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Positively α-Far Sets and Existence Results for Generalized Perturbed Sweeping Processes

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Dedicated to the memory of Jean Jacques Moreau.

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We consider the general class of positively α-far sets, introduced in [29], which contains strictly the class of uniformly prox-regular sets and the class of uniformly subsmooth sets. We provide some conditions to assure the uniform subsmoothness, and thus the positive α-farness, of the inverse images under a differentiable mapping. Then, we take advantage of the properties of this class to study the generalized perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in F(t,x(t)) + g(x(t))N\left(C(t), h(x(t))\right) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in h^{-1}\left(C(T_0)\right), \end{cases}$$

where $g\colon X \to \mathcal{L}(Y;X)$, $h\colon X \to Y$ are two functions, X, Y are two separable Hilbert spaces and the sets $C(t)$ belong to the class of positively α-far sets. This differential inclusion includes the classical perturbed sweeping process as well as complementarity dynamical systems. Our study is achieved by approximating the given differential inclusion with maximally perturbed differential inclusions which, under certain compactness conditions, converges to an absolutely continuous solution. Moreover, this approach allows us to get existence for evolution inclusions of the form

$$\begin{cases} -\dot{x}(t) \in \partial f(t,x(t)) + F(t,x(t)) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0, \end{cases}$$

where $[T_0,T]$ is a fixed interval with $0 \le T_0 < T$, $f\colon [T_0,T] \times X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, not necessarily convex. Here $\partial f(t,\cdot)$ denotes the Clarke subdifferential of the function $f(t,\cdot)$ and $F\colon [T_0,T] \times X \rightrightarrows X$ is a perturbation term.

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1. Introduction

In this paper we study the class of positively α-far sets, which is the class of closed sets for which the origin is kept uniformly positively far from the Clarke subdifferential of the distance function, say $\partial d(\cdot, S)$, in some tube $\{x\colon 0 < d(x,S) < \rho\}$ for some $\rho \in (0,+\infty]$, and its applications to some differential inclusions. This class was introduced in [29], as a localization of the class of *subdifferentially behaved* sets in [27], to study some differential inclusions of sweeping process type in finite dimensions. We will show that this notion of set includes several ones as paraconvexity, uniformly prox-regularity and uniformly subsmoothness. Then, we will develop some properties and characterizations of positively α-far sets. Finally, we will take advantage of the properties of this class to show existence of solutions for some differential inclusions.

Let X and Y be two separable Hilbert spaces and $C\colon [T_0,T] \rightrightarrows Y$ be an absolutely continuous set-valued map with positively α-far values. We are interested in the differential inclusion

$$\begin{cases} -\dot{x}(t) \in F(t,x(t)) + g(x(t))N\left(C(t),h(x(t))\right) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in h^{-1}\left(C(T_0)\right), \end{cases} \tag{1}$$

where $g\colon X \to \mathcal{L}(Y;X)$ and $h\colon X \to Y$ are two functions and $N(C(t),\cdot)$ denotes the Clarke normal cone. We call the differential inclusion (1) *generalized perturbed sweeping process* (GSP) because when $X = Y$, $h \equiv \mathrm{Id}$ and $g \equiv Dh$ we get the classical perturbed sweeping process. The motivation to introduce (1) is that several differential inclusions can be recast into a GSP. For example, perturbed sweeping process, complementarity dynamical systems (CDS), projected dynamical systems [14], some differential variational inequalities [49, Section 2.5], etc. The perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in N(C(t),x(t)) + F(t,x(t)) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{2}$$

is well known in the literature and has been studied by several authors. Its importance comes from the study of problems in mechanics of elastoplastic materials [47], non-smooth mechanics [13], dynamics of systems with inelastic shocks [41], modeling and simulation of switched electrical circuits [1], crowd motion modeling [36], etc. When $F \equiv 0$ and the sets $C(t)$ are convex, the system above is referred to as a *sweeping process* because it can be visualized as a point $x(t)$ moving inside $C(t)$ and being pushed by the boundary of this convex set when contact is established. It was introduced in the seventies by Moreau in a series of papers [43, 44, 45, 46] to study contact problems in mechanical systems. Since then, many improvements have been made by weakening the convexity assumption and by considering the perturbed version of (2). In [17], Castaing dealt with sweeping process associated with sets $C(t) = S + v(t)$, where S is a fixed closed nonconvex set, and v is a mapping with finite variation. Then, Valadier [56] extended the study of the absolutely continuous case to a more general situation of complements of convex sets in finite dimension. Afterward, Benabdellah [6] and Colombo and Goncharov [22] proved independently and almost at the same time the existence of solutions for general nonconvex sets $C(t)$ in $\mathbb{R}^n$ moving in a Lipschitz continuous way. Extensions of existence and uniqueness of solution of (2) when $C(t)$ is a uniformly prox-regular set of a Hilbert space X, moving in a Lipschitz continuous way and in an absolutely continuous way, have been obtained by Colombo and Goncharov [22] and Bounkhel and Thibault [12] (see also the paper [55] by Thibault for a regularization of the process in such a context). In the case where the uniformly prox-regular sets $C(t)$ move with bounded variation we refer the reader to the paper [26] by Edmond and Thibault.

The study of the differential inclusions which are perturbations of sweeping processes began, as far as we know, with [30]. In this paper, Henry, to deal with planning procedures in mathematical economy, introduced the differential inclusion

$$\begin{cases} -\dot{x}(t) \in P_{T_C(x(t))}(F(x(t))) \quad \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C, \end{cases}$$

where $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an upper semicontinuous set-valued map with nonempty compact convex values, C is a fixed closed convex set, $T_C(\cdot)$ is the tangent cone to C and $P_{T_C(x(t))}$ denotes the projection mapping into the closed convex set $T_C(x(t))$. This differential inclusion has been also considered by Cornet [24] with a Clarke tangentially regular set C of $\mathbb{R}^n$, reducing the problem as in [30] to the existence of a solution of the perturbed differential inclusion

$$\begin{cases} -\dot{x}(t) \in N(C, x(t)) + F(x(t)) \quad \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C. \end{cases}$$

Since then the case of moving sets $C(t)$ has been developed. We refer to Monteiro-Marques [40], Castaing, Duc Ha and Valadier [18] and Castaing and Monteiro-Marques [19] for the study of perturbed sweeping processes in the form of (2) in the cases where all the sets $C(t)$ are either convex or complements of open convex sets. For the case of general nonconvex closed sets of $\mathbb{R}^n$ we refer to Thibault [54] (see also the paper of Haddad, Jourani and Thibault [29] for reduction of sweeping process to unconstrained differential inclusions). Several other papers deal with perturbed sweeping processes, in the Hilbert setting, under uniform prox-regularity assumptions, such as the works of Bounkhel and Thibault [12], Edmond and Thibault [26], Mazade and Thibault [37, 38] and Sene and Thibault [53] (see [11] for a general account on sweeping process and the prox-regularity).

Another interesting example of (1) can be obtained from complementarity dynamical systems (CDS)

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) + g(x(t))u(t), \\ y(t) &= H(t, x(t), u(t)), \\ 0 \leq y(t) &\perp u(t) \geq 0. \end{aligned} \tag{3}$$

which can be written as a generalized perturbed sweeping process. CDS have been object of strong interest because of their applications in various fields like mechanics, electrical circuits, transportation science, control systems, etc (see [15] and the references therein). The CDS formalism includes the so-called linear complementarity systems (LCS), widely used to deal with some electrical problems (see [1]), and the so called gradient complementarity system (GCS) which corresponds to the particular case $g(x) \equiv [Dh(x)]^*$ for every $x \in X$. The CDS has been studied in [15] where the authors consider an "input-output property" to perform a change of state variable allowing them to write (3) as a GCS which is transformed into a perturbed sweeping process. These transformations are made by using the identity

$$I_{C(t)}\left(h(x)\right) = I_{h^{-1}(C(t))}(x) \quad \forall x \in X,$$

showing that the sets $h^{-1}\left(C(t)\right)$ are r-uniformly prox-regular and using a non-smooth chain rule, for which they must assume some regularity and constraint qualification conditions on h. Then, existence is obtained from known results in the literature of sweeping process [26, 54]. We will follow this path but only in the particular case of the GCS, where we show that the sets $h^{-1}\left(C(t)\right)$ are equi-uniformly subsmooth and then we apply our existence results. In the general case we will transform the CDS into a GSP and we apply our existence results. This avoids the assumption of a special structure on the functions g and h, as the input-output property in [15].

We also deal with perturbations of evolution inclusions associated to time dependent subdifferentials, i.e., with differential inclusions of the form

$$\begin{cases} -\dot{x}(t) \in \partial f(t, x(t)) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0, \end{cases} \tag{4}$$

where $[T_0, T]$ is still a fixed interval with $0 \leq T_0 < T$, $f\colon [T_0, T] \times X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function and X is a real separable Hilbert space. Here $\partial f(t, \cdot)$ denotes the Clarke subdifferential of the function $f(t, \cdot)$ and $F\colon [T_0, T] \times X \rightrightarrows X$ is a perturbation term. These differential inclusions, which include the perturbed sweeping process (by taking $f(t, \cdot)$ as the indicator function of the sets $C(t)$), has been considered by several authors because they appear naturally in differential variational inequalities, optimization, optimal control theory, mechanics, etc. When $f(t, \cdot)$ is a proper, lower semicontinuous and convex function, it has been studied principally by Bénilan and Brézis [7], Brézis [16], Attouch and Damlamian [2], Aubin and Cellina [3], Moreau [43, 44, 45, 46], Watanabe [58], Yotsutami [59], Hu and Papageorgiou [32], Barbu [5], etc. (see [32, Chapter II] and the references therein for a broader discussion on the subject and some applications to parabolic optimal control problems). In the non-convex case the results are less abundant; we can mention, for instance, the works of Guillaume [28] for convex composite functions, Marcellin and Thibault [35] for primal lower nice functions and Qin and Xue [51] for regular perturbed convex functions among others.

Therefore, the aim of this paper is to study the class of positively α-far sets and apply its properties to get the existence of solutions for the generalized sweeping process (1), the perturbed sweeping process (2), the complementarity dynamical systems (3) and the differential inclusion (4) in the following cases:

(a) Subgradient evolution inclusions: $f(t,\cdot)$ is lower semicontinuous and $F(t,\cdot)$ is upper semicontinuous from X into X_w with nonempty closed and convex values.

(b) Subgradient perturbed sweeping processes: $f(t,x)$ is lower semicontinuous and $F(t,x) = N(C(t),x)$.

The hypotheses used in this paper are rather standard to ensure existence of solutions of the differential inclusion (4) when $F(t,x) = N(C(t),x)$. We assume either ball-inf-compactness of $f(t,\cdot)$ or ball-compactness of the sets $C(\cdot)$. Indeed, this standardization comes from the existence of solutions of optimization problems of the form

$$\min\{f(x)\colon x \in C\}, \tag{5}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function and C is a closed subset of some Banach space X. The usual assumptions ensuring existence of solutions to this problem are either the inf-compactness of f or the compactness of C. If in addition $\bar{x}$ is a solution to this problem and f is locally Lipschitz around $\bar{x}$, then

$$0 \in \partial f(\bar{x}) + N(C,\bar{x}). \tag{6}$$

Our approach is based on approximating the given differential inclusion with maximally perturbed differential inclusions which, under certain compactness conditions, converges to an absolutely continuous solution. This method can be fruitful in many other situations but we will limit ourselves to using it in the cases previously mentioned.

The paper is organized as follows. In Section 2 we recall several concepts of non-smooth and variational analysis which are involved through the paper. In Section 3 we present the definition of the class of positively α-far sets, then we show that this class includes the classes of paraconvex sets, uniformly prox-regular sets and uniformly subsmooth sets and also we give conditions to assure the uniform subsmoothness of the inverse images under a differentiable mapping. In Section 4, for sake of readability, we collect the hypotheses used along the paper. In Section 5 we give some results needed to prove the main results. In Section 6, after obtaining an existence result for the GSP (1) we obtain the existence for the perturbed sweeping process (2). In Sections 7 and 8 we prove the existence of solutions for (4) in the cases (a) and (b) referred above, respectively. Finally, in Section 9 we make an application to CDS.

2. Preliminaries

From now on X stands for a separable Hilbert space whose norm is denoted by $\|\cdot\|_X$. The closed ball centered at x with radius r, is defined by $\bar{B}(x,r) = \{u \in X : \|u - x\| \leq r\}$ and the closed unit ball is denoted by $\mathbb{B}_X$. The notation X_w stands for X equipped with the weak topology and $x_n \rightharpoonup x$ stands for the weak convergence of a sequence $(x_n)_n$ to x.
The Clarke normal cone (see [20]) is defined as the negative polar of the Clarke tangent cone. Recall that a vector $h \in X$ is in the Clarke tangent cone $T_C(S,x)$ to the set S at $x \in S$ when for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in S converging to x and every sequence of positive numbers $\{t_n\}_{n\in\mathbb{N}}$ converging to 0, there exists some sequence $\{h_n\}_{n\in\mathbb{N}}$ in X converging to h such that $x_n + t_n h_n \in S$ for all $n \in \mathbb{N}$. This cone is closed and convex and its negative polar $N(S,x)$ is the Clarke normal cone to S at $x \in S$, that is,

$$N(S,x) = \{v \in X\colon \langle v, h\rangle \leq 0 \ \forall h \in T_C(S,x)\}.$$

As usual, $N(S,x) = \emptyset$ if $x \notin S$. Through that normal cone, the Clarke subdifferential of the function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial f(x) := \{v \in X\colon (v,-1) \in N\left(\operatorname{epi} f, (x, f(x))\right)\},$$

where $\operatorname{epi} f := \{(y,r) \in X \times \mathbb{R}\colon f(y) \leq r\}$ is the epigraph of f. When the function is finite and locally Lipschitzian around x, the Clarke subdifferential is characterized (see [20]) in the following simple and amenable way

$$\partial f(x) = \{v \in X\colon \langle v, h\rangle \leq f^\circ(x;h) \ \forall h \in X\},$$

where

$$f^\circ(x;h) := \limsup_{(t,x')\to(0^+,x)} t^{-1}\left[f(x'+th) - f(x')\right],$$

is the generalized directional derivative of the locally Lipschitzian function f at x in the direction $h \in X$. The function $f^\circ(x;\cdot)$ is in fact the support function of $\partial f(x)$. That characterization easily yields that the Clarke subdifferential of any locally Lipschitzian function has the important property of upper semicontinuity from X into X_w (see Proposition 2.3). The equality (see [20])

$$N(S,x) = \overline{\mathbb{R}_+ \partial d(\cdot,S)(x)} \quad \text{for } x \in S,$$

gives the expression of the Clarke normal cone in terms of the distance function. This will allow us to take advantage, in the next sections, of the upper semicontinuity of the Clarke subdifferential of the particular Lipschitzian function $d(\cdot,S)$. As usual, it will be convenient to write below

$$\partial d(x,S) \ \text{ in place of } \partial d(\cdot,S)(x).$$

The following definitions related to set-valued mappings will be needed in the sequel.

Definition 2.1. Let $([T_0, T], \Sigma, \mu)$ the Lebesgue measure space and $\Phi\colon [T_0, T] \rightrightarrows X$. We say that Φ is measurable if the set

$$\Phi^{-1}(C) := \{t \in [T_0, T]\colon \Phi(t) \cap C \neq \emptyset\} \in \Sigma$$

for any closed subset $C \subset X$.

Definition 2.2. A set-valued map $\Psi\colon X \rightrightarrows X$ is said to be upper semicontinuous from X into X_w if the set

$$\Psi^{-1}(C) := \{x \in X\colon \Psi(x) \cap C \neq \emptyset\}$$

is norm-closed for any C weakly closed of X.

An important example of upper semicontinuous from X into X_w is the Clarke subdifferential of a locally Lipschitz function.

Proposition 2.3 ([9, Proposition 5.2.7]). *Let $f\colon X \to \mathbb{R}$ be a locally Lipschitz function. Then $\partial f\colon X \rightrightarrows X$ is upper semicontinuous from X into X_w.*

Let $A\colon D(A) \subset X \rightrightarrows X$ be a maximal monotone operator. Then $S(t)x = \lim_n \left(I + \frac{t}{n}A\right)^{-n} x$ exists for each $x \in \overline{D(A)}$ and uniformly for t in every compact in $\mathbb{R}_+$ (see [57] for more details). In addition, $\{S(t)\colon \overline{D(A)} \to \overline{D(A)}, t \geq 0\}$ defines a semigroup called the *semigroup generated* by $-A$ on $\overline{D(A)}$. If for each $t > 0$, $S(t)$ is a compact operator we say that the semigroup generated by $-A$ on $\overline{D(A)}$ is *compact*. As an example, the following proposition characterizes the compactness of the semigroup generated for subdifferentials (see [57]).

Proposition 2.4 (Konishi-Brézis). *If the operator $A\colon D(A) \subset X \rightrightarrows X$ is defined by $Ax = \partial\varphi(x)$ for each $x \in D(A) = D(\partial\varphi)$, where $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ is a proper, l.s.c., convex function, then the following conditions are equivalent:*

- *For each $k > 0$, the level set $\{x \in X\colon \|x\|^2 + \varphi(x) \leq k\}$ is relatively compact in X.*
- *The semigroup generated by $-A$ on $\overline{D(A)}$ is compact.*

Let $f \in L^1([T_0, T], X)$ be a given function and consider the differential inclusion

$$\begin{cases} \dfrac{dx}{dt}(t) \in -Ax(t) + f(t) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in \overline{D(A)}. \end{cases} \tag{7}$$

Our main tool is the following perturbed version of (7) which will be used to approximate the solutions of (1):

$$\begin{cases} \dfrac{dx}{dt}(t) \in -Ax(t) + f(t) & \text{a.e. } t \in [T_0, T], \\ f(t) \in F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in \overline{D(A)}, \end{cases} \tag{8}$$

where $F\colon [T_0, T] \times \overline{D(A)} \rightrightarrows X$ is a given set-valued map. The next theorem, which is a consequence of [10, Theorem 1] and [7, Corollary 1.3], asserts the existence of solutions for (8) when $D(A) = X$.

Theorem 2.5. *Let $A\colon X \to X$ be a single valued maximal monotone operator such that $-A$ generates a compact semigroup on $\overline{D(A)} := X$ and let $F\colon [T_0, T] \times \overline{D(A)} \rightrightarrows X$ be a set-valued map with closed and convex values satisfying*

– *For each $x \in X$, $F(\cdot, x)$ is measurable.*
– *For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from X into X_w.*
– *There exist $\gamma, \delta \in L^1(T_0, T)$ such that*

$$\begin{aligned} |F(t,x)| := {} & \sup\{\|w\| \colon w \in F(t,x)\} \\ & \leq \gamma(t)\|x\| + \delta(t) \quad \text{a.e. } t \in [T_0, T], x \in X. \end{aligned}$$

Then there exists at least a solution of (8).

Finally, let $A\colon D(A) \subset X \rightrightarrows X$ be a maximal monotone operator such that $-A$ generates a compact semigroup $S_A(\cdot)$ on $\overline{D(A)}$. We recall that for $\lambda > 0$ $S_{\lambda A}(t) = S_A(\lambda t)$ thus $-\lambda A$ also generates a compact semigroup.

3. Positively α-far sets

This section is devoted to the study of the class of positively α-far sets. For $\rho \in (0, +\infty]$ let us define the open ρ-tube around the set S by

$$U_\rho(S) := \{x \in X \colon 0 < d(x, S) < \rho\}.$$

Definition 3.1. Let $\alpha \in (0, 1]$ and $\rho \in (0, +\infty]$. Let S be a nonempty closed subset of X with $S \neq X$. We say that the Clarke subdifferential of the distance function $d(\cdot, S)$ keeps the origin α-far-off on the open ρ-tube around S provided

$$\alpha \leq \inf_{x \in U_\rho(S)} d(0, \partial d(x, S)). \tag{9}$$

Also, if E is a given nonempty set, we say that the family $(S(t))_{t \in E}$ is positively α-far if every set $S(t)$ satisfies (9) with the same $\alpha > 0$ and $\rho > 0$, for some ρ.

The class of these sets will be called the class of *positively α-far sets*. It was already proved that this class is strictly bigger than the class of uniformly prox-regular sets (see [29] and Example 3.10 below). Since the notion of positively

α-farness involves the Clarke subdifferential of the distance, which can be characterized via the formula (see [31])

$$\partial d(x,S) = \bigcap_{\gamma>0} \overline{\mathrm{co}}\left(\frac{x-P_S^{\gamma}(x)}{d_S(x)}\right), \tag{10}$$

where $P_S^{\gamma}(x) = \{z \in S\colon \|x-z\| < d(x,S)+\gamma\}$, we can get the following proposition.

Proposition 3.2. *Let S be a closed subset of X and $\rho > 0$.*

1. *Assume that the following property holds: For every $x \in U_\rho(S)$ there exists $\gamma(x)$ such that for all $\gamma \in]0,\gamma(x)[$*

$$u_1^*, u_2^* \in \frac{x-P_S^{\gamma}(x)}{d(x,S)} \quad \Rightarrow \quad \langle u_1^*, u_2^*\rangle \geq \alpha^2 + \theta(\gamma,x), \tag{11}$$

where $\lim_{\gamma\downarrow 0}\theta(\gamma,x) = 0$ for all $x \in U_\rho(S)$. Then the origin is kept positively α-far from the Clarke subdifferential of the distance function $d(\cdot,S)$ on $U_\rho(S)$.

2. *Assume that the origin is kept positively α-far from the Clarke subdifferential of the distance function $d(\cdot,S)$ on $U_\rho(S)$. Then the following property holds:*

$$\forall x \in U_\rho(S) \quad u_1^*, u_2^* \in \partial d(x,S) \quad \Rightarrow \quad \langle u_1^*, u_2^*\rangle \geq 2\alpha^2 - 1. \tag{12}$$

Proof. *1.* Fix $x \in U_\rho(S)$. If $u^* \in \mathrm{co}\left(\frac{x-P_S^{\gamma}(x)}{d_S(x)}\right)$ then $u^* = \sum_{i=1}^n \lambda_i u_i^*$ for some $u_i^* \in \frac{x-P_S^{\gamma}(x)}{d_S(x)}$ and $\lambda_i \geq 0$ for $i = 1,\ldots,n$ with $\sum_{i=1}^n \lambda_i = 1$ and some $n \in \mathbb{N}$. Thus,

$$\begin{aligned}
\|u^*\|^2 &= \left\langle \sum_{i=1}^n \lambda_i u_i^*, \sum_{j=1}^n \lambda_j u_j^* \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \left\langle u_i^*, u_j^* \right\rangle \\
&\geq \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \left(\alpha^2 + \theta(\gamma,x)\right) \\
&\geq \alpha^2 + \theta(\gamma,x).
\end{aligned}$$

Hence, (11) holds for $u_1^*, u_2^* \in \mathrm{co}\left(\frac{x-P_S^{\gamma}(x)}{d(x,S)}\right)$, and, since the weak and strong closure of convex sets coincide, (11) holds for $u_1^*, u_2^* \in \overline{\mathrm{co}}\left(\frac{x-P_S^{\gamma}(x)}{d(x,S)}\right)$. Therefore, due to the formula (10), if $x^* \in \partial d(x,S)$

$$\|x^*\|^2 \geq \alpha^2 + \theta(\gamma,x),$$

for all $\gamma < \gamma(x)$, and taking $\gamma \downarrow 0$, we obtain that $\|x^*\| \geq \alpha$.

2. Take $x \in U_\rho(S)$ and $u_1^*, u_2^* \in \partial d(x,S)$. Since $u_1^*, u_2^* \in \partial d(x,S)$ we obtain $\frac{u_1^*+u_2^*}{2} \in \partial d(x,S)$. Thus,

$$\begin{aligned}\alpha^2 &\leq \left\| \frac{u_1^*+u_2^*}{2} \right\|^2 \\ &= \frac{\|u_1^*\|^2}{4} + \frac{1}{2}\langle u_1^*, u_2^* \rangle + \frac{\|u_1^*\|^2}{4} \\ &\leq \frac{1}{4} + \frac{1}{2}\langle u_1^*, u_2^* \rangle + \frac{1}{4}\end{aligned}$$

which entails (12). □

The next example shows that the inequality in relation (12) is attained.

Example 3.3 ([29]). Consider the set $S = \{(x,y) \in \mathbb{R}^2 \colon y \geq -|x|\}$. For $(r,s) \notin S$ we have $d\left((r,s),S\right) = \frac{\sqrt{2}}{2}\left||r|+s\right|$ and

$$\partial d\left((r,s),S\right) = \begin{cases} \frac{\sqrt{2}}{2}(-1,-1) & \text{if } r > 0, \\ \left\{ \frac{\sqrt{2}}{2}(1-2\lambda,-1) \in \mathbb{R}^2 \colon \lambda \in [0,1] \right\} & \text{if } r = 0, \\ \frac{\sqrt{2}}{2}(1,-1) & \text{if } r < 0, \end{cases}$$

therefore the origin is kept positively $\frac{\sqrt{2}}{2}$-far from the Clarke subdifferential of the distance function on $X \backslash S$. Then, if $(0,s) \notin S$, $u_1^* = \frac{\sqrt{2}}{2}(-1,-1)$, $u_2^* = \frac{\sqrt{2}}{2}(1,-1)$ belong to $\partial d\left((0,s),S\right)$ and satisfy $\langle u_1^*, u_2^*\rangle = 0$, attaining the equality in (12).

Let S be a closed set and $\rho > 0$. We recall that S is ρ-uniformly prox-regular [50, Definition 2.4] if for all $x \in S$ and all $v \in N(S,x)$ with $\|v\| < 1$ x is the unique nearest point of S to $x + \rho v$, i.e.,

$$x = P_S(x + \rho v).$$

Here $P_S(u)$ denotes the unique nearest point of S to u. It has been shown [50, Corollary 3.8] that S is ρ-uniformly prox-regular if and only if the Clarke subdifferential of the distance $\partial d(\cdot,S)$ is a singleton for every $x \in U_\rho(S)$. Hence, using Proposition 3.2, we can give the following characterization of ρ-uniformly prox-regularity.

Corollary 3.4. *Let S be a closed subset of X with $S \neq X$ and $\rho > 0$. Then S is ρ-uniformly prox-regular if and only if the origin is kept positive 1-far from the Clarke subdifferential of the distance function $d(\cdot,S)$ on $U_\rho(S)$.*

Proof. The necessity was shown in [29, Proposition 2]. For the sufficiency we take $x \in U_\rho(S)$ and $u_1^*, u_2^* \in \partial d(x,S)$. Then, due to Proposition 3.2, we have

$\langle u_1^*, u_2^* \rangle = 1$ and therefore

$$\|u_1^* - u_1^*\|^2 = \|u_1^*\|^2 - 2\langle u_1^*, u_2^* \rangle + \|u_1^*\|^2 \le 1 - 2 + 1 = 0,$$

Thus, $\partial d(x,S)$ is a singleton for every $x \in U_\rho(S)$ which proves (by using [50, Corollary 3.8]) that S is ρ-uniformly prox-regular. $\square$

The next proposition gives a characterization of the concept of positively α-farness in terms of the existence of a function called pseudo-gradient. We use the well-known Lau theorem [34] which asserts that the set of those points which have a nearest point to any fixed closed subset is a dense set.

Proposition 3.5. *Let S be a closed subset of X and let $\rho > 0$ and $\alpha > 0$. Then the following assertions are equivalent:*

1. *The origin is kept positively α-far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$.*
2. *For all $\varepsilon \in]0, \alpha[$, there exists a locally Lipschitz function $V : X \setminus S \mapsto X$ such that*

$$\forall x \in U_\rho(S), \quad \|V(x)\| \le 1 + \varepsilon, \qquad \inf_{x^* \in \partial d(x,S)} \langle x^*, V(x) \rangle \ge (\alpha - \varepsilon).$$

3. *For all $\varepsilon \in]0, \alpha[$, there exists a locally Lipschitz function $V : X \setminus S \mapsto X$ such that*

$$\forall x \in U_\rho(S), \quad \|V(x)\| \le 1 + \varepsilon,$$
$$\langle u - P_S(u), V(u) \rangle \ge (\alpha - \varepsilon) d(u,S) \quad \forall u \in U_\rho(S) \cap D.$$

Here D is the dense set of those points which have a nearest point to S.

Proof. *1.* $\Rightarrow$ *2.*: This implication is an adaptation of the proof of Theorem 4.1 in [25]. The equivalence *2.* $\Leftrightarrow$ *3.* follows from the characterization of the Clarke's subdifferential of the distance function (see [21, Theorem 1.6.1] and [21, Theorem 2.6.1])

$$\partial d(x,S) = \overline{\mathrm{co}} \left\{ w\text{-} \lim_{i \to +\infty} \frac{x_i - P_S(x_i)}{\|x_i - P_S(x_i)\|} \; : \; D \ni x_i \to x \right\},$$

Finally the implication *2.* $\Rightarrow$ *1.* is direct. $\square$

Remark 3.6. The equivalence *1.* $\Longleftrightarrow$ *2.* holds true in any Banach space.

The following proposition gives another characterization of positively $\alpha-$far sets.

Proposition 3.7. *Let $S \subset X$ be a closed set, $\alpha \in]0, 1[$ and $\rho > 0$.*

(*i*) *If the origin is kept positively α-far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $U_\rho(S)$, then*

$$\forall x \in U_\rho(S),\ \forall x^* \in \partial d(x,S); \quad d(x - x^* d(x,S), S) \le d(x,S)\sqrt{1-\alpha^2}. \tag{13}$$

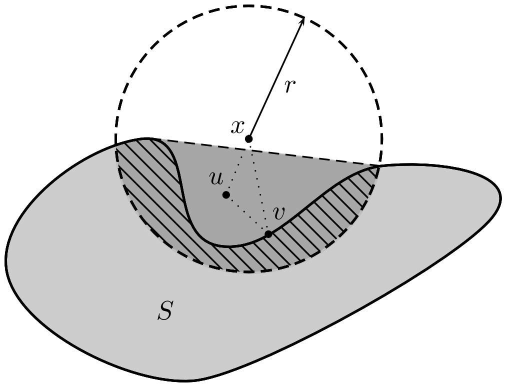

Figure 3.1: Geometrical interpretation of Lemma 3.8.

(*ii*) *Conversely, if* (13) *is satisfied then the origin is kept positively* $1-\sqrt{1-\alpha^2}$*-far from the Clarke subdifferential of the distance function* $d(\cdot, S)$ *on* $U_\rho(S)$.

The proof of the proposition is based on the following geometrical lemma.

Lemma 3.8. *Let* $r > 0$ *and* $x \in X$ *be such that* $S \cap B(x,r) \neq \emptyset$. *Then*

$$\forall u \in co\big(S \cap B(x,r)\big),\ \exists v \in S \cap B(x,r); \quad \|u-v\|^2 + \|x-u\|^2 \leq r^2. \tag{14}$$

Consequently,

$$\forall u \in \overline{co}\big(S \cap B(x,r)\big), \quad d(u,S) \leq \sqrt{r^2 - \|x-u\|^2}. \tag{15}$$

Proof of Lemma 3.8. The second assertion follows directly from the first one. Let us prove (14). By contradiction, assume that there exists $u \in \mathrm{co}\,(S \cap B(x,r))$ such that for all $v \in S \cap B(x,r)$

$$\|u-v\|^2 + \|x-u\|^2 > r^2.$$

Then, for all $v \in S \cap B(x,r)$

$$\begin{aligned} r^2 &\geq \|x-v\|^2 \\ &= \|x-u+u-v\|^2 \\ &= \|x-u\|^2 + 2\langle x-u, u-v\rangle + \|u-v\|^2 \\ &> 2\langle x-u, u-v\rangle + r^2, \end{aligned}$$

So, using the last inequality and passing to the convex hull, we obtain

$$\langle x-u, u-v\rangle < 0 \quad \text{for all } v \in \mathrm{co}\,(S \cap B(x,r)),$$

which is a contradiction because $u \in \mathrm{co}\,(S \cap B(x,r))$. □

Now we proceed to prove Proposition 3.7.

Proof. The assertion *ii)* is obvious. Let us prove *i)*. Let $x \in U_\rho(S)$ and $x^* \in \partial d(x,S)$ or equivalently $x^* = \frac{x-u}{d(x,S)}$, for some $u \in \bigcap_{\gamma>0} \overline{\text{co}} P_S^\gamma(x)$ (see (10)). Then, for all $\gamma > 0$, $u = x - x^* d(x,S) \in \overline{\text{co}} P_S^\gamma(x)$. So, by (15),

$$d(x - x^* d(x,S), S) \leq \sqrt{(d(x,S)+\gamma)^2 - \|x^*\|^2 d^2(x,S)}$$

and since the origin is kept positively α-far from the Clarke subdifferential of the distance function $d(\cdot,S)$ on $U_\rho(S)$, we get

$$d(x - x^* d(x,S), S) \leq \sqrt{(d(x,S)+\gamma)^2 - \alpha^2 d^2(x,S)}.$$

As $\gamma > 0$ is arbitrary, we obtain

$$d(x - x^* d(x,S), S) \leq d(x,S)\sqrt{1-\alpha^2}. \qquad \square$$

Now, we show that the class of positively α-far sets strictly includes the class of uniformly subsmooth sets introduced in [4]. Let C be a closed subset of X. We say that C is *uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|, \tag{16}$$

holds for all $x_1, x_2 \in C$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(C, x_i) \cap \mathbb{B}_X$ for $i = 1, 2$. Also, if E is a given nonempty set, we say that the family $(C(t))_{t\in E}$ is *equi-uniformly subsmooth*, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that (16) holds for each $t \in E$, and all $x_1, x_2 \in C(t)$ satisfying $\|x_1 - x_2\| < \delta$ and all $x_i^* \in N(C(t), x_i) \cap \mathbb{B}_X$ for $i = 1, 2$.

Proposition 3.9. *Assume that S is uniformly subsmooth. Then, for all $\varepsilon \in]0,1[$ there exists $\rho \in]0,+\infty[$ such that the origin is kept positively $\sqrt{1-\varepsilon}$-far from the Clarke subdifferential of the distance function $d(\cdot,S)$ on the open ρ-tube $U_\rho(S) = \{y \in H : 0 < d(y,S) < \rho\}$, i.e.,*

$$\sqrt{1-\varepsilon} \leq \inf_{y \in U_\rho(S)} d\left(0, \partial d(y,S)\right).$$

Proof. Let $\varepsilon \in]0,1[$ and define $\rho = \min\{\frac{\delta}{4}, \frac{1}{2}\}$, where δ is given by the uniform subsmoothness of S for ε. Fix $x \in U_\rho(S)$. Due to Proposition 3.2, it is sufficient to show that for all $\gamma < \min\left\{\frac{(1-\varepsilon)^2}{100}\frac{d(x,S)^4}{(2+d(x,S))^4}, \frac{\delta}{4}\right\}$ the following property holds:

$$u_1^*, u_2^* \in \frac{x - P_S^\gamma(x)}{d(x,S)} \quad \Rightarrow \quad \langle u_1^*, u_2^* \rangle \geq 1 - \varepsilon - 10\sqrt{\gamma}\left(\frac{2}{d(x,S)} + 1\right)^2. \tag{17}$$

For $i \in \{1,2\}$ we take $u_i^* = \frac{x - z_i}{d(x,S)}$ with $z_i \in P_S^\gamma(x)$, i.e., $z_i \in S$ and $\|x - z_i\| \leq d(x,S) + \gamma$. Next, we apply Ekeland's variational principle [9, Theorem 2.1.3] to the function $u \mapsto I_S(u) + \|u - x\|$, to obtain the existence of $u_i \in S$ such that

(i) $\|z_i - u_i\| \leq \sqrt{\gamma}$,

(ii) $\|u_i - x\| + \sqrt{\gamma}\|z_i - u_i\| \leq \|z_i - x\|$,
(iii) the function $u \mapsto I_S(u) + \|u - x\| + \sqrt{\gamma}\|u - u_i\|$ attains a minimum at $u_i \in S$.

Next, using exact penalization [21, Proposition 1.6.3], (iii) is equivalent to the fact that the function $u \mapsto \left(1 + 2\sqrt{\gamma}\right) d(u, S) + \|u - x\| + \sqrt{\gamma}\|u - u_i\|$ attains a minimum at u_i. Note that $u_i \neq x$ because $u_i \in S$ and $x \notin S$. Hence,

$$0 \in (1 + 2\sqrt{\gamma})\, \partial d(u_i, S) + \frac{u_i - x}{\|u_i - x\|} + \sqrt{\gamma}\mathbb{B}.$$

Therefore, there exists $b_i \in \mathbb{B}$ such that $v_i^* = \frac{x - u_i}{\|x - u_i\|} + \sqrt{\gamma} b_i \in \left(1 + 2\sqrt{\gamma}\right) \partial d(u_i, S)$ for $i = 1, 2$. Then, for $i = 1, 2$

$$u_i^* - v_i^* = \frac{x - u_i}{d(x, S)} - \frac{x - u_i}{\|x - u_i\|} + \frac{u_i - z_i}{d(x, S)} - \sqrt{\gamma} b_i.$$

Thus,

$$\|u_i^* - v_i^*\| \leq \frac{\gamma}{d(x, S)} + \frac{\sqrt{\gamma}}{d(x, S)} + \sqrt{\gamma} \leq \sqrt{\gamma}\left(\frac{2}{d(x, S)} + 1\right)$$

Moreover, as a result of (ii), $\|u_1 - u_2\| \leq \|u_1 - x\| + \|u_2 - x\| \leq 2d(x, S) + 2\gamma < \delta$ and consequently, by uniform subsmoothness of S,

$$\langle v_1^* - v_2^*, u_1 - u_2 \rangle \geq -\varepsilon\, (1 + 2\sqrt{\gamma})\, \|u_1 - u_2\|. \tag{18}$$

Next, on the one hand

$$\begin{aligned}
&\langle v_1^* - v_2^*, u_1 - u_2 \rangle \\
\leq\ & \left\langle \frac{x - u_1}{\|x - u_1\|} - \frac{x - u_2}{\|x - u_2\|}, u_1 - u_2 \right\rangle + 2\sqrt{\gamma}\|u_1 - u_2\| \\
=\ & (\|x - u_1\| + \|x - u_2\|)\, [-1 + \langle v_1^* - \sqrt{\gamma} b_1, v_2^* - \sqrt{\gamma} b_2 \rangle] + 2\sqrt{\gamma}\|u_1 - u_2\| \\
\leq\ & (-1 + 5\sqrt{\gamma} + \langle v_1^*, v_2^* \rangle)\, (\|x - u_1\| + \|x - u_2\|)\,.
\end{aligned}$$

On the other hand,

$$-\varepsilon\, (1 + 2\sqrt{\gamma})\, \|u_1 - u_2\| \geq -\varepsilon\, (1 + 2\sqrt{\gamma})\, (\|x - u_1\| + \|x - u_2\|)\,.$$

Therefore, combining these last two inequalities with (18) and dividing by $(\|x - u_1\| + \|x - u_2\|)$, we obtain

$$-1 + 5\sqrt{\gamma} + \langle v_1^*, v_2^* \rangle \geq -\varepsilon\, (1 + 2\sqrt{\gamma})\,,$$

which entails

$$\langle v_1^*, v_2^* \rangle \geq 1 - \varepsilon(1 + 2\sqrt{\gamma}) - 5\sqrt{\gamma}. \tag{19}$$

On the other hand, we notice that

$$\begin{aligned}\langle v_1^*, v_2^*\rangle &= \langle v_1^* - u_1^*, v_2^* - u_2^*\rangle + \langle v_1^* - u_1^*, u_2^*\rangle + \langle u_1^*, v_2^* - u_2^*\rangle + \langle u_1^*, u_2^*\rangle \\ &\leq \gamma\left(\frac{2}{d(x,S)} + 1\right)^2 + 2\sqrt{\gamma}\left(\frac{2}{d(x,S)} + 1\right)\left(\frac{\gamma}{d(x,S)} + 1\right) + \langle u_1^*, u_2^*\rangle \\ &\leq 3\sqrt{\gamma}\left(\frac{2}{d(x,S)} + 1\right)^2 + \langle u_1^*, u_2^*\rangle .\end{aligned}$$

Finally, using this last inequality and (19), we get

$$\begin{aligned}\langle u_1^*, u_2^*\rangle &\geq \langle v_1^*, v_2^*\rangle - 3\sqrt{\gamma}\left(\frac{2}{d(x,S)} + 1\right)^2 \\ &\geq 1 - \varepsilon(1 + 2\sqrt{\gamma}) - 5\sqrt{\gamma} - 3\sqrt{\gamma}\left(\frac{2}{d(x,S)} + 1\right)^2 \\ &\geq 1 - \varepsilon - 10\sqrt{\gamma}\left(\frac{2}{d(x,S)} + 1\right)^2,\end{aligned}$$

which proves (17). □

Now, with the following example, we proceed to show that the class of positively α-far sets is strictly bigger than the class of uniformly prox-regular sets and the class of subsmooth sets.

Example 3.10. Consider the set $S = \{(x, y) \in \mathbb{R}^2 \colon y \geq -|x|\}$. Then the origin is kept positively $\frac{\sqrt{2}}{2}$-far from the Clarke subdifferential of the distance function $d(\cdot, S)$ on $X \backslash S$ but S is not uniformly prox-regular neither uniformly subsmooth.

The class of positively α-far sets contains also that of paraconvex sets by Michael [39]. Following Michael, a set S is α-paraconvex for some $\alpha \in [0, 1[$ if, whenever $x \in X$ and $r > 0$ are such that $d(x, S) < r$, then

$$d(u, S) \leq \alpha r \quad \forall\, u \in \operatorname{co}[B(x, r) \cap S].$$

This implies immediately that

$$\forall \rho > 0,\ \forall x \in U_\rho(S),\ \forall \gamma > 0, \quad \alpha(d(x, S) + \gamma) \geq d(u, S) \quad \forall u \in \operatorname{co}P_S^\gamma(x). \tag{20}$$

This remark allows us to get the following proposition.

Proposition 3.11. *Let $S \subset X$ be a closed set which is α-paraconvex for some $\alpha \in [0, 1[$. Then for each $\rho > 0$, the Clarke subdifferential of the distance function $d(\cdot, S)$ keeps the origin $(1 - \alpha)$-far-off on the open ρ-tube around S.*

Proof. Let $\rho > 0$ and $x \in U_\rho(S)$. First note that $d(0, \partial d(x, S)) = \frac{\|x-u\|}{d(x,S)}$ for some $u \in \bigcap_{\gamma>0} \overline{\operatorname{co}} P_S^\gamma(x)$. As $u \in \overline{\operatorname{co}} P_S^\gamma(x)$ for all $\gamma > 0$, relation (20) ensures that

$$\|x - u\| \geq d(x, S) - d(u, S) \geq d(x, S) - \alpha(d(x, S) + \gamma).$$

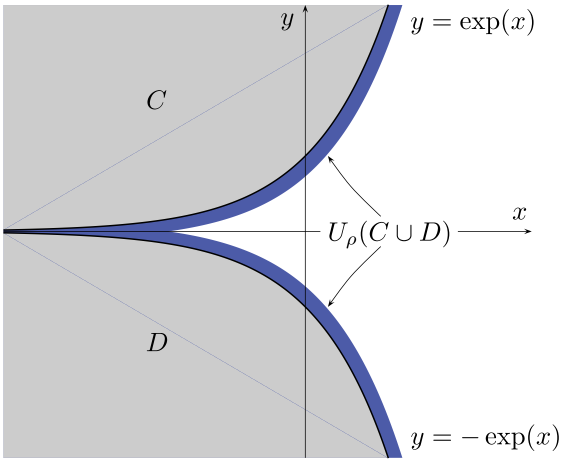

Figure 3.2: The union of two convex sets is not positively α-far.

Thus

$$d(0, \partial d(x, S)) \geq (1-\alpha) - \frac{\gamma\alpha}{d(x, S)} \quad \forall \gamma > 0,$$

which completes the proof. □

Remark 3.12. 1. The union of two convex sets is not positively α-far. Indeed, consider $C = \{(x, y) \in \mathbb{R}^2 \colon \exp(x) \leq y\}$ and $D = \{(x, y) \in \mathbb{R}^2 \colon y \leq -\exp(x)\}$. Then $C \cup D$ is not positively α-far. To see this, define the function $f(x)$ as the unique solution of the equation $y + \exp(2y) = x$. Thus, if $x \to -\infty$ $f(x) \to -\infty$ and $d((x, 0), C \cup D) = \sqrt{2}\exp(f(x))$. Moreover, $x^* = \left(\frac{\exp(f(x))}{\sqrt{2}}, 0\right) \in \partial d((x, 0), C \cup D)$ and

$$\|x^*\| = \frac{d((x, 0), C \cup D)}{2} = \frac{\exp(f(x))}{\sqrt{2}} \to 0 \quad \text{if } x \to -\infty,$$

which shows that $C \cup D$ is not positively α-far.

2. It will be very interesting to know, probably under some constraint qualification, when the intersection of two positively $\alpha-$far sets remains positively $\beta-$far, for some $\beta \in]0, 1[$. But, under some constraint qualification, it is not difficult to prove [48, Theorem 0.2.5] that the intersection of uniformly subsmooth sets remains uniformly subsmooth, and therefore positively α-far. Indeed, let C_1 and C_2 be two uniformly subsmooth sets and assume that there is $\beta > 0$ such that

$$N(C_1 \cap C_2, x) \cap \mathbb{B} \subset N(C_1, x) \cap \beta\mathbb{B} + N(C_2, x) \cap \beta\mathbb{B} \quad \text{for all } x \in C_1 \cap C_2, \quad (21)$$

called in [23] as "normal cone intersection property". Then $C_1 \cap C_2$ is uniformly subsmooth and therefore positively α-far. Indeed, let $\varepsilon > 0$ and define $\delta = \min\{\delta_1, \delta_2\}$, where $\delta_1 > 0$ and $\delta_2 > 0$ are given, respectively, by the uniform subsmoothnes of C_1 and C_2 for $\frac{\varepsilon}{2\beta}$. Take $x_1, x_2 \in C_1 \cap C_2$ with $\|x_1 - x_2\| < \delta$ and

$x_i^* \in N(C_1 \cap C_2, x_i)$ for $i = 1, 2$. Thus, due to (21), there are $u_i^* \in N(C_i, x_1) \cap \beta\mathbb{B}$ and $v_i^* \in N(C_i, x_2) \cap \beta\mathbb{B}$ for $i = 1, 2$, such that $x_1^* = u_1^* + u_2^*$ and $x_2^* = v_1^* + v_2^*$. Therefore, by uniformly subsmoothnes of C_1 and C_2,

$$\langle u_1^* - v_1^*, x_1 - x_2 \rangle \geq -\frac{\varepsilon}{2}\|x_1 - x_2\|,$$
$$\langle u_2^* - v_2^*, x_1 - x_2 \rangle \geq -\frac{\varepsilon}{2}\varepsilon\|x_1 - x_2\|,$$

which implies $\langle x_1^* - x_1^*, x_1 - x_2 \rangle \geq -\varepsilon\|x_1 - x_2\|$, i.e, $C_1 \cap C_2$ is uniformly subsmooth.

3.1. Preservation of subsmoothness under inverse images

Let $h\colon X \to Y$ be a differentiable mapping from a Hilbert space X into another Hilbert space Y. In this subsection we give some conditions to assure the uniform subsmoothness of the family $(h^{-1}(D(t)))_{t\in[T_0,T]}$, where $(D(t))_{t\in[T_0,T]}$ is a given family.

Let $C \subset X$, $D \subset Y$ be two closed sets and $\bar{x} \in h^{-1}(D) \cap C$. We say that the inverse image set $h^{-1}(D) \cap C$ has the *normal cone inverse image property* at $\bar{x} \in h^{-1}(D) \cap C$ with respect to the Clarke normal cone [23] if there exists some $k > 0$ and some neighborhood U of $\bar{x}$ such that for all $x \in U \cap (h^{-1}(D) \cap C)$

$$N\left(h^{-1}(D) \cap C, x\right) \cap \mathbb{B}_X \subset Dh(x)^* \left(N(D, h(x)) \cap k\mathbb{B}_Y\right) + N(C, x). \qquad (22)$$

Also, we say that the inverse image set $h^{-1}(D) \cap C$ has the *uniform normal cone inverse image property* (UNCIIP) if there exists some $k > 0$ such that (22) holds for all $x \in h^{-1}(D) \cap C$. The next proposition gives some sufficient conditions to assure the UNCIIP and the Lipschitz continuity of the set-valued map $w \rightrightarrows h^{-1}(D - w) \cap C$ which will be useful in the Section 9.

Proposition 3.13. *Let $C \subset X$ and $D \subset Y$ be two closed convex sets and $h\colon X \to Y$ be a differentiable function. Consider the following statements:*

(*i*) *There exists $k > 0$ such that for all $x \in C$*

$$\mathbb{B}_Y \subset Dh(x)\left(T(x, C) \cap k\mathbb{B}_X\right) - D$$

(*ii*) *There exists $k > 0$ such that for all $w \in Y$ and all $x \in C$*

$$d\left(x, h^{-1}(D - w) \cap C\right) \leq kd\left(h(x) + w, D\right).$$

(*iii*) *There exists $k > 0$ such that for all $w \in Y$ and all $x \in h^{-1}(D - w) \cap C$*

$$\mathbb{B}_Y \subset T(h(x) + w, D) - kDh(x)\left(T(x, C) \cap \mathbb{B}_X\right).$$

(*iv*) *There exists $k > 0$ such that for all $w \in Y$ and all $x \in h^{-1}(D - w) \cap C$*

$$N\left(h^{-1}(D - w) \cap C, x\right) \cap \mathbb{B}_X \subset Dh(x)^*\left[N(D, h(x) + w) \cap k\mathbb{B}_Y\right] + N(C, x).$$

Then the following hold.

1. $(ii) \Leftrightarrow (iii) \Rightarrow (iv)$
2. *If D is a cone,* $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$
3. *Under* (ii) *the set-valued map* $w \rightrightarrows h^{-1}(D-w) \cap C$ *is k'-Lipschitz continuous for every* $k' > k$.

Proof. 1. $(ii) \Rightarrow (iii)$: Take $b \in \mathbb{B}_Y$ and $x \in h^{-1}(D-w) \cap C$. Then, by (ii), for all $t > 0$

$$d\left(x, h^{-1}(D-w-tb) \cap C\right) \leq kd\left(h(x)+w+tb, D\right).$$

Thus, for every $\varepsilon > 0$, there exists $v_\varepsilon(t) \in h^{-1}(D-w-tb) \cap C$ such that

$$\begin{aligned}\|x - v_\varepsilon(t)\| &\leq d\left(x, h^{-1}(D-w-tb) \cap C\right) + t\varepsilon \\ &\leq kd\left(h(x)+w+tb, D\right) + t\varepsilon \\ &\leq kt\|b\| + t\varepsilon \\ &\leq t\left(k+\varepsilon\right).\end{aligned}$$

Hence, since $v_\varepsilon(t) \in h^{-1}(D-w-tb) \cap C$,

$$\begin{aligned} tb &\in D - w - h(v_\varepsilon(t)) \\ &\subset D - w - h(x) - Dh(x)\left(v_\varepsilon(t) - x\right) + \theta\left(\|v_\varepsilon(t) - x\|, \varepsilon\right) \\ &\subset D - w - h(x) - t\left(k+\varepsilon\right)Dh(x)\left(T(x,C)\cap \mathbb{B}_X\right) + \eta\left(t,\varepsilon\right),\end{aligned}$$

for some mappings $\theta(\cdot,\varepsilon)$ and $\eta(\cdot,\varepsilon)$ with $\lim_{s\downarrow 0} s^{-1}\theta(s,\varepsilon) = 0$ and $\lim_{s\downarrow 0} s^{-1}\eta(s,\varepsilon) = 0$ in Y. From where we obtain

$$\mathbb{B}_Y \subset \frac{D-w-h(x)}{t} - \left(k+\varepsilon\right)Dh(x)\left(T(x,C)\cap \mathbb{B}_X\right) + \frac{\eta\left(t,\varepsilon\right)}{t},$$

then, taking $t \downarrow 0$, we get

$$\mathbb{B}_Y \subset \overline{T(h(x)+w, D) - \left(k+\varepsilon\right)Dh(x)\left(T(x,C)\cap \mathbb{B}_X\right)}.$$

Next, consider the set-valued map $M\colon z \rightrightarrows T(h(x)+w, D) - (k+\varepsilon)\,Dh(x)z$ and define the closed convex set

$$E = \{(z,y) \in T(x,C)\cap \mathbb{B}_X \times Y : y \in M(z)\}.$$

Hence, $P_X(E) = T(x,C) \cap \mathbb{B}_X$ is bounded and

$$P_Y(E) = T(h(x), D) - \left(k+\varepsilon\right)Dh(x)\left(T(x,C)\cap \mathbb{B}_X\right).$$

Next, using Robinson's Lemma [52, Lemma 1] we obtain that $P_Y(E)$ is closed which implies (iii).

(iii) ⇒ *(ii)*: By contradiction, suppose that for every $n \in \mathbb{N}$ there exist $x_n \in C$ and $w_n \in Y$ such that

$$d\left(x_n, h^{-1}(D - w_n) \cap C\right) > nd\left(h(x_n) + w_n, D\right). \tag{23}$$

For every $n \in \mathbb{N}$, consider the function $f_n(x) = d(h(x) + w_n, D) + I_C(x)$. Then, applying Ekeland's variational principle [9, Theorem 2.1.2] with $\varepsilon_n = d(h(x_n) + w_n, D)$ and $\lambda_n = n\varepsilon_n$, there exists $u_n \in C$ such that

(*a*) $\|u_n - x_n\| \leq \lambda_n$,
(*b*) $f_n(u_n) + \frac{1}{n} \leq f_n(x_n)$,
(*c*) the function $u \mapsto f_n(u) + \frac{1}{n}\|u - u_n\|$ attains a minimum at $u = u_n$.

We claim that $h(u_n) + w_n \notin D$. Indeed, if $h(u_n) + w_n \in D$, due to (*a*) and (23), we have

$$\begin{aligned}\|u_n - x_n\| &\leq nd(h(x_n) + w_n, D)\\ &< d\left(x_n, h^{-1}(D - w_n) \cap C\right)\\ &\leq \|u_n - x_n\|,\end{aligned}$$

which is a contradiction, hence $h(u_n) + w_n \notin D$. Next, on the one hand, as a result of (*c*), we obtain the existence $y_n^* \in X$ with $\|y_n^*\| = 1$ and $y_n^* \in \partial d\left(h(u_n) + w_n, D\right)$ and $b_n \in \mathbb{B}_X$ such that $0 \in Dh(u_n)^*\left(y_n^*\right) + \frac{1}{n}b_n + N(C, u_n)$. This last inclusion implies that

$$\left\langle Dh(u_n)^*\left(y_n^*\right) + \frac{1}{n}b_n, c\right\rangle \geq 0 \quad \forall c \in T(u_n, C), \tag{24}$$

and since $y_n^* \in \partial d(h(u_n) + w_n, D)$ and $h(u_n) + w_n \notin D$, $y_n^* \in N(D, v_n)$ with v_n the projection of $h(u_n) + w_n$ into D. On the other hand, due to *(iii)*, for every $b \in \mathbb{B}_Y$ there exist $c_n \in T(u_n, C) \cap k\mathbb{B}_X$ and $d_n \in T(v_n, D)$ such that $b = -Dh(u_n)c_n + d_n$. Thus, by virtue of (24),

$$\begin{aligned}\langle y_n^*, b\rangle &= \langle y_n^*, -Dh(u_n)c_n + d_n\rangle\\ &= \langle -Dh(u_n)^*\left(y_n^*\right), c_n\rangle + \langle y_n^*, d_n\rangle\\ &\leq \left\langle \frac{1}{n}b_n, c_n\right\rangle + \langle y_n^*, d_n\rangle\\ &\leq \frac{k}{n},\end{aligned}$$

from where we get that $\|y_n^*\| \leq \frac{k}{n}$, which is a contradiction. Therefore *(ii)* holds.

(iii) ⇒ *(iv)*: For every $x \in X$ we have the following identity

$$I_{h^{-1}(D-w)\cap C}(x) = I_D(h(x) + w) + I_C(x).$$

Then, due to *(iii)*, we can apply the chain rule for nonsmooth functions [42, Theorem 3.4.1] to obtain

$$N\left(h^{-1}(D - w) \cap C, x\right) = Dh(x)^*N(D, h(x) + w) + N(C, x) \quad \text{for all } x \in h^{-1}(D - w) \cap C.$$

Next, take $x^* \in N\left(h^{-1}(D-w) \cap C, x\right) \cap \mathbb{B}_X$ so $x^* = Dh(x)^* y^* + z^*$ for some $y^* \in N(D, h(x)+w)$ and $z^* \in N(C, x)$. Then, by (iii), for every $v \in \mathbb{B}_Y$ there exist $w^* \in T(h(x)+w, D)$ and $u \in T(x, C) \cap \mathbb{B}_X$ such that $v = w^* - kDh(x)u$. Hence,

$$\begin{aligned}\langle y^*, v\rangle &= \langle y^*, w^* - kDh(x)u\rangle \\ &= \langle y^*, w^*\rangle - k\left\langle Dh(x)^* y^*, u\right\rangle \\ &= \langle y^*, w^*\rangle - k\left\langle x^* - z^*, u\right\rangle \\ &\leq k,\end{aligned}$$

which implies that $\|y^*\| \leq k$. Therefore (iv) holds.

Assume that D is a cone. We proceed to prove that $(i) \Rightarrow (ii)$: By contradiction, suppose that for every $n \in \mathbb{N}$ there exist $x_n \in C$ and $w_n \in Y$ such that

$$d\left(x_n, h^{-1}(D - w_n) \cap C\right) > nd\left(h(x_n) + w_n, D\right). \tag{25}$$

For every $n \in \mathbb{N}$ consider the function $f_n(x) = d(h(x) + w_n, D) + I_C(x)$. Then, applying Ekeland's variational principle [9, Theorem 2.1.2] for every $n \in \mathbb{N}$, with $\varepsilon_n = d(h(x_n) + w_n, D)$ and $\lambda_n = n\varepsilon_n$, there exists $u_n \in C$ such that

(a) $\|u_n - x_n\| \leq \lambda_n$,
(b) $f_n(u_n) + \frac{1}{n} \leq f_n(x_n)$,
(c) the function $u \mapsto f_n(u) + \frac{1}{n}\|u - u_n\|$ attains a minimum at $u = u_n$.

We claim that $h(u_n) + w_n \notin D$. Indeed, if $h(u_n) + w_n \in D$, due to (a) and (25), we have

$$\|u_n - x_n\| \leq nd(h(x_n) + w_n, D) < d\left(x_n, h^{-1}(D - w_n) \cap C\right) \leq \|u_n - x_n\|,$$

which is a contradiction, hence $h(u_n) + w_n \notin D$. Next, on the one hand, as a result of (c), we obtain the existence of $y_n^* \in X$ with $\|y_n^*\| = 1$ and $y_n^* \in \partial d\left(h(u_n) + w_n, D\right)$ and $b_n \in \mathbb{B}_X$ such that $0 \in Dh(u_n)^*\left(y_n^*\right) + \frac{1}{n}b_n + N(C, u_n)$. This last inclusion implies that

$$\left\langle Dh(u_n)^*\left(y_n^*\right) + \frac{1}{n}b_n, c\right\rangle \geq 0 \quad \forall c \in T(u_n, C), \tag{26}$$

and since D is a cone $\langle y_n^*, d\rangle \leq 0$ for every $d \in D$. On the other hand, due to (ii), for every $b \in \mathbb{B}_Y$ there exist $c_n \in T(u_n, C) \cap k\mathbb{B}_X$ and $d_n \in D$ such that $b = -Dh(u_n)c_n + d_n$. Thus, by virtue of (26),

$$\begin{aligned}\langle y_n^*, b\rangle &= \langle y_n^*, -Dh(u_n)c_n + d_n\rangle \\ &= \left\langle -Dh(u_n)^*\left(y_n^*\right), c_n\right\rangle + \langle y_n^*, d_n\rangle \\ &\leq \left\langle \frac{1}{n}b_n, c_n\right\rangle + \langle y_n^*, d_n\rangle \\ &\leq \frac{k}{n},\end{aligned}$$

from which we get that $\|y_n^*\| \leq \frac{k}{n}$, and this is a contradiction because $\|y_n^*\| = 1$. Therefore (i) holds.

Consider $w_1, w_2 \in Y$ and $x \in h^{-1}(D - w_1) \cap C$. Then, due to (ii),

$$d\left(x, h^{-1}(D - w_2) \cap C\right) \leq kd(h(x) + w_2, D) \leq k\|w_1 - w_2\|, \tag{27}$$

Thus, for $\varepsilon < k' - k$ there exists $y_\varepsilon \in h^{-1}(D - w_2) \cap C$ such that $\|x - y_\varepsilon\| \leq d(x, h^{-1}(D - w_2) \cap C) + \varepsilon\|w_1 - w_2\|$. Therefore, as a result of (27), $\|x - y_\varepsilon\| \leq (k + \varepsilon)\|w_1 - w_2\|$, which implies that

$$h^{-1}(D - w_1) \cap C \subset h^{-1}(D - w_2) \cap C + k'\|w_1 - w_2\|\mathbb{B}_X. \qquad \square$$

Now we give some conditions to assure the equi-uniform subsmoothness of inverse images of a given family of sets under a differentiable mapping $h\colon X \to Y$. The first statement of the next proposition was stated in [48] without a proof.

Proposition 3.14. *Let $h\colon X \to Y$ be a continuously differentiable mapping with Dh uniformly continuous.*

1. *Assume that the function h is Lipschitz, the family $(D(t))_{t\in[T_0,T]} \subset Y$ is equi-uniformly subsmooth and UNCIIP holds for all $D(t)$, $t \in [T_0, T]$ with the same $k > 0$. Then the family $(h^{-1}(D(t)))_{t\in[T_0,T]}$ is equi-uniformly subsmooth and, hence, positively α-far for some $\alpha > 0$.*
2. *Assume that the set $D(t) \subset Y$ is convex for every $t \in [T_0, T]$ and UNCIIP holds for all $D(t)$, $t \in [T_0, T]$ with the same $k > 0$. Then the family $(h^{-1}(D(t)))_{t\in[T_0,T]}$ is equi-uniformly subsmooth and, hence, positively α-far for some $\alpha > 0$.*
3. *Assume that the function h is Lipschitz, the set $D \subset Y$ is uniformly subsmooth, $d\colon [T_0, T] \to Y$ is a function and UNCIIP holds for all $D - d(t)$, $t \in [T_0, T]$ with the same $k > 0$. Then the family $(h^{-1}(D - d(t)))_{t\in[T_0,T]}$ is equi-uniformly subsmooth and, hence, positively α-far for some $\alpha > 0$.*
4. *Assume that the set $D \subset Y$ is convex, $d\colon [T_0, T] \to Y$ is a function and UNCIIP holds for all $D - d(t)$, $t \in [T_0, T]$ with the same $k > 0$. Then the family $(h^{-1}(D - d(t)))_{t\in[T_0,T]}$ is equi-uniformly subsmooth and, hence, positively α-far for some $\alpha > 0$.*

Proof. *1.* Since h is Lipschitz, there exist $L > 0$ and $\delta_1 > 0$ such that

$$\|x - y\| < \delta_1 \text{ then } \|h(x) - h(y)\| \leq L\|x - y\|.$$

For $\varepsilon > 0$ take $\delta = \min\{\frac{\delta_3}{L}, \delta_2, \delta_1\}$, where δ_3 is given by the equi-uniform subsmoothness of $(D(t))_{t\in[T_0,T]}$ for $\frac{\varepsilon}{2kL}$, δ_2 is given by the uniform continuity of Dh for $\frac{\varepsilon}{4k}$. Next, fix $t \in [T_0, T]$ and take $u_1, u_2 \in h^{-1}(D(t))$ with $\|u_1 - u_2\| < \delta$ and $x_i^* \in N(h^{-1}(D(t)), u_i) \cap \mathbb{B}_X$ for $i = 1, 2$. We have to prove that

$$\langle x_1^* - x_2^*, u_1 - u_2\rangle \geq -\varepsilon\|u_1 - u_2\|. \tag{28}$$

Indeed, by the UNCIIP, there exist $y_i^* \in N(D(t), h(u_i)) \cap \mathbb{B}_Y$ for $i = 1, 2$ such that $x_i^* = kDh(u_i)^* y_i^*$ for $i = 1, 2$. Then, since $\|h(u_1) - h(u_2)\| \leq L\|u_1 - u_2\| < \delta_3$, we can use the equi-uniform subsmoothness of $D(t)$ to get

$$\langle y_1^* - y_2^*, h(u_1) - h(u_2)\rangle \geq -\frac{\varepsilon}{2kL}\|h(u_1) - h(u_2)\|. \tag{29}$$

Next, since $\|u_2 + s(u_1 - u_2) - u_1\| < \delta_2$ for all $s \in [0, 1]$ and due to the uniform continuity of Dh we have

$$\|Dh\left(u_2 + s(u_1 - u_2)\right) - Dh(u_1)\| \leq \frac{\varepsilon}{4k}.$$

Therefore, using this last inequality

$$\begin{aligned}
&\langle ky_1^*, h(u_1) - h(u_2)\rangle \\
= &\left\langle ky_1^*, \int_0^1 Dh\left(u_2 + s(u_1 - u_2)\right)(u_1 - u_2)ds\right\rangle \\
= &\langle ky_1^*, Dh(u_1)(u_1 - u_2)\rangle \\
&\quad + \left\langle ky_1^*, \int_0^1 \left[Dh\left(u_2 + s(u_1 - u_2)\right) - Dh(u_1)\right](u_1 - u_2)ds\right\rangle \\
\leq &\langle ky_1^*, Dh\left(u_1\right)\left(u_1 - u_2\right)\rangle + k \cdot \frac{\varepsilon}{4k}\|u_1 - u_2\| \\
= &\langle kDh\left(u_1\right)^* y_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{4}\|u_1 - u_2\| \\
= &\langle x_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{4}\|u_1 - u_2\|
\end{aligned}$$

Therefore,

$$\langle ky_1^*, h(u_1) - h(u_2)\rangle \leq \langle x_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{4}\|u_1 - u_2\|,$$

and similarly,

$$\langle ky_2^*, h(u_2) - h(u_1)\rangle \leq \langle x_2^*, u_2 - u_1\rangle + \frac{\varepsilon}{4}\|u_1 - u_2\|,$$

Next, by adding up these two inequalities we obtain

$$\langle ky_1^* - ky_2^*, h(u_1) - h(u_2)\rangle \leq \langle x_1^* - x_2^*, u_1 - u_2\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\|,$$

which combined with (29) and the Lipschitzianity of h entails (28), i.e., the equi-uniform subsmoothness of the family $(h^{-1}(D(t)))_{t\in[T_0,T]}$.

2. For $\varepsilon > 0$ take δ given by the uniform continuity of Dh for $\frac{\varepsilon}{2k}$. Next, fix $t \in [T_0, T]$ and take $u_1, u_2 \in h^{-1}(D(t))$ with $\|u_1 - y_2\| < \delta$ and $x_i^* \in N\left(h^{-1}\left(D(t)\right), u_i\right) \cap \mathbb{B}_X$ for $i = 1, 2$. We have to prove that

$$\langle x_1^* - x_2^*, u_1 - u_2\rangle \geq -\varepsilon\|u_1 - u_2\|. \tag{30}$$

Indeed, by the UNCIIP, there exist $y_i^* \in N(D(t), h(u_i)) \cap \mathbb{B}_Y$ for $i = 1, 2$ such that $x_i^* = kDh(u_i)^* y_i^*$ for $i = 1, 2$. Then, due to the convexity of $D(t)$, we obtain

$$\langle y_1^* - y_2^*, h(u_1) - h(u_2)\rangle \geq 0. \tag{31}$$

Next, since $\|u_2 + s(u_1 - u_2) - u_1\| < \delta_2$ for all $s \in [0, 1]$ and due to the uniform continuity of Dh, we have

$$\|Dh\,(u_2 + s(u_1 - u_2)) - Dh(u_1)\| \leq \frac{\varepsilon}{2k}.$$

Therefore, using this last inequality

$$\begin{aligned}
&\langle ky_1^*, h(u_1) - h(u_2)\rangle \\
=\;& \left\langle ky_1^*, \int_0^1 Dh\,(u_2 + s(u_1 - u_2))\,(u_1 - u_2)ds \right\rangle \\
=\;& \langle ky_1^*, Dh(u_1)(u_1 - u_2)\rangle \\
&\qquad + \left\langle ky_1^*, \int_0^1 \left[Dh\,(u_2 + s(u_1 - u_2)) - Dh(u_1)\right](u_1 - u_2)ds \right\rangle \\
\leq\;& \langle ky_1^*, Dh\,(u_1)\,(u_1 - u_2)\rangle + k \cdot \frac{\varepsilon}{2k}\|u_1 - u_2\| \\
=\;& \langle kDh\,(u_1)^*\, y_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\| \\
=\;& \langle x_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\|
\end{aligned}$$

Therefore,

$$\langle ky_1^*, h(u_1) - h(u_2)\rangle \leq \langle x_1^*, u_1 - u_2\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\|,$$

and similarly,

$$\langle ky_2^*, h(u_2) - h(u_1)\rangle \leq \langle x_2^*, u_2 - u_1\rangle + \frac{\varepsilon}{2}\|u_1 - u_2\|,$$

Next, by adding up these two inequalities we obtain

$$\langle ky_1^* - ky_2^*, h(u_1) - h(u_2)\rangle \leq \langle x_1^* - x_2^*, u_1 - u_2\rangle + \varepsilon\|u_1 - u_2\|,$$

which combined with (31) entails (30), i.e., the equi-uniformly subsmoothness of $(h^{-1}(D)(t))$.

3. It follows from the proof of Part *1* and the formula

$$N(D - d(t), x) = N(D, x + d(t)) \quad \text{for all } t \in [T_0, T]. \tag{32}$$

4. It follows from the proof of Part *2* and the formula (32). □

The next corollary will be useful in Section 9.

Corollary 3.15. *Let $h\colon X \to Y$ be a continuously differentiable mapping with Dh uniformly continuous and let $d\colon [T_0, T] \to Y$ be a function. Assume that there exists $k > 0$ such that*

$$\mathbb{B}_Y \subset Dh(x)k\mathbb{B}_X - K \quad \text{for all } x \in X,$$

where $K \subset Y$ is a closed convex cone. Then the family $\left(h^{-1}\left(K - d(t)\right)\right)_{t\in[T_0,T]}$ is equi-uniformly subsmooth, hence, positively α-far and the set-valued map $t \rightrightarrows h^{-1}\left(K - d(t)\right)$ is absolutely continuous over $[T_0, T]$.

Proof. For the first assertion, by Proposition 3.14, it is sufficient to prove that the set $h^{-1}\left(K - d(t)\right)$ satisfies the UNCIIP for all $t \in [T_0, T]$ with the same k which follows directly from the first statement of Proposition 3.13. The second assertion follows from the third statement of Proposition 3.13 combined with the absolute continuity of d. □

4. Technical assumptions

For the sake of readability, in this section we collect the hypothesis used along this paper.

The hypothesis on the perturbation set-valued mapping $F\colon [T_0, T] \times X \rightrightarrows X$ are:

(H_1^F) For each $x \in X$, $F(\cdot, x)$ is measurable.
(H_2^F) For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from X into X_w.
(H_3^F) There exist $\alpha, \beta \in L^1(T_0, T)$ such that

$$|F(t, x)| := \sup\{\|w\|\colon w \in F(t, x)\} \le \alpha(t)\|x\| + \beta(t) \quad \text{for all } x \in X \text{ and a.e. } t \in [T_0, T].$$

On the set-valued mapping $C\colon [T_0, T] \rightrightarrows Y$ we consider the following ones:

(H_1) α_0-farness: There exist two constants $\alpha_0 \in (0, 1]$ and $\rho \in (0, +\infty)$ such that, for a.e. $t \in [T_0, T]$, the origin is kept α_0-far from the Clarke subdifferential of the distance function $d(\cdot, C(t))$ on the open ρ-tube $U_\rho(C(t)) = \{x \in Y\colon 0 < d(x, C(t)) < \rho\}$, i.e.,

$$0 < \alpha_0 \le \inf_{x\in U_\rho(C(t))} d(0, \partial d(x, C(t)))\text{a.e. } t \in [T_0, T].$$

(H_2) Absolute continuity: There exists an absolutely continuous function $\zeta\colon [T_0, T] \to \mathbb{R}_+$, with $\dot{\zeta} \in L^1(T_0, T)$, such that

$$|d(x, C(t)) - d(x, C(s))| \le |\zeta(t) - \zeta(s)|,$$

for all $x \in Y$ and all $s, t \in [T_0, T]$.

(H_3) Ball-compactness: For all $r > 0$ and $t \in [T_0, T]$ the set $C(t) \cap r\mathbb{B}$ is compact in Y.

The assumptions on the mappings $g\colon X \to \mathcal{L}(Y; X)$ and $h\colon X \to Y$ are:

(H_3') For all $r > 0$ and $t \in [T_0, T]$ the set $h^{-1}(C(t)) \cap r\mathbb{B}$ is compact in X.

(H_4) – The function g is continuous with

$$\|g(x)\| \leq M_1\|x\| + M_2 \quad \text{for all } x \in X,$$

where $M_1 \geq 0$ and $M_2 > 0$.

– $h\colon X \to Y$ is a differentiable function with $\sup_{x\in X} \|Dh(x)\| \leq M$ for some $M > 0$.

– There exists $\lambda > 0$ such that for all $x \in X$

$$\langle Dh(x)g(x)h, h\rangle \geq \lambda\|h\|^2 \quad \forall h \in Y.$$

Finally, the assumptions on the function $f\colon [T_0, T] \times X \to \mathbb{R} \cup \{+\infty\}$ are:

(H_1^f) Sublinear growth: There exist $\gamma, \delta \in L^1(T_0, T)$ such that

$$\partial f(t, x) \subset (\gamma(t)\|x\| + \delta(t))\,\mathbb{B},$$

for all $x \in X$ and a.e. $t \in [T_0, T]$.

(H_2^f) Measurability: For each $x \in X$, $\partial f(\cdot, x)$ is measurable.

(H_3^f) For every $r > 0$ there exists an absolutely continuous function $\zeta_r\colon [T_0, T] \to \mathbb{R}_+$ with $\dot{\zeta}_r \in L^1(T_0, T)$ and such that

$$|f(t, x) - f(s, x)| \leq |\zeta_r(t) - \zeta_r(s)|,$$

for all $x \in r\mathbb{B}$ and all $s, t \in [T_0, T]$.

(H_4^f) Ball-inf-compactness: For all $r > 0$ and $\lambda \in \mathbb{R}$ there exists $t^* \in [T_0, T]$ such that the set $\{f(t^*, \cdot) \leq \lambda\} \cap r\mathbb{B}$ is compact in X.

5. Preparatory Lemmas

In this section we gather some lemmas needed in the sequel. The next lemma is a consequence of Zagrodny's approximate mean value [60, Theorem 4.3].

Lemma 5.1. *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function satisfying $\partial f(x) \subset k\mathbb{B}$ for all $x \in B(\bar{x}, r)$. Then f is Lipschitz on $B(\bar{x}, r)$ with constant k.*

Since $-d(\cdot, S)$ has a directional derivative that coincides with the Clarke's directional derivative of $-d(\cdot, S)$ whenever $x \notin S$ (see [8]), we obtain the following lemma.

Lemma 5.2. *Let $S \subset X$ be a closed set, $x \notin S$ and $v \in X$. Then*

$$\lim_{h\downarrow 0} \frac{d(x + hv, S) - d(x, S)}{h} = \min_{y^*\in\partial d(x,S)} \langle y^*, v\rangle\,.$$

The next two statements are well known compactness criteria which will be used to prove Theorem 5.9.

Proposition 5.3 ([3, Theorem 0.3.4]). *Let us consider a sequence of absolutely continuous function x_k from an interval I of $\mathbb{R}$ to a Banach space X satisfying*

(*i*) $\forall t \in I$, $\{x_k(t)\}_k$ *is a relatively compact subset of* X,
(*ii*) *there exists a positive function* $c \in L^1(I)$ *such that, for almost all* $t \in I$, $\|x_k'(t)\| \leq c(t)$.

Then there exist a subsequence (again denoted by x_k) converging to an absolutely continuous function x from I to X in the sense that

(*i*) x_k *converges uniformly to* x *over compact subsets of* I,
(*ii*) x_k' *converges weakly to* x' *in* $L^1(I, X)$.

Lemma 5.4 ([41, Theorem 0.2.2]). *Let $(x_n)_n$ be a sequence of functions $x_n\colon [T_0, T] \to X$ which is uniformly bounded in norm and variation, i.e.*

$$\sup_{n\in\mathbb{N}} |x_n|_\infty \leq K_1 \quad \text{and} \quad \sup_{n\in\mathbb{N}} \operatorname{var}(x_n) \leq K_2$$

for some constants $K_1, K_2 > 0$. Then there exists a subsequence (again denoted by) $(x_n)_{n\in\mathbb{N}}$ and a bounded variation function $x\colon [T_0, T] \to X$ such that

$$\operatorname{var}(x) \leq K_2 \quad \text{and} \quad x_n(t) \to x(t) \quad \text{weakly as } n \to \infty \quad \text{for all } t \in [T_0, T].$$

If all x_n are right-continuous, then for every $T_1 \in]0, T]$ and every continuous $\phi\colon [T_0, T] \to X$

$$\lim_{n\to\infty} \int_{]0,T_1]} \phi \cdot dx_n = \int_{]0,T_1]} \phi \cdot dx + \phi(0) \cdot \left[x^+(0) - x(0)\right] - \phi(T_1) \cdot \left[x^+(T_1) - x(T_1)\right],$$

where dx_n and dx denotes the differential measure associated with x_n and x, respectively.

We will use the following special form of the Convergence Theorem in [3, Theorem 1.4.1].

Lemma 5.5. *Let $F\colon I \times X \rightrightarrows X$ be a set-valued map with closed and convex values satisfying*

(*i*) *For each* $x \in X$, $F(\cdot, x)$ *is measurable,*
(*ii*) *For a.e.* $t \in I$, $F(t, \cdot)$ *is upper semicontinuous from* X *into* X_w.

Let $x_n, f_n\colon I \to X$ $(n \in \mathbb{N})$ be measurable functions such that x_n converges a.e. on I to a function $x\colon I \to X$ and f_n converges weakly in $L^1(I, X)$ to $f\colon I \to X$. If $f_n(t) \in F(t, x_n(t)) + r_n(t)\mathbb{B}$, for all $n \in \mathbb{N}$ and almost all $t \in I$. If $r_n(t)$ converges to 0 in $L^1(I)$ then $f(t) \in F(t, x(t))$ a.e. on I.

To get a priori bounds on the solutions of differential inclusions we need the following consequence of Grönwall's inequality.

Lemma 5.6 ([3, Proposition 2.4.1]). *Let $I = [T_0, T]$, α and u be continuous functions and β be a nonnegative integrable function, all defined on I. Assume that α is non-increasing and that*

$$u(t) \leq \alpha(t) + \int_{T_0}^{t} \beta(s)u(s)ds \quad t \in [T_0, T].$$

Then

$$u(t) \leq \alpha(t) \exp\left(\int_{T_0}^{t} \beta(s)ds\right) \quad t \in [T_0, T].$$

The next result says that to have invariance of the solutions it is sufficient to have it in small intervals.

Lemma 5.7. *Let $F\colon [T_0, T] \times X \rightrightarrows X$ and $C\colon [T_0, T] \rightrightarrows X$ be two set-valued maps. Let a, b be two positive numbers with $T_0 \leq a < b \leq T$ and let us consider the differential inclusion*

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & a.e.\ t \in [a, b], \\ x(\tau) = y. \end{cases} \qquad \mathcal{P}(a, b, y)$$

Let $\eta > 0$ be a positive number and assume that $x_0 \in C(T_0)$. Suppose that for every $\tau \in [T_0, T)$, with $\tau + \eta \leq T_0$, and every $y \in C(\tau)$ there exists at least one solution of $\mathcal{P}(\tau, \tau + \eta, y)$ with $x(t) \in C(t)$ for all $t \in [\tau, \tau + \eta]$. Then there exists at least one solution of $\mathcal{P}(T_0, T, x_0)$ satisfying $x(t) \in C(t)$ for all $t \in [T_0, T]$.

Proof. Fix, like in [29], an integer N such that $(T - T_0)/N < \eta$ and fix also the subdivision of $[T_0, T]$ with $T_0, T_1, \ldots, T_N = T$ given, for each $k \in \{0, 1, \ldots, N\}$, by

$$T_k = T_0 + \frac{k}{N}(T - T_0).$$

For $k = 1$, since $T_1 - T_0 < \eta$, there exists a solution $x^1\colon [T_0, T_1] \to X$ of $\mathcal{P}(T_0, T, x_0)$ with $x^1(t) \in C(t)$ for all $t \in [T_0, T_1]$. Inductively, for $k = 2, \ldots, N$, as $T_k - T_{k-1} < \eta$ and $x^{k-1}(T_{k-1}) \in C(T_{k-1})$, let $x^k\colon [T_{k-1}, T_k] \to X$ be a solution of $\left(\mathcal{P}\left(T_{k-1}, T_k, x^{k-1}(T_{k-1})\right)\right)$ with $x^k(t) \in C(t)$ for all $t \in [T_{k-1}, T_k]$. Finally, we define $x(t) = x^k(t)$ over $[T_{k-1}, T_k]$ for $k = 1, \ldots, N$, which is a solution of $\mathcal{P}(T_0, T, x_0)$. □

The next result gives some properties of the distance function composed with differentiable functions along a curve.

Lemma 5.8. *Assume that X, Y are two separable Hilbert spaces. Let $x\colon [T_0, T] \to X$ be an absolutely continuous function, let $C\colon [T_0, T] \rightrightarrows Y$ be a set-valued map with nonempty closed values satisfying (H_2) and $h\colon X \to Y$ be a differentiable function with bounded derivative. Then*

1. *The function $t \to d(h(x(t)), C(t))$ is absolutely continuous over $[T_0, T]$.*
2. *For all $t \in]T_0, T[$, where $\dot{\zeta}(t)$ exists,*

$$\begin{aligned} &\limsup_{s\downarrow 0} \frac{d(h\left(x(t+s)\right), C(t+s)) - d(h\left(x(t)\right), C(t))}{s} \\ &\leq |\dot{\zeta}(t)| + \limsup_{s\downarrow 0} \frac{d(h\left(x(t+s)\right), C(t)) - d(h\left(x(t)\right), C(t))}{s}. \end{aligned} \tag{33}$$

3. *For all $t \in]T_0, T[$, where $\dot{x}(t)$ exists,*

$$\begin{aligned} &\limsup_{s\downarrow 0} \frac{d(h\left(x(t+s)\right), C(t)) - d(h\left(x(t)\right), C(t))}{s} \\ &\leq \max_{y^*\in\partial d(h(x(t)),C(t))} \left\langle y^*, Dh(x(t))\, \dot{x}(t) \right\rangle . \end{aligned} \tag{34}$$

4. *For all $t \in \{t \in]T_0, T[: h\left(x(t)\right) \notin C(t)\}$, where $\dot{x}(t)$ exists, $d(h\left(x(\cdot)\right), C(t))$ has a directional derivative at t and*

$$\begin{aligned} &\lim_{s\downarrow 0} \frac{d(h\left(x(t+s)\right), C(t)) - d(h\left(x(t)\right), C(t))}{s} \\ &= \min_{y^*\in\partial d(h(x(t)),C(t))} \left\langle y^*, Dh(x(t))\, \dot{x}(t) \right\rangle . \end{aligned}$$

5. *For every $x \in X$ the set-valued map $t \mapsto \partial d(h(x), C(t))$ is measurable.*

Proof. Let $\psi\colon [T_0, T] \to \mathbb{R}$ be the function defined by $\psi(t) := d(h\left(x(t)\right), C(t))$.

1. It follows directly from (H_2) and the boundedness of the derivative of h.

2. Let $t \in]T_0, T[$ be such that $\dot{\zeta}(t)$ exists. Then

$$\begin{aligned} &\frac{\psi(t+s) - \psi(t)}{s} \\ &= \frac{d(h(x(t+s)), C(t+s)) - d(h(x(t+s)), C(t))}{s} \\ &\qquad + \frac{d(h(x(t+s)), C(t)) - d(h(x(t)), C(t))}{s} \\ &\leq \frac{|\zeta(t+s) - \zeta(t)|}{s} + \frac{d(h(x(t+s)), C(t)) - d(h(x(t)), C(t))}{s}, \end{aligned}$$

and taking superior limit, we get (33).

3. Let $t \in]T_0, T[$ be such that $\dot{x}(t)$ exists. Let $s_n \to 0^+$ be such

$$\begin{aligned} &\limsup_{s\downarrow 0^+} \frac{d(h(x(t+s)), C(t)) - d(h(x(t)), C(t))}{s} \\ &= \lim_{n\to+\infty} \frac{d(h(x(t+s_n)), C(t)) - d(h(x(t)), C(t))}{s_n}. \end{aligned}$$

By virtue of Lebourg's Mean Value Theorem [21, Theorem 2.2.4], there exist $z_n \in]h(x(t)), h(x(t+s_n))[$ and $\xi_n \in \partial d(z_n, C(t))$ such that

$$\begin{aligned} &\frac{d(h(x(t+s_n)), C(t)) - d(h(x(t)), C(t))}{s_n} \\ = &\left\langle \xi_n, \frac{h(x(t+s_n)) - h(x(t))}{s_n} \right\rangle, \end{aligned}$$

since $\|\xi_n\| \le 1$, there is a subsequence (without relabeling) of $\{\xi_n\}_n$ such that $\xi_n \rightharpoonup \xi \in \partial d(h(x(t)), C(t))$, and taking limit in the last equality we get (34).

4. Let $t \in \{t \in]T_0, T[: x(t) \notin C(t)\}$ where $\dot{x}(t)$ exists. Then, for $s > 0$ small enough,

$$\begin{aligned} &\frac{d(h(x(t+s)), C(t)) - d(h(x(t)), C(t))}{s} \\ = &\frac{d(h(x(t)) + sDh(x(t))\dot{x}(t) + s\varepsilon(s,t)), C(t)) - d(h(x(t)), C(t))}{s} \\ = &\frac{d(h(x(t)) + sDh(x(t))\dot{x}(t), C(t)) - d(h(x(t)), C(t))}{s} + \eta(s,t), \end{aligned}$$

for some mappings $\varepsilon(\cdot, t)$ and $\eta(\cdot, t)$ with $\lim_{s\downarrow 0} \varepsilon(s,t) = 0$ and $\lim_{s\downarrow 0} \eta(s,t) = 0$. Then, using Lemma 5.2, we get

$$\begin{aligned} &\lim_{s\downarrow 0} \frac{d(h(x(t+s)), C(t)) - d(h(x(t)), C(t))}{s} \\ = &\lim_{s\downarrow 0} \frac{d(h(x(t)) + sDh(x(t))\dot{x}(t), C(t)) - d(h(x(t)), C(t))}{h} \\ = &\min_{y^* \in \partial d(h(x(t)), C(t))} \langle y^*, Dh(x(t))\dot{x}(t) \rangle . \end{aligned}$$

5. Let $\Gamma(t) := \partial d(h(x), C(t))$. Then, Γ takes weakly compact and convex values. Fixing any $d \in Y$, by virtue of [32, Proposition 2.2.39], it is enough to verify that the support function $t \mapsto \sigma(d, \Gamma(t))$ is measurable, where

$$\sigma(d, \Gamma(t)) := \sup_{v \in \Gamma(t)} \langle v, d \rangle.$$

Recalling that $\sigma(d, \Gamma(t)) = d^\circ_{C(t)}(h(x); d)$ and fixing a countable dense subset Δ of $\mathbb{B}$, we have

$$\begin{aligned} &d^\circ_{C(t)}(h(x); d) \\ = &\inf_{n\in\mathbb{N}} \sup_{s \in]0, \frac{1}{n}[, y \in \frac{1}{n}\mathbb{B}} \frac{1}{s} \left[d(y + h(x) + sd, C(t)) - d(y + h(x), C(t)) \right] \\ = &\inf_{n\in\mathbb{N}} \sup_{s \in]0, \frac{1}{n}[\cap\mathbb{Q}, y \in \frac{1}{n}\Delta} \frac{1}{s} \left[d(y + h(x) + sd, C(t)) - d\left(y + h(x), C(t)\right) \right]. \end{aligned}$$

This and the continuity of the function $(t,x) \mapsto d(h(x), C(t))$, due to the continuity of h and (H_2), guarantees the desired measurability of $t \mapsto \sigma(d, \Gamma(t))$ which finishes the proof. $\square$

The next result is the cornerstone of our approach to approximate solutions of differential inclusions.

Theorem 5.9. *Let $A\colon X \to X$ be a maximal monotone operator with $\|Ax\| \leq c\|x\| + d$ for all $x \in X$ for some $c, d \geq 0$ such that $-A$ generates a compact semigroup on X. For $i = 1,2$, let $F_i\colon [T_0, T] \times X \rightrightarrows X$ be a set-valued map with closed and convex values satisfying (H_1^F), (H_2^F) and $\left(H_3^F\right)$ for some functions $\alpha_i, \beta_i \in L^1(T_0, T)$ for $i = 1,2$. Let $\mu_n \in L^1(T_0,T)$ be a sequence of nonnegative functions which converges to $\mu \in L^1(T_0,T)$ with $\mu_n\alpha_2, \mu_n\beta_2 \in L^1(T_0,T)$ for $i = 1,2$. Let us consider a sequence $\{x_n\}_n$ of solutions of the differential inclusion (whose existence is guaranteed by Theorem 2.5)*

$$\begin{cases} -\dot{x}_n(t) \in \dfrac{1}{n} A x_n(t) + F_1(t, x_n(t)) + \mu_n(t) F_2(t, x_n(t)) & \text{a.e. } t \in [T_0, T], \\ x_n(T_0) = x_0, \end{cases} \tag{35}$$

Assume that $\forall t \in [T_0,T]$, $\{x_n(t)\}_n$ is a relatively compact subset of X. Then there exists a subsequence (again denoted by $(x_n)_n$) converging to an absolutely continuous function x solution of

$$\begin{cases} -\dot{x}(t) \in F_1(t, x(t)) + \mu(t) F_2(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0, \end{cases} \tag{36}$$

which satisfies

$$\|x(t)\| \leq d(t) := \exp\left(\int_{T_0}^t \tilde{\alpha}(s) ds\right)\left(\|x_0\| + \int_{T_0}^t \tilde{\beta}(s) ds\right) \quad t \in [T_0, T],$$

$$\|\dot{x}(t)\| \leq c(t) := \tilde{\alpha}(t) \exp\left(\int_{T_0}^t \tilde{\alpha}(s) ds\right)\left(\|x_0\| + \int_{T_0}^t \tilde{\beta}(s) ds\right) + \tilde{\beta}(t)$$
$$\text{a.e. } t \in [T_0, T],$$

where $\tilde{\alpha} := \alpha_1 + \mu\alpha_2$ and $\tilde{\beta} := \beta_1 + \mu\beta_2$.

Proof. For $n \in \mathbb{N}$ we define $\alpha_n := \alpha_1 + \mu_n\alpha_2 + \frac{c}{n}$, $\beta_n := \beta_1 + \mu_n\beta_2 + \frac{d}{n}$ and

$$c_n(t) := \alpha_n(t) \exp\left(\int_{T_0}^t \alpha_n(s) ds\right)\left(\|x_0\| + \int_{T_0}^t \beta_n(s) ds\right) + \beta_n(t)$$
$$\text{a.e. } t \in [T_0, T].$$

Due to (H_3^F), we have

$$\|\dot{x}_n(t)\| \leq \alpha_n(t)\|x_n(t)\| + \beta_n(t) \quad \text{a.e. } t \in [T_0, T].$$

Then, by virtue of Lemma 5.6, for all $t \in [T_0, T]$

$$\|x_n(t)\| \leq d_n(t) := \left(\|x_0\| + \int_{T_0}^{t} \beta_n(s)ds\right) \exp\left(\int_{T_0}^{t} \alpha_n(s)ds\right). \tag{37}$$

Thus, using the last two inequalities, for a.e. $t \in [T_0, T]$

$$\|\dot{x}_n(t)\| \leq \alpha_n(t)\|x_n(t)\| + \beta_n(t) \leq c_n(t). \tag{38}$$

Therefore, as a result of Proposition 5.3, there exists a subsequence (again denoted by x_n) converging to an absolutely continuous function x from $[T_0, T]$ in the sense that

(i) x_n converges uniformly to x over $[T_0, T]$,
(ii) x_n' converges weakly to x' in $L^1([T_0, T], X)$.

Moreover, by (H_1^F), (H_2^F) and Lemma 5.5, it follows directly that x satisfies (36). □

6. Set-valued perturbed generalized sweeping processes

Let X and Y be two separable Hilbert spaces. Let $C\colon [T_0, T] \rightrightarrows Y$ and $F\colon [T_0, T] \times X \rightrightarrows X$ be two set-valued maps with nonempty closed values and nonempty closed convex values, respectively. In this section we show existence of solutions for the generalized sweeping process

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + g(x(t))N\left(C(t), h(x(t))\right) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in h^{-1}\left(C(T_0)\right), \end{cases} \tag{39}$$

where $g\colon X \to \mathcal{L}(Y; X)$ and $h\colon X \to Y$ are two functions. To achieve this goal we will use Lemma 5.8 and Theorem 5.9 to construct approximations which, under an appropriate compactness conditions, converges to a solution of (39).

Theorem 6.1. *Let $C\colon [T_0, T] \rightrightarrows Y$ be a set-valued map satisfying (H_1), (H_2), (H_3') and (H_4) and let $F\colon [T_0, T] \times X \rightrightarrows X$ be a set-valued map with nonempty closed and convex values satisfying the conditions (H_1^F), (H_2^F) and (H_3^F). Then there exists at least one solution of (39) satisfying*

$$\|\dot{x}(t)\| \leq \dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t} \beta(s)ds\right) + M_2\dot{\sigma}(t)\int_{T_0}^{t} \mu(s)ds + \beta(t) + M_2\mu(t)$$
$$\textit{a.e. } t \in [T_0, T],$$

where η, σ and μ are functions from $[T_0, T]$ to $\mathbb{R}_+$ defined by the following *formulas:*

$$\eta(t) = M\dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t}\beta(s)ds\right) + M\beta(t) + |\dot{\zeta}(t)|,$$

$$\sigma(t) = \exp\left(\int_{T_0}^{t}(\alpha(s)+M_1)\,ds\right),$$

$$\mu(t) = MM_2\frac{\eta(t)}{\alpha_0^2\lambda} + MM_2\frac{\dot{\sigma}(t)}{\alpha_0^2\lambda}\int_{T_0}^{t}\frac{MM_2\eta(s)}{\alpha_0^2\lambda}\exp\left(MM_2\int_{s}^{t}\frac{\dot{\sigma}(\tau)}{\alpha_0^2\lambda}d\tau\right)ds.$$

Proof. Without loss of generality we assume that $MM_2 \geq 1$. Under the framework of Theorem 5.9, let us consider $F_1(t,x) = F(t,x)$, $F_2(t,x) = g(x)\partial d(h(x),C(t))$ and μ_n defined by

$$\mu_n(t) := MM_2\frac{\eta_n(t)}{\alpha_0^2\lambda} + MM_2\frac{\dot{\sigma}_n(t)}{\alpha_0^2\lambda}\int_{T_0}^{t}\frac{MM_2\eta_n(s)}{\alpha_0^2\lambda}\exp\left(MM_2\int_{s}^{t}\frac{\dot{\sigma}_n(\tau)}{\alpha_0^2\lambda}d\tau\right)ds,$$

where σ_n and η_n are functions from $[T_0,T]$ to $\mathbb{R}_+$ defined by the formulas:

$$\sigma_n(t) := \exp\left[\int_{T_0}^{t}\left(\alpha(s)+M_1+\frac{c}{n}\right)ds\right]$$

$$\eta_n(t) := M\dot{\sigma}_n(t)\left(\|x_0\| + \int_{T_0}^{t}\left[\beta(s)+\frac{d}{n}\right]ds\right) + M\left(\beta(t)+\frac{d}{n}\right) + |\dot{\zeta}(t)|.$$

Lemma 6.2. *For all $n \in \mathbb{N}$*

$$MM_2\eta_n(t) + M\,M_2\,\dot{\sigma}_n(t)\int_{T_0}^{t}\mu_n(s)ds = \lambda\alpha_0^2\mu_n(t) \quad a.e\ t \in [T_0,T].$$

As a result of Lemma 5.8, F_1, F_2 and μ_n satisfy the conditions of Theorem 5.9 with $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\alpha_2 = M_1$ and $\beta_2 = M_2$. Let $(x_n)_n$ be a sequence of solutions of (35) with $x_n(T_0) = x_0 \in h^{-1}(C(T_0))$. Then there exist $f_n, g_n \in L^1([T_0,T],X)$ such that for a.e. $t \in [T_0,T]$ $f_n(t) \in F(t,x_n(t))$, $g_n(t) \in \partial d(h(x_n(t)),C(t))$ and $-\dot{x}_n(t) = \frac{1}{n}Ax_n(t) + f_n(t) + \mu_n(t)g(x_n(t))\,g_n(t)$. Moreover, due to (37) and (38),

$$\|x_n(t)\| \leq d_n(t) := \sigma_n(t)\left(\|x_0\| + \int_{T_0}^{t}\left(\beta(s)+\frac{d}{n}\right)ds + M_2\int_{T_0}^{t}\mu_n(s)ds\right) \quad \text{for } t \in [T_0,T], \tag{40}$$

$$\|\dot{x}_n(t)\| \leq c_n(t) := \dot{\sigma}_n(t)\left(\|x_0\| + \int_{T_0}^{t}\left(\beta(s)+\frac{d}{n}\right)ds + M_2\int_{T_0}^{t}\mu_n(s)ds\right) + \beta(t) + \frac{d}{n} + M_2\mu_n(t) \quad \text{a.e. } t \in [T_0,T]. \tag{41}$$

Let $\eta > 0$ be such that

$$\left(\lambda\alpha_0^2 + MM_2\right)\int_{T_0}^{T_0+\eta}\mu_n(s)ds < \rho. \tag{42}$$

Step 1. First, we prove the theorem under the condition $T - T_0 < \eta$.
Suppose that for all $n \in \mathbb{N}$ $h(x_n(t)) \in C(t)$ for all $t \in [T_0, T]$. Hence, in view of (H_3') and (40), for all $t \in [T_0, T]$ $\{x_n(t)\}_n$ is a relatively compact subset of X, then, Theorem 5.9 asserts that there exists a subsequence (again denoted by) $x_n(t)$ converging to an absolutely continuous function x solution of

$$\begin{cases} -\dot{x}(t) \in F(t, x(t)) + \mu(t)g(x(t))\partial d(h\left(x(t)\right), C(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

with $h\left(x(t)\right) \in C(t)$ for all $t \in [T_0, T]$. Therefore $\mu(t)\partial d(h\left(x(t)\right), C(t)) \subset N(C(t), h\left(x(t)\right))$ for all $t \in [T_0, T]$ which implies that x is a solution of (39).
So it remains to prove that for all $n \in \mathbb{N}$, $h\left(x_n(t)\right) \in C(t)$ for all $t \in [T_0, T]$. We begin by showing that $\psi_n(t) := d(h\left(x_n(t)\right), C(t)) < \rho$ for all $t \in [T_0, T]$. Indeed, due to (H_2) and Lemma 5.8, the function ψ_n is absolutely continuous. Hence, for a.e. $t \in [T_0, T]$

$$\begin{aligned} \psi_n'(t) &\leq |\dot{\zeta}(t)| + \max_{y^* \in \partial d(h(x_n(t)), C(t))} \left\langle y^*, h'\left(x_n(t)\right)\dot{x}_n(t)\right\rangle, \\ &\leq |\dot{\zeta}(t)| + M\|\dot{x}_n(t)\| \\ &\leq |\dot{\zeta}(t)| + Mc_n(t) \\ &= \eta_n(t) + MM_2\dot{\sigma}_n(t)\int_{T_0}^{t}\mu_n(s)ds + MM_2\mu_n(t) \\ &\leq \left(\lambda\alpha_0^2 + MM_2\right)\mu_n(t). \end{aligned}$$

Thus, by virtue of (42), for all $t \in [T_0, T]$

$$\begin{aligned} d(h\left(x_n(t)\right), C(t)) &= d(h\left(x_0\right), C(T_0)) + \int_{T_0}^{t}\psi_n'(s)ds \\ &\leq d(h\left(x_0\right), C(T_0)) + \left(\lambda\alpha_0^2 + MM_2\right)\int_{T_0}^{t}\mu_n(s)ds \\ &< \rho. \end{aligned}$$

Next we prove that $h\left(x_n(t)\right) \in C(t)$ for all $t \in [T_0, T]$. Indeed, the set

$$\Omega_n = \{t \in [T_0, T] \colon h\left(x_n(t)\right) \notin C(t)\} = \psi_n^{-1}\left(]0, +\infty[\right)$$

is open in $[T_0, T]$. We claim, as in [54], that Ω_n is empty. In fact, let us suppose that $\Omega_n \neq \emptyset$ and put

$$\tau_- = \inf\{t > T_0 \colon h\left(x_n(t)\right) \notin C(t)\}.$$

Then $\psi_n(\tau_-) = 0$, because of the continuity of ψ_n and, since Ω_n is open, there exists τ_+ such that $]\tau_-, \tau_+[\subset \Omega_n$. By virtue of Lemma 5.8, (40), (41) and (H_1), for a.e. $t \in]\tau_-, \tau_+[$

$$\begin{aligned}
\psi_n'(t) &\leq |\dot{\zeta}(t)| + \min_{y^* \in \partial d(h(x_n(t)), C(t))} \langle y^*, Dh\left(x_n(t)\right) \dot{x}_n(t) \rangle, \\
&= |\dot{\zeta}(t)| + \min_{y^* \in \partial d(h(x_n(t)), C(t))} \Big\langle y^*, Dh\left(x_n(t)\right) \Big(-\frac{1}{n} A x_n(t) - f_n(t) \\
&\qquad\qquad - \mu_n(t) g(x_n(t)) g_n(t) \Big) \Big\rangle, \\
&\leq |\dot{\zeta}(t)| + \frac{M}{n}\left(c\|x_n(t)\| + d\right) + M\left(\alpha(t)\|x_n(t)\| + \beta(t)\right) \\
&\qquad + \min_{y^* \in \partial d(h(x_n(t)), C(t))} \langle y^*, -\mu_n(t) Dh\left(x_n(t)\right) g(x_n(t)) g_n(t) \rangle, \\
&= |\dot{\zeta}(t)| + M\left(\alpha(t) + \frac{c}{n}\right) \|x_n(t)\| + M\left(\beta(t) + \frac{d}{n}\right) \\
&\qquad + \mu_n(t) \min_{y^* \in \partial d(h(x_n(t)), C(t))} \langle y^*, -Dh\left(x_n(t)\right) g(x_n(t)) g_n(t) \rangle, \\
&\leq |\dot{\zeta}(t)| + M\left(\alpha(t) + \frac{c}{n}\right) d_n(t) + M\left(\beta(t) + \frac{d}{n}\right) - \lambda \mu_n(t) \|g_n(t)\|^2, \\
&\leq |\dot{\zeta}(t)| + M\left(\alpha(t) + \frac{c}{n}\right) d_n(t) + M\left(\beta(t) + \frac{d}{n}\right) - \lambda \alpha_0^2 \mu_n(t), \\
&= \eta_n(t) + M M_2 \dot{\sigma}_n(t) \int_{T_0}^t \mu_n(s) ds - \lambda \alpha_0^2 \mu_n(t) \\
&\leq 0,
\end{aligned}$$

where the last inequality is due to the definition of μ_n and Lemma 6.2. Therefore, for almost all $t \in]\tau_-, \tau_+[$ we have $\psi_n'(t) \leq 0$ which implies

$$\psi_n(t) \leq \psi_n(\tau_-) = 0 \quad \text{ for every } t \in]\tau_-, \tau_+[,$$

which is a contradiction with $]\tau_-, \tau_+[\subset \Omega_n$. So $\Omega_n = \emptyset$ which means that $h\left(x_n(t)\right) \in C(t)$ for all $t \in [T_0, T]$ and finishes the first step.

Step 2. In the general case, without any restriction on the length of T, it is sufficient to use Step 1 with Lemma 5.7. $\square$

Taking $X = Y$, $h \equiv \mathrm{Id}$ and $g \equiv Dh^*$ we obtain, as a consequence of Theorem 6.1, the existence of solutions for the perturbed sweeping process (2) in the non-prox-regular setting which, as a result of Proposition 3.9, greatly extends [48, Theorem 1.2.1]. The next result should be understood as an intermediary one between [12, Theorem 4.2] for prox-regular sets in Hilbert spaces without compactness and [54, Theorem 4.3] for closed sets in finite dimensional Hilbert spaces. It is noteworthy that unlike [12, Theorem 4.2] we do not suppose any kind of strong compactness on the perturbation F.

Corollary 6.3. *Let $C\colon [T_0,T] \rightrightarrows X$ be a set-valued map satisfying (H_1)–(H_3) and let $F\colon [T_0,T]\times X \rightrightarrows X$ be a set-valued map with nonempty closed and convex values satisfying the conditions (H_1^F), (H_2^F) and (H_3^F). Then there exists at least one solution of (2) satisfying*

$$\|\dot{x}(t)\| \leq \frac{1+\alpha_0^2}{\alpha_0^2}\left(\eta(t)+\dot{\sigma}(t)\int_{T_0}^{t}\frac{\eta(s)}{\alpha_0^2}\exp\left(\frac{\sigma(t)}{\alpha_0^2}-\frac{\sigma(s)}{\alpha_0^2}\right)ds\right)-|\dot{\zeta}(t)| \quad \text{a.e. } t\in[T_0,T],$$

where η and σ are functions from $[T_0,T]$ to $\mathbb{R}_+$ defined by the following formulas:

$$\eta(t) = \dot{\sigma}(t)\left(\|x_0\|+\int_{T_0}^{t}\beta(s)ds\right)+\beta(t)+|\dot{\zeta}(t)|,$$
$$\sigma(t) = \exp\left(\int_{T_0}^{t}\alpha(s)ds\right).$$

In the case of the convex sweeping process our approach is flexible enough to recover the existence and uniqueness of solutions.

Theorem 6.4. *Let $C\colon [T_0,T] \rightrightarrows X$ be a set-valued map with closed convex values satisfying (H_2). Then there exists a unique absolutely continuous solution of the sweeping process*

$$\begin{cases} -\dot{x}(t)\in N(C(t),x(t)) & \text{a.e. } t\in[T_0,T],\\ x(T_0)=x_0\in C(T_0). \end{cases} \tag{43}$$

which satisfies $\|\dot{x}(t)\|\leq|\dot{\zeta}(t)|$ a.e. $t\in[T_0,T]$.

Proof. Let $(x_n)_n$ be the sequence of functions defined in the proof of Theorem 6.3 with $X=Y$, $F\equiv 0$, $h=\mathrm{Id}$ and $g=Dh^*$. By virtue of Lemma 5.4, (40) and (41), there exists a subsequence (again denoted by) $(x_n)_n$ and a bounded variation function $x\colon [T_0,T]\to X$ such that $x_n(t)\rightharpoonup x(t)$ for all $t\in[T_0,T]$, $x_n\rightharpoonup x$ in $L^1([T_0,T],X)$ and $\dot{x}_n\rightharpoonup\dot{x}$ in $L^1([T_0,T],X)$. Under these conditions it is not difficult to show that the function x is absolutely continuous. Furthermore, since $C(t)$ is convex and $x_n(t)\rightharpoonup x(t)$ $\forall t\in[T_0,T]$, we obtain that $x(t)\in C(t)$ for all $t\in[T_0,T]$. Moreover, since $x_n(t)\in C(t)$ for all $t\in[T_0,T]$, we have that $-\dot{x}_n(t)-\frac{1}{n}Ax_n(t)\in N(C(t),x_n(t))$ for a.e. $t\in[T_0,T]$, which entails that for a.e. $t\in[T_0,T]$

$$\begin{aligned}\langle -\dot{x}_n(t), y-x_n(t)\rangle &\leq \left\langle \frac{1}{n}Ax_n, y-x_n(t)\right\rangle\\ &\leq \frac{1}{n}\left(c\|x_n(t)\|+d\right)\left(\|y\|+\|x_n(t)\|\right) \quad \text{for all } y\in C(t).\end{aligned}$$

Let $M \geq \sup_{n\in\mathbb{N}} \|x_n\|_\infty$ and define $C_M(t) := C(t) \cap M\mathbb{B}$. Then

$$\langle -\dot{x}_n(t), y - x_n(t)\rangle \leq \frac{2}{n}\,(cM+d)\,M \quad \text{for all } y \in C_M(t).$$

Next, taking supremum over $C_M(t)$ and integrating over T_0 and T,

$$\frac{1}{2}\|x_n(T)\|^2 - \frac{1}{2}\|x_0\|^2 + \int_{T_0}^{T} \sup_{y\in C_M(t)} \langle -\dot{x_n}(t), y\rangle\, dt \leq \frac{2}{n}\,(cM+d)\,M(T-T_0).$$

Therefore, by virtue of the weak convergence of $\dot{x}_n$ to $\dot{x}$ and [33, Lemma 3],

$$\begin{aligned}
&\frac{1}{2}\|x(T)\|^2 - \frac{1}{2}\|x_0\|^2 + \int_{T_0}^{T} \sup_{y\in C_M(t)} \langle -\dot{x}(t), y\rangle\, dt \\
\leq\; &\liminf_{n\to+\infty} \frac{1}{2}\|x_n(T)\|^2 - \frac{1}{2}\|x_0\|^2 + \liminf_{n\to+\infty} \int_{T_0}^{T} \sup_{y\in C_M(t)} \langle -\dot{x}_n(t), y\rangle\, dt \\
\leq\; &\liminf_{n\to+\infty} \left(\frac{1}{2}\|x_n(T)\|^2 - \frac{1}{2}\|x_0\|^2 + \int_{T_0}^{T} \sup_{y\in C_M(t)} \langle -\dot{x}_n(t), y\rangle\, dt \right) \\
\leq\; &\liminf_{n\to+\infty} \frac{2}{n}\,(cM+d)\,M(T-T_0) \\
=\; &0.
\end{aligned}$$

From which we obtain

$$\frac{1}{2}\|x(T)\|^2 - \frac{1}{2}\|x_0\|^2 + \int_{T_0}^{T} \sup_{y\in C_M(t)} \langle -\dot{x}(t), y\rangle\, dt \leq 0,$$

which turns out to be equivalent to

$$\int_{T_0}^{T} \Delta(t)dt := \int_{T_0}^{T} \left(\langle \dot{x}(t), x(t)\rangle + \sup_{y\in C_M(t)} \langle -\dot{x}(t), y\rangle \right) dt \leq 0.$$

Therefore, noting that $x(t) \in C_M(t)$ for all $t \in [T_0, T]$, we get that $\Delta(t) \geq 0$ a.e. $t \in [T_0, T]$. Consequently, $\Delta(t) = 0$ for a.e. $t \in [T_0, T]$, which gives for a.e. $t \in [T_0, T]$

$$\langle -\dot{x}(t), y - x(t)\rangle \leq 0 \quad y \in C_M(t).$$

Finally, since this inequality is valid for all M bigger enough, we conclude that x is a solution of (43). The uniqueness of solutions follows directly from the definition of normal cone. $\square$

7. Subgradient evolution inclusions

Let $F\colon [T_0,T]\times X \rightrightarrows X$ be a set-valued map with nonempty closed and convex values satisfying the hypothesis (H_1^F), (H_2^F) and (H_3^F) and $f\colon [T_0,T]\times H \to \mathbb{R}$ be a function such that for a.e. $t\in[T_0,T]$ the function $x\mapsto f(t,x)$ is Lipschitz on bounded sets. In this section we study the existence of solutions of the following differential inclusion:

$$\begin{cases} -\dot{x}(t) \in \partial f(t,x(t)) + F(t,x(t)) & \text{a.e. } t\in[T_0,T], \\ x(T_0) = x_0, \end{cases} \tag{44}$$

Before establishing the main result of this section we need the following lemma.

Lemma 7.1. *Let $x\colon [T_0,T]\to X$ be an absolutely continuous function and suppose that (H_1^f) and (H_3^f) hold. Put $r=\sup_{t\in[T_0,T]}\|x(t)\|+1$. Then the function $\psi(t) := f(t,x(t))$ is absolutely continuous over $[T_0,T]$ and for all $t\in]T_0,T[$, where $\dot{x}(t)$ exists and $\gamma(t),\delta(t)<+\infty$,*

$$\limsup_{h\downarrow 0}\frac{f(t+h,x(t+h))-f(t,x(t))}{h} \le |\dot{\zeta}_r(t)| + \max_{y^*\in\partial f(t,x(t))}\langle y^*,\dot{x}(t)\rangle\,,$$

where $\dot{\zeta}_r$ is the function given by (H_3^f).

Proof. The first statement follows directly from (H_3^f). For the second one, take $t\in]T_0,T[$ where $\dot{x}(t)$ exists and $\gamma(t),\delta(t)<+\infty$. Let $h_n\to 0^+$ be such that

$$\limsup_{h\downarrow 0}\frac{f(t+h,x(t+h))-f(t,x(t))}{h} = \lim_{n\to+\infty}\frac{f(t+h_n,x(t+h_n))-f(t,x(t))}{h_n}.$$

Using Lebourg's Mean Value Theorem [21, Theorem 2.2.4], for every $h_n>0$ there exists $z_n\in]x(t),x(t+h_n)[$ and $\xi_n\in\partial f(t,z_n)$ such that

$$f(t,x(t+h_n)) - f(t,x(t)) = \langle \xi_n, x(t+h_n)-x(t)\rangle\,.$$

Then, by virtue of (H_3^f),

$$\begin{aligned} &\frac{f(t+h_n,x(t+h_n))-f(t,x(t))}{h_n} \\ &= \frac{f(t+h_n,x(t+h_n))-f(t,x(t+h_n))}{h_n} + \frac{f(t,x(t+h_n))-f(t,x(t))}{h_n} \\ &\le \frac{|\zeta_r(t+h_n)-\zeta_r(t)|}{h_n} + \left\langle \xi_n, \frac{x(t+h_n)-x(t)}{h_n}\right\rangle. \end{aligned} \tag{45}$$

Moreover, since $\xi_n\in\partial f(t,z_n)$, in view of (H_1^f), we have

$$\|\xi_n\| \le \gamma(t)\|z_n\| + \delta(t),$$

using this and the fact that the sequence $\{z_n\}_n$ belongs to the ball $r\mathbb{B}$ we get that the sequence $\{\xi_n\}_n$ is bounded. Hence there is a subsequence of $\{\xi_n\}_n$, without relabeling, such that ξ_n converges weakly in X to some $\xi \in \partial f(t, x(t))$. Consequently, taking superior limit in (45) we get the desired inequality. □

In the next theorem we prove existence of solutions for (44). We emphasize that the hypothesis considered on F are rather standard as well as (H_1^f)-(H_3^f). The new feature here is the possibility to include a non-convex function $f(t,\cdot)$, in contrast with the previous works of Hu and Papageorgiou [32, Chapter 2] where the function $f(t,\cdot)$ must be proper, convex and lower semicontinuous. Of course, there is a price to pay: (H_3^f) is a stronger condition than the considered in, for example, [32, Theorem 1.11].

Theorem 7.2. *Suppose, in addition to (H_1^F)–(H_3^F) and (H_1^f)–(H_4^f), that $\gamma, \delta \in L^\infty(T_0, T)$ and $\alpha, \beta \in L^1(T_0, T)$ or $\alpha, \beta, \gamma, \delta \in L^2(T_0, T)$. Then, there exists at least one solution of (44) satisfying*

$$\|x(t)\| \leq \exp\left(\int_{T_0}^t \tilde{\alpha}(s)ds\right)\left(\|x_0\| + \int_{T_0}^t \tilde{\beta}(s)ds\right) \quad t \in [T_0, T],$$

$$\|\dot{x}(t)\| \leq \tilde{\alpha}(t)\exp\left(\int_{T_0}^t \tilde{\alpha}(s)ds\right)\left(\|x_0\| + \int_{T_0}^t \tilde{\beta}(s)ds\right) + \tilde{\beta}(t) \quad a.e.\ t \in [T_0, T],$$

where $\tilde{\alpha}(t) = \alpha(t) + \gamma(t)$ and $\tilde{\beta}(t) = \beta(t) + \delta(t)$.

Proof. Under the framework of Theorem 5.9, let us consider $F_1(t, x) = F(t, x)$, $F_2(t, x) = \partial f(t, x)$ and $\mu_n(t) = 1$. Due to Lemma 5.8 F_1, F_2 and μ_n satisfy the conditions of Theorem 5.9. Let $(x_n)_n$ be a sequence of solutions of (35). Then

$$\|x_n(t)\| \leq d_n(t) := \sigma_n(t)\left(\|x_0\| + \int_{T_0}^t \beta_n(s)ds\right) \quad \text{for } t \in [T_0, T], \tag{46}$$

$$\|\dot{x}_n(t)\| \leq c_n(t) := \dot{\sigma}_n(t)\left(\|x_0\| + \int_{T_0}^t \beta_n(s)ds\right) + \beta_n(t) \quad \text{a.e. } t \in [T_0, T], \tag{47}$$

where σ_n and β_n are defined by $\sigma_n(t) = \exp\left(\int_{T_0}^t \left(\alpha(s) + \gamma(s) + \frac{c}{n}\right) ds\right)$ and $\beta_n(t) = \beta(t) + \delta(t) + \frac{d}{n}$ for $t \in [T_0, T]$. We wish to prove that for all $t \in [T_0, T]$, $\{x_n(t)\}_n$ is a relatively compact subset of X. To do this we use (H_4^f). Indeed, define $\psi_n(t) = f(t, x_n(t))$, by Lemma 7.1 this function is absolutely continuous and for a.e. $t \in [T_0, T]$

$$\begin{aligned}
\psi_n'(t) &\leq |\dot{\zeta}_r(t)| + \max_{y^* \in \partial f(t, x_n(t))} \langle y^*, \dot{x}_n(t)\rangle \\
&\leq |\dot{\zeta}_r(t)| + \max_{y^* \in \partial f(t, x_n(t))} \|y^*\| \times \|\dot{x}_n(t)\| \\
&\leq |\dot{\zeta}_r(t)| + (\gamma(t)\|x_n(t)\| + \delta(t))\,\|\dot{x}_n(t)\| \\
&\leq |\dot{\zeta}_r(t)| + (\gamma(t)d_n(t) + \delta(t))\,c_n(t),
\end{aligned}$$

where the last inequality is due to (46) and (47) and ζ_r is given by (H_3^f) with $r > d_n(t)$ for all $n \in \mathbb{N}$. Therefore

$$
\begin{aligned}
f(t, x_n(t)) &= f(T_0, x_0) + \int_{T_0}^{t} \psi_n'(s)ds \\
&\leq f(T_0, x_0) + \int_{T_0}^{t} \Big(|\dot{\zeta}_r(s)| + (\gamma(s)d_n(s) + \delta(s))\, c_n(s) \Big)\, ds, \\
&\leq L(T_0, T, x_0),
\end{aligned}
$$

where

$$
L(T_0, T, x_0) = f(T_0, x_0) + \int_{T_0}^{T} \Big(|\dot{\zeta}_r(s)| + (\gamma(s)d_1(s) + \delta(s))\, c_1(s) \Big)\, ds,
$$

is a finite constant independent of n. Furthermore, for every $t \in [T_0, T]$

$$
\begin{aligned}
f(t^*, x_n(t)) &= f(t^*, x_n(t)) - f(t, x_n(t)) + f(t, x_n(t)) \\
&\leq |\zeta_r(t^*) - \zeta_r(t)| + L(T_0, T, x_0) \\
&\leq \|\dot{\zeta}_r\|_{L^1(T_0,T)} + L(T_0, T, x_0).
\end{aligned}
$$

Thus, the bounded family $\{x_n(t)\}_n$ belong to the fixed sublevel $f(t^*, \cdot)$. Hence, due to (H_4^f), for all $t \in [T_0, T]$ $\{x_n(t)\}_n$ is relatively compact in X. Consequently, applying Theorem 5.9, there exists a subsequence (again denoted by) $x_n(t)$ converging to an absolutely continuous function x which is solution of (44). $\square$

8. Subgradient perturbed sweeping processes

Let $f\colon [T_0, T] \times X \to \mathbb{R}$ be a function such that for a.e. $t \in [T_0, T]$ the function $x \to f(t, x)$ is lower semicontinuous. In this section we study the existence of solutions of the following differential inclusion:

$$
\begin{cases}
-\dot{x}(t) \in \partial f(t, x(t)) + N(C(t), x(t)) & \text{a.e. } t \in [T_0, T], \\
x(T_0) = x_0 \in C(T_0).
\end{cases}
\tag{48}
$$

This differential inclusion can be seen as the dynamic analogue of the first order optimality condition (6) whose critical points, in the time-independent case, correspond to stationary points of the problem (5). We show existence of solutions of (48) under standard conditions in the study of optimization problems, namely, ball-compactness of the sets $C(\cdot)$ or ball-inf-compactness of the function f.

The first result is the existence of solution under ball-compactness of the set $C(t)$ for all $t \in [T_0, T]$ which is a straightforward consequence of Theorem 6.3.

Theorem 8.1. *Assume that* (H_1)*–*(H_3) *and* (H_1^f)*–*(H_2^f) *hold. Then there exists at least one solution of the problem* (48) *which satisfies for a.e.* $t \in [T_0, T]$

$$\|\dot{x}(t)\| \leq \frac{1+\alpha_0^2}{\alpha_0^2}\left(\eta(t) + \dot{\sigma}(t)\int_{T_0}^t \frac{\eta(s)}{\alpha_0^2}\exp\left(\frac{\sigma(t)}{\alpha_0^2} - \frac{\sigma(s)}{\alpha_0^2}\right)ds\right) - |\dot{\zeta}(t)|, \quad (49)$$

where η *and* σ *are functions from* $[T_0, T]$ *to* $\mathbb{R}_+$ *defined by the following formulas:*

$$\eta(t) = \dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^t \delta(s)ds\right) + \delta(t) + |\dot{\zeta}(t)|,$$
$$\sigma(t) = \exp\left(\int_{T_0}^t \gamma(s)ds\right).$$

The next statement asserts that we can interchange the ball-compactness of the set-valued map C by the ball-inf-compactness of some sublevel of the function $f(t,\cdot)$ for some $t \in [T_0, T]$.

Theorem 8.2. *Suppose, in addition to* (H_1)*–*(H_2) *and* (H_1^f)*–*(H_4^f)*, that* $\gamma, \delta \in L^\infty(T_0, T)$ *or* $\gamma, \delta, \dot{\zeta} \in L^2(T_0, T)$*. Then there exists at least one solution of* (48) *which satisfies* (49).

Proof. We define the function μ from $[T_0, T]$ to $\mathbb{R}_+$:

$$\mu(t) = \frac{\eta(t)}{\alpha_0^2} + \frac{\dot{\sigma}(t)}{\alpha_0^2}\int_{T_0}^t \frac{\eta(s)}{\alpha_0^2}\exp\left(\int_s^t \frac{\dot{\sigma}(\tau)}{\alpha_0^2}d\tau\right)ds,$$

where η and σ as in the statement of the theorem. We have the following formula:

$$\eta(t) + \dot{\sigma}(t)\int_{T_0}^t \mu(s)ds = \alpha_0^2\mu(t) \quad \text{a.e. } t \in [T_0, T]. \quad (50)$$

Let $\eta_0 > 0$ be such that

$$\left(1+\alpha_0^2\right)\int_{T_0}^{T_0+\eta_0} \mu(s)ds < \rho. \quad (51)$$

Step 1. First, we prove the theorem under the condition $T - T_0 < \eta_0$.
Let us consider the following unconstrained differential inclusion:

$$\begin{cases} -\dot{x}(t) \in \partial f(t, x(t)) + \mu(t)\partial d(x(t), C(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0). \end{cases} \quad (52)$$

By virtue of Theorem 7.2, there exists at least one solution of the problem (52) satisfying

$$\|x(t)\| \leq d(t) := \sigma(t)\left(\|x_0\| + \int_{T_0}^{t}\delta(s)ds\right) + \int_{T_0}^{t}\mu(s)ds \quad \text{for } t \in [T_0, T],$$

$$\|\dot{x}(t)\| \leq c(t) := \dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t}\delta(s)ds + \int_{T_0}^{t}\mu(s)ds\right) + \delta(t) + \mu(t)$$
$$\text{a.e. } t \in [T_0, T].$$

Therefore, there exist $f, g \in L^1([T_0, T], X)$ such that

$$\begin{cases} -\dot{x}(t) = f(t) + \mu(t)g(t) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

with $f(t) \in \partial f(t, x(t))$ and $g(t) \in \partial d(x(t), C(t))$ for a.e. $t \in [T_0, T]$. We claim that $d(x(t), C(t)) < \rho$ for all $t \in [T_0, T]$. Indeed, let be $\psi(t) = d(x(t), C(t))$. Then, due to (H_2) and Lemma 5.8, ψ is absolutely continuous. Hence, for a.e. $t \in [T_0, T]$

$$\begin{aligned} \psi'(t) &\leq |\dot{\zeta}(t)| + \max_{y^* \in \partial d(x(t), C(t))} \langle y^*, \dot{x}(t)\rangle, \\ &\leq |\dot{\zeta}(t)| + \|\dot{x}(t)\| \\ &\leq |\dot{\zeta}(t)| + c(t) \\ &\leq \left(1 + \alpha_0^2\right)\mu(t). \end{aligned}$$

In view of (51), this implies

$$d(x(t), C(t)) = d(x_0, C(T_0)) + \int_{T_0}^{t}\psi'(s)ds \leq \left(1 + \alpha_0^2\right)\int_{T_0}^{t}\mu(s)ds < \rho$$
$$\text{for all } t \in [T_0, T].$$

Next we prove that $x(t) \in C(t)$ for all $t \in [T_0, T]$. The set

$$\Omega = \{t \in [T_0, T] \colon x(t) \notin C(t)\} = \psi^{-1}\left(]0, +\infty[\right)$$

is open in $[T_0, T]$. We claim, as in [54], that the set Ω is in fact empty. Indeed, let us suppose that $\Omega \neq \emptyset$ and put

$$\tau_- = \inf\{t > T_0 \colon x(t) \notin C(t)\}.$$

Then $\psi(\tau_-) = 0$, because of continuity of ψ, and, since Ω is open, there exists τ_+ such that $]\tau_-, \tau_+[\subset \Omega$. By virtue of Lemma 5.8 and (H_1), for a.e. $t \in]\tau_-, \tau_+[$

$$\begin{aligned} \psi'(t) &\leq |\dot{\zeta}(t)| + \min_{y^* \in \partial d(x(t), C(t))} \langle y^*, \dot{x}(t)\rangle \\ &\leq |\dot{\zeta}(t)| + \gamma(t)\|x(t)\| + \delta(t) - \mu(t)\min_{y^* \in \partial d(x(t), C(t))} \langle y^*, g(t)\rangle \\ &\leq \eta(t) + \dot{\sigma}(t)\int_{T_0}^{t}\mu(s)ds - \alpha_0^2\mu(t) \\ &= 0, \end{aligned}$$

where the last equality is due to (50). Therefore, for almost all $t \in]\tau_-, \tau_+[$ we have $\psi'(t) \leq 0$ which implies

$$\psi(t) \leq \psi(\tau_-) = 0 \quad \text{for every } t \in]\tau_-, \tau_+[,$$

which is a contradiction with $]\tau_-, \tau_+[\subset \Omega$. So $\Omega = \emptyset$ which means that $x(t) \in C(t)$ for all $t \in [T_0, T]$. Consequently, $\partial d(x(t), C(t)) \subset N(C(t), x(t))$ for all $t \in [T_0, T]$ which implies that x is a solution of (48).

Step 2. In the general case, without any restriction on the length of T, it is sufficient to use Step 1 together Lemma 5.7. □

9. An application to complementarity dynamical systems

In this section, we apply the preceding theory to complementarity dynamical systems and give sufficient conditions for the existence of solutions. A complementarity dynamical system (CDS) consists of ordinary differential equations coupled to complementarity conditions which can be specified by functions $F\colon [T_0, T] \times X \to Y$, $g\colon X \to \mathcal{L}(Y; X)$ and $H\colon X \to Y$. The defining equations for the CDS corresponding to F, g and H are

$$\begin{aligned} \dot{x}(t) &= F(t, x(t)) + g(x(t))u(t), \\ y(t) &= H(t, x(t), u(t)), \\ K \ni y(t) &\perp u(t) \in K^*, \end{aligned} \tag{53}$$

where $K \subset Y$ is a closed convex cone and $K^* = \{d \in Y \colon \langle v, d\rangle \geq 0 \, \forall v \in K\}$ denotes the dual cone of K. The third line in (53) is a complementarity relation between $y(t)$ and $u(t)$ which are forced to remain always orthogonal one to each other. This fact can be expressed in an equivalent way as

$$K \ni y(t) \perp u(t) \in K^* \quad \Leftrightarrow \quad -u(t) \in N\left(K, y(t)\right). \tag{54}$$

The aim of this section is to show the existence of solutions for the CDS (53) by using the results obtained in Section 6. For this purpose we will make the following assumption:

Assumption 9.1. The system (53) satisfies: $H(t, x(t), u(t)) = h(x(t)) + d(t)$ where $h\colon X \to Y$ is a differentiable function and $d\colon [T_0, T] \to Y$ is an absolutely continuous function.

Under Assumption 9.1 we can write (53) as a GSP. Indeed, due to (54) and the identity

$$N\left(K, h(x(t)) + d(t)\right) = N\left(K - d(t), h(x(t))\right),$$

(53) is formally equivalent to

$$-\dot{x}(t) \in -F(t, x(t)) + g(x(t)) N\left(C(t), h(x(t))\right), \tag{55}$$

where $C(t) = K - d(t)$, which is a generalized sweeping process. In order to apply the Theorem 6.1, we make the following assumption:

Assumption 9.2.

(i) h is a differentiable function with $\sup_{x\in X}\|Dh(x)\| \leq M$, for some $M > 0$.
(ii) g is a continuous function satisfying

$$\|g(x)\| \leq M_1\|x\| + M_2 \quad \text{for all } x \in X.$$

(iii) There exists $\lambda > 0$ such that for all $x \in X$

$$\langle Dh(x)g(x)h, h\rangle \geq \lambda\|h\|^2 \quad \forall h \in Y.$$

Theorem 9.3. *In addition to Assumptions* 9.1 *and* 9.2*, assume that F satisfies (H_1^F), (H_2^F) and (H_3^F) and $C(t) := K - d(t)$ satisfies (H_3'). Then for every x_0 with $h(x_0) + d(T_0) \in K$ there exists at least one absolutely continuous solution x of* (53) *satisfying $x(T_0) = x_0$ and*

$$\|\dot{x}(t)\| \leq \dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t}\beta(s)ds\right) + M_2\dot{\sigma}(t)\int_{T_0}^{t}\mu(s)ds + \beta(t) + M_2\mu(t)$$
$$a.e.\ t \in [T_0, T],$$

where η, σ and μ are functions from $[T_0, T]$ to $\mathbb{R}_+$ defined by the following formulas:

$$\eta(t) = M\dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t}\beta(s)ds\right) + M\beta(t) + |\dot{d}(t)|,$$
$$\sigma(t) = \exp\left(\int_{T_0}^{t}(\alpha(s) + M_1)\,ds\right),$$
$$\mu(t) = MM_2\frac{\eta(t)}{\lambda} + MM_2\frac{\dot{\sigma}(t)}{\lambda}\int_{T_0}^{t}\frac{MM_2\eta(s)}{\lambda}\exp\left(MM_2\int_{s}^{t}\frac{\dot{\sigma}(\tau)}{\lambda}d\tau\right)ds.$$

Proof. Let us consider the set $C(t) = K - d(t)$ for every $t \in [T_0, T]$. It is clear that $C(t)$ is convex so satisfies (H_1). Also, since d is absolutely continuous with $\dot{d} \in L^1(T_0, T)$, $C(t)$ satisfies (H_2) with $\zeta(t) = d(t)$. Due to Assumption 9.2, the hypothesis (H_4) is obviously satisfied. Therefore, the result follows from Theorem 6.1. □

Now we consider the so called gradient complementarity dynamical systems (GCS) which correspond to the particular case where $g \equiv [Dh]^*$. In order to obtain the existence of solutions we will make the following assumption:

Assumption 9.4. The mapping h is differentiable with uniformly continuous derivative Dh, $g \equiv [Dh]^*$ and there exists $k > 0$ such that

$$\mathbb{B}_Y \subset Dh(x)k\mathbb{B}_X - K \quad \text{for all } x \in h^{-1}(K).$$

Under this assumption, we transform (55) into a perturbed sweeping process from which we obtain the existence from Corollary 6.3. Indeed, consider the following identity:

$$I_{h^{-1}(K-d(t))}(x) = I_K(h(x)+d(t)) \quad \text{for all } x \in X, t \in [T_0, T].$$

Then, this inequality and Assumption 9.4 allow us to apply the chain rule for nonsmooth functions [42, Theorem 3.41] to obtain

$$N\left(h^{-1}(K-d(t)), x\right) = [Dh(x)]^* N(K-d(t), h(x)).$$

Therefore, (55) can be rewritten as

$$-\dot{x}(t) \in -F(t, x(t)) + N\left(h^{-1}(K-d(t)), x(t)\right).$$

Next, due to Corollary 3.15, the family $(h^{-1}(K-d(t)))_{t\in[T_0,T]}$ is equi-uniformly subsmooth, thus positively α_0-far for some $\alpha_0 < 1$, and the set-valued map $C(t) = h^{-1}(K-d(t))$ satisfies

$$|d(x, C(t)) - d(x, C(s))| \leq k'|d(t) - d(s)| \quad \text{for all } x \in X \text{ and } s, t \in [T_0, T],$$

with $k' > k$. Therefore, from Corollary 6.3 we get the following result which extends [14, Theorem 1] and [15, Theorem 4.3].

Theorem 9.5. *In addition to the Assumptions* 9.1 *and* 9.4*, assume that F satisfies (H_1^F), (H_2^F) and (H_3^F) and $C(t) := h^{-1}(K-d(t))$ satisfies (H_3). Then, for every x_0 with $h(x_0)+d(T_0) \in K$, there exists at least one absolutely continuous solution x of* (53) *satisfying $x(T_0) = x_0$ and*

$$\|\dot{x}(t)\| \leq \frac{1+\alpha_0^2}{\alpha_0^2}\left(\eta(t) + \dot{\sigma}(t)\int_{T_0}^{t} \frac{\eta(s)}{\alpha_0^2}\exp\left(\frac{\sigma(t)}{\alpha_0^2} - \frac{\sigma(s)}{\alpha_0^2}\right)ds\right) - k'|\dot{d}(t)|$$
$$a.e. \ t \in [T_0, T],$$

where η and σ are functions from $[T_0, T]$ to $\mathbb{R}_+$ defined by the following formulas:

$$\eta(t) = \dot{\sigma}(t)\left(\|x_0\| + \int_{T_0}^{t}\beta(s)ds\right) + \beta(t) + k'|\dot{d}(t)|,$$
$$\sigma(t) = \exp\left(\int_{T_0}^{t}\alpha(s)ds\right).$$

Remark 9.6. Theorem 9.5 extends [15, Theorem 4.3] to Hilbert spaces and weakens the regularity of the function h from the class C^2 to merely differentiable functions with uniformly continuous differential.

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