsto \gamma(x - p_1) - \gamma(x - p_2)$ is non-decreasing in direction $p_2 - p_1$ on each straight line parallel to $\langle p_1, p_2\rangle$. In particular, if $z \in \langle p_1, p_1 + t\rangle \cap ((p_2 + C) \setminus \langle p_2, p_2 + s\rangle)$, then $\gamma(w - p_1) - \gamma(w - p_2) > \gamma(z - p_1) - \gamma(z - p_2)$ for all $w \in [z, z + (p_2 - p_1)\rangle \setminus \{z\}$. This holds since $\left[\frac{z-p_2}{\gamma(z-p_2)}, \frac{w-p_1}{\gamma(w-p_1)}\right] \not\subseteq S(0, 1)$ (see Lemma 4.4), because $\frac{z-p_2}{\gamma(z-p_2)} \in (s, t)$, but $\frac{w-p_1}{\gamma(w-p_1)} \notin [s, t]$. It follows that $\langle p_1, p_1 + t\rangle$ is a supporting line of $(p_1 + C) \cap (p_2 + C)$ as well as of $H_{p_1,p_2,\leq\alpha}$. But for $\lambda = \frac{\gamma(p_2-p_1)-\alpha}{\gamma(p_2-p_1)+\gamma(p_1-p_2)}$ we obtain $\lambda p_1 + (1 - \lambda)p_2 \in (p_1, p_2) \cap H_{p_1,p_2,\leq\alpha}$, a contradiction. $\square$

Proof of Theorem 4.11. The implication $(a)\Longrightarrow(c)$ is known from Euclidean geometry, $(c)\Longrightarrow(b)$ and $(c)\Longrightarrow(d)$ are trivial. For $(d)\Longrightarrow(a)$, we refer to [101].

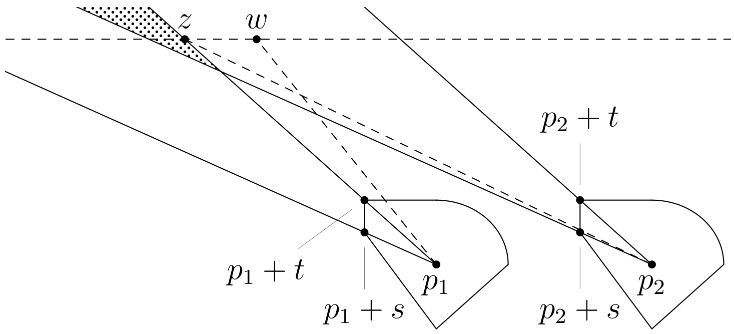


Figure 4.3: Proof of Lemma 4.12.

We prove the remaining implication $\neg(a) \Longrightarrow \neg(b)$ by considering two cases.

Case 1: γ is strictly convex. By [101] (see also Proposition 4.10), there exist distinct points $p_1, p_2 \in \mathbb{R}^d$ such that $\operatorname{bis}(p_1, p_2)$ is not convex. Hence there exist $a, b \in \operatorname{bis}(p_1, p_2)$ and $c \in [a, b]$ such that $c \notin \operatorname{bis}(p_1, p_2)$; say, $\gamma(c - p_1) > \gamma(c - p_2)$ (otherwise we exchange p_1 and p_2). By continuity of γ, there exists a small $\lambda_0 > 0$ such that

$$\gamma((c + \lambda_0(p_1 - p_2)) - p_1) > \gamma((c + \lambda_0(p_1 - p_2)) - p_2). \tag{1}$$

By Lemma 4.5, $\gamma((a + \lambda_0(p_1 - p_2)) - p_1) < \gamma((a + \lambda_0(p_1 - p_2)) - p_2)$ and $\gamma((b + \lambda_0(p_1 - p_2)) - p_1) < \gamma((b + \lambda_0(p_1 - p_2)) - p_2)$. Thus,

$$\begin{aligned} \alpha := \max\{&\gamma((a + \lambda_0(p_1 - p_2)) - p_1) - \gamma((a + \lambda_0(p_1 - p_2)) - p_2), \\ &\gamma((b + \lambda_0(p_1 - p_2)) - p_1) - \gamma((b + \lambda_0(p_1 - p_2)) - p_2)\} < 0, \end{aligned}$$

and both $a + \lambda_0(p_1 - p_2)$ and $b + \lambda_0(p_1 - p_2)$ belong to the sublevel set $H_{p_1,p_2,\leq\alpha}$. By (1), $c + \lambda_0(p_1 - p_2) \notin H_{p_1,p_2,\leq\alpha}$. However, $c \in [a, b]$ shows that $c + \lambda_0(p_1 - p_2) \in [a + \lambda_0(p_1 - p_2), b + \lambda_0(p_1 - p_2)]$. Thus, $H_{p_1,p_2,\leq\alpha}$ is not convex.

Case 2: γ is not strictly convex. Now $S(0, 1)$ contains a non-degenerate segment $[s, t]$. Let $V := \operatorname{lin}\{s, t\}$. The generalized Minkowski plane $(V, \gamma|_V)$ is not strictly convex, too, and Lemma 4.12 gives distinct points $p_1, p_2 \in V$ and $\alpha \in (-\gamma|_V(p_1 - p_2), 0)$ such that the sublevel set

$$H^V_{p_1,p_2,\leq\alpha} := \{x \in V \mid \gamma|_V(x - p_1) - \gamma|_V(x - p_2) \leq \alpha\}$$

in $(V, \gamma|_V)$ is not convex. Then the corresponding set $H_{p_1,p_2,\leq\alpha}$ in $\mathbb{R}^d$ is not convex as well, because $H_{p_1,p_2,\leq\alpha} \cap V = H^V_{p_1,p_2,\leq\alpha}$. $\square$

In the Euclidean plane every hyperbola and every ellipse with the same foci have orthogonal tangents at their common points; see, e.g., [30, pp. 5–6]. Here we present a related result for generalized Minkowski planes with polygonal balls (see [20, Theorem 9] for the subcase of polygonal norms and Figure 4.4 for an illustration). A common notion in normed spaces is that of *Birkhoff orthogonality* which is an extension of usual orthogonality in Hilbert spaces. The

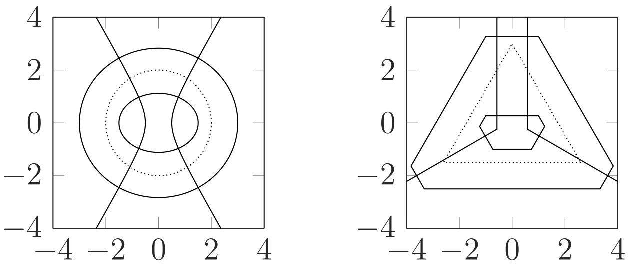


Figure 4.4: Confocal ellipses and hyperbolas intersect Birkhoff orthogonally: Euclidean norm (left), polygonal gauge (right). The corresponding unit circles are shown as dotted curves.

analogous definition for generalized Minkowski planes $(\mathbb{R}^2, \gamma)$ reads as follows. We say that $x \in \mathbb{R}^2$ is *Birkhoff orthogonal* to $y \in \mathbb{R}^2$ if $\gamma(x) \leq \gamma(x + \lambda y)$ for all $\lambda \in \mathbb{R}$. Obviously, x is Birkhoff orthogonal to y if and only if αx is Birkhoff orthogonal to βy, where $\alpha > 0$ and $\beta \in \mathbb{R}$. Therefore we can speak about Birkhoff orthogonality of a ray to a straight line. Geometrically, x is Birkhoff orthogonal to y if and only if $\{x + \lambda y \mid \lambda \in \mathbb{R}\}$ is a supporting line of $B(0, \gamma(x))$ at x.

The following theorem extends [20, Theorem 9]. It clarifies the behaviour of hyperbolas $H_{p_1,p_2,=\beta}$ and *bifocal ellipses*

$$E_{p_1,p_2,=\alpha} := \left\{ x \in \mathbb{R}^2 \,\middle|\, \gamma(x - p_1) + \gamma(x - p_2) = \alpha \right\}$$

at intersection points, when the unit ball $B(0, 1)$ of the underlying generalized Minkowski plane $(\mathbb{R}^2, \gamma)$ is a polygon. Then γ is a piecewise linear function. More precisely, if F is a 1-face of $B(0, 1)$, then γ is linear on the cone $C_F := \{\lambda x \mid \lambda \geq 0, x \in F\}$. This yields a dissection of the plane into sets $R_{F,F'} := (p_1 + C_F) \cap (p_2 + C_{F'})$, where F, F' are 1-faces of $B(0, 1)$. On each of the regions $R_{F,F'}$, the functions $\gamma(\cdot - p_1)$ and $\gamma(\cdot - p_2)$ are affine functions, and so are $f = \gamma(\cdot - p_1) + \gamma(\cdot - p_2)$ and $g = \gamma(\cdot - p_1) - \gamma(\cdot - p_2)$.

Theorem 4.13. *Let $(\mathbb{R}^2, \gamma)$ be a generalized Minkowski plane such that $B(0, 1)$ is a polygon. Fix distinct points $p_1, p_2 \in \mathbb{R}^2$ and a number $\beta \in [-\gamma(p_1 - p_2), \gamma(p_2 - p_1)]$. Assume that $H_{p_1,p_2,=\beta}$ is a simple curve, and choose $\alpha > 0$ large enough. Then every intersection point $x_0 \in H_{p_1,p_2,=\beta} \cap E_{p_1,p_2,=\alpha}$ is the initial point of a ray $[x_0, x_0 + u\rangle \subseteq H_{p_1,p_2,=\beta}$ which is Birkhoff orthogonal to any tangent line of $E_{p_1,p_2,=\alpha}$ at x_0.*

Proof. For

$$\alpha > \max \left\{ f(x) \mid x \in R_{F,F'} \text{ for some bounded region } R_{F,F'} \right\},$$

the ellipse $E_{p_1,p_2,=\alpha}$ is a subset of the union of the unbounded regions $R_{F,F'}$. Let $x_0 \in H_{p_1,p_2,=\beta} \cap E_{p_1,p_2,=\alpha}$ be fixed. Then x_0 is contained in an unbounded region $R_{F_0,F_0'}$.

The set $R_{F_0,F_0'}$ is unbounded only if $F_0 \cap F_0' \neq \emptyset$. If $F_0 = F_0'$, then g is constant on $R_{F_0,F_0'} = (p_1 + C_{F_0}) \cap (p_2 + C_{F_0'})$, see Lemma 4.3. But then $g|_{R_{F_0,F_0'}} \equiv g(x_0) = \beta$, $R_{F_0,F_0'} \subseteq H_{p_1,p_2,=\beta}$, and $H_{p_1,p_2,=\beta}$ is not a simple curve. Thus, $F_0 \cap F_0' = \{u\}$.

Next we show that $[x_0, x_0 + u\rangle \subseteq H_{p_1,p_2,=\beta}$; i.e., g is constant on $[x_0, x_0 + u\rangle$. Let $y \in [x_0, x_0 + u\rangle$. Choose $w_1, w_2 \in [p_1, p_1 + u\rangle$, $z_1, z_2 \in [p_2, p_2 + u\rangle$ such that

$$\gamma(x_0 - p_1) = \gamma(w_1 - p_1), \qquad \gamma(y - p_1) = \gamma(w_2 - p_1),$$
$$\gamma(x_0 - p_2) = \gamma(z_1 - p_2), \qquad \gamma(y - p_2) = \gamma(z_2 - p_2).$$

Since both $\gamma(\cdot - p_1)$ and $\gamma(\cdot - p_2)$ are affine functions on $(p_1 + C_{F_0})$ and $(p_2 + C_{F_0'})$, respectively, both $\{w_1, w_2, y, x_0\} \subseteq p_1 + C_{F_0}$ and $\{z_1, z_2, y, x_0\} \subseteq p_2 + C_{F_0'}$ are the vertex sets of (possibly degenerate) parallelograms. Thus $y - x_0 = w_2 - w_1 = z_2 - z_1$ and

$$\begin{aligned} g(x_0) &= \gamma(x_0 - p_1) - \gamma(x_0 - p_2) \\ &= \gamma(w_1 - p_1) - \gamma(z_1 - p_2) \\ &= \gamma(w_1 - p_1) + \gamma(w_2 - w_1) - \gamma(z_1 - p_2) - \gamma(z_2 - z_1) \\ &= \gamma(w_2 - p_1) - \gamma(z_2 - p_2) \\ &= \gamma(y - p_1) - \gamma(y - p_2) \\ &= g(y). \end{aligned}$$

Therefore, $[x_0, x_0 + u\rangle \subseteq H_{p_1,p_2,=\beta}$.

Note that u is Birkhoff orthogonal to (the directions of) the segments F_0 and F_0', since $F_0 \cap F_0' = \{u\}$.

The function $f = \gamma(\cdot - p_1) + \gamma(\cdot - p_2)$ is an affine function on $(p_1 + C_{F_0}) \cap (p_2 + C_{F_0'})$, whose gradient vector coincides with the sum of the gradient vectors of $\gamma(\cdot - p_1)$ and of $\gamma(\cdot - p_2)$. In other words, the slope of the level sets of f restricted to $R_{F_0,F_0'} = (p_1 + C_{F_0}) \cap (p_2 + C_{F_0'})$ lies between the slopes of F_0 and F_0', which are the slopes of the level sets of f at the neighboring regions R_{F_0,F_0} and $R_{F_0',F_0'}$. In particular, every tangent line of $E_{p_1,p_2,=\alpha} = f_{=f(x_0)}$ at $x_0 \in R_{F_0,F_0'}$ has a slope between those of F_0 and F_0'. Since u is Birkhoff orthogonal to (the directions of) F_0 and F_0', the ray $[x_0, x_0 + u\rangle$ is Birkhoff orthogonal to every tangent line of $E_{p_1,p_2,=\alpha}$ at x_0. □

Remark 4.14. The assumption of $B(0,1)$ being a polygon is necessary in Theorem 4.13. For instance, let $\gamma : \mathbb{R}^2 \to \mathbb{R}$, $\gamma(\xi_1, \xi_2) = \sqrt{4\xi_1^2 + 3\xi_2^2} + \xi_1$, and let $p_1 = (1, 0)$, $p_2 = (-1, 0)$. Then there is a hyperbola $H_{p_1,p_2,=\beta}$ such that no half-tangent at $x_0 \in H_{p_1,p_2,=\beta}$ is Birkhoff orthogonal to a tangent line of $E_{p_1,p_2,=\alpha}$ at x_0, where $\alpha = \alpha(x_0) = \gamma(x_0 - p_1) + \gamma(x_0 - p_2)$. We choose $\beta = -2$ and obtain $H_{p_1,p_2,=-2} = \{(0, \xi_2) \mid \xi_2 \in \mathbb{R}\}$. That is, at every point $x_0 \in H_{p_1,p_2,=-2}$ half-tangents of $H_{p_1,p_2,=-2}$ have direction $(0, 1)$ or $(0, -1)$. By definition, $(0, 1)$ is Birkhoff orthogonal to (η_1, η_2) iff $\eta_1 + \sqrt{3}\eta_2 = 0$, and $(0, -1)$ is Birkhoff orthogonal to (η_1, η_2) iff $\eta_1 - \sqrt{3}\eta_2 = 0$. Since $B(0,1)$ and $\{p_1, p_2\}$ are symmetric with respect to the horizontal axis, we restrict our considerations to half-tangents with direction $(0,1)$. Assume that there exists a point $x_0 = (0, \xi_2)$ such that $(0,1)$ is Birkhoff orthogonal to the tangent direction of $E_{p_1,p_2,=\alpha}$. In other words, the tangent line of $E_{p_1,p_2,=\alpha}$ at x_0 is parallel to $\left\{(\eta_1, \eta_2) \,\middle|\, \eta_1 + \sqrt{3}\eta_2 = 0\right\}$. The function γ is differentiable at each point $(\xi_1, \xi_2) \neq (0,0)$ with derivative $\left(\frac{4\xi_1}{\sqrt{4\xi_1^2+3\xi_2^2}} + 1, \frac{3\xi_2}{\sqrt{4\xi_1^2+3\xi_2^2}}\right)$. Therefore, the function $f = \gamma(\cdot - p_1) + \gamma(\cdot - p_2)$ is differentiable at x_0 with derivative $\left(2, \frac{6\xi_2}{\sqrt{4+3\xi_2^2}}\right)$. If the Birkhoff orthogonality assumption is true, then this derivative can be written as $\lambda\left(1, \sqrt{3}\right)$ with $\lambda \in \mathbb{R}$. The equality of the first coordinates implies $\lambda = 2$, but $\frac{6\xi_2}{\sqrt{4+3\xi_2^2}} < 2\sqrt{3}$ for all $\xi_2 \in \mathbb{R}$. This contradiction completes the proof. □

5. Cassini Curves

Let there be given two foci p_1, p_2 in the Euclidean plane. The locus of all points whose *product of distances* to p_1 and p_2 is a positive constant α^2 is called a *Cassini curve with foci p_1 and p_2*. If 2δ denotes the distance between p_1 and p_2, then the ratio α/δ basically determines the shape of the Cassini curve: for $\alpha > \delta$ the curve is a single loop in form of a convex curve (roughly like an ellipse) or star-shaped (like a dog bone), and for $\alpha < \delta$ it consists of two convex loops; the borderline case $\alpha = \delta$ yields the famous *lemniscate of Bernoulli*. This interesting class of curves is named after the famous astronomer G. D. Cassini who proposed (in opposition to Kepler) such curves in 1680 as planetary orbits; see [9, p. 149]. The class of Cassini curves has close relations to and overlaps with various famous curve classes, like sinusoidal spirals (whose tangents move with constant angular speed if the polar angle does the same), spirics of Perseus and hippopedes (as plane sections of tori), n-lemniscates (see below) and others. For this background material we refer to the monographs [41, 23, 21, 39, 8, 78], and to further books cited therein. More related to our considerations here is the paper [48], in which various geometric properties of bifocal Cassini curves in Minkowski planes are derived. In [79] the concept of bifocal Cassini curves is modified for the Minkowskian space-time plane, and also the investigation of the multifocal generalizations is proposed there. Figure 5.1 presents bifocal Cassini curves for the same unit circles as used already in Figures 3.1, 3.2, and 4.1.

As in case of ellipses, we can obviously introduce multifocal Cassini curves. A plane algebraic curve C of order $2n$ is called an *n-focal Cassini curve or n-lemniscate* if the product of the Euclidean distances of each point x from C to n given points $p_1, \ldots, p_n \in \mathbb{R}^2$ (again called *foci* of C) is equal to the n-th power of a given number $\alpha > 0$, the *radius* of C. The equation of an n-lemniscate in Cartesian coordinates is given by

$$\|x - p_1\| \cdot \ldots \cdot \|x - p_n\| = \alpha^n,$$

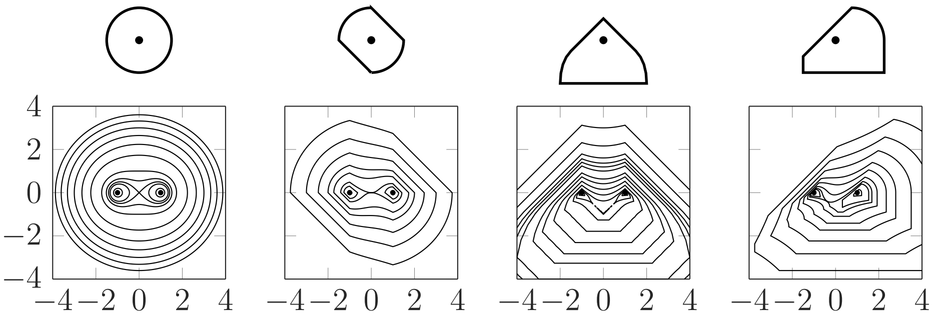


Figure 5.1: Four unit circles and corresponding bifocal Cassini curves with foci $(1,0)$ and $(-1,0)$.

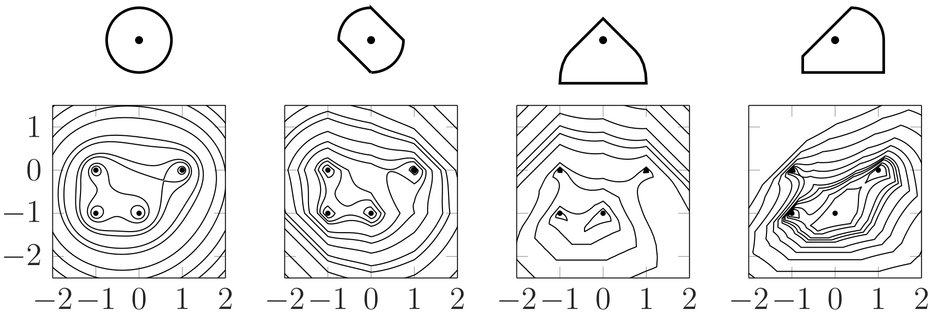


Figure 5.2: Four unit circles and corresponding 4-focal Cassini curves with foci $(1,0)$, $(-1,0)$, $(-1,-1)$ and $(0,-1)$.

and these curves can be conveniently studied also in the complex plane. Thus, circles occur as the 1-lemniscates, the classical Cassini curves are precisely the 2-lemniscates, and with this concept an n-lemniscate can be seen as level curve of a polynomial (see again [23]).

We collect now geometric results obtained for n-lemniscates and their higher-dimensional analogues. The deeper geometric study of n-lemniscates started perhaps with the papers [96, 83, 86]. It is shown there that n-lemniscates are n-circular curves and a special case of equipotential curves (see also our Section 3), regarding some aspects having a similar behaviour as in the classical case $n = 2$; if, e.g., the radius is very small, they form small convex components around the foci, and for large radii they become one-component convex curves. In case there exist singularities, these occur only within the convex hull of the n foci, and all normals at regular points (constructions of which are given) meet this convex hull. Classical theorems on peripherical angles of bifocal Cassini curves carry over to $n > 2$. Our Figure 5.2 shows 4-focal Cassini curves, again for unit circles corresponding to those in Figures 3.1, 3.2, 4.1, and 5.1.

In [84] analogues of these results are verified for two-dimensional analogues in 3-space: these are surfaces of order $2n$, and again singularities can only occur within the convex hull of the foci, which is analogously met by all surface normals at regular surface points. With the help of circular disks that cover the n foci, in [97] and [75] it is clarified when whole planar n-lemniscates or their disjoint components close to the foci are convex. In [74] inequalities for the curvature and properties of orthogonal trajectories and perpendicular bisectors of chords of n-lemniscates are derived; the latter again have to meet the convex hull of the foci (compare with our Theorem 3.2 and [87] for multifocal ellipses and their higher-dimensional analogues). And in [76] generalizations of n-lemniscates (called equipotentials) are studied, immediately in arbitrary dimensions (see also [96]). The authors of [18] show how the containment of an n-lemniscate in another n-lemniscate with some additional property guarantees already that the two occuring distance products are equal. A similar containment relation between n-lemniscates and circular disks of "proportional size" is established in [13]. Continuing this, in [81] estimates on radii of circles contained in n-lemniscates and on areas of n-lemniscates of given radius are presented.

In 1897, Hilbert showed the following (Hilbert's lemniscate theorem; see, e.g., [98, 44]): if K is a compact set in the plane and U is a neighborhood of K, then there is an n-lemniscate which lies within U but encloses K. Due to this theorem and various generalizations of it, n-lemnicates became very important for approximation theory and numerical analysis. (One should mention that there are further related concepts in these disciplines which are also based on products of distances. One example is the concept of Leja points; see, for instance, [68].) We continue with some more recent contributions in the spirit of approximation theory. In [54] Hilbert's theorem was extended to touching systems of curves. More precisely, it is shown that an approximating n-lemniscate, due to Hilbert always situated "between" two Jordan curves which are close to each other, still exists when these two curves touch each other in finitely many points. In [37], Hausdorff approximation of Jordan curves by n-lemniscates is studied in terms of the moduli of smoothness of the conformal mapping of the Jordan curve to the exterior of the unit disk and its inverse. Analytic approximation of smooth closed curves by n-lemniscates is presented in [66, 67], using also algorithms based on least squares. This approach is also compared with capabilities of classical harmonic approximation. In [16] Cassini curves are used to investigate the location of zeros in hypergeometric polynomials. The paper [24] is even related to precise parametric representations. Namely, many such representations are known for the sphere and the torus, but it is less trivial to find parametric representations for spheres with $k > 2$ handles in 3-space. In [24] such representations are presented for any $k > 2$, with the help of surfaces which are two-dimensional analogues of multifocal Cassini curves.

In 1947, Brauer [6] introduced Cassini ovals for determining inclusion regions in the complex plane for the eigenvalues of an arbitrary matrix A in $\mathbb{C}^{n\times n}, n > 1$. More precisely, let $A = (a_{ij})_{i,j=1}^{n}$ be a square matrix of order n over the complex field, and let $\varrho_i = \sum_{j=1}^{n} |a_{ij}| - |a_{ii}|$ and $\sigma_j = \sum_{i=1}^{n} |a_{ij}| - |a_{jj}|$. Then each characteristic root of A lies in at least one of the $\frac{1}{2}n(n-1)$ Cassini ovals

$|z - a_{ii}|\,|z - a_{jj}| \leq \varrho_i \varrho_j$, $i \neq j$, and, analogously, $|z - a_{ii}|\,|z - a_{jj}| \leq \sigma_i \sigma_j$, $i \neq j$. Later Brauer extended this concept to stochastic matrices [7]. Then [92] proved that (in comparison with the n Gershgorin row disks, introduced in 1937 as sets whose union also contains the set of eigenvalues of A) Brauer's Cassini ovals have better properties for the restriction of eigenvalues than Gershgorin disks. This holds since their unions are usually proper subsets of the unions of Gershgorin disks. More material in this direction is presented in the monograph [93]. A very simple proof of Brauer's inclusion property was presented in [60], thus yielding the known name "Brauer–Ostrowski theorem". Later this theorem was generalized in many directions. For example, related distance-function inequalities were generalized in [53]. In 1994, [105] published a slight correction of Brauer's widely cited theorem using a directed graph defined for A; it shows that, when $n > 2$, it is possible that indeed Gershgorin disks are needed for the purpose of the theorem. In [29] the whole concept was successfully extended to operator matrices. Melman [50] showed that for constant main diagonals the spectrum of general complex matrices is even contained in a subset of Brauer's set, which again is the union of Cassini ovals and can be easily computed for Toeplitz matrices. For polynomials with complex coefficients, the same author determined two Cassini ovals containing all their zeros, and he showed that under certain conditions these are proper subsets of the corresponding Gershgorin and Brauer sets [51]. Weakly related to all these considerations, [11] generalized classical theorems from function theory referring to holomorphic functions defined on the interiors of circles to multiply connected domains bounded by n-focal Cassini curves. Namely, using the representation of such Cassini curves via polynomials of degree $n > 0$, Schwarz's lemma, Jensen's formula, Hadamard's three-circles theorem and other statements are extended to such more general domains.

Cassini curves have many applications in natural and technical sciences. The special case of dog-bone shape is used to model particles in biology [15], and Cassini curves also occur in optics [28] and the light field theory [45]. Further applications (referring to radar systems, sensor fields, nuclear science, modelling evolutionary processes and population distribution, optimization of fuel tanks, and other fields) are nicely collected in the final section of [34].

Due to the large variety of shapes that n-lemniscates and their analogues in higher dimensions can take, these curves and surfaces are certainly interesting also for computer scientists (with motivations from computer aided geometric design, but also again from approximation theory). We survey some recent results in this direction. Given a planar region (e.g., a topological annulus, or an unbounded region with hole) into which a grid should be placed such that the grid lines could be interpreted as isoparametric lines. The author of [102] constructed a polynomial mapping between fundamental regions and topologically equivalent regions to approximate the whole region to be gridded, but without methods like usual $1:1$ parametrization. He used geometric properties of families of non-intersecting suitable n-lemniscates to solve this. The paper [40] is dominated by analogous aims and methods, as well as by approximation tasks which again are based on n-lemniscates. The three papers [61, 64, 1] are mainly devoted to CAGD, and they are similar in nature: analogues of n-lemniscates (i.e., *multifocal Cassini surfaces*) in 3-space are studied and applied. Besides other goals, the authors concentrate on the control of the procedures of coalescing/splitting the connected components of these surfaces when the radius of the lemniscates continuously changes. The authors want to make sure that these surfaces become a potentially useful tool for CAGD. They present their results for special values of n.

The following subsections present new results on multifocal Cassini curves and their higher-dimensional analogues, called *Cassini level and sublevel sets*, for gauges (and norms).

5.1. Starshapedness

Recall that a function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *coercive* if $f(x) \to \infty$ for $\|x\| \to \infty$, where $\|\cdot\|$ denotes the Euclidean norm. The following lemma says that under a convexity assumption coercivity of functions $f : \mathbb{R}^d \to \mathbb{R}$ coincides with the boundedness of their set of minimizers; see also [69, Corollary 8.7.1]. Here, boundedness is meant with respect to the norm induced by the Euclidean unit ball $B = \left\{x \in \mathbb{R}^d \,\middle|\, \langle x \,|\, x\rangle \leq 1\right\}$.

Lemma 5.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function such that the set of minimizers of f is non-empty and bounded. Then the following statements are true.*

(a) *For every $\beta \in \mathbb{R}$, the sublevel set $f_{\leq\beta}$ is bounded.*
(b) *There exists $\varrho \geq 0$ such that $f(\lambda_1 x) < f(\lambda_2 x)$ whenever $x \in \mathbb{R}^d \setminus \varrho B$ and $1 \leq \lambda_1 < \lambda_2$.*

Proof. (a) Assume that $f_{\leq\beta_0}$ is unbounded for some β_0, and (without loss of generality) that 0 is a minimizer of f. Then there is a sequence $(x_n)_{n=1}^{\infty} \subseteq f_{\leq\beta_0}$ such that $\|x_n\| \geq n$ for $n \geq 1$. By compactness of $\mathrm{bd}(B)$, we can assume that $\left(\frac{x_n}{\|x_n\|}\right)_{n=1}^{\infty}$ is a convergent sequence whose limit shall be denoted by $x_0 \in \mathrm{bd}(B)$. Then the ray $\{\lambda x_0 \mid \lambda \geq 1\}$ is a subset of $f_{\leq f(0)}$, the set of minimizers of f, because

$$\begin{aligned}
f(\lambda x_0) &= \lim_{n\to\infty} f\left(\lambda \frac{x_n}{\|x_n\|}\right)\\
&= \lim_{n\to\infty}\left(f\left(\left(\frac{\lambda}{\|x_n\|}x_n\right) + \left(1-\frac{\lambda}{\|x_n\|}\right)0\right)\right)\\
&\leq \lim_{n\to\infty}\left(\frac{\lambda}{\|x_n\|}f(x_n) + \left(1-\frac{\lambda}{\|x_n\|}\right)f(0)\right)\\
&\leq \lim_{n\to\infty}\frac{\lambda}{\|x_n\|}\beta_0 + \lim_{n\to\infty}\left(1-\frac{\lambda}{\|x_n\|}\right)f(0)\\
&= f(0).
\end{aligned}$$

(b) Assume that, contrary to our claim, there are sequences $(y_n)_{n=1}^\infty \subseteq \mathbb{R}^d$, $\|y_n\| > n$, $(\lambda_{1,n})_{n=1}^\infty$, $(\lambda_{2,n})_{n=1}^\infty \subseteq \mathbb{R}$ such that $1 \leq \lambda_{1,n} < \lambda_{2,n}$ and $f(\lambda_{1,n} y_n) \geq f(\lambda_{2,n} y_n)$. Then the convex function

$$[0, \infty) \to \mathbb{R}, \qquad \lambda \mapsto f\left(\lambda \frac{y_n}{\|y_n\|}\right),$$

is monotonically non-increasing on the intervall $[0, \lambda_{1,n} \|y_n\|] \supseteq [0, n]$. As above, we can assume that $\left(\frac{y_n}{\|y_n\|}\right)_{n=1}^\infty$ is a convergent sequence whose limit shall be denoted by $y_0 \in \mathrm{bd}(B)$. The convex function

$$[0, \infty) \to \mathbb{R}, \qquad \lambda \mapsto f(\lambda y_0),$$

is also monotonically non-increasing because, for $0 \leq \lambda_1 < \lambda_2$, we have

$$f(\lambda_1 y_0) - f(\lambda_2 y_0) = \lim_{n\to\infty} \underbrace{\left(f\left(\lambda_1 \frac{y_n}{\|y_n\|}\right) - f\left(\lambda_2 \frac{y_n}{\|y_n\|}\right)\right)}_{\geq 0 \text{ for } \lambda_2 < n} \geq 0.$$

Therefore $f(\lambda y_0) \leq f(0 y_0) = f(0)$, which implies that $f_{\leq f(0)}$ is unbounded. This contradicts (a). $\square$

Lemma 5.2. *Let $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}$ be convex and coercive functions, and define $f : \mathbb{R}^d \to \mathbb{R}$, $f(x) := f_1(x) \cdot \ldots \cdot f_n(x)$. Then there is a number $\alpha_0 \in \mathbb{R}$ such that the sublevel sets $f_{\leq \alpha}$, $\alpha \geq \alpha_0$, are star-shaped with respect to $x_0 = 0$.*

Proof. By Lemma 5.1, there exist numbers ϱ_1 and ϱ_2 such that $\bigcup_{i=1}^n (f_i)_{\leq 0} \subseteq \varrho_1 B$ and $\lambda \mapsto f_i(\lambda x)$ is monotonically increasing on $[1, \infty)$ for all $i \in \{1, \ldots, n\}$ and $x \in \mathbb{R}^d \setminus \varrho_2 B$. Define $\varrho_0 := \max\{\varrho_1, \varrho_2\}$ and $\alpha_0 := \max\{f(x) \mid x \in \varrho_0 B\}$. Let $\alpha \geq \alpha_0$. We will show that, for all $x \in f_{\leq\alpha}$ and $\mu \in [0, 1]$, we have $\mu x \in f_{\leq\alpha}$.

Case 1. $\mu x \in \varrho_0 B$. Then $\mu x \in \varrho_0 B \subseteq f_{\leq \alpha_0} \subseteq f_{\leq\alpha}$.

Case 2. $\mu x \in \mathbb{R}^d \setminus \varrho_0 B$. By choice of ϱ_0, we have $f_i(\mu x) > 0$ and $f_i(\mu x) = f_i(1 \cdot \mu x) \leq f_i\left(\frac{1}{\mu}\mu x\right) = f_i(x)$ for all $i \in \{1, \ldots, n\}$. It follows that

$$f(\mu x) = f_1(\mu x) \cdot \ldots \cdot f_n(\mu x) \leq f_1(x) \cdot \ldots \cdot f_n(x) = f(x) \leq \alpha,$$

and hence $\mu x \in f_{\leq\alpha}$. This completes the proof. $\square$

A special case of this result refers to starshapedness of n-focal Cassini sets where each focus is even endowed with its own gauge.

Theorem 5.3. *Let $\gamma_1, \ldots, \gamma_n : \mathbb{R}^d \to \mathbb{R}$ be gauges and let $p_1, \ldots, p_n \in \mathbb{R}^d$ be arbitrary points. Moreover, define $f : \mathbb{R}^d \to \mathbb{R}$, $f(x) := \gamma_1(x - p_1) \cdot \ldots \cdot \gamma_n(x - p_n)$, and let x_0 be fixed. Then there is a number $\alpha_0 \in \mathbb{R}$ such that the sublevel sets $f_{\leq\alpha}$, $\alpha \geq \alpha_0$, are star-shaped with respect to x_0.*

Proof. Apply Lemma 5.2 to $f_i := \gamma_i(\cdot + x_0 - p_i)$, $i \in \{1, \ldots, n\}$, and obtain the starshapedness of the sublevel sets $\bar{f}_{\leq\alpha}$, $\alpha \geq \alpha_0$, with respect to 0 where $\bar{f}$ is defined as the pointwise product of the functions f_i. Observe that $f = \bar{f}(\cdot - x_0)$, and thus $f_{\leq\alpha} = \bar{f}_{\leq\alpha} + x_0$. □

In normed spaces, bifocal Cassini sets can be decomposed into two star-shaped sets and, therefore, have at most two connected components.

Lemma 5.4. *Let $p_1, p_2 \in \mathbb{R}^d$, and $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}$ be a norm. Define $f : \mathbb{R}^d \to \mathbb{R}$, $f(x) = \|x - p_1\| \|x - p_2\|$, and let V_1, V_2 be the Voronoi cells associated to $\{p_1, p_2\}$ as in Theorem 4.9. Then, for each $\alpha \geq 0$ and $i \in \{1, 2\}$, the set $f_{\leq\alpha} \cap V_i$ is star-shaped with respect to p_i.*

Proof. Without loss of generality, let $i = 1$. Let $x \in V_1$, $\mu \in [0, 1]$. We have

$$\begin{aligned}
f(\mu x + (1-\mu)p_1) &= \|\mu x + (1-\mu)p_1 - p_1\| \|\mu x + (1-\mu)p_1 - p_2\| \\
&= \mu \|x - p_1\| \|x - p_2 - (1-\mu)(x - p_1)\| \\
&\leq \mu \|x - p_1\| (\|x - p_2\| + (1-\mu) \|x - p_1\|) \\
&\leq \mu \|x - p_1\| (\|x - p_2\| + (1-\mu) \|x - p_2\|) \\
&= \mu(2-\mu) \|x - p_1\| \|x - p_2\| \\
&\leq f(x).
\end{aligned}$$

Thus, $\mu x + (1-\mu)p_1 \in f_{\leq f(x)}$. □

Theorem 5.2 shows that for large parameters Cassini sets are star-shaped. For small parameters, e.g. zero, this is clearly wrong, because those Cassini sets are not even connected. However, connectedness does not imply starshapedness, see [48, Remark 15]. The next theorem refers to starshapedness of n-focal Cassini sets with small parameters, again based on possibly different gauges.

Theorem 5.5. *Let $\gamma_1, \ldots, \gamma_n : \mathbb{R}^d \to \mathbb{R}$, $n \geq 2$, be gauges, let $p_1, \ldots, p_n \in \mathbb{R}^d$ be pairwise different points, let $f : \mathbb{R}^d \to \mathbb{R}$ be defined by $f(x) = \prod_{i=1}^n \gamma_i(x - p_i)$, and let*

$$\varrho_i = \frac{1}{n} \min \left\{ \frac{\gamma_i(z)}{\gamma_j(-z)} \gamma_j(p_i - p_j) \,\middle|\, z \in \mathbb{R}^d \setminus \{0\}, \, j \in \{1, \ldots, n\} \setminus \{i\} \right\}$$

for $1 \leq i \leq n$. Then, for every $\alpha \geq 0$ and $i \in \{1, \ldots, n\}$, $f_{\leq\alpha} \cap B_{\gamma_i}(p_i, \varrho_i)$ is star-shaped with respect to p_i, and the open balls $\operatorname{int}(B_{\gamma_i}(p_i, \varrho_i))$, $1 \leq i \leq n$, are mutually disjoint.

Moreover, if $\alpha \in [0, \prod_{i=1}^n \varrho_i)$, then $f_{\leq\alpha} \subseteq \bigcup_{i=1}^n \operatorname{int}(B_{\gamma_i}(p_i, \varrho_i))$, so that $f_{\leq\alpha}$ splits into exactly n star-shaped connected components $f_{\leq\alpha} \cap \operatorname{int}(B_{\gamma_i}(p_i, \varrho_i))$, $1 \leq i \leq n$.

Note that ϱ_i is well-defined and positive, because, for every $j \in \{1, \ldots, n\} \setminus \{i\}$,

$$\min \left\{ \frac{\gamma_i(z)}{\gamma_j(-z)} \,\middle|\, z \in \mathbb{R}^d \setminus \{0\} \right\} = \min \left\{ \frac{\gamma_i(z)}{\gamma_j(-z)} \,\middle|\, z \in S_{\gamma_1}(0, 1) \right\} > 0$$

is the minimum of a continuous function with positive values on the compact set $S_{\gamma_1}(0,1)$.

Proof. *Step 1: $f_{\leq\alpha} \cap B_{\gamma_i}(p_i, \varrho_i)$ is star-shaped with respect to p_i.*

It is enough to show that

$$f(\mu x + (1-\mu)p_i) < f(x) \quad \text{for all } x \in B_{\gamma_i}(p_i, \varrho_i) \setminus \{p_i\}\,,\ \mu \in (0,1).$$

So let x and μ be fixed and consider $j \in \{1, \dots, n\} \setminus \{i\}$. The inclusion $x \in B_{\gamma_i}(p_i, \varrho_i)$ gives

$$\gamma_i(x - p_i) \leq \varrho_i \leq \frac{1}{n}\frac{\gamma_i(x-p_i)}{\gamma_j(p_i - x)}\gamma_j(p_i - p_j).$$

We estimate

$$\begin{aligned}
&\gamma_j((\mu x + (1-\mu)p_i) - p_j)\\
&= \gamma_j((1-\mu)(p_i - x) + (x - p_j))\\
&\leq (1-\mu)\gamma_j(p_i - x) + \gamma_j(x - p_j)\\
&= \left(\mu^{-\frac{1}{n-1}} - 1\right)\left(\mu^{\frac{1}{n-1}} + \mu^{\frac{2}{n-1}} + \ldots + \mu^{\frac{n-1}{n-1}}\right)\gamma_j(p_i - x) + \gamma_j(x - p_j)\\
&< \left(\mu^{-\frac{1}{n-1}} - 1\right)(n-1)\gamma_j(p_i - x) + \gamma_j(x - p_j)\\
&= \left(\mu^{-\frac{1}{n-1}} - 1\right)\left(n\frac{\gamma_j(p_i - x)}{\gamma_i(x - p_i)}\gamma_i(x - p_i) - \gamma_j(p_i - x)\right) + \gamma_j(x - p_j)\\
&\leq \left(\mu^{-\frac{1}{n-1}} - 1\right)\left(n\frac{\gamma_j(p_i - x)}{\gamma_i(x - p_i)}\varrho_i - \gamma_j(p_i - x)\right) + \gamma_j(x - p_j)\\
&\leq \left(\mu^{-\frac{1}{n-1}} - 1\right)(\gamma_j(p_i - p_j) - \gamma_j(p_i - x)) + \gamma_j(x - p_j)\\
&\leq \left(\mu^{-\frac{1}{n-1}} - 1\right)\gamma_j(x - p_j) + \gamma_j(x - p_j)\\
&= \mu^{-\frac{1}{n-1}}\gamma_j(x - p_j).
\end{aligned}$$

We conclude

$$\begin{aligned}
&f(\mu x + (1-\mu)p_i)\\
&= \gamma_i((\mu x + (1-\mu)p_i) - p_i)\prod_{j\in\{1,\ldots,n\}\setminus\{i\}}\gamma_j((\mu x + (1-\mu)p_i) - p_j)\\
&< \mu\gamma_i(x - p_i)\prod_{j\in\{1,\ldots,n\}\setminus\{i\}}\mu^{-\frac{1}{n-1}}\gamma_j(x - p_j)\\
&= \mu\mu^{-1}\prod_{j=1}^{n}\gamma_j(x - p_j)\\
&= f(x).
\end{aligned}$$

This finishes Step 1.

Step 2: $\operatorname{int}(B_{\gamma_i}(p_i,\varrho_i)) \cap \operatorname{int}\left(B_{\gamma_j}(p_j,\varrho_j)\right) = \emptyset$ *for* $1 \le i < j \le n$.

Assume that, contrary to our claim, there exists a point $x \in \operatorname{int}(B_{\gamma_i}(p_i,\varrho_i)) \cap \operatorname{int}\left(B_{\gamma_j}(p_j,\varrho_j)\right)$; i.e., $\gamma_i(x-p_i) < \varrho_i$ and $\gamma_j(x-p_j) < \varrho_j$. Note that $x \ne p_j$, because otherwise

$$\gamma_i(p_j-p_i) < \varrho_i \le \frac{1}{n}\frac{\gamma_i(p_j-p_i)}{\gamma_j(p_i-p_j)}\gamma_j(p_i-p_j) = \frac{1}{n}\gamma_i(p_j-p_i),$$

which is impossible. We obtain

$$\begin{aligned}
&\gamma_i(p_j-p_i)\\
&\le \gamma_i(p_j-x)+\gamma_i(x-p_i)\\
&= \gamma_j(x-p_j)\frac{\gamma_i(p_j-x)}{\gamma_j(x-p_j)}+\gamma_i(x-p_i)\\
&< \varrho_j\frac{\gamma_i(p_j-x)}{\gamma_j(x-p_j)}+\varrho_i\\
&\le \left(\frac{1}{n}\frac{\gamma_j(x-p_j)}{\gamma_i(p_j-x)}\gamma_i(p_j-p_i)\right)\frac{\gamma_i(p_j-x)}{\gamma_j(x-p_j)}+\frac{1}{n}\frac{\gamma_i(p_j-p_i)}{\gamma_j(p_i-p_j)}\gamma_j(p_i-p_j)\\
&= \frac{2}{n}\gamma_i(p_j-p_i).
\end{aligned}$$

This contradiction completes Step 2.

Step 3: $f_{\le\alpha} \subseteq \bigcup_{i=1}^n \operatorname{int}(B_{\gamma_i}(p_i,\varrho_i))$ *if* $\alpha \in [0,\prod_{i=1}^n \varrho_i)$.

Assume that $x \in \mathbb{R}^d \setminus \bigcup_{i=1}^n \operatorname{int}(B_{\gamma_i}(p_i,\varrho_i))$. Then $\gamma_i(x-p_i) \ge \varrho_i$ for $i \in \{1,\dots,n\}$, and therefore $f(x) \ge \prod_{i=1}^n \varrho_i > \alpha$ and $x \notin f_{\le\alpha}$. This completes the proof. □

The following example illustrates a bifocal situation of Theorem 5.5 for a specific gauge.

Example 5.6. Figure 5.3 depicts the unit ball of a gauge γ on $\mathbb{R}^2$ and level sets of the function $f = \gamma(\cdot - p_1)\gamma(\cdot - p_2)$ where $p_1 = (-1,0)$, $p_2 = (1,0)$. We obtain $\varrho_1 = \varrho_2 = \frac{1}{3}$, and Theorem 5.5 yields a decomposition of $f_{\le\alpha}$ into $n=2$ star-shaped components if $f_{\le\alpha} \subseteq \operatorname{int}(B(p_1,\varrho_1)) \cup \operatorname{int}(B(p_2,\varrho_2))$. This leads to the sharp inequality $\alpha < \frac{96}{169}$ here, the latter number being the function value at the local minimum $\left(\frac{5}{13}, -\frac{12}{13}\right)$ of f.

Lemma 5.4 and Theorem 5.5 partially answer the following problem.

Problem 5.7. *For given points $x, p_1, \dots, p_n$ in a generalized Minkowski space $\left(\mathbb{R}^d,\gamma\right)$, does there exist a local minimizer x_0 of the function $f = \prod_{i=1}^n \gamma(\cdot - p_i)$ such that $[x,x_0] \subseteq f_{\le f(x)}$?*

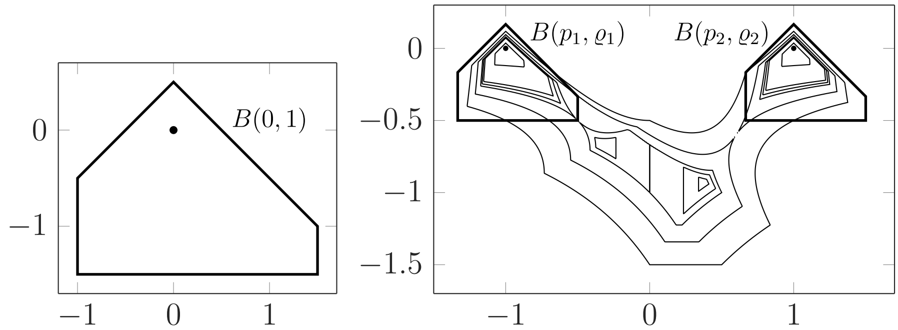


Figure 5.3: Illustration of Theorem 5.5: sublevel sets of $f = \gamma(\cdot - p_1)\gamma(\cdot - p_2)$ which are contained in $\mathrm{int}(B(p_1, \varrho_1)) \cup \mathrm{int}(B(p_2, \varrho_2))$ consist of $n = 2$ star-shaped components. Here $p_1 = (-1, 0)$, $p_2 = (1, 0)$, and $\varrho_1 = \varrho_2 = \frac{1}{3}$.

Weakly related to starshapedness is the following proposition. We present it here, since it also describes geometric properties of level sets.

Proposition 5.8. *Let $p_1, \ldots, p_n$, $n \geq 1$, be points of a generalized Minkowski plane $(\mathbb{R}^2, \gamma)$. If the gauge γ is not strictly convex, then there is a level set of the function $f = \prod_{i=1}^n \gamma(\cdot - p_i)$ which contains a non-degenerate segment.*

Proof. If γ is not strictly convex, there is a non-degenerate segment $[s,t] \subseteq S(0,1)$. The cone $\bigcap_{i=1}^n (p_i + C)$, where $C := \{\lambda u \mid u \in [s,t], \lambda \geq 0\}$, has non-empty interior, and $\gamma(\cdot - p_i)$ is an affine function on this cone, $i \in \{1, \ldots, n\}$. In particular, each function $\gamma(\cdot - p_i)$ is constant on segments which are parallel to $[s,t]$ and contained in $\bigcap_{i=1}^n (p_i + C)$, and so is f. $\square$

5.2. Exoticism

The connectedness structure of Cassini sets can be surprisingly exotic, even in the bifocal planar case.

Example 5.9. Consider the monotonically increasing and concave function

$$h : [-2, 4] \to [0, 4], \qquad h(\xi) = \begin{cases} \xi + 2, & \xi \in [-2, 1], \\ 4\sqrt{\xi} - \xi, & \xi \in [1, 4]. \end{cases}$$

Let F be a closed set satisfying $\{1, 4\} \subseteq F \subseteq [1, 4]$, define

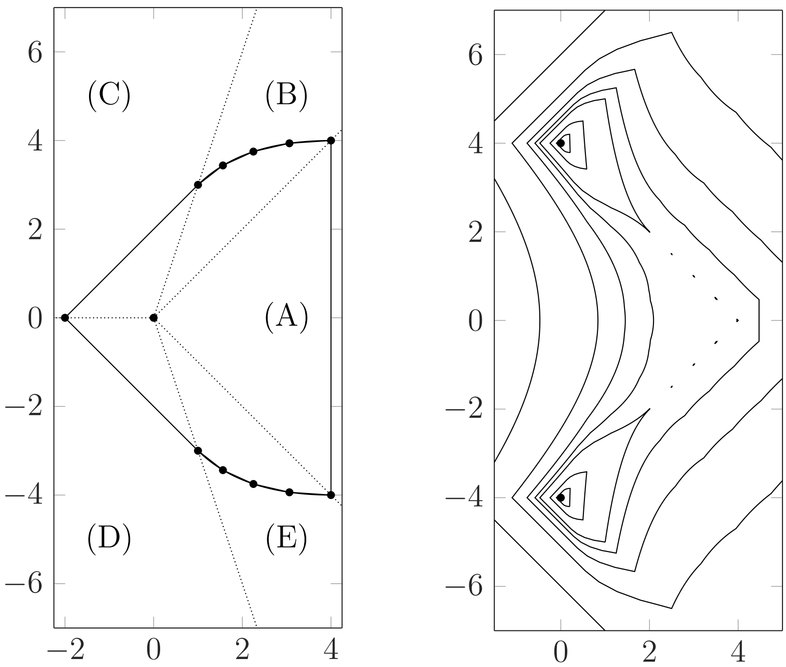
$$B_F := \mathrm{co}\left(\left\{(\xi_1, \pm h(\xi_1)) \in \mathbb{R}^2 \,\middle|\, \xi_1 \in [-2, 1] \cup F\right\}\right),$$

and let γ_F be the gauge on $\mathbb{R}^2$ with unit ball B_F. For $p_1 = (0, 4)$ and $p_2 = (0, -4)$, consider

$$f^F(x) := \gamma_F(x - p_1)\gamma_F(x - p_2).$$

Then

$$f^F_{\leq 1} \cap ((2, \infty) \times \mathbb{R}) = f^F_{=1} \cap ((2, \infty) \times \mathbb{R}) = \left\{(2\sqrt{\eta}, \pm(4 - 2\sqrt{\eta})) \mid \eta \in F\right\}.$$



(a) B_F and $B_{[1,4]}$. The reader should notice that the two sets differ only very slightly.

(b) Level sets of the function f^F.

Figure 5.4: Illustration of Example 5.9 with $F = \left\{1, \frac{25}{16}, \frac{9}{4}, \frac{49}{16}, 4\right\}$.

In particular, $f^F_{=1} \cap ((2,\infty) \times \mathbb{R})$ consists of local minimizers of f^F. Figure 5.4 illustrates an example. There are some interesting special cases of this construction:

(a) If $F = [1,4]$, then $f^F_{\leq 1}$ is connected but not star-shaped.

(b) F can have uncountably many connected components. For example, this is the case when F is a fractal set, e.g., if F is the classical Cantor set, scaled to $[1,4]$. Then $f^F_{\leq 1} \cap ((2,\infty) \times \mathbb{R})$ splits into uncountably many connected components, which are singletons.

Proof. *Step 1. We consider $F = [1,4]$.* We write γ instead of $\gamma_{[1,4]}$. Let $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Depending on the position of $\frac{(\xi_1,\xi_2)}{\gamma(\xi_1,\xi_2)}$ in $S_\gamma(0,1)$ (see Figure 5.4(a)), we obtain

$$\gamma(\xi_1,\xi_2) = \begin{cases} \frac{\xi_1}{4} & \text{if } -\xi_1 \leq \xi_2 \leq \xi_1 \quad \text{(type A)}, \\ \frac{(\xi_1+\xi_2)^2}{16\xi_1} & \text{if } \xi_1 \leq \xi_2 \leq 3\xi_1 \quad \text{(type B)}, \\ \frac{-\xi_1+\xi_2}{2} & \text{if } \max\{0, 3\xi_1\} \leq \xi_2 \quad \text{(type C)}, \\ \frac{-\xi_1-\xi_2}{2} & \text{if } \xi_2 \leq \min\{0, -3\xi_1\} \quad \text{(type D)}, \\ \frac{(\xi_1-\xi_2)^2}{16\xi_1} & \text{if } -3\xi_1 \leq \xi_2 \leq -\xi_1 \quad \text{(type E)}. \end{cases}$$

We have to show that the function $f = f^{[1,4]}$ satisfies

$$f(\xi_1, \xi_2) \begin{cases} > 1 & \text{for } \xi_1 > 2,\ |\xi_2| \neq 4 - \xi_1, \\ = 1 & \text{for } \xi_1 > 2,\ |\xi_2| = 4 - \xi_1. \end{cases}$$

Since both $B_{[1,4]}$ and $\{p_1, p_2\}$ are symmetric with respect to the horizontal axis, we can assume that $\xi_2 \geq 0$. So let $(\xi_1, \xi_2) \in (2, \infty) \times [0, \infty)$ be fixed. The following cases cover the whole situation.

Case 1: $\xi_2 \geq 4$. We have $\gamma((\xi_1, \xi_2) - p_1) = \gamma(\xi_1, \xi_2 - 4) > \frac{1}{2}$ because $\xi_1 > 2$, and we have $\gamma((\xi_1, \xi_2) - p_2) = \gamma(\xi_1, \xi_2 + 4) \geq 2$ because $\xi_2 + 4 \geq 8$. Thus, $f(\xi_1, \xi_2) > 1$.

Case 2: $\xi_2 < \xi_1 - 4$. Then $\xi_1 > 4$ and $f(\xi_1, \xi_2) = \gamma(\xi_1, \xi_2 - 4)\gamma(\xi_1, \xi_2 + 4) = \frac{\xi_1}{4}\frac{\xi_1}{4} > 1$ (both lengths are of type A).

Case 3: $3\xi_1 - 4 < \xi_2 < 4$. Now $\gamma(\xi_1, \xi_2 - 4) = \frac{\xi_1}{4}$ (type A) and $\gamma(\xi_1, \xi_2 + 4) = \frac{-\xi_1 + (\xi_2 + 4)}{2}$ (type C). Hence, $f(\xi_1, \xi_2) = \frac{\xi_1(-\xi_1 + \xi_2 + 4)}{8} > \frac{\xi_1(-\xi_1 + (3\xi_1 - 4) + 4)}{8} = \frac{\xi_1^2}{4} > 1$.

Case 4: $\xi_1 + \xi_2 < 4$. In this case $\gamma(\xi_1, \xi_2 - 4) = \frac{(\xi_1 - (\xi_2 - 4))^2}{16\xi_1}$ (type E) and $\gamma(\xi_1, \xi_2 + 4) = \frac{(\xi_1 + (\xi_2 + 4))^2}{16\xi_1}$ (type B). Thus,

$$f(\xi_1, \xi_2) = \left(1 + \frac{(\xi_1 + \xi_2 - 4)(\xi_1 - \xi_2 - 4)}{16\xi_1}\right)^2 > 1,$$

because $\xi_1 + \xi_2 - 4 < 0$ as well as $\xi_1 - \xi_2 - 4 < 0$.

Case 5: $\xi_1 - 4 \leq \xi_2 \leq 3\xi_1 - 4$, $\xi_1 + \xi_2 \geq 4$ *and* $\xi_2 < 4$. We obtain $\gamma(\xi_1, \xi_2 - 4) = \frac{\xi_1}{4}$ (type A) and $\gamma(\xi_1, \xi_2 + 4) = \frac{(\xi_1 + (\xi_2 + 4))^2}{16\xi_1}$ (type B). This gives $f(\xi_1, \xi_2) = \frac{(\xi_1 + \xi_2 + 4)^2}{64}$. Hence $f(\xi_1, \xi_2) = 1$ if $\xi_1 + \xi_2 = 4$, and $f(\xi_1, \xi_2) > 1$ if $\xi_1 + \xi_2 > 4$.

Step 2. We consider arbitrary F. The inclusion $F \subseteq [1, 4]$ implies $B_F \subseteq B_{[1,4]}$, $\gamma_F(\cdot) \geq \gamma_{[1,4]}(\cdot)$ and $f^F(\cdot) \geq f^{[1,4]}(\cdot) = f(\cdot)$. Hence, by Step 1, it is enough to show that, for all $\eta \in [1, 4]$, $f^F(2\sqrt{\eta}, \pm(4 - 2\sqrt{\eta})) = f(2\sqrt{\eta}, \pm(4 - 2\sqrt{\eta}))$ if and only if $\eta \in F$. In fact, by symmetry with respect to the horizontal axis it remains prove that

$$f^F(2\sqrt{\eta}, 4 - 2\sqrt{\eta}) = f(2\sqrt{\eta}, 4 - 2\sqrt{\eta}) \quad \text{if and only if } \eta \in F.$$

As in Step 1, we obtain $\gamma_F(2\sqrt{\eta}, (4 - 2\sqrt{\eta}) - 4) = \gamma(2\sqrt{\eta}, (4 - 2\sqrt{\eta}) - 4) = \frac{2\sqrt{\eta}}{4} = \frac{\sqrt{\eta}}{2}$ (type A), and our claim is equivalent to

$$\gamma_F(2\sqrt{\eta}, (4 - 2\sqrt{\eta}) + 4) = \gamma(2\sqrt{\eta}, (4 - 2\sqrt{\eta}) + 4) \quad \text{if and only if } \eta \in F.$$

The last equation is satisfied if and only if the ray

$$[(0, 0), (2\sqrt{\eta}, (4 - 2\sqrt{\eta}) + 4)\rangle = [(0, 0), (\sqrt{\eta}, 4 - \sqrt{\eta})\rangle$$

meets the boundaries of B_F and of $B_{[1,4]}$ at the same point. The intersection point with $\mathrm{bd}(B_{[1,4]})$ is $(\eta, 4\sqrt{\eta} - \eta) = (\eta, h(\eta))$ (see Step 1, type B). The same intersection is obtained with $\mathrm{bd}(B_F)$ if and only if $(\eta, h(\eta)) \in \mathrm{bd}(B_F)$. By definition of B_F, this is equivalent to $\eta \in F$. This finishes the proof. $\square$

Problem 5.10. *Let $p_1, \ldots, p_n$ be points in a generalized Minkowski space $\left(\mathbb{R}^d, \gamma\right)$, and define $f = \prod_{i=1}^n \gamma(\cdot - p_i)$.*

(*a*) *How large can level sets $f_{=\alpha}$ be (nowhere dense, Lebesgue measure zero, Hausdorff dimension)?*

(*b*) *How small are the following sets (countable, Lebesgue measure zero, meager)?*

$$\{\alpha \in \mathbb{R} \mid f_{=\alpha} \textit{ has infinitely many connected components}\},$$
$$\{\alpha \in \mathbb{R} \mid f_{=\alpha} \textit{ is not a disjoint union of finitely many closed curves}\}$$
$$\textit{(if } d = 2\textit{)},$$
$$\{\alpha \in \mathbb{R} \mid f_{=\alpha} \textit{ is not a union of finitely many closed curves}\}$$
$$\textit{(if } d = 2\textit{)}.$$

Of course, one can consider these problems for special cases, such as for $d = 2$ or $n = 2$.

In the bifocal case, level sets $f_{=\alpha}$ are nowhere dense, as the following theorem shows.

Theorem 5.11. *Let p_1 and p_2 be distinct points in a generalized Minkowski space $\left(\mathbb{R}^d, \gamma\right)$, and define $f = \gamma(\cdot - p_1)\gamma(\cdot - p_2)$. Then, for each $\alpha \geq 0$, the interior of the level set $f_{=\alpha}$ is empty.*

Proof. If $\alpha = 0$, then $f_{=\alpha} = \{p_1, p_2\}$ and the claim is trivial. Let $x \in \mathbb{R}^d \setminus \{p_1, p_2\}$ be an arbitrary point.

Case 1. Every neighborhood of x contains points of $\mathbb{R}^d \setminus (B(p_1, \gamma(x - p_1)) \cup B(p_2, \gamma(x - p_2)))$. That is, in every neighborhood of x, there is a point y with $f(y) = \gamma(y - p_1)\gamma(y - p_2) > \gamma(x - p_1)\gamma(x - p_2) = f(x) =: \alpha$. Therefore x is not an interior point of $f_{=\alpha}$.

Case 2. There is a neighborhood $U = \mathrm{int}(B(x, \varepsilon))$ of x which does not contain any points of $\mathbb{R}^d \setminus (B(p_1, \gamma(x - p_1)) \cup B(p_2, \gamma(x - p_2)))$. That is, $U \subseteq B(p_1, \gamma(x - p_1)) \cup B(p_2, \gamma(x - p_2))$. Let H_i be a closed half-space supporting $B(p_i, \gamma(x - p_i))$ at x, $i \in \{1, 2\}$. Then U is contained in the union $H_1 \cup H_2$ of half-spaces whose bounding hyperplanes meet at $x \in U$. It follows that $\mathrm{bd}(H_1) = \mathrm{bd}(H_2)$ is the unique supporting hyperplane of $B(p_i, \gamma(x - p_i))$ at x, and $H_i \cap U = B(p_i, \gamma(x - p_i)) \cap U$, $i \in \{1, 2\}$. Hence x lies in the relative interiors of respective $(d - 1)$-faces of these balls. Then the restrictions of $\gamma(\cdot - p_1)$ and $\gamma(\cdot - p_2)$ to every sufficiently small neighborhood V of x are non-constant affine functions, whose product f cannot be constant on V. $\square$

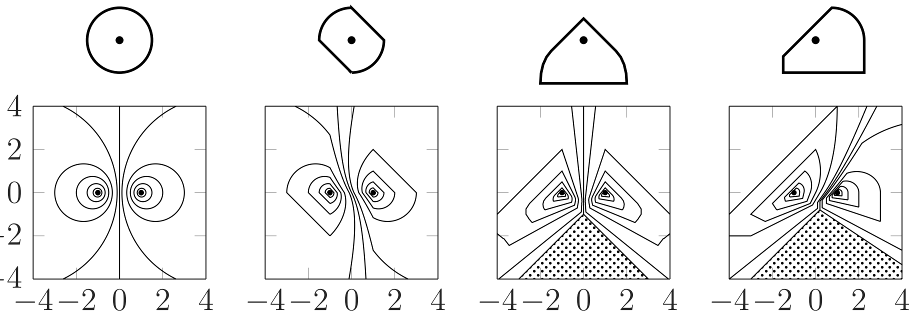


Figure 6.1: Four unit circles and corresponding Apollonius circles with foci $(1, 0)$ and $(-1, 0)$.

6. The circles of Apollonius

Given two fixed foci p_1, p_2 in the Euclidean plane, the locus of all points whose two distances from p_1 and p_2 are in a constant ratio $\alpha > 0$ is the so-called *circle of Apollonius*, going back even to Aristotle (see [12, §§ 6.6, 6.8] as well as [63, § 18.3]). It is immediate that this figure can be generalized to an *Apollonius sphere* in higher dimensions. We have $\alpha = 1$ precisely for the perpendicular bisector of p_1 and p_2, and if α varies over the positive reals, the bundle of Apollonius circles of p_1, p_2 presents a *hyperbolic (or Apollonius) pencil of coaxial circles* (cf. [12, § 6.5] and [63, § 29]). Figure 6.1 depicts Apollonius circles for the unit balls also used in Figures 3.1, 3.2, 4.1, 5.1, and 5.2, for norms see also Figures 4.11 a and 4.11 b in [89].

Such pencils of Euclidean circles are important in Möbius geometry and occur also as special bundles of hyperbolic lines in Poincaré's models of the hyperbolic plane (see [12, § 16.7] and [63, §§ 56, 57]). Furthermore, they occur in physics (e.g., vibrations in circular drumheads), combined also with the inversion in circles as useful tool [55, 56]. With the help of Apollonius circles well-known optimization problems can be solved [99], and the Apollonius metric, which is closely related to the hyperbolic metric, is based on them [2, 36]. Using the hyperbolic cosine law of sides, the authors of [103] proved that Apollonius circles in the hyperbolic plane are smooth and strictly convex curves, and that in spherical geometry they need not be convex. It is natural to ask for analogues of Apollonius circles when the foci p_1, p_2 are, more generally, replaced by (e.g., convex) sets. For foci with the shape of circles see [43], but nothing seems to be done in this direction in a more general setting. Clearly, Apollonius circles are also interesting from the viewpoint of location science (see [57, § 3.2]).

Although multifocality makes no sense when considering constant *ratio* of distances, Apollonius circles were also studied for three given foci p_1, p_2, p_3, these circles also naturally occuring in triples then (each circle created by two of the points). Namely, the following concept is natural: given three non-collinear points p_1, p_2, p_3 forming a triangle T and three corresponding positive real numbers $\alpha_1, \alpha_2, \alpha_3$. We ask for points whose distance ratio to p_1, p_2, p_3 equals $\alpha_1 : \alpha_2 : \alpha_3$, thus lying in the intersection of all three corresponding Apollonius circles. It turns out that this intersection can consist of two points, of one point or of none, and then these three circles determine an elliptic, parabolic or hyperbolic coaxial pencil, respectively. In the first case, these two intersection points are called *isodynamic points of* T, and their affine hull is the *Lemoine line of* T. For recent results in this direction we refer to [32, 77, 35]. The three-dimensional analogue (again with a pair of isodynamic points as intersection of six spheres) was investigated in [5].

Another way of considering more than two foci to generalize the concept of Apollonius circles was presented in [85]. With respect to two sets consisting of n foci each, so-called Apollonian curves are introduced. They are defined as locus of all points in the plane which guarantee a constant ratio between the distance product to the first n foci and that to the second n foci (for $n = 1$ thus yielding the classical Apollonius circle). Several geometric properties of these curves are proved in [85].

Closer to our topic here, it was proved in [14] that a Banach space is Hilbert if for any pair of fixed points p_1, p_2 and any $\alpha > 0$ the locus of all points whose two distances from p_1 and p_2 yield a constant ratio α is a sphere.

Presenting from now on new results on higher-dimensional analogues of Apollonius circles for gauges, we first prove an extension of the last fact to generalized Minkowski spaces $\left(\mathbb{R}^d, \gamma\right)$ (see Theorem 6.2). Given two distinct points $p_1, p_2 \in \mathbb{R}^d$ and a ratio $\alpha > 0$, we consider the *Apollonius level set* and the *Apollonius sublevel set* defined by

$$A_{p_1,p_2,=\alpha} := \left\{x \in \mathbb{R}^d \,\middle|\, \gamma(x - p_1) = \alpha\gamma(x - p_2)\right\}$$

and

$$A_{p_1,p_2,\leq\alpha} := \left\{x \in \mathbb{R}^d \,\middle|\, \gamma(x - p_1) \leq \alpha\gamma(x - p_2)\right\},$$

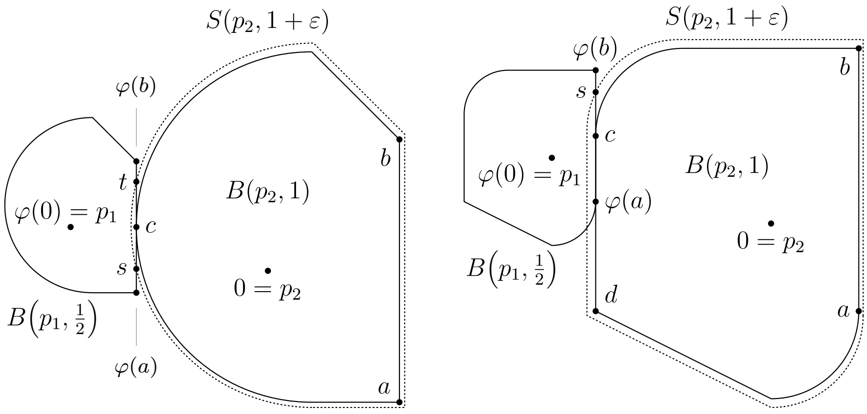
respectively. The set $A_{p_1,p_2,=\alpha}$ is clearly the analogue of a circle of Apollonius. It is enough to consider $\alpha \in (0, 1]$, because $A_{p_1,p_2,=\alpha} = A_{p_2,p_1,=\frac{1}{\alpha}}$. Clearly, we have $A_{p_1,p_2,=1} = \operatorname{bis}(p_1, p_2)$.

Lemma 6.1. *Let p_1, p_2 be two distinct points of a generalized Minkowski space $\left(\mathbb{R}^d, \gamma\right)$ and let $\alpha \in (0, 1]$. Then $A_{p_1,p_2,\leq\alpha}$ is star-shaped with respect to p_1. Moreover, if $\alpha < 1$, then $A_{p_1,p_2,\leq\alpha}$ is bounded; namely,*

$$A_{p_1,p_2,\leq\alpha} \subseteq B\left(p_1, \frac{\alpha}{1-\alpha}\,\gamma(p_1 - p_2)\right).$$

Proof. Application of Lemma 4.8 to p_1, p_2 and α says that $A_{p_1,p_2,\leq\alpha} = V_1^{\alpha,\cdot}$ is star-shaped with respect to p_1. If $\alpha < 1$, then every $x \in A_{p_1,p_2,\leq\alpha}$ satisfies

$$\gamma(x - p_1) \leq \alpha\gamma(x - p_2) \leq \alpha(\gamma(x - p_1) + \gamma(p_1 - p_2)).$$



(a) Case 1.

(b) Case 2.

Figure 6.2: Proof of Lemma 6.3.

Thus, $\gamma(x - p_1) \leq \frac{\alpha}{1-\alpha}\,\gamma(p_1 - p_2)$, and in turn

$$A_{p_1,p_2,\leq\alpha} \subseteq B\left(p_1, \frac{\alpha}{1-\alpha}\,\gamma(p_1 - p_2)\right). \qquad \square$$

Our main claim on Apollonius level sets complements Proposition 4.10 and Theorem 4.11 and extends Euclidean results from [85] to gauges.

Theorem 6.2. *The following are equivalent for every generalized Minkowski space* $\left(\mathbb{R}^d, \gamma\right)$, $d \geq 2$.

(a) *The gauge* γ *is a norm induced by an inner product.*

(b) *For any two distinct points* $p_1, p_2 \in \mathbb{R}^d$ *and any* $\alpha \in (0,1)$, *the Apollonius level set* $A_{p_1,p_2,=\alpha}$ *is a sphere in* $\left(\mathbb{R}^d, \gamma\right)$.

(c) *For any two distinct points* $p_1, p_2 \in \mathbb{R}^d$ *and any* $\alpha \in (0,1)$, *the Apollonius sublevel set* $A_{p_1,p_2,\leq\alpha}$ *is convex.*

We prepare the proof of this theorem by an additional lemma.

Lemma 6.3. *Let* $(\mathbb{R}^2, \gamma)$ *be a generalized Minkowski plane that is not strictly convex. Then there exist distinct points* $p_1, p_2 \in \mathbb{R}^2$ *and* $\alpha \in (0,1)$ *such that* $A_{p_1,p_2,\leq\alpha}$ *is not convex.*

Proof. Let $[a, b]$ be a maximal (under inclusion) non-degenerate segment in $S(0,1)$, and let L be the supporting line of $B(0,1)$ parallel to and different from $\langle a, b\rangle$. We investigate two cases (see Figure 6.2).

Case 1: $B(0,1) \cap L$ *is a singleton* $\{c\}$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the homothety with factor $\frac{1}{2}$ that maps $\frac{a+b}{2}$ onto c; i.e., $\varphi(x) = \frac{1}{2}\left(x - \frac{a+b}{2}\right) + c$. We consider $p_1 := \varphi(0)$ and $p_2 := 0$.

Since $S(p_2, 1)$ meets the straight line $L = \langle\varphi(a), \varphi(b)\rangle$ only in $c = \frac{\varphi(a)+\varphi(b)}{2}$, $S(p_2, 1+\varepsilon)$ meets the segment $[\varphi(a), \varphi(b)]$ in exactly two points s, t such that

$$c \in (s, t)$$

if $\varepsilon > 0$ is chosen small enough. Now we put $\alpha = \frac{1}{2(1+\varepsilon)}$. Then

$$s, t \in A_{p_1,p_2,\leq\alpha},$$

because $s, t \in [\varphi(a), \varphi(b)] \cap S(p_2, 1+\varepsilon) \subseteq S\left(p_1, \frac{1}{2}\right) \cap S(p_2, 1+\varepsilon)$, which gives

$$\gamma(s - p_1) = \frac{1}{2} = \alpha(1+\varepsilon) = \alpha\gamma(s - p_2).$$

In the same way $\gamma(t - p_1) = \alpha\gamma(t - p_2)$, and in turn $s, t \in A_{p_1,p_2,\leq\alpha}$. However,

$$c \notin A_{p_1,p_2,\leq\alpha},$$

since

$$\gamma(c - p_1) = \frac{1}{2} = \alpha(1+\varepsilon) \nleq \alpha = \alpha\gamma(c - p_2).$$

The properties $s, t \in A_{p_1,p_2,\leq\alpha}$, $c \notin A_{p_1,p_2,\leq\alpha}$ and $c \in (s, t)$ show that $A_{p_1,p_2,\leq\alpha}$ is not convex.

Case 2: $B(0, 1) \cap L$ is a non-degenerate segment $[c, d]$. We may suppose that a, b, c, d are in cyclic order on $S(0, 1)$.

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homothety with factor $\frac{1}{2}$ such that

$$c \in (\varphi(a), \varphi(b)) \quad \text{and} \quad \varphi(a) \in (c, d).$$

We consider $p_1 := \varphi(0)$ and $p_2 := 0$.

If $\varepsilon > 0$ is chosen small enough, then $S(p_2, 1+\varepsilon)$ meets the relatively open segment $(c, \varphi(b))$ in a point s; in particular,

$$c \in (\varphi(a), s).$$

We put $\alpha = \frac{1}{2}$. Note that

$$s \in \operatorname{int}(A_{p_1,p_2,\leq\alpha}),$$

because $\gamma(s - p_1) = \frac{1}{2} < \alpha(1+\varepsilon) = \alpha\gamma(s - p_2)$ and γ is continuous. On the other hand, $\varphi(a), c \in A_{p_1,p_2,\leq\alpha}$, since

$$\gamma(\varphi(a) - p_1) = \gamma(c - p_1) = \frac{1}{2} = \alpha = \alpha\gamma(\varphi(a) - p_2) = \alpha\gamma(c - p_2).$$

In fact,

$$\varphi(a), c \in \operatorname{bd}(A_{p_1,p_2,\leq\alpha}),$$

because $\varphi(a)$ and c are accumulation points of $\mathrm{int}(B(p_2,1))$, and every $x \in \mathrm{int}(B(p_2,1)) \subseteq \mathbb{R}^2 \setminus B\left(p_1, \frac{1}{2}\right)$ satisfies

$$\gamma(x-p_1) > \frac{1}{2} = \alpha > \alpha\gamma(x-p_2);$$

i.e., $x \notin A_{p_1,p_2,\leq\alpha}$.

The inclusions $s \in \mathrm{int}(A_{p_1,p_2,\leq\alpha})$, $\varphi(a), c \in \mathrm{bd}(A_{p_1,p_2,\leq\alpha})$ and $c \in (\varphi(a), s)$ show that $A_{p_1,p_2,\leq\alpha}$ is not convex. □

Proof of Theorem 6.2. The classical theorem on Apollonius circles proves the implication $(a) \Longrightarrow (b)$. Lemma 6.1 gives $(b) \Longrightarrow (c)$. We prove $\neg(a) \Longrightarrow \neg(c)$ by considering two cases.

Case 1: γ is strictly convex. By [101] (see also Proposition 4.10), there exist distinct points $p_1, p_2 \in \mathbb{R}^d$ such that $\mathrm{bis}(p_1,p_2)$ is not convex. Hence there exist $a, b \in \mathrm{bis}(p_1,p_2)$ and $c \in [a,b]$ such that $c \notin \mathrm{bis}(p_1,p_2)$; say, $\gamma(c-p_1) > \gamma(c-p_2)$. By continuity of γ, there exists a small $\lambda_0 > 0$ such that

$$\gamma((c+\lambda_0(p_1-p_2))-p_1) > \gamma((c+\lambda_0(p_1-p_2))-p_2). \tag{2}$$

By Lemma 4.5, $\gamma((a+\lambda_0(p_1-p_2))-p_1) < \gamma((a+\lambda_0(p_1-p_2))-p_2)$ and $\gamma((b+\lambda_0(p_1-p_2))-p_1) < \gamma((b+\lambda_0(p_1-p_2))-p_2)$. Thus,

$$\alpha := \max\left\{\frac{\gamma((a+\lambda_0(p_1-p_2))-p_1)}{\gamma((a+\lambda_0(p_1-p_2))-p_2)}, \frac{\gamma((b+\lambda_0(p_1-p_2))-p_1)}{\gamma((b+\lambda_0(p_1-p_2))-p_2)}\right\} < 1,$$

and both $a+\lambda_0(p_1-p_2)$ and $b+\lambda_0(p_1-p_2)$ belong to the Apollonius sublevel set $A_{p_1,p_2,\leq\alpha}$. By (2), $c+\lambda_0(p_1-p_2) \notin A_{p_1,p_2,\leq\alpha}$. However, $c \in [a,b]$ shows that $c+\lambda_0(p_1-p_2) \in [a+\lambda_0(p_1-p_2), b+\lambda_0(p_1-p_2)]$. Hence $A_{p_1,p_2,\leq\alpha}$ is not convex.

Case 2: γ is not strictly convex. Now $S(0,1)$ contains a non-degenerate segment $[s,t]$. Let $V := \mathrm{lin}\,\{s,t\}$. The generalized Minkowski plane $(V, \gamma|_V)$ is not strictly convex, too, and Lemma 6.3 gives distinct points $p_1, p_2 \in V$ and $\alpha \in (0,1)$ such that the Apollonius sublevel set

$$A^V_{p_1,p_2,\leq\alpha} := \{x \in V \mid \gamma|_V(x-p_1) \leq \alpha\gamma|_V(x-p_2)\}$$

in $(V, \gamma|_V)$ is not convex. Then the corresponding set $A_{p_1,p_2,\leq\alpha}$ in $\mathbb{R}^d$ is not convex as well, because $A_{p_1,p_2,\leq\alpha} \cap V = A^V_{p_1,p_2,\leq\alpha}$. □

We finish this section by the following

Proposition 6.4. *Let p_1, p_2 be distinct points of a generalized Minkowski plane $(\mathbb{R}^2, \gamma)$. If the gauge γ is not strictly convex, then there is a level $\alpha > 0$ such that $A_{p_1,p_2,=\alpha}$ contains a non-degenerate segment.*

Proof. If γ is not strictly convex, there is a non-degenerate segment $[s,t] \subseteq S(0,1)$. The cone $(p_1 + C) \cap (p_2 + C)$, where $C := \{\lambda u \mid u \in [s,t], \lambda \geq 0\}$, has non-empty interior, and $\gamma(\cdot - p_i)$ is an affine function on this cone, $i \in \{1,2\}$. In particular, each function $\gamma(\cdot - p_i)$ is constant on segments which are parallel to $[s,t]$ and contained in $(p_1 + C) \cap (p_2 + C)$, and so is $\frac{\gamma(\cdot - p_1)}{\gamma(\cdot - p_2)}$. $\square$

7. Concluding remarks

The present paper reports on ongoing fruitful research. For example, one may ask if Propositions 5.8 and 6.4 can be extended to further characterizations of non-strict convexity of generalized Minkowski planes. Complementing results on strict convexity (see, e.g., Theorems 3.5 and 4.6), the question occurs whether there are similar results regarding smoothness of γ, that is, the uniqueness of supporting hyperplanes at each boundary point of $B(0,1)$.

And in the spirit of inverse problems, one can pose questions of the following type: Let $\star$ denote either addition, subtraction, multiplication, or division. What are characterizing properties of the non-degenerate level sets of $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \gamma(x - p_1) \star \gamma(x - p_2)$? Given a curve in $\mathbb{R}^2$, what assumptions have to be made such that there exists a gauge γ on $\mathbb{R}^2$ for which the given curve is a level set of f? Clearly, analogues of these questions can be posed for higher dimensions and multifocality (where it is appropriate).

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