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Multibody Dynamics with Unilateral Constraints and Dry Friction: How the Contact Dynamics Approach May Handle Coulomb's Law Indeterminacies?

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Dedicated to the memory of Jean Jacques Moreau.

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For the numerical simulation of mechanical systems subjected to unilateral constraints, the *Contact Dynamics* approach has been developped by J. J. Moreau since the mid 80's. The core idea consists in applying a discrete contact law $\mathcal{S}$ to the left "free" velocity computed at each time-step. The mapping $\mathcal{S}$ should of course mimic the behavior of the system in case of contact. But, when dry friction occurs, the dynamics may exhibit indeterminacies of Painlevé's paradoxes type. Then a natural question arises: how a deterministic discrete contact law may handle such phenomena? An answer has been given through numerical experiments by J. J. Moreau and is subtanciated in this paper by introducing the notions of *asymptotic consistency* of the discrete contact law with respect to Coulomb's friction and *asymptotic indeterminacy* of the scheme.

Keywords: Multibody dynamics, unilateral constraints, Coulomb's friction, time-stepping scheme, asymptotic consistency, indeterminacy

1. Introduction

Let us consider a system of rigid bodies with d degrees of freedom. We denote by $q \in \mathbb{R}^d$ its representative point in generalized coordinates. Starting from Lagrange's equations for conservative discrete mechanical systems, the dynamics is described by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right) = \frac{\partial \mathcal{L}}{\partial q} + F(t, q, \dot q), \qquad \mathcal{L} = \frac{1}{2}\langle \dot q, M(q)\dot q\rangle - \mathcal{V}(q), \tag{1}$$

where $M(q)$ is the inertia operator of the system, $\mathcal{V}$ is a smooth convex potential, F allows to take into account external forces which do not derive from a potential, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product of $\mathbb{R}^d$.

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Let us assume now that the motion is subjected to unilateral constraints characterized by the non penetration condition

$$g(q) \leq 0$$

with a smooth (at least C^1) function g. If a contact occurs at some instant $t > 0$, i.e. $g(q(t)) = 0$, a reaction is applied to the system. Moreover, if $\dot{q}(t \pm 0)$ exist, we obtain

$$\langle \dot{q}(t-0), \nabla g(q(t)) \rangle \geq 0, \qquad \langle \dot{q}(t+0), \nabla g(q(t)) \rangle \leq 0 \tag{2}$$

and the velocity may be discontinuous at t. Hence, the convenient mathematical framework is the space of functions of bounded variations for $u = \dot{q}$ and the Ordinary Differential Equation (1) is replaced by a Measure Differential Equation

$$M(q)du = f(t, q, u)dt + r, \tag{3}$$

where du is the Stieljes measure associated to u,

$$f(t, q, u) = -\mathcal{V}'(q) + F(t, q, u) - \frac{1}{2}\left(dM(q) \cdot u\right) u,$$

and r is a $\mathbb{R}^d$-vector valued measure describing the contact forces and contact percussions ([22, 12, 15, 16, 11, 17]).

Let us assume furthermore soft and frictional contact. Then

$$u(t+0) \in T(q(t)) = \left(\mathbb{R}\nabla g(q(t))\right)^{\perp} \quad \text{(soft contact)} \tag{4}$$

for all $t > 0$ such that $g(q(t)) = 0$ and the reaction belongs to the so-called friction cone $C(q(t))$.

In the classical case of isotropic friction, $C(q(t))$ reduces to a revolution cone around the inner normal to the set of admissible configurations at $q(t)$. If the kinetic metric coincides with the Euclidean one (i.e. $M(q) \equiv m\mathrm{Id}_{\mathbb{R}^d}$, $m > 0$), Coulomb's law implies that

$$-u(t+0) \in \mathrm{Proj}\left(T(q(t)), \partial\psi_{C(q(t))}(r)\right) \tag{5}$$

where $\psi_{C(q(t))}$ is the indicatrix function of the friction cone (see [3, 13, 15, 4] for instance). Whenever $u(t+0) \neq u(t-0)$ the Measure Differential Equation (3) yields

$$\begin{aligned} & u(t+0) \in \left(u(t-0) + M^{-1}(q(t))\, C(q(t))\right) \cap T(q(t)) \\ = \ & \left(u(t-0) + C(q(t))\right) \cap T(q(t)). \end{aligned}$$

Hence velocity jumps may occur only in case of collisions and (4)–(5) yields

$$u(t+0) = \mathrm{Proj}\left(0, \left(u(t-0) + C(q(t))\right) \cap T(q(t))\right) \tag{6}$$

for all $t > 0$ such that $g\left(q(t)\right) = 0$ and $\langle u(t-0), \nabla g\left(q(t)\right)\rangle > 0$.

In the general case of a non trivial inertia operator, we still define $u(t+0)$ as the Argmin of the kinetic norm of all the admissible right velocities in case of collisions, i.e. for all $t > 0$ such that $g\left(q(t)\right) = 0$ and $\langle u(t-0), \nabla g\left(q(t)\right)\rangle > 0$

$$u(t+0) = \mathrm{Proj}_{q(t)}\left(0, \left(u(t-0) + M^{-1}\left(q(t)\right) C\left(q(t)\right)\right) \cap T\left(q(t)\right)\right)$$

where $\mathrm{Proj}_{q(t)}$ denotes the projection relatively to the kinetic metric at $q(t)$ ([13, 15]).

But velocity jumps without collision may also occur if $M^{-1}\left(q(t)\right) C\left(q(t)\right) \cap T\left(q(t)\right) \neq \{0\}$. Such a behavior can be easily exhibited when we study the motion of a slender rod in contact at one edge with a rigid horizontal obstacle: the norm of the contact force may tend to $+\infty$ at some instant t_s and the smooth sliding regime ends at t_s with a velocity jump, leading to the famous Painlevé's paradoxes. This kind of phenomenon is known since the end of the 19th century [7, 19, 20] and has been interpreted as an inconsistency of Coulomb's law by several prominent scientists [8, 10, 6, 21]. Nevertheless the possibility of *tangential shocks*, i.e. velocity jumps without collision for tangential contacts, first proposed by L. Lecornu [9] in 1905, has been proved to be in agreement with physical observations by H. Beghin [1] but yields to indeterminacies. This model problem enters the more general framework of one-dimensional friction, which has been fully studied by J. J. Moreau in [15]: whenever $g\left(q(t)\right) = 0$, $\langle u(t-0), \nabla g\left(q(t)\right)\rangle = 0$ and $M^{-1}\left(q(t)\right) C\left(q(t)\right) \cap T\left(q(t)\right) \neq \{0\}$, J. J. Moreau proved that there exists a subset $\mathcal{A}\left(q(t), u(t-0)\right) \subset \mathbb{R}^d$, containing $u(t-0)$ but not reduced to this single point, such that any value of $u(t+0)$ in $\mathcal{A}\left(q(t), u(t-0)\right)$ solves the problem.

For the numerical simulation of multibody dynamics with unilateral constraints, the *Contact Dynamics* approach has been introduced and developped by J. J. Moreau both in the frictionless and frictional cases (see [13, 14, 15] or for a more recent presentation [17]). The core idea of this approach is to avoid any regularization of the non penetration condition and friction law and to build a time-stepping scheme by integrating directly the Measure Differential Equation describing the dynamics on the time-steps $[t_i, t_{i+1}]$ with an impulsional form of the contact law at t_{i+1}. In the frictional case it leads to

$$q_{i+1} = q_i + hu_i, \quad M(q_{i+1})(u_{i+1} - u_i) = hf(t_{i+1}, q_{i+1}, u_i) + r_{i+1}, \quad h = t_{i+1} - t_i$$

with

$$r_{i+1} \in C(q_{i+1})$$

and u_{i+1} satisfying some discrete contact law which has to be introduced.

Starting from the definition of $C(q_{i+1})$ we get

$$u_{i+1} = u_i + hM^{-1}(q_{i+1}) f(t_{i+1}, q_{i+1}, u_i), \quad r_{i+1} = 0$$

if $g(q_{i+1}) < 0$. By interpreting $v_{i+1} \stackrel{\text{def}}{=} u_i + hM^{-1}(q_{i+1})f(t_{i+1}, q_{i+1}, u_i)$ as the left "free" velocity at t_{i+1} ([13, 15]), one may consider that a discrete collision occurs at t_{i+1} if $g(q_{i+1}) \geq 0$ and $\langle v_{i+1}, \nabla g(q_{i+1})\rangle > 0$ and u_{i+1} is defined with a discrete version of (6), i.e.

$$u_{i+1} = \mathrm{Proj}_{q_{i+1}}\left(0, \left(v_{i+1} + M^{-1}(q_{i+1})C(q_{i+1})\right) \cap T(q_{i+1})\right). \tag{7}$$

On the contrary, when $g(q_{i+1}) \geq 0$ and $\langle v_{i+1}, \nabla g(q_{i+1})\rangle < 0$, the left "free" velocity points inward and we may choose $u_{i+1} = v_{i+1}$.

Finally, when $g(q_{i+1}) \geq 0$ and $\langle v_{i+1}, \nabla g(q_{i+1})\rangle = 0$, one may consider that we have a discrete tangential contact at t_{i+1}. If furthermore $M^{-1}(q_{i+1})C(q_{i+1}) \cap T(q_{i+1}) = \{0\}$, we get $u_{i+1} \in v_{i+1} + M^{-1}(q_{i+1})C(q_{i+1}) \cap T(q_{i+1}) = v_{i+1}$. Otherwise there is not any natural choice for u_{i+1} since Coulomb's law leads to indeterminacies.

Of course we could simply propose u_{i+1} as any random value in $\mathcal{A}(q_{i+1}, v_{i+1})$. Such a choice satisfies Coulomb's law at the discrete level but is not well adapted to implementation. Indeed which procedure will make this choice? In order to avoid such difficulties, it seems more convenient to compute u_{i+1} by applying some formula.

In [13, 14, 15], J. J. Moreau proposed $u_{i+1} = v_{i+1}$. It follows that the last three cases can be gathered and we simply get $u_{i+1} = \mathcal{S}(q_{i+1}, v_{i+1})$ with

$$\mathcal{S}(q_{i+1}, v_{i+1}) = v_{i+1} \quad \text{if } g(q_{i+1}) < 0 \text{ or } g(q_{i+1}) \geq 0 \text{ and } \langle v_{i+1}, \nabla g(q_{i+1})\rangle \leq 0,$$
$$\mathcal{S}(q_{i+1}, v_{i+1}) = \mathrm{Proj}_{q_{i+1}}\left(0, \left(v_{i+1} + M^{-1}(q_{i+1})C(q_{i+1})\right) \cap T(q_{i+1})\right) \text{ otherwise.}$$

A convergence result has been established for this scheme by M. Monteiro-Marques in [11] when $M(q) \equiv \mathrm{Id}_{\mathbb{R}^d}$. As already explained, in such a case we have $C\,(q(t)) \cap T\,(q(t)) = M^{-1}\,(q(t))\,C\,(q(t)) \cap T\,(q(t)) = \{0\}$ for all $t > 0$ such that $g\,(q(t)) = 0$ and the basic description of the dynamics implies the continuity of the velocities at t.

But this scheme has also been implemented when $M(q) \not\equiv \mathrm{Id}_{\mathbb{R}^d}$ (see [13, 15, 18] for instance) and then a natural question arises: will this kind of time-stepping algorithms lead at the limit, when the time-step tends to zero, to the indeterminacies of Coulomb's law?

Some answer has been given by J. J. Moreau in [15]: by performing several simulations for various time-steps he showed that the approximated trajectories exhibit the plurality of solutions given by Coulomb's law. Since the time-stepping scheme mimics at the discrete level a deterministic contact law, the limit trajectories will satisfy in case of convergence the following property

$$u(t+0) \in \lim_{\varepsilon \to 0} \left\{\mathcal{S}(q_\varepsilon, u_\varepsilon^-);\ q_\varepsilon \in B(q, \varepsilon),\ u_\varepsilon^- \in B\left(u(t-0), \varepsilon\right)\right\}$$

for any $t > 0$. Hence we may recover the indeterminacies of Coulomb's law if, for all (q, u^-) such that $g(q) = 0$, $M^{-1}(q)C(q) \cap T(q) \neq \{0\}$ and $\langle u^-, \nabla g(q)\rangle = 0$, we have

$$\lim_{\varepsilon \to 0} d_H \left(\mathcal{A}_\varepsilon(q, u^-), \mathcal{A}(q, u^-) \right) = 0 \tag{8}$$

where d_H denotes the Hausdorff distance and

$$\mathcal{A}_\varepsilon(q, u^-) = \left\{ \mathcal{S}(q_\varepsilon, u_\varepsilon^-);\ q_\varepsilon \in B(q, \varepsilon),\ u_\varepsilon^- \in B\left(u^-, \varepsilon\right) \right\}.$$

Reminding that the Hausdorff distance between two subsets A and B of $\mathbb{R}^d$ is defined as

$$d_H(A, B) = \max\left(e(A, B), e(B, A)\right)$$

where

$$e(A, B) = \sup_{a \in A} \operatorname{dist}(a, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\| \quad \text{(the excess of } A \text{ from } B\text{)},$$
$$e(B, A) = \sup_{b \in B} \operatorname{dist}(b, A) = \sup_{b \in B} \inf_{a \in A} \|b - a\| \quad \text{(the excess of } B \text{ from } A\text{)},$$

relation (8) can be decomposed as

$$\lim_{\varepsilon \to 0} \sup \left\{ \operatorname{dist}\left(\mathcal{S}(q_\varepsilon, u_\varepsilon^-), \mathcal{A}(q, u^-) \right);\ q_\varepsilon \in B(q, \varepsilon),\ u_\varepsilon^- \in B\left(u^-, \varepsilon\right) \right\} = 0$$

and

$$\lim_{\varepsilon \to 0} \sup \left\{ \operatorname{dist}\left(\bar{v}, \mathcal{A}_\varepsilon(q, u^-) \right);\ \bar{v} \in \mathcal{A}(q, u^-) \right\} = 0.$$

The former property means that the limit trajectories satisfy Coulomb's law, and can be interpreted as an *asymptotic consistency* of the discrete contact law with respect to Coulomb's friction. The latter property means that the plurality of solutions given by Coulomb's law can be observed as limits of the approximate solutions and can be interpreted as an *asymptotic indeterminacy* of the scheme.

The paper is organized as follows. In the next section we introduce the notations and assumptions used in our mathematical study. Then in Section 3 we will prove that the *asymptotic consistency* property is satisfied by any time-stepping algorithm based on the *Contact Dynamics* approach. In Section 4, we will prove that the *asymptotic indeterminacy* property is not always true and depends on the evolution of the mappings $q_\varepsilon \mapsto C(q_\varepsilon)$ and $q_\varepsilon \mapsto \nabla g(q_\varepsilon)$ in a neighborhood of the contact point. Finally Section 5 will be devoted to some concluding remarks.

2. Notations and assumptions

Let us assume the following regularity properties for M and g:

(H1) the mapping M is of class C^1 from $\mathbb{R}^d$ to the set of symmetric positive definite $d \times d$ matrices;

(H2) the function g belongs to $C^1(\mathbb{R}^d)$, ∇g is locally Lipschitz continuous and does not vanish in a neighbourhood of $\{q \in \mathbb{R}^d; g(q) = 0\}$.

We define the set of admissible configurations by

$$K = \left\{ q \in \mathbb{R}^d;\ g(q) \le 0 \right\}.$$

Then, for all $q \in \partial K$, the inner unit normal to K at q is given by

$$n(q) = -\frac{\nabla g(q)}{\|\nabla g(q)\|}$$

and the tangent plane to K at q is

$$T(q) = \left\{ v \in \mathbb{R}^d; \langle v, n(q) \rangle = 0 \right\}.$$

We consider the general framework of isotropic or anisotropic friction by letting

$$C(q) = \mathbb{R}^+ \left(n(q) + D_1(q) \right) \quad \forall q \in \partial K$$

where $D_1(q)$ satisfies

(H3) for all $q \in \mathbb{R}^d$, $D_1(q)$ is a closed, bounded, convex subset of $\mathbb{R}^d$ such that $0 \in D_1(q)$ and the multivalued mapping $q \mapsto D_1(q)$ is Hausdorff continuous. Furthermore, $\nabla g(q) \notin \mathrm{Span}\left(D_1(q)\right)$ for all $q \in \mathbb{R}^d$ such that $\nabla g(q) \neq 0$.

For any $q \in \partial K$ there exists a neighborhood $V_q = \overline{B}(q, r_q)$ with $r_q > 0$ such that $\|\nabla g(q')\| \ge \frac{1}{2} \|\nabla g(q)\|$ for all $q' \in V_q$ and we define the mapping

$$n : \begin{cases} V_q \to \mathbb{R}^d \\ q' \mapsto n(q') = -\dfrac{\nabla g(q')}{\|\nabla g(q')\|} \end{cases}$$

It follows that we can extend the definition of the tangent plane and the friction cone to V_q by

$$T(q') = \left\{ v \in \mathbb{R}^d;\ \langle v, n(q') \rangle = 0 \right\},$$
$$C(q') = \mathbb{R}^+ \left(n(q') + D_1(q') \right) \quad \forall q' \in V_q \setminus \mathrm{Int}(K).$$

Since V_q is a compact subset of $\mathbb{R}^d$, we infer from assumption (H1) that the kinetic metric is equivalent to the Euclidean one on V_q and there exist $\beta_{V_q} \ge \alpha_{V_q} > 0$ such that

$$\alpha_{V_q} \|v\|^2 \le \|v\|^2_{q'} \stackrel{\text{def}}{=} \langle v, M(q') v \rangle \le \beta_{V_q} \|v\|^2 \quad \forall q' \in V_q,\ \forall v \in \mathbb{R}^d.$$

It follows that

$$\frac{1}{\beta_{V_q}} \|v\|^2 \le \langle v, M^{-1}(q') v \rangle \le \frac{1}{\alpha_{V_q}} \|v\|^2 \quad \forall q' \in V_q,\ \forall v \in \mathbb{R}^d.$$

By using also assumption (H2), and possibly reducing r_q, we obtain that the mappings $q' \mapsto n(q')$, $q' \mapsto M(q')$, $q' \mapsto M^{-1}(q')$ are Lipschitz continuous on V_q and we will denote by L_{n,V_q}, L_{M,V_q} and L_{M^{-1},V_q} their Lipschitz constant on V_q.

Moreover let us define C_q by

$$C_q = \min_{w \in D_1(q)} \|n(q) + w\| .$$

With (H3) we know that $n(q) \notin \mathrm{Span}\,(D_1(q))$ and $D_1(q)$ is compact. Thus $C_q > 0$. Then for all $q' \in V_q$ and for all $w' \in D_1(q')$, we let $w = \mathrm{Proj}\,(D_1(q), w')$ and we have

$$\begin{aligned} C_q \leq & \ \|n(q) + w\| \leq \|n(q') + w'\| + \|n(q) - n(q')\| + \|w - w'\| \\ \leq & \ \|n(q') + w'\| + L_{n,V_q}\|q' - q\| + d_H\left(D_1(q'), D_1(q)\right). \end{aligned}$$

Possibly modifying once again r_q, we obtain

$$L_{n,V_q}\|q' - q\| \leq \frac{C_q}{4}, \quad d_H\left(D_1(q'), D_1(q)\right) \leq \frac{C_q}{4} \quad \forall q' \in V_q,$$

and

$$\min_{w' \in D_1(q')} \|n(q') + w'\| \geq \frac{C_q}{2} \quad \forall q' \in V_q.$$

In the one dimensional friction case, the model problem of Painlevé's example (see for instance [19, 20, 15, 5, 2]) shows that if for some instant $t > 0$ the position and left velocity of the system are given by (q, u^-) with $q \in \partial K$, $\langle u^-, n(q)\rangle = 0$ and $M^{-1}(q)C(q) \cap T(q) \neq \{0\}$, then Coulomb's law yields to indeterminacies and there exists a subset $\mathcal{A}(q, u^-)$ such that any right velocity $u^+ \in \mathcal{A}(q, u^-)$ solves the problem. Since friction is a dissipative phenomenon and $\dim\left(\mathrm{Span}\,(D_1(q))\right) = 1$ (one dimensional friction case), we infer that $\mathcal{A}(q, u^-)$ is included into a segment. Indeed let $D_{1,\mathrm{adm}}(q)$ be defined by

$$D_{1,\mathrm{adm}}(q) = \left\{w \in D_1(q);\ \left\langle n(q), M^{-1}(q)\left(n(q) + w\right)\right\rangle = 0\right\}.$$

Then $M^{-1}(q)C(q) \cap T(q) \neq \{0\}$ implies that $D_{1,\mathrm{adm}}(q) \neq \emptyset$. Let $w \in D_{1,\mathrm{adm}}(q)$. We have $w \neq 0$ and $\mathbb{R}w \subset \mathrm{Span}\,(D_1(q))$. Hence $\mathbb{R}w = \mathrm{Span}\,(D_1(q))$ and, for any $w' \in D_1(q) \setminus \{w\}$, there exists $t \in \mathbb{R} \setminus \{1\}$ such that $w' = tw$. It follows that

$$\left\langle n(q), M^{-1}(q)\left(n(q) + w'\right)\right\rangle = (1 - t)\left\langle n(q), M^{-1}(q)n(q)\right\rangle \neq 0.$$

We may conclude that $D_{1,\mathrm{adm}}(q)$ is reduced to a single point, that we denote as $\tilde{w}$. Moreover $u^+ \in \left(u^- + M^{-1}(q)C(q)\right) \cap T(q)$, so

$$u^+ = u^- + \lambda M^{-1}(q)\left(n(q) + \tilde{w}\right), \quad \lambda \geq 0$$

and the dissipativity of Coulomb's friction yields

$$\|u^+\|_q \leq \|u^-\|_q,$$

i.e.

$$0 \leq \lambda \leq \frac{\max\left(-2\langle u^-, \tilde{w}\rangle, 0\right)}{\langle n(q) + \tilde{w}, M^{-1}(q)\left(n(q) + \tilde{w}\right)\rangle}.$$

But a more precise study has been performed by J. J. Moreau in [15] and he showed that

$$\mathcal{A}(q,u^-) = \begin{cases} \{u^-, \tilde{u}\} & \text{if } \min_{w\in D_1(q)} \langle n(q), M^{-1}(q)\,(n(q)+w)\rangle < 0, \\ [u^-, \tilde{u}] & \text{if } \min_{w\in D_1(q)} \langle n(q), M^{-1}(q)\,(n(q)+w)\rangle = 0 \end{cases}$$

with

$$\tilde{u} = \mathrm{Proj}_q\left(0, \left(u^- + M^{-1}(q)C(q)\right) \cap T(q)\right).$$

Let us consider now a time-stepping scheme constructed by using the *Contact Dynamics* approach. Then the right discrete velocity at the end of each time step $[t_i, t_{i+1}]$ is given by some formula

$$u_{i+1} = \mathcal{S}(q_{i+1}, v_{i+1})$$

where the mapping $\mathcal{S}$ is a deterministic discrete contact law satisfying Coulomb's friction in the following sense

$$\begin{aligned} \mathcal{S}(q_{i+1}, v_{i+1}) = v_{i+1} \ \text{ if } g(q_{i+1}) < 0, \\ \mathcal{S}(q_{i+1}, v_{i+1}) = v_{i+1} \ \text{ if } g(q_{i+1}) \geq 0 \text{ and } \langle v_{i+1}, n(q_{i+1})\rangle > 0, \\ \mathcal{S}(q_{i+1}, v_{i+1}) = \tilde{u}_{i+1} \ \text{ if } g(q_{i+1}) \geq 0 \text{ and } \langle v_{i+1}, n(q_{i+1})\rangle < 0, \end{aligned} \tag{9}$$

with

$$\tilde{u}_{i+1} = \mathrm{Proj}_{q_{i+1}}\left(0, \left(v_{i+1} + M^{-1}(q_{i+1})C(q_{i+1})\right) \cap T(q_{i+1})\right)$$

and, in case of discrete tangential contact, i.e. if $g(q_{i+1}) \geq 0$ and $\langle v_{i+1}, n(q_{i+1})\rangle = 0$

$$\begin{aligned} \mathcal{S}(q_{i+1}, v_{i+1}) = v_{i+1} \ \text{ if } M^{-1}(q_{i+1})C(q_{i+1}) \cap T(q_{i+1}) = \{0\}, \\ \mathcal{S}(q_{i+1}, v_{i+1}) \in \mathcal{A}(q_{i+1}, v_{i+1}) \ \text{ if } M^{-1}(q_{i+1})C(q_{i+1}) \cap T(q_{i+1}) \neq \{0\}. \end{aligned} \tag{10}$$

Motivated by the study of the asymptotic properties of the discrete contact law in the vicinity of frictional catastrophes, we will assume from now on that (q, u^-) is such that $g(q) = 0$, $\langle u^-, n(q)\rangle = 0$ and $M^{-1}(q)C(q) \cap T(q) \neq \{0\}$. We will assume furthermore that $\dim(\mathrm{Span}(D_1(q))) = 1$ and that, possibly reducing r_q we have also $\dim(\mathrm{Span}(D_1(q'))) = 1$ for all $q' \in V_q = \overline{B}(q, r_q)$.

In the next Section we prove that any mapping $\mathcal{S}$ satisfying (9)–(10) is *asymptotically consistent* with respect to Coulomb's law, i.e.

$$\lim_{\varepsilon\to 0} \sup\left\{\mathrm{dist}\left(\mathcal{S}(q_\varepsilon, u_\varepsilon^-), \mathcal{A}(q,u^-)\right);\ q_\varepsilon \in B(q,\varepsilon), u_\varepsilon^- \in B\left(u^-, \varepsilon\right)\right\} = 0. \tag{11}$$

3. Asymptotic consistency

Let us prove (11) by a contradiction argument. Hence let us assume that there exists $\delta > 0$ such that for all $\varepsilon' > 0$ there exists $\varepsilon \in (0, \varepsilon')$ such that $e\left(\mathcal{A}_\varepsilon(q, u^-), \mathcal{A}(q, u^-)\right) > \delta$, i.e.

$$\sup \left\{ \text{dist}\left(\mathcal{S}(q_\varepsilon, u_\varepsilon^-), \mathcal{A}(q, u^-)\right);\ q_\varepsilon \in B(q, \varepsilon),\ u_\varepsilon^- \in B\left(u^-, \varepsilon\right) \right\} > \delta. \tag{12}$$

First we observe that, for all $\varepsilon > 0$, there exists $q_\varepsilon \in B(q, \varepsilon) \cap \text{Int}(K)$ and then, for any $u_\varepsilon^- \in B(u^-, \varepsilon)$, we have $\mathcal{S}(q_\varepsilon, u_\varepsilon^-) = u_\varepsilon^-$. It follows that

$$\text{dist}\left(\mathcal{S}(q_\varepsilon, u_\varepsilon^-), \mathcal{A}(q, u^-)\right) \leq \|u_\varepsilon^- - u^-\| < \varepsilon.$$

The same inequality holds whenever $\mathcal{S}(q_\varepsilon, u_\varepsilon^-) = u_\varepsilon^-$ and we infer that (12) implies that for all $\varepsilon' \in (0, \delta)$ there exists $\varepsilon \in (0, \varepsilon')$ and $(q_\varepsilon, u_\varepsilon^-) \in B(q, \varepsilon) \times B(u^-, \varepsilon)$ such that

$$\mathcal{S}(q_\varepsilon, u_\varepsilon^-) \neq u_\varepsilon^-, \quad \text{dist}\left(\mathcal{S}(q_\varepsilon, u_\varepsilon^-), \mathcal{A}(q, u^-)\right) \geq \delta.$$

With the definition of $\mathcal{S}(q_\varepsilon, u_\varepsilon^-)$ it follows that $q_\varepsilon \notin \text{Int}(K)$ and either $\langle u_\varepsilon^-, n(q_\varepsilon) \rangle < 0$ or $\langle u_\varepsilon^-, n(q_\varepsilon) \rangle = 0$ and $M^{-1}(q_\varepsilon) C(q_\varepsilon) \cap T(q_\varepsilon) \neq \{0\}$. In both cases $\mathcal{S}(q_\varepsilon, u_\varepsilon^-)$ belongs to the segment $[u_\varepsilon^-, \tilde{u}_\varepsilon]$ with

$$\tilde{u}_\varepsilon = \text{Proj}_{q_\varepsilon}\left(0, \left(u_\varepsilon^- + M^{-1}(q_\varepsilon) C(q_\varepsilon)\right) \cap T(q_\varepsilon)\right).$$

Hence we can build a sequence $(\varepsilon_n)_{n \geq 1}$ decreasing to zero such that, for all $n \geq 1$, we have

$$\varepsilon_n \in (0, \min(r_q, \delta)), \quad q_{\varepsilon_n} \notin \text{Int}(K), \quad \left\langle u_{\varepsilon_n}^-, n(q_{\varepsilon_n}) \right\rangle \leq 0 \tag{13}$$

and $u_{\varepsilon_n}^+ \stackrel{\text{def}}{=} \mathcal{S}(q_{\varepsilon_n}, u_{\varepsilon_n}^-)$ satisfies

$$u_{\varepsilon_n}^+ \in [u_{\varepsilon_n}^-, \tilde{u}_{\varepsilon_n}] \setminus \{u_{\varepsilon_n}^-\}, \quad \text{dist}\left(u_{\varepsilon_n}^+, \mathcal{A}(q, u^-)\right) \geq \delta. \tag{14}$$

Let $n \geq 1$. We can decompose $\tilde{u}_{\varepsilon_n}$ as follows

$$\tilde{u}_{\varepsilon_n} = u_{\varepsilon_n}^- + \lambda_{\varepsilon_n} M^{-1}(q_{\varepsilon_n}) \left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right),$$

with $w_{\varepsilon_n} \in D_1(q_{\varepsilon_n})$ and $\lambda_{\varepsilon_n} > 0$. Now let us define $\mu_{\varepsilon_n} \geq 0$ by

$$\mu_{\varepsilon_n} = \frac{-\left\langle u_{\varepsilon_n}^-, n(q_{\varepsilon_n}) \right\rangle}{\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n}) n(q_{\varepsilon_n}) \rangle}.$$

By construction we have

$$u_{\varepsilon_n}^- + \mu_{\varepsilon_n} M^{-1}(q_{\varepsilon_n}) n(q_{\varepsilon_n}) \in \left(u_{\varepsilon_n}^- + M^{-1}(q_{\varepsilon_n}) C(q_{\varepsilon_n})\right) \cap T(q_{\varepsilon_n})$$

and thus

$$\begin{aligned}\|\tilde{u}_{\varepsilon_n}\|_{q_{\varepsilon_n}} &\le \left\|u^-_{\varepsilon_n} + \mu_{\varepsilon_n} M^{-1}(q_{\varepsilon_n}) n(q_{\varepsilon_n})\right\|_{q_{\varepsilon_n}} \\ &\le \|u^-_{\varepsilon_n}\|_{q_{\varepsilon_n}} + \mu_{\varepsilon_n} \left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n}) n(q_{\varepsilon_n})\right\rangle^{1/2} \\ &\le \|u^-_{\varepsilon_n}\|_{q_{\varepsilon_n}} + \frac{\left|\left\langle u^-_{\varepsilon_n}, n(q_{\varepsilon_n})\right\rangle\right|}{\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n}) n(q_{\varepsilon_n})\rangle^{1/2}} \le 2\sqrt{\beta_V}\|u^-_{\varepsilon_n}\|.\end{aligned}$$

We infer that

$$\begin{gathered}\lambda_{\varepsilon_n} \|n(q_{\varepsilon_n}) + w_{\varepsilon_n}\| = \left\|M(q_{\varepsilon_n})(\tilde{u}_{\varepsilon_n} - u^-_{\varepsilon_n})\right\| \\ \le \sqrt{\beta_V}\|\tilde{u}_{\varepsilon_n} - u^-_{\varepsilon_n}\|_{q_{\varepsilon_n}} \le 3\beta_V \|u^-_{\varepsilon_n}\|.\end{gathered}$$

Since $q_{\varepsilon_n} \in B(q, \varepsilon_n) \subset V_q$, we have also

$$\frac{C_q}{2} \le \|n(q_{\varepsilon_n}) + w_{\varepsilon_n}\|$$

and we may conclude that

$$0 < \lambda_{\varepsilon_n} \le \frac{6\beta_V}{C_q}\|u^-_{\varepsilon_n}\| \le \frac{6\beta_V}{C_q}\left(\|u^-\| + \varepsilon_n\right) \quad \forall n \ge 1.$$

Next we define $u_{\varepsilon_n} = u^-_{\varepsilon_n} + \lambda_{\varepsilon_n} M^{-1}(q)\left(n(q) + \bar{w}_{\varepsilon_n}\right)$ with $\bar{w}_{\varepsilon_n} = \mathrm{Proj}\left(D_1(q), w_{\varepsilon_n}\right)$. For all $n \ge 1$

$$\begin{aligned}&\tilde{u}_{\varepsilon_n} - u_{\varepsilon_n} \\ &= \lambda_{\varepsilon_n}\left(M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right) - M^{-1}(q)\left(n(q) + \bar{w}_{\varepsilon_n}\right)\right) \\ &= \lambda_{\varepsilon_n}\left(\left(M^{-1}(q_{\varepsilon_n}) - M^{-1}(q)\right)\left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right) + M^{-1}(q)\left(n(q_{\varepsilon_n}) - n(q)\right)\right. \\ &\qquad \left. + M^{-1}(q)\left(w_{\varepsilon_n} - \bar{w}_{\varepsilon_n}\right)\right).\end{aligned}$$

So

$$\|\tilde{u}_{\varepsilon_n} - u_{\varepsilon_n}\| \le \lambda_{\varepsilon_n}\phi(q_{\varepsilon_n})$$

where the mapping ϕ is defined on V_q by

$$\begin{aligned}\phi(q') = L_{M^{-1},V_q}\|q' - q\|&\left(1 + d_H\left(D_1(q'), D_1(q)\right) + \max_{w \in D_1(q)}\|w\|\right) \\ &+ \left\|M^{-1}(q)\right\|\left(L_{n,V_q}\|q' - q\| + d_H\left(D_1(q'), D_1(q)\right)\right) \quad \forall q' \in V_q.\end{aligned} \tag{15}$$

The sequence $(\lambda_{\varepsilon_n})_{n \ge 1}$ being bounded, assumption (H3) implies that

$$\lim_{n \to +\infty}\|\tilde{u}_{\varepsilon_n} - u_{\varepsilon_n}\| = 0.$$

It follows that $(\tilde{u}_{\varepsilon_n})_{n \ge 1}$ is bounded.

Moreover, reminding that $D_1(q)$ is a compact subset of $\mathbb{R}^d$, possibly extracting a subsequence still denoted $(\varepsilon_n)_{n\geq 1}$, we have the following convergences

$$\lambda_{\varepsilon_n} \to \lambda_* \geq 0, \quad \bar{w}_{\varepsilon_n} \to \bar{w}_* \in D_1(q)$$

and

$$u_{\varepsilon_n} \to u_* \stackrel{\text{def}}{=} u^- + \lambda_* M^{-1}(q)\,(n(q) + \bar{w}_*)\,.$$

Hence

$$\tilde{u}_{\varepsilon_n} \to u_*. \tag{16}$$

Furthermore, $\tilde{u}_{\varepsilon_n} \in T(q_{\varepsilon_n})$, i.e. $\langle \tilde{u}_{\varepsilon_n}, n(q_{\varepsilon_n})\rangle = 0$ for all $n \geq 1$. By using the continuity of the mapping $q' \mapsto n(q')$, we infer that

$$\langle u_*, n(q)\rangle = 0. \tag{17}$$

We distinguish now two cases.

Case 1: $\min_{w\in D_1(q)} \langle n(q), M^{-1}(q)\,(n(q)+w)\rangle < 0$. Then possibly reducing r_q, the same property holds in V_q, i.e.

$$\min_{w'\in D_1(q')} \big\langle n(q'), M^{-1}(q')\,(n(q')+w')\big\rangle < 0 \quad \forall q' \in V_q.$$

Indeed, let $\bar{w} \in D_1(q)$ be such that $\langle n(q), M^{-1}(q)\,(n(q)+\bar{w})\rangle < 0$. For any $q' \in V_q$ we define $\bar{w}' = \operatorname{Proj}(D_1(q'), \bar{w})$. Then we have

$$\begin{aligned}
&\big|\big\langle n(q'), M^{-1}(q')\,(n(q')+\bar{w}')\big\rangle - \big\langle n(q), M^{-1}(q)\,(n(q)+\bar{w})\big\rangle\big| \\
\leq\ & \|n(q') - n(q)\|\,\big\|M^{-1}(q)\,(n(q)+\bar{w})\big\| \\
&\quad + \big\|M^{-1}(q')\,(n(q')+\bar{w}') - M^{-1}(q)\,(n(q)+\bar{w})\big\| \\
\leq\ & L_{n,V_q}\|q'-q\|\,\big\|M^{-1}(q)\big\|\left(1 + \max_{w\in D_1(q)}\|w\|\right) + \phi(q').
\end{aligned}$$

Hence

$$\begin{aligned}
&\min_{w'\in D_1(q')} \big\langle n(q'), M^{-1}(q')\,(n(q')+w')\big\rangle \leq \big\langle n(q'), M^{-1}(q')\,(n(q')+\bar{w}')\big\rangle \\
\leq\ & \big\langle n(q), M^{-1}(q)\,(n(q)+\bar{w})\big\rangle \\
&\quad + L_{n,V_q}\|q'-q\|\,\big\|M^{-1}(q)\big\|\left(1 + \max_{w\in D_1(q)}\|w\|\right) + \phi(q')
\end{aligned}$$

and assumption (H3) gives the announced result.

It follows that $\mathcal{A}(q,u^-) = \{u^-, \tilde{u}\}$. Moreover for all $q_\varepsilon \in V_q$ such that $g(q_\varepsilon) \geq 0$ we have $M^{-1}(q_\varepsilon)C(q_\varepsilon) \cap T(q_\varepsilon) \neq \{0\}$ and for all u_ε^- such that $\langle n(q_\varepsilon), u_\varepsilon^-\rangle = 0$ we have $\mathcal{S}(q_\varepsilon, u_\varepsilon^-) \in \mathcal{A}(q_\varepsilon, u_\varepsilon^-) = \{u_\varepsilon^-, \tilde{u}_\varepsilon\}$. Then, with (13)–(14) we obtain that $\mathcal{S}(q_{\varepsilon_n}, u_{\varepsilon_n}^-) = u_{\varepsilon_n}^+ = \tilde{u}_{\varepsilon_n}$ for all $n \geq 1$ and, possibly extracting a subsequence, $\lim_{n\to+\infty} u_{\varepsilon_n}^+ = u_*$. In the following Proposition we prove that $u_* = \tilde{u}$ which gives a contradiction and allows us to conclude that (11) is satisfied.

Proposition 3.1. *We have* $u_* = \tilde{u}$.

Proof. We already have obtained that $u_* \in (u^- + M^{-1}(q)C(q)) \cap T(q)$. Hence it remains only to prove that

$$\langle -u_*, v - u_* \rangle_q \leq 0 \quad \forall v \in \big(u^- + M^{-1}(q)C(q)\big) \cap T(q)$$

i.e.

$$\big\langle -u_*, M(q)(u^- - u_*) + \lambda \tilde{w} \big\rangle \leq 0 \quad \forall \lambda \geq 0,\ \forall \tilde{w} \in D_{1,\mathrm{adm}}(q),$$

where we recall that

$$D_{1,\mathrm{adm}}(q) = \big\{ w \in D_1(q);\ \big\langle n(q), M^{-1}(q)\left(n(q) + w\right)\big\rangle = 0 \big\}\,.$$

Since $\min_{w \in D_1(q)} \langle n(q), M^{-1}(q)\left(n(q) + w\right)\rangle < 0$, there exists $\bar{w} \in D_1(q)$ such that $\langle n(q), M^{-1}(q)\left(n(q) + \bar{w}\right)\rangle < 0$ and we let

$$\bar{C} = -\big\langle n(q), M^{-1}(q)\left(n(q) + \bar{w}\right)\big\rangle > 0.$$

Let $\rho \in (0,1)$ and $\tilde{w} \in D_{1,\mathrm{adm}}(q)$. We define $w_\rho = \rho \bar{w} + (1-\rho)\tilde{w}$. Reminding that $D_1(q)$ is a convex subset of $\mathbb{R}^d$, we have $w_\rho \in D_1(q)$. Furthermore

$$\big\langle n(q), M^{-1}(q)\left(n(q) + w_\rho\right)\big\rangle = \rho \big\langle n(q), M^{-1}(q)\left(n(q) + \bar{w}\right)\big\rangle = -\rho\bar{C} < 0.$$

For any $n \geq 1$, we denote by w_{ρ,ε_n} the projection of w_ρ on $D_1(q_{\varepsilon_n})$, i.e. $w_{\rho,\varepsilon_n} = \mathrm{Proj}\left(D_1(q_{\varepsilon_n}), w_\rho\right)$. Then

$$\begin{aligned}
&\big\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\rho,\varepsilon_n}\right)\big\rangle \\
= \ &\big\langle n(q), M^{-1}(q)\left(n(q) + w_\rho\right)\big\rangle \\
&\quad + \big\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\rho,\varepsilon_n}\right) - M^{-1}(q)\left(n(q) + w_\rho\right)\big\rangle \\
&\quad + \big\langle n(q_{\varepsilon_n}) - n(q), M^{-1}(q)\left(n(q) + w_\rho\right)\big\rangle \\
\leq \ & -\rho\bar{C} + \big\| M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\rho,\varepsilon_n}\right) - M^{-1}(q)\left(n(q) + w_\rho\right)\big\| \\
&\quad + L_{n,V_q}\|q_{\varepsilon_n} - q\|\, \big\|M^{-1}(q)\big\| \left(1 + \max_{w \in D_1(q)} \|w\|\right).
\end{aligned}$$

By using the definition of ϕ introduced in (15), we get

$$\begin{aligned}
&\big\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\rho,\varepsilon_n}\right)\big\rangle \\
\leq \ & -\rho\bar{C} + \phi(q_{\varepsilon_n}) + L_{n,V_q}\|q_{\varepsilon_n} - q\|\, \big\|M^{-1}(q)\big\| \left(1 + \max_{w \in D_1(q)} \|w\|\right).
\end{aligned}$$

We infer that there exists $N_\rho \geq 1$ such that, for all $n \geq N_\rho$ we have

$$\big\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\rho,\varepsilon_n}\right)\big\rangle \leq -\frac{\rho\bar{C}}{2}.$$

Now let $\lambda \geq 0$. For all $n \geq N_\rho$, there exists $\mu_{\lambda,\varepsilon_n} \geq 0$ such that

$$\big\langle u^-_{\varepsilon_n}, n(q_{\varepsilon_n})\big\rangle + \mu_{\lambda,\varepsilon_n}\big\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n}) n(q_{\varepsilon_n})\big\rangle \\ + \lambda\big\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\rho,\varepsilon_n}\right)\big\rangle = 0.$$

With (H3) we know that $0 \in D_1(q_{\varepsilon_n})$ for all $n \geq N_\rho$. Hence $u^-_{\varepsilon_n} + M^{-1}(q_{\varepsilon_n})((\mu_{\lambda,\varepsilon_n} + \lambda) n(q_{\varepsilon_n}) + \lambda w_{\rho,\varepsilon_n})$ belongs to $\big(u^-_{\varepsilon_n} + M^{-1}(q_{\varepsilon_n}) C(q_{\varepsilon_n})\big) \cap T(q_{\varepsilon_n})$. It follows that, for all $n \geq N_\rho$ we have

$$\big\langle -\tilde{u}_{\varepsilon_n}, M(q_{\varepsilon_n})(u^-_{\varepsilon_n} - \tilde{u}_{\varepsilon_n}) + \lambda w_{\rho,\varepsilon_n}\big\rangle \leq 0$$

and

$$\big\langle -\tilde{u}_{\varepsilon_n}, M(q_{\varepsilon_n})(u^-_{\varepsilon_n} - \tilde{u}_{\varepsilon_n}) + \lambda w_\rho\big\rangle \\ \leq \langle \tilde{u}_{\varepsilon_n}, \lambda(w_{\rho,\varepsilon_n} - w_\rho)\rangle \leq \lambda \|\tilde{u}_{\varepsilon_n}\| d_H\left(D_1(q_{\varepsilon_n}), D_1(q)\right).$$

Reminding that the sequence $(\tilde{u}_{\varepsilon_n})_{n\geq 1}$ is bounded, we can pass to the limit as n tends to $+\infty$ and we get

$$\big\langle -u_*, M(q)(u^- - u_*) + \lambda w_\rho\big\rangle \leq 0 \quad \forall \lambda \geq 0,\ \forall \tilde{w} \in D_{1,\mathrm{adm}}(q),\ \forall \rho \in (0,1).$$

Finally, we let ρ tend to zero to conclude. □

Case 2: $\min_{w\in D_1(q)} \langle n(q), M^{-1}(q)\left(n(q) + w\right)\rangle = 0$. Then $\mathcal{A}(q, u^-) = [u^-, \tilde{u}]$ with

$$\tilde{u} = \mathrm{Proj}_q\left(0, \left(u^- + M^{-1}(q) C(q)\right) \cap T(q)\right),$$

i.e.

$$\tilde{u} = u^- + \tilde{\lambda} M^{-1}(q)\left(n(q) + \tilde{w}\right), \qquad \tilde{\lambda} = \frac{\max\left(-\langle u^-, \tilde{w}\rangle, 0\right)}{\langle n(q) + \tilde{w}, M^{-1}(q)\left(n(q) + \tilde{w}\right)\rangle}$$

and $\tilde{w}$ is the unique vector belonging to $D_{1,\mathrm{adm}}(q)$. It follows that $\bar{v} \in \mathcal{A}(q, u^-)$ if and only if $\bar{v} = u^- + \lambda M^{-1}(q)\left(n(q) + \tilde{w}\right)$ with $0 \leq \lambda \leq \tilde{\lambda}$.

Let us recall that $\mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n}) = u^+_{\varepsilon_n} \in [u^-_{\varepsilon_n}, \tilde{u}_{\varepsilon_n}] \setminus \{u^-_{\varepsilon_n}\}$ for all $n \geq 1$. Since the sequences $(u^-_{\varepsilon_n})_{n\geq 1}$ and $(\tilde{u}_{\varepsilon_n})_{n\geq 1}$ are bounded, we infer that $(u^+_{\varepsilon_n})_{n\geq 1}$ is also bounded. Moreover, for all $n \geq 1$, we can rewrite $u^+_{\varepsilon_n}$ as

$$u^+_{\varepsilon_n} = \gamma_{\varepsilon_n} u^-_{\varepsilon_n} + (1 - \gamma_{\varepsilon_n})\tilde{u}_{\varepsilon_n} \quad \text{with } \gamma_{\varepsilon_n} \in [0, 1).$$

For any limit point u^+_* of $(u^+_{\varepsilon_n})_{n\geq 1}$, there exists a subsequence, still denoted $(\varepsilon_n)_{n\geq 1}$, such that

$$\tilde{u}_{\varepsilon_n} = u^-_{\varepsilon_n} + \lambda_{\varepsilon_n} M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right) \to u_* = u^- + \lambda_* M^{-1}(q)\left(n(q) + \bar{w}_*\right)$$

with

$$\lambda_{\varepsilon_n} \to \lambda_* \geq 0, \qquad w_{\varepsilon_n} \to \bar{w}_* \in D_1(q), \qquad \langle u_*, n(q)\rangle = 0$$

and

$$\gamma_{\varepsilon_n} \to \gamma_* \in [0, 1], \qquad u^+_{\varepsilon_n} \to u^+_* = \gamma_* u^- + (1 - \gamma_*) u_*.$$

Furthermore

Proposition 3.2. *We have $u_* \in [u^-, \tilde{u}]$.*

Proof. Let us distinguish two subcases.

Case 2.1: there exists $\rho > 0$ such that, for all $N \geq 1$, there exists $n \geq N$ such that

$$\left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right)\right\rangle \geq \rho.$$

Then possibly extracting another subsequence, still denoted $(\varepsilon_n)_{n\geq 1}$ we may assume without loss of generality that the previous property holds for all $n \geq 1$. It follows that

$$\tilde{u}_{\varepsilon_n} = u^-_{\varepsilon_n} + \lambda_{\varepsilon_n} M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right)$$

with

$$0 < \lambda_{\varepsilon_n} = \frac{-\left\langle u^-_{\varepsilon_n}, n(q_{\varepsilon_n})\right\rangle}{\left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right)\right\rangle} \leq \frac{1}{\rho}\left|\left\langle u^-_{\varepsilon_n}, n(q_{\varepsilon_n})\right\rangle\right|$$

for all $n \geq 1$. By passing to the limit as n tends to $+\infty$, we get

$$\lim_{n\to+\infty}\left\langle u^-_{\varepsilon_n}, n(q_{\varepsilon_n})\right\rangle = \left\langle u^-, n(q)\right\rangle = 0$$

and $\lim_{n\to+\infty}\lambda_{\varepsilon_n} = \lambda_* = 0$. It follows that $u_* = u^-$.

Case 2.2: for any $\rho > 0$ there exists $N_\rho \geq 1$ such that, for all $n \geq N_\rho$ we have

$$\left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right)\right\rangle < \rho.$$

For all $n \geq 1$ we have

$$\left\langle -\tilde{u}_{\varepsilon_n}, v - \tilde{u}_{\varepsilon_n}\right\rangle_{q_{\varepsilon_n}} \leq 0 \quad \forall v \in \left(u^-_{\varepsilon_n} + M^{-1}(q_{\varepsilon_n})C(q_{\varepsilon_n})\right) \cap T(q_{\varepsilon_n}).$$

We may choose

$$v = u^-_{\varepsilon_n} + \mu_{\varepsilon_n} M^{-1}(q_{\varepsilon_n}) n(q_{\varepsilon_n})$$

with

$$\mu_{\varepsilon_n} = \frac{-\left\langle u^-_{\varepsilon_n}, n(q_{\varepsilon_n})\right\rangle}{\left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n}) n(q_{\varepsilon_n})\right\rangle}$$

and we get

$$\left\langle \tilde{u}_{\varepsilon_n}, w_{\varepsilon_n}\right\rangle \leq 0 \quad \forall n \geq 1.$$

Let us recall the definition of $\bar{w}_{\varepsilon_n}$ as $\bar{w}_{\varepsilon_n} = \operatorname{Proj}\left(D_1(q), w_{\varepsilon_n}\right)$. For all $n \geq 1$ we have

$$\left\langle \tilde{u}_{\varepsilon_n}, \bar{w}_{\varepsilon_n}\right\rangle \leq \left\langle \tilde{u}_{\varepsilon_n}, \bar{w}_{\varepsilon_n} - w_{\varepsilon_n}\right\rangle \leq \|\tilde{u}_{\varepsilon_n}\| d_H\left(D_1(q_{\varepsilon_n}), D_1(q)\right)$$

and

$$\begin{aligned}
&\left|\left\langle n(q), M^{-1}(q)\left(n(q) + \bar{w}_{\varepsilon_n}\right)\right\rangle - \left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right)\right\rangle\right| \\
\leq\ & \|n(q) - n(q_{\varepsilon_n})\|\, \left\|M^{-1}(q)\left(n(q) + \bar{w}_{\varepsilon_n}\right)\right\| \\
&\qquad + \left\|M^{-1}(q)\left(n(q) + \bar{w}_{\varepsilon_n}\right) - M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w_{\varepsilon_n}\right)\right\| \\
\leq\ & L_{n,V_q}\|q - q_{\varepsilon_n}\|\, \left\|M^{-1}(q)\right\|\left(1 + \max_{w\in D_1(q)}\|w\|\right) + \phi(q_{\varepsilon_n})
\end{aligned}$$

where ϕ has been introduced in (15). Now let $\rho > 0$. For any $n \geq N_\rho$ we obtain

$$\begin{aligned}
&\left\langle n(q), M^{-1}(q)\,(n(q) + \bar{w}_{\varepsilon_n})\right\rangle \\
&\leq \rho + L_{n,V_q}\|q - q_{\varepsilon_n}\|\,\left\|M^{-1}(q)\right\| \left(1 + \max_{w\in D_1(q)} \|w\|\right) + \phi(q_{\varepsilon_n}).
\end{aligned}$$

Reminding that $(\bar{w}_{\varepsilon_n})_{n\geq 1}$ converges to $\bar{w}_* \in D_1(q)$ we have

$$\lim_{n\to+\infty} \langle \tilde{u}_{\varepsilon_n}, \bar{w}_{\varepsilon_n}\rangle = \langle u_*, \bar{w}_*\rangle \leq 0$$

and

$$\begin{aligned}
&\lim_{n\to+\infty} \left\langle n(q), M^{-1}(q)\,(n(q) + \bar{w}_{\varepsilon_n})\right\rangle \\
&= \left\langle n(q), M^{-1}(q)\,(n(q) + \bar{w}_*)\right\rangle \leq \rho \quad \forall \rho > 0.
\end{aligned}$$

We infer that $\langle n(q), M^{-1}(q)\,(n(q) + \bar{w}_*)\rangle \leq 0$ and finally $\langle n(q), M^{-1}(q)(n(q) + \bar{w}_*)\rangle = 0$. Thus $\bar{w}_* \in D_{1,\mathrm{adm}}(q) = \{\tilde{w}\}$ and

$$\begin{aligned}
\langle u_*, \bar{w}_*\rangle &= \langle u_*, \tilde{w}\rangle = \langle u^-, \tilde{w}\rangle + \lambda_* \left\langle M^{-1}(q)\,(n(q) + \tilde{w})\,, \tilde{w}\right\rangle \\
&= \langle u^-, \tilde{w}\rangle + \lambda_* \left\langle M^{-1}(q)\,(n(q) + \tilde{w})\,, n(q) + \tilde{w}\right\rangle \leq 0.
\end{aligned}$$

It follows that $\langle u^-, \tilde{w}\rangle \leq 0$ and

$$\langle u_*, \tilde{w}\rangle = (\lambda_* - \tilde{\lambda})\left\langle M^{-1}(q)\,(n(q) + \tilde{w})\,, n(q) + \tilde{w}\right\rangle \leq 0.$$

Hence $\lambda_* \in [0, \tilde{\lambda}]$ and we may conclude that $u_* \in [u^-, \tilde{u}]$. □

The previous result implies that all the limit points of $\left(\mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n}) = u^+_{\varepsilon_n}\right)_{n\geq 1}$ belong to the segment $[u^-, \tilde{u}] = \mathcal{A}(q, u^-)$. Reminding that $(u^+_{\varepsilon_n})_{n\geq 1}$ is bounded we obtain

$$\mathrm{dist}\left(u^+_{\varepsilon_n}, [u^-, \tilde{u}]\right) \to_{n\to+\infty} 0$$

which gives a contradiction with (14).

4. Asymptotic indeterminacy of the scheme

Let us study now the excess of $\mathcal{A}(q, u^-)$ from $\mathcal{A}_\varepsilon(q, u^-)$. Once again we distinguish two cases.

Case 1: $\min_{w\in D_1(q)} \left(\langle n(q), M^{-1}(q)\,(n(q) + w)\rangle\right) < 0$. Then

Proposition 4.1. *We have*

$$\lim_{\varepsilon\to 0} e\left(\mathcal{A}(q, u^-), \mathcal{A}_\varepsilon(q, u^-)\right) = \lim_{\varepsilon\to 0} \sup\left\{\mathrm{dist}\left(\bar{v}, \mathcal{A}_\varepsilon(q, u^-)\right);\ \bar{v} \in \mathcal{A}(q, u^-)\right\} = 0.$$

Proof. Since $\mathcal{A}(q,u^-) = \{u^-, \tilde{u}\}$, we only need to prove that

$$\lim_{\varepsilon\to 0} \mathrm{dist}\big(u^-, \mathcal{A}_\varepsilon(q,u^-)\big) = 0$$

and

$$\lim_{\varepsilon\to 0} \mathrm{dist}\big(\tilde{u}, \mathcal{A}_\varepsilon(q,u^-)\big) = 0.$$

The former property is almost obvious. Indeed for all $\varepsilon > 0$, there exists $q_\varepsilon \in B(q,\varepsilon) \cap \mathrm{Int}(K)$ and then, for any $u_\varepsilon^- \in B(u^-,\varepsilon)$, we have $\mathcal{S}(q_\varepsilon, u_\varepsilon^-) = u_\varepsilon^-$. It follows that

$$\mathrm{dist}\big(u^-, \mathcal{A}_\varepsilon(q,u^-)\big) \le \big\|u^- - \mathcal{S}(q_\varepsilon, u_\varepsilon^-)\big\| = \|u^- - u_\varepsilon^-\| < \varepsilon.$$

In order to check that the latter property holds, we observe that for all $\varepsilon \in (0, r_q)$, there exists $(q_\varepsilon, u_\varepsilon^-) \in B(q,\varepsilon) \times B(u^-,\varepsilon)$ such that $q_\varepsilon \notin \mathrm{Int}(K)$ and $\langle u_\varepsilon^-, n(q_\varepsilon)\rangle < 0$. Indeed we may consider $q_\varepsilon = q$ and $u_\varepsilon^- = u^- - \frac{\varepsilon}{2\|M^{-1}(q)n(q)\|} M^{-1}(q)n(q)$. Then, by definition of $\mathcal{S}$ we obtain $\mathcal{S}(q_\varepsilon, u_\varepsilon^-) = \tilde{u}_\varepsilon$. By using the same arguments as in Section 3, we know that $(\tilde{u}_\varepsilon)_{r_q > \varepsilon > 0}$ is a bounded sequence and all its convergent subsequences admit $\tilde{u}$ for limit (see Proposition 3.1). It follows that

$$\lim_{\varepsilon\to 0} \tilde{u}_\varepsilon = \tilde{u}$$

and

$$\lim_{\varepsilon\to 0} \mathrm{dist}\big(\tilde{u}, \mathcal{A}_\varepsilon(q,u^-)\big) \le \lim_{\varepsilon\to 0} \|\tilde{u} - \tilde{u}_\varepsilon\| = 0. \qquad \square$$

Case 2: $\min_{w\in D_1(q)} \langle n(q), M^{-1}(q)\,(n(q)+w)\rangle = 0$. Then $\mathcal{A}(q,u^-) = [u^-, \tilde{u}]$. If $\tilde{u} = u^-$ we get immediately that $e(\mathcal{A}(q,u^-), \mathcal{A}_\varepsilon(q,u^-))$ tends to zero as ε tends to zero. Indeed, for any $\varepsilon > 0$ we may choose $q_\varepsilon \in B(q,\varepsilon) \cap \mathrm{Int}(K)$ and for any $u_\varepsilon^- \in B(u^-,\varepsilon)$ we have $\mathcal{S}(q_\varepsilon, u_\varepsilon^-) = u_\varepsilon^-$ so

$$e\big(\mathcal{A}(q,u^-), \mathcal{A}_\varepsilon(q,u^-)\big) = \mathrm{dist}\big(u^-, \mathcal{A}_\varepsilon(q,u^-)\big) \le \big\|u^- - \mathcal{S}(q_\varepsilon, u_\varepsilon^-)\big\| < \varepsilon.$$

Otherwise, whenever $\tilde{u} \ne u^-$, the study of $e(\mathcal{A}(q,u^-), \mathcal{A}_\varepsilon(q,u^-))$ is much more involved and depends strongly on the evolution of the mappings $q' \mapsto n(q')$ and $q' \mapsto D_1(q')$ in a neighborhood of the contact point. More precisely, we will prove that

Proposition 4.2. *Let us assume that*

(H4) *for any $\varepsilon \in (0, r_q)$ there exists $q_\varepsilon \in B(q,\varepsilon) \setminus (\mathrm{Int}(K) \cup \{q\})$ such that*

$$\min_{w\in D_1(q_\varepsilon)} \big\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon)+w)\big\rangle > 0.$$

Then for any $\bar{v} \in \mathcal{A}(q,u^-) = [u^-, \tilde{u}]$, there exists at least a sequence $(\varepsilon_n)_{n\ge 1}$, decreasing to zero and a sequence $(q_{\varepsilon_n}, u_{\varepsilon_n}^-)_{n\ge 1}$ such that $(q_{\varepsilon_n}, u_{\varepsilon_n}^-) \in B(q,\varepsilon_n) \times B(u^-,\varepsilon_n)$ for all $n \ge 1$ and $(\mathcal{S}(q_{\varepsilon_n}, u_{\varepsilon_n}^-))_{n\ge 1}$ converges to $\bar{v}$.

On the contrary, if (H4) *does not hold and if the restriction of the mapping* $\mathcal{S}$ *to* $\{(q', u') \in (B(q, r_q) \setminus \mathrm{Int}(K)) \times \mathbb{R}^d;\ \langle u', n(q') \rangle = 0\}$ *is continuous at* (q, u^-)*, then for any* $\bar{v} \in [u^-, \tilde{u}] \setminus \{u^-, \tilde{u}, \mathcal{S}(q, u^-)\}$*,* $\mathrm{dist}\,(\bar{v}, \mathcal{A}_\varepsilon(q, u^-))$ *does not tend to zero as* ε *tends to zero.*

Proof. Let us assume that (H4) holds and let us prove that, for any $\bar{v} \in \mathcal{A}(q, u^-) = [u^-, \tilde{u}]$, there exists at least a sequence $(\varepsilon_n)_{n \geq 1}$, decreasing to zero and a sequence $(q_{\varepsilon_n}, u^-_{\varepsilon_n})_{n \geq 1}$ such that $(q_{\varepsilon_n}, u^-_{\varepsilon_n}) \in B(q, \varepsilon_n) \times B(u^-, \varepsilon_n)$ for all $n \geq 1$ and $(\mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n}))_{n \geq 1}$ converges to $\bar{v}$.

If $\bar{v} = u^-$ the result is obvious: for any sequence $(\varepsilon_n)_{n \geq 1}$ decreasing to zero, we may choose $q_{\varepsilon_n} \in B(q, \varepsilon_n) \cap \mathrm{Int}(K)$ and $u^-_{\varepsilon_n} \in B(u^-, \varepsilon_n)$.

Now, let us assume that $\tilde{u} \neq u^-$. It follows that

$$\tilde{u} = u^- + \tilde{\lambda} M^{-1}(q)\,(n(q) + \tilde{w}), \qquad \tilde{\lambda} = \frac{\max\,(-\langle u^-, \tilde{w} \rangle, 0)}{\langle n(q) + \tilde{w}, M^{-1}(q)\,(n(q) + \tilde{w}) \rangle}$$

with $\tilde{\lambda} > 0$ and $D_{1,\mathrm{adm}} = \{\tilde{w}\}$. Hence $\langle u^-, \tilde{w} \rangle < 0$. Let $\bar{v} \in [u^-, \tilde{u}] \setminus \{u^-\}$, i.e.

$$v = u^- + \lambda M^{-1}(q)\,(n(q) + \tilde{w}), \quad 0 < \lambda \leq \tilde{\lambda}.$$

Since (H4) holds we know that for all $\varepsilon \in (0, r_q)$ there exists $q_\varepsilon \in B(q, \varepsilon) \setminus (\mathrm{Int}(K) \cup \{q\})$ such that

$$\min_{w \in D_1(q_\varepsilon)} \big\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon) + w) \big\rangle > 0.$$

Let us begin with a technical result.

Lemma 4.3. *Possibly reducing* r_q*,* $\mathrm{Argmin}_{w \in D_1(q_\varepsilon)} \langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon) + w) \rangle$ *is reduced to a single point for all* $q_\varepsilon \in B(q, \varepsilon)$ *with* $\varepsilon \in (0, r_q)$*, and we denote this single point as* $\hat{w}_\varepsilon$*. Furthermore*

$$\lim_{\varepsilon \to 0} \hat{w}_\varepsilon = \tilde{w}.$$

Proof. Let $\varepsilon \in (0, r_q)$ and $q_\varepsilon \in B(q, \varepsilon)$. We define $\tilde{w}'_\varepsilon = \mathrm{Proj}(D_1(q_\varepsilon), \tilde{w})$. With assumptions (H1)–(H3) we have $\lim_{\varepsilon \to 0} \tilde{w}'_\varepsilon = \tilde{w}$ and

$$\lim_{\varepsilon \to 0} \big\langle n(q_\varepsilon), M^{-1}(q_\varepsilon) \tilde{w}'_\varepsilon \big\rangle = \big\langle n(q), M^{-1}(q) \tilde{w} \big\rangle = -\big\langle n(q), M^{-1}(q) n(q) \big\rangle < 0.$$

Hence, possibly reducing r_q, we have

$$\big\langle n(q_\varepsilon), M^{-1}(q_\varepsilon) \tilde{w}'_\varepsilon \big\rangle \leq -\frac{1}{2} \big\langle n(q), M^{-1}(q) n(q) \big\rangle < 0 \quad \forall q_\varepsilon \in B(q, \varepsilon),\ \forall \varepsilon \in (0, r_q).$$

Since $D_1(q_\varepsilon)$ is a compact subset of $\mathbb{R}^d$, $\mathrm{Argmin}_{w \in D_1(q_\varepsilon)} \langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon) + w) \rangle$ is not empty and for all $\hat{w}_\varepsilon \in \mathrm{Argmin}_{w \in D_1(q_\varepsilon)} \langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon) + w) \rangle$ we have

$$\big\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon) + \hat{w}_\varepsilon) \big\rangle \leq \big\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon) + \tilde{w}'_\varepsilon) \big\rangle.$$

It follows that

$$\left\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\hat{w}_\varepsilon \right\rangle \le -\frac{1}{2}\left\langle n(q), M^{-1}(q)n(q)\right\rangle < 0 \quad \forall q_\varepsilon \in B(q,\varepsilon),\ \forall \varepsilon \in (0, r_q)$$

and $0 \notin \mathrm{Argmin}_{w \in D_1(q_\varepsilon)} \langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon)+w)\rangle$ for all $q_\varepsilon \in B(q,\varepsilon)$ and for all $\varepsilon \in (0, r_q)$.

Now let $\varepsilon \in (0, r_q)$, $q_\varepsilon \in B(q,\varepsilon)$ and $\hat{w}_\varepsilon \in \mathrm{Argmin}_{w \in D_1(q_\varepsilon)}\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)(n(q_\varepsilon)+ w)\rangle$. Then $\mathbb{R}\hat{w}_\varepsilon \subset \mathrm{Span}\,(D_1(q_\varepsilon))$. Reminding that $\dim(\mathrm{Span}\,(D_1(q_\varepsilon))) = 1$ we get $\mathbb{R}\hat{w}_\varepsilon = \mathrm{Span}\,(D_1(q_\varepsilon))$ and, for any $\hat{w} \in D_1(q_\varepsilon) \setminus \{\hat{w}_\varepsilon\}$ we have $\hat{w} = t\hat{w}_\varepsilon$ with $t \in \mathbb{R} \setminus \{1\}$. It follows that

$$\begin{aligned}
&\left\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon)+\hat{w})\right\rangle \\
=\ &\left\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon)+\hat{w}_\varepsilon)\right\rangle + (t-1)\left\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\hat{w}_\varepsilon\right\rangle \\
=\ &\min_{w \in D_1(q)}\left\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon)+w)\right\rangle + (t-1)\left\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\hat{w}_\varepsilon\right\rangle \\
\neq\ &\min_{w \in D_1(q)}\left\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon)+w)\right\rangle.
\end{aligned}$$

We may conclude that $\mathrm{Argmin}_{w \in D_1(q_\varepsilon)} \langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon)+w)\rangle$ is reduced to $\{\hat{w}_\varepsilon\}$.

Moreover, by definition of $\hat{w}_\varepsilon$, we have

$$\left\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon)+\hat{w}_\varepsilon)\right\rangle \le \left\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon)+\tilde{w}'_\varepsilon)\right\rangle$$

for all $q_\varepsilon \in B(q,\varepsilon)$ and for all $\varepsilon \in (0, r_q)$. Let us define now $\check{w}_\varepsilon = \mathrm{Proj}\,(D_1(q), \hat{w}_\varepsilon)$ for all $q_\varepsilon \in B(q,\varepsilon)$ and for all $\varepsilon \in (0, r_q)$. Then, $(\check{w}_\varepsilon)_{r_q>\varepsilon>0}$ is bounded and $\lim\limits_{\varepsilon\to 0} \hat{w}_\varepsilon - \check{w}_\varepsilon = 0$. It follows that $(\hat{w}_\varepsilon)_{r_q>\varepsilon>0}$ is also bounded and for any convergent subsequence $(\hat{w}_{\varepsilon_n})_{n\ge1}$, with $(\varepsilon_n)_{n\ge1}$ decreasing to zero, admitting $\hat{w}_*$ as limit we have

$$\check{w}_{\varepsilon_n} \to \hat{w}_* \in D_1(q)$$

and

$$\begin{aligned}
\left\langle n(q), M^{-1}(q)\,(n(q)+\hat{w}_*)\right\rangle &= \lim_{n\to+\infty}\left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\,(n(q_{\varepsilon_n})+\hat{w}_{\varepsilon_n})\right\rangle \\
\le \lim_{n\to+\infty}\left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n})+\tilde{w}'_{\varepsilon_n}\right)\right\rangle &= \left\langle n(q), M^{-1}(q)\,(n(q)+\tilde{w})\right\rangle = 0.
\end{aligned}$$

It follows that $\hat{w}_* = \tilde{w}$ and finally $\lim_{\varepsilon\to0}\hat{w}_\varepsilon = \tilde{w}$. □

Now let $\varepsilon' \in (0, r_q)$ and $q_{\varepsilon'} \in B(q,\varepsilon') \setminus (\mathrm{Int}(K) \cup \{q\})$ be such that

$$\begin{aligned}
&\left\langle n(q_{\varepsilon'}), M^{-1}(q_{\varepsilon'})\,(n(q_{\varepsilon'})+\hat{w}_{\varepsilon'})\right\rangle \\
=\ &\min_{w \in D_1(q_{\varepsilon'})}\left\langle n(q_{\varepsilon'}), M^{-1}(q_{\varepsilon'})\,(n(q_{\varepsilon'})+w)\right\rangle > 0.
\end{aligned}$$

We define $u^-_{\varepsilon'}$ as follows

$$\begin{aligned} u^-_{\varepsilon'} &\stackrel{\text{def}}{=} u^- - \left\langle u^-, n(q_{\varepsilon'})\right\rangle n(q_{\varepsilon'}) - \lambda \left\langle n(q_{\varepsilon'}), M^{-1}(q_{\varepsilon'}) \left(n(q_{\varepsilon'}) + \hat{w}_{\varepsilon'}\right)\right\rangle n(q_{\varepsilon'}) \\ &= u^- + \rho_{\varepsilon'} n(q_{\varepsilon'}) \end{aligned}$$

with

$$\rho_{\varepsilon'} = -\left\langle u^-, n(q_{\varepsilon'})\right\rangle - \lambda \left\langle n(q_{\varepsilon'}), M^{-1}(q_{\varepsilon'}) \left(n(q_{\varepsilon'}) + \hat{w}_{\varepsilon'}\right)\right\rangle.$$

By construction we have $\left\langle u^-_{\varepsilon'}, n(q_{\varepsilon'})\right\rangle < 0$ and thus

$$\mathcal{S}(q_{\varepsilon'}, u^-_{\varepsilon'}) = u^+_{\varepsilon'} = \mathrm{Proj}_{q_{\varepsilon'}}\left(0, \left(u^-_{\varepsilon'} + M^{-1}(q_{\varepsilon'}) C(q_{\varepsilon'})\right) \cap T(q_{\varepsilon'})\right).$$

It follows that

$$u^+_{\varepsilon'} = u^-_{\varepsilon'} + \lambda_{\varepsilon'} M^{-1}(q_{\varepsilon'}) \left(n(q_{\varepsilon'}) + w_{\varepsilon'}\right)$$

with $\lambda_{\varepsilon'} > 0$, $w_{\varepsilon'} \in D_1(q_{\varepsilon'})$ such that

$$\left\langle u^+_{\varepsilon'}, n(q_{\varepsilon'})\right\rangle = 0 \tag{18}$$

and

$$\left\langle -u^+_{\varepsilon'}, M(q_{\varepsilon'})(u^-_{\varepsilon'} - u^+_{\varepsilon'}) + \lambda' w'\right\rangle = \left\langle -u^+_{\varepsilon'}, \lambda' w' - \lambda_{\varepsilon'} w_{\varepsilon'}\right\rangle \le 0 \tag{19}$$

for all $\lambda' \ge 0$, for all $w' \in D_1(q_{\varepsilon'})$ such that $u^-_{\varepsilon'} + \lambda' M^{-1}(q_{\varepsilon'}) \left(n(q_{\varepsilon'}) + w'\right) \in T(q_{\varepsilon'})$. With (18) we get

$$\begin{aligned} 0 = {} & \left\langle u^-_{\varepsilon'}, n(q_{\varepsilon'})\right\rangle + \lambda_{\varepsilon'} \left\langle n(q_{\varepsilon'}), M^{-1}(q_{\varepsilon'}) \left(n(q_{\varepsilon'}) + w_{\varepsilon'}\right)\right\rangle \\ = {} & -\lambda \left\langle n(q_{\varepsilon'}), M^{-1}(q_{\varepsilon'}) \left(n(q_{\varepsilon'}) + \hat{w}_{\varepsilon'}\right)\right\rangle + \lambda_{\varepsilon'} \left\langle n(q_{\varepsilon'}), M^{-1}(q_{\varepsilon'}) \left(n(q_{\varepsilon'}) + w_{\varepsilon'}\right)\right\rangle \\ \ge {} & (\lambda_{\varepsilon'} - \lambda) \left\langle n(q_{\varepsilon'}), M^{-1}(q_{\varepsilon'}) \left(n(q_{\varepsilon'}) + \hat{w}_{\varepsilon'}\right)\right\rangle \end{aligned}$$

and we infer that

$$\lambda \ge \lambda_{\varepsilon'}. \tag{20}$$

Let us choose now $w' = 0$ and

$$\lambda' = \frac{-\left\langle u^-_{\varepsilon'}, n(q_{\varepsilon'})\right\rangle}{\left\langle n(q_{\varepsilon'}), M^{-1}(q_{\varepsilon'}) n(q_{\varepsilon'})\right\rangle}$$

in (19). We obtain

$$\left\langle u^+_{\varepsilon'}, w_{\varepsilon'}\right\rangle \le 0. \tag{21}$$

Next we choose $w' = \hat{w}_{\varepsilon'}$, $\lambda' = \lambda$ and we have

$$\left\langle -u^+_{\varepsilon'}, \lambda \hat{w}_{\varepsilon'} - \lambda_{\varepsilon'} w_{\varepsilon'}\right\rangle \le 0. \tag{22}$$

With the same arguments as in Section 3, we obtain that $(u^+_{\varepsilon'})_{r_q>\varepsilon'>0}$ is bounded and admits convergent subsequences, denoted as $(u^+_{\varepsilon'_n})_{n\geq 1}$, with $(\varepsilon'_n)_{n\geq 1}$ decreasing to zero, such that

$$\lambda_{\varepsilon'_n} \to \lambda_* \geq 0, \quad w_{\varepsilon'_n} \to \bar{w}_* \in D_1(q)$$

with

$$u^+_{\varepsilon'_n} \to u_* = u^- + \lambda_* M^{-1}(q)\,(n(q) + \bar{w}_*) \in T(q).$$

Now passing to the limit in (21)–(22) and using Lemma 4.3 we get

$$\langle u_*, \bar{w}_* \rangle \leq 0, \quad \langle u_*, \lambda\tilde{w} - \lambda_*\bar{w}_* \rangle \geq 0.$$

If $\lambda_* = 0$ we have $u_* = u^-$ and $\langle u_*, \lambda\tilde{w}\rangle = \lambda\langle u^-, \tilde{w}\rangle \geq 0$ which is absurd since $\tilde{\lambda} \geq \lambda > 0$. It follows that $\lambda_* > 0$ and $\bar{w}_* \in D_{1,\mathrm{adm}}(q) = \{\tilde{w}\}$. Hence

$$\langle u_*, \tilde{w} \rangle \leq 0, \quad \langle u_*, (\lambda - \lambda_*)\tilde{w} \rangle \geq 0$$

which implies that $\lambda \leq \lambda_*$. By using (20) we have $\lambda \geq \lambda_*$ and finally $\lambda = \lambda_*$. Since $(u^+_{\varepsilon'})_{r_q>\varepsilon'>0}$ is bounded we infer that

$$\lim_{\varepsilon'\to 0} u^+_{\varepsilon'} = \bar{v}. \tag{23}$$

In order to conclude let us define now $h(\varepsilon') = \varepsilon' + |\rho_{\varepsilon'}|$ for all $\varepsilon' \in (0, r_q)$. By using Lemma 4.3, we know that $\lim_{\varepsilon'\to 0} h(\varepsilon') = 0$. It follows that we may build successively $\varepsilon'_n \in (0, r_q)$ for $n = 1$, $n = 2$, $n = 3$ and so on such that the sequences $(\varepsilon'_n)_{n\geq 1}$ and $\big(\varepsilon_n \overset{\text{def}}{=} h(\varepsilon'_n)\big)_{n\geq 1}$ decrease to zero. By construction we have

$$(q_{\varepsilon'_n}, u^-_{\varepsilon'_n}) \in B(q, \varepsilon'_n) \times \overline{B}\left(u^-, |\rho_{\varepsilon'_n}|\right) \subset B(q, \varepsilon_n) \times B(u^-, \varepsilon_n)$$

and we let $q_{\varepsilon_n} \overset{\text{def}}{=} q_{\varepsilon'_n}$ and $u^-_{\varepsilon_n} \overset{\text{def}}{=} u^-_{\varepsilon'_n}$ for all $n \geq 1$. Then $\mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n}) = u^+_{\varepsilon'_n}$ and with (23) we obtain that $\big(\mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n})\big)_{n\geq 1}$ converges to $\bar{v}$.

Let us assume now that (H4) does not hold, i.e. there exists $\varepsilon_q \in (0, r_q)$ such that

$$\min_{w\in D_1(q_\varepsilon)} \big\langle n(q_\varepsilon), M^{-1}(q_\varepsilon)\,(n(q_\varepsilon) + w)\big\rangle \leq 0 \quad \forall q_\varepsilon \in B(q, \varepsilon_q) \setminus \mathrm{Int}(K). \tag{24}$$

If $\tilde{u} = u^-$ the result is obvious. Otherwise $\tilde{u} \neq u^-$ and we consider $\bar{v} \in [u^-, \tilde{u}] \setminus \{u^-, \tilde{u}\}$, i.e.

$$\bar{v} = u^- + \lambda M^{-1}(q)\,(n(q) + \tilde{w})\,, \quad \text{with } 0 < \lambda < \tilde{\lambda}.$$

If moreover $\bar{v} \neq \mathcal{S}(q, u^-)$ we will prove that $\mathrm{dist}\,(\bar{v}, \mathcal{A}_\varepsilon(q, u^-))$ does not tend to zero as ε tends to zero by contradiction. So we assume that, for all $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$ we have

$$\mathrm{dist}\,\big(\bar{v}, \mathcal{A}_\varepsilon(q, u^-)\big) \leq \delta.$$

Let $\delta = \frac{1}{6}\min\left(\|\bar v - u^-\|, \|\bar v - \tilde u\|, \|\bar v - \mathcal{S}(q,u^-)\|\right)$. Then there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0,\varepsilon_\delta)$, there exists $(q_\varepsilon, u^-_\varepsilon) \in B(q,\varepsilon) \times B(u^-,\varepsilon)$ such that

$$\left\|\bar v - \mathcal{S}(q_\varepsilon, u^-_\varepsilon)\right\| = \|\bar v - u^+_\varepsilon\| \le \frac{1}{3}\min\left(\|\bar v - u^-\|, \|\bar v - \tilde u\|, \left\|\bar v - \mathcal{S}(q,u^-)\right\|\right).$$

Now let $(\varepsilon_n)_{n\ge1}$ be a decreasing sequence to zero. It follows that there exists $N_\delta \ge 1$ such that, for all $n \ge N_\delta$ we have $\varepsilon_n \in (0, \min(\varepsilon_q, \varepsilon_\delta))$ and there exists $(q_{\varepsilon_n}, u^-_{\varepsilon_n}) \in B(q,\varepsilon_n) \times B(u^-,\varepsilon_n)$ such that

$$\|\bar v - u^+_{\varepsilon_n}\| \le \frac{1}{3}\min\left(\|\bar v - u^-\|, \|\bar v - \tilde u\|, \left\|\bar v - \mathcal{S}(q,u^-)\right\|\right) \quad \forall n \ge N_\delta.$$

With (24) for any $n \ge N_\delta$ such that $q_{\varepsilon_n} \notin \mathrm{Int}(K)$ we have

$$\min_{w \in D_1(q_{\varepsilon_n})} \left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n})\left(n(q_{\varepsilon_n}) + w\right)\right\rangle \le 0.$$

By definition of $\mathcal{S}$ we obtain

$$\mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n}) \in \{u^-_{\varepsilon_n}, \tilde u_{\varepsilon_n}\}$$

for all $n \ge N_\delta$, for all $q_{\varepsilon_n} \in B(q,\varepsilon_n)$ and for all $u^-_{\varepsilon_n} \in B(u^-,\varepsilon_n)$ such that $g(q_{\varepsilon_n}) < 0$ or $g(q_{\varepsilon_n}) \ge 0$ and $\left\langle u^-_{\varepsilon_n}, n(q_{\varepsilon_n})\right\rangle \ne 0$. Moreover if $g(q_{\varepsilon_n}) \ge 0$ and $\left\langle u^-_{\varepsilon_n}, n(q_{\varepsilon_n})\right\rangle = 0$ we have

$$u^+_{\varepsilon_n} = \mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n}) \in \mathcal{A}(q_{\varepsilon_n}, u^-_{\varepsilon_n}).$$

The continuity of the restriction of $\mathcal{S}$ to $\{(q',u') \in B(q,r_q) \times \mathbb{R}^d;\ g(q') \ge 0,\ \langle u', n(q')\rangle = 0\}$ at (q,u^-) implies that there exists $r'_{(q,u^-)} > 0$ such that, for all $(q',u') \in B(q, r'_{(q,u^-)}) \times B(u^-, r'_{(q,u^-)})$ such that $g(q') \ge 0$ and $\langle u', n(q')\rangle = 0$ we have

$$\left\|\mathcal{S}(q,u^-) - \mathcal{S}(q',u')\right\| \le \frac{1}{3}\left\|\bar v - \mathcal{S}(q,u^-)\right\|.$$

By applying the next lemma we infer that there exists $N'_\delta \ge N_\delta$ such that, for all $n \ge N'_\delta$, for all $q_{\varepsilon_n} \in B(q,\varepsilon_n)$ and for all $u^-_{\varepsilon_n} \in B(u^-,\varepsilon_n)$, we have

$$\|\bar v - u^+_{\varepsilon_n}\| = \left\|\bar v - \mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n})\right\| \ge \frac{2}{3}\min\left(\|\bar v - u^-\|, \|\bar v - \tilde u\|, \left\|\bar v - \mathcal{S}(q,u^-)\right\|\right)$$

which yields a contradiction.

Lemma 4.4. *Let us assume that* $\mathcal{N} \overset{\text{def}}{=} \left\{n \ge N_\delta; \mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n}) = \tilde u_{\varepsilon_n} \ne u^-_{\varepsilon_n}\right\}$ *is not bounded. Then*

$$\lim_{n \to +\infty, n \in \mathcal{N}} \tilde u_{\varepsilon_n} = \tilde u.$$

Proof. By using the same notations as in Section 3, for all $n \in \mathcal{N}$, we have

$$\mathcal{S}(q_{\varepsilon_n}, u^-_{\varepsilon_n}) = \tilde{u}_{\varepsilon_n} = u^-_{\varepsilon_n} + \lambda_{\varepsilon_n} M^{-1}(q_{\varepsilon_n}) \left(n(q_{\varepsilon_n}) + w_{\varepsilon_n} \right)$$

with $\lambda_{\varepsilon_n} > 0$ and $w_{\varepsilon_n} \in D_1(q_{\varepsilon_n})$ and we already know that $(\tilde{u}_{\varepsilon_n})_{n \in \mathcal{N}}$ is bounded. Moreover, with (24) we know also that $D_{1,\mathrm{adm}}(q_{\varepsilon_n}) \neq \emptyset$ and we consider $\tilde{w}_{\varepsilon_n} \in D_{1,\mathrm{adm}}(q_{\varepsilon_n})$ for all $n \geq N_\delta$. Let $\lambda > 0$. For all $n \in \mathcal{N}$ we define

$$\mu_{\varepsilon_n} = \frac{-\left\langle u_{\varepsilon_n}, n(q_{\varepsilon_n}) \right\rangle}{\left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n}) n(q_{\varepsilon_n}) \right\rangle} \geq 0$$

and

$$v = u^-_{\varepsilon_n} + M^{-1}(q_{\varepsilon_n}) \left((\mu_{\varepsilon_n} + \lambda) n(q_{\varepsilon_n}) + \lambda \tilde{w}_{\varepsilon_n} \right).$$

By construction $v \in \left(u^-_{\varepsilon_n} + M^{-1}(q_{\varepsilon_n}) C(q_{\varepsilon_n}) \right) \cap T(q_{\varepsilon_n})$ and it follows that

$$\left\langle -\tilde{u}_{\varepsilon_n}, M(q_{\varepsilon_n}) \left(u^-_{\varepsilon_n} - \tilde{u}_{\varepsilon_n} \right) + \lambda \tilde{w}_{\varepsilon_n} \right\rangle \leq 0.$$

Hence

$$\left\langle -\tilde{u}_{\varepsilon_n}, \lambda \tilde{w}_{\varepsilon_n} - \lambda_{\varepsilon_n} w_{\varepsilon_n} \right\rangle \leq 0 \quad \forall \lambda > 0,\ \forall \tilde{w}_{\varepsilon_n} \in D_{1,\mathrm{adm}}(q_{\varepsilon_n}),\ \forall n \in \mathcal{N}.$$

By dividing by λ and letting $\lambda \to +\infty$ we obtain

$$\left\langle -\tilde{u}_{\varepsilon_n}, \tilde{w}_{\varepsilon_n} \right\rangle \leq 0 \quad \forall \tilde{w}_{\varepsilon_n} \in D_{1,\mathrm{adm}}(q_{\varepsilon_n}),\ \forall n \in \mathcal{N}. \tag{25}$$

Furthermore, the sequence $(\tilde{w}_{\varepsilon_n})_{n \geq N_\delta}$ admits $\tilde{w}$ for limit as n tends to $+\infty$. Indeed, let $\bar{w}'_{\varepsilon_n} = \mathrm{Proj}\left(D_1(q), \tilde{w}_{\varepsilon_n} \right)$ for all $n \geq N_\delta$. We have

$$\| \tilde{w}_{\varepsilon_n} - \bar{w}'_{\varepsilon_n} \| \leq d_H \left(D_1(q_{\varepsilon_n}), D_1(q) \right) \quad \forall n \geq N_\delta$$

and $D_1(q)$ is a compact subset of $\mathbb{R}^d$. Hence $(\tilde{w}_{\varepsilon_n})_{n \geq 1}$ is a bounded sequence. Moreover, possibly extracting a subsequence, still denoted $(\varepsilon_n)_{n \geq N_\delta}$ we have

$$\bar{w}'_{\varepsilon_n} \to \bar{w}'_* \in D_1(q).$$

Then $\lim_{n \to +\infty} \tilde{w}_{\varepsilon_n} = \bar{w}'_*$. But $\tilde{w}_{\varepsilon_n} \in D_{1,\mathrm{adm}}(q_{\varepsilon_n})$ i.e.

$$\left\langle n(q_{\varepsilon_n}), M^{-1}(q_{\varepsilon_n}) \left(n(q_{\varepsilon_n}) + \tilde{w}_{\varepsilon_n} \right) \right\rangle = 0 \quad \forall n \geq N_\delta$$

and at the limit as n tends to $+\infty$ we get

$$\left\langle n(q), M^{-1}(q) \left(n(q) + \bar{w}'_* \right) \right\rangle = 0.$$

We infer that $\bar{w}'_* \in D_{1,\mathrm{adm}}(q) = \{\tilde{w}\}$ and $\bar{w}'_* = \tilde{w}$. It follows that $\tilde{w}$ is the unique limit point of the bounded sequence $(\tilde{w}_{\varepsilon_n})_{n \geq N_\delta}$ and we infer that the whole sequence converges to $\tilde{w}$.

Let us prove now that $\lim_{n \to +\infty, n \in \mathcal{N}} \tilde{u}_{\varepsilon_n} = \tilde{u}$ by contradiction. Hence let us assume that there exists $\tilde{\delta} > 0$ such that, for all $N \geq 1$ there exists $n \in \mathcal{N}$ such that $n \geq N$ and $\|\tilde{u}_{\varepsilon_n} - \tilde{u}\| > \tilde{\delta}$. Then there exists a subsequence $(\tilde{\varepsilon}_n = \varepsilon_{\varphi(n)})_{n \geq 1}$ such that φ is an increasing mapping from $\{n \in \mathbb{N};\ n \geq 1\}$ into $\{n \in \mathbb{N};\ n \in \mathcal{N}\}$ and $\|\tilde{u}_{\tilde{\varepsilon}_n} - \tilde{u}\| > \tilde{\delta}$ for all $n \geq 1$.

By using the same arguments as in Section 3, we know that any limit point u_* of $(\tilde{u}_{\tilde{\varepsilon}_n})_{n \geq 1}$ belongs to $(u^- + M^{-1}(q)C(q)) \cap T(q)$ (see (16)–(17)). Moreover, with Proposition 3.2 we have $u_* \in [u^-, \tilde{u}]$ i.e.

$$u_* = u^- + \lambda_* M^{-1}(q)\left(n(q) + \tilde{w}\right), \quad 0 \leq \lambda_* \leq \tilde{\lambda}.$$

With (25) we get also

$$\langle u_*, \tilde{w} \rangle \geq 0.$$

Hence

$$\begin{aligned} &\langle u^-, \tilde{w} \rangle + \lambda_* \left\langle M^{-1}(q)\left(n(q) + \tilde{w}\right), \tilde{w} \right\rangle \\ &= (\lambda_* - \tilde{\lambda}) \left\langle n(q) + \tilde{w}, M^{-1}(q)\left(n(q) + \tilde{w}\right) \right\rangle \geq 0. \end{aligned}$$

It follows that $\tilde{\lambda} = \lambda_*$ and $u_* = \tilde{u}$. Since $(\tilde{u}_{\tilde{\varepsilon}_n})_{n \geq 1}$ is bounded, we may conclude that the whole sequence converges to $\tilde{u}$ which is absurd. $\square$

We infer

Proposition 4.5. *Let us assume that $\tilde{u} \neq u^-$. Then we have*

$$\lim_{\varepsilon \to 0} e\left(\mathcal{A}(q, u^-), \mathcal{A}_\varepsilon(q, u^-)\right) = 0 \quad \textit{if (H4) holds,}$$

$$\lim_{\varepsilon \to 0} e\left(\mathcal{A}(q, u^-), \mathcal{A}_\varepsilon(q, u^-)\right) \neq 0$$

if (H4) does not hold and the restriction of the mapping $\mathcal{S}$ to $\left\{(q', u') \in (B(q, r_q) \setminus \mathrm{Int}(K)) \times \mathbb{R}^d;\ \langle u', n(q') \rangle = 0\right\}$ is continuous at (q, u^-).

Proof. The second part of the result is obvious and we only need to check that

$$\lim_{\varepsilon \to 0} e\left(\mathcal{A}(q, u^-), \mathcal{A}_\varepsilon(q, u^-)\right) = \lim_{\varepsilon \to 0} \sup \left\{\mathrm{dist}\left(\bar{v}, \mathcal{A}_\varepsilon(q, u^-)\right);\ \bar{v} \in [u^-, \tilde{u}]\right\} = 0$$

when $\tilde{u} \neq u^-$ and (H4) holds. But Proposition 4.2 implies that

$$\lim_{\varepsilon \to 0} \mathrm{dist}\left(\bar{v}, \mathcal{A}_\varepsilon(q, u^-)\right) = 0$$

for all $\bar{v} \in \mathcal{A}(q, u^-) = [u^-, \tilde{u}]$ and the compactness of $\mathcal{A}(q, u^-)$ allows us to conclude. $\square$

5. Concluding remarks

Let us emphasize that the results of the two previous sections have been established for any time-stepping scheme satisfying (9)–(10). Hence they are true in particular for the algorithm proposed by J. J. Moreau in [13, 14, 15] which corresponds to the choice $\mathcal{S}(q,u^-) = \mathcal{S}_{JJM}(q,u^-) \stackrel{\text{def}}{=} u^-$ whenever $g(q) \geq 0$ and $\langle u^-, \nabla g(q)\rangle = 0$, i.e.

$$\mathcal{S}_{JJM}(q,u^-) = u^- \ \text{ if } u^- \in T_K(q), \qquad \mathcal{S}_{JJM}(q,u^-) = \tilde{u} \text{ otherwise},$$

with

$$T_K(q) = \begin{cases} \mathbb{R}^d & \text{if } g(q) < 0, \\ \left\{v \in \mathbb{R}^d;\ \langle v, \nabla g(q)\rangle \leq 0\right\}, & \text{if } g(q) \geq 0. \end{cases}$$

It follows that the discrete contact law does not lead to any velocity jump in case of tangential contact and is in agreement with Painlevé's heuristic principle "deux surfaces solides qui, à un instant t, dans des conditions données, ne presseraient pas l'une sur l'autre si elles étaient parfaitement polies, ne réagissent pas davantage l'une sur l'autre quand elles sont rugueuses" ([20]).

Nevertheless the asymptotic behavior of this discrete contact law still presents indeterminacies of Painlevé's paradoxes type. One may argue that in Case 1 we do not have any paradox since $\tilde{u}$ appears as the limit of right velocities u_ε^+ only when $u_\varepsilon^+ = \tilde{u}_\varepsilon$: with $\mathcal{S}_{JJM}$ it implies that $u_\varepsilon^- \notin T_K(q_\varepsilon)$ and we have a collision. But in Case 2, when $\tilde{u} \neq u^-$ and (H4) holds, we have proved that all $\bar{v} \in [u^-, \tilde{u}] \setminus \{u^-, \tilde{u}\}$ is the limit of a sequence of post collision velocities $(\mathcal{S}_{JJM}(q_{\varepsilon_n}, u_{\varepsilon_n}^-))_{n\geq 1}$, with $(\varepsilon_n)_{n\geq 1}$ decreasing to zero and $(q_{\varepsilon_n}, u_{\varepsilon_n}^-) \in B(q,\varepsilon_n) \times B(u^-, \varepsilon_n)$ such that $u_{\varepsilon_n}^- \notin T_K(q_{\varepsilon_n})$ for all $n \geq 1$. Despite $\mathcal{S}_{JJM}(q_{\varepsilon_n}, u_{\varepsilon_n}^-)$ is given by the Argmin of the kinetic norm of the admissible right velocities at $(q_{\varepsilon_n}, u_{\varepsilon_n}^-)$ for all $n \geq 1$, the limit $\bar{v}$ is not the Argmin of the kinetic norm of all the admissible right velocities at (q, u^-), which may appear as a real paradox from the mechanical point of view.

On the contrary, when $\tilde{u} \neq u^-$ and (H4) does not hold, we have the same result as in Case 1 since the restriction of $\mathcal{S}_{JJM}$ to $\{(q', u') \in (\mathbb{R}^d \setminus \mathrm{Int}(K)) \times \mathbb{R}^d; \langle u', \nabla g(q')\rangle = 0\}$ is continuous, and the only limit right velocities are $\tilde{u}$ and u^-. By using Lemma 4.3 it is clear that the mapping

$$\gamma : q' \mapsto \min_{w \in D_1(q')} \left\langle n(q'), M^{-1}(q')\,(n(q') + w)\right\rangle$$

is continuous at q. We may observe that $\gamma(q) > 0$ means that the friction cone $C(q)$ is above the plane $M(q)T(q)$ while $\gamma(q) < 0$ (resp. $\gamma(q) = 0$) means that $C(q)$ crosses (resp. is tangent to) the plane $M(q)T(q)$. Hence Case 1 and Case 2 can be reformulated as $\gamma(q) < 0$ and $\gamma(q) = 0$ respectively and Case 2 appears as a limit case from the geometrical point of view. Furthermore when (H4) holds we have

$$\forall \varepsilon \in (0, r_q) \quad \exists q_\varepsilon \in B(q,\varepsilon) \setminus (\mathrm{Int}(K) \cup \{q\}) \quad / \quad \gamma(q_\varepsilon) > 0$$

and the asymptotic behavior of J. J. Moreau's scheme exhibits indeterminacies. But limit velocities $\bar{v} \in [\tilde{u}, u^-] \setminus \{\tilde{u}, u^-\}$ are obtained only for sequences $(q_{\varepsilon_n}, u^-_{\varepsilon_n})_{n\geq 1}$ such that $\gamma(q_{\varepsilon_n}) > 0$ (see the proof of Proposition 4.2).

We may observe that all these results are also valid when we choose $\mathcal{S}(q, u^-) = \mathcal{S}_{min}(q, u^-) \stackrel{\text{def}}{=} \tilde{u} = \text{Proj}_q\left(0, (u^- + M^{-1}(q)C(q)) \cap T(q)\right)$ for all $(q, u^-) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $g(q) \geq 0$ and $\langle u^-, \nabla g(q)\rangle = 0$. Then, for all (q, u^-) such that $q \notin \text{Int}(K)$, $u^- \in T(q)$ and $\gamma(q) = 0$, we get

$$\mathcal{S}_{min}(q, u^-) = u^- + \tilde{\lambda} M^{-1}(q)\,(n(q) + \tilde{w})\,,$$

$$\tilde{\lambda} = \frac{\max\left(-\langle u^-, \tilde{w}\rangle, 0\right)}{\langle n(q) + \tilde{w}, M^{-1}(q)\,(n(q) + \tilde{w})\rangle}$$

with $D_{1,\text{adm}} = \{\tilde{w}\}$. It follows that $\langle u^-, \tilde{w}\rangle < 0$ if and only if $\tilde{u} \neq u^-$ and a straighforward adaptation of Lemma 4.4 shows that the restriction of $\mathcal{S}_{min}$ to $\big\{(q', u') \in \big(\mathbb{R}^d \setminus \text{Int}(K)\big) \times \mathbb{R}^d;\ \langle u', \nabla g(q')\rangle = 0\big\}$ is continuous at (q, u^-) when (H4) does not hold and $\tilde{u} \neq u^-$.

Indeed, let us argue by contradiction. So we assume that there exists a sequence $(q'_{\varepsilon_n}, u'_{\varepsilon_n})_{n\geq 1}$ such that $(\varepsilon_n)_{n\geq 1}$ decreases to zero, $(q'_{\varepsilon_n}, u'_{\varepsilon_n}) \in B(q, \varepsilon_n) \times B(u^-, \varepsilon_n)$, $g(q'_{\varepsilon_n}) \geq 0$, $\big\langle u'_{\varepsilon_n}, n(q'_{\varepsilon_n})\big\rangle = 0$ for all $n \geq 1$, and $\big(\mathcal{S}_{min}(q'_{\varepsilon_n}, u'_{\varepsilon_n})\big)_{n\geq 1}$ does not converge to $\mathcal{S}_{min}(q, u^-)$. Since (H4) does not hold, there exists $\varepsilon_q > 0$ such that $\gamma(q') \leq 0$ for all $q' \in B(q, \varepsilon_q) \setminus \text{Int}(K)$. It follows that there exists $N \geq 1$ such that, for all $n \geq N$, $D_{1,\text{adm}}(q'_{\varepsilon_n}) \neq \emptyset$.

Then, for all $n \geq N$ we get

$$\mathcal{S}_{min}(q'_{\varepsilon_n}, u'_{\varepsilon_n}) = u'_{\varepsilon_n} + \lambda'_{\varepsilon_n} M^{-1}(q'_{\varepsilon_n})\left(n(q'_{\varepsilon_n}) + \tilde{w}'_{\varepsilon_n}\right), \tag{26}$$

with $\tilde{w}'_{\varepsilon_n} \in D_{1,\text{adm}}(q'_{\varepsilon_n})$ and $\lambda'_{\varepsilon_n} \geq 0$. For all $n \geq N$ we let $\bar{w}_{\varepsilon_n} = \text{Proj}\left(D_1(q), \tilde{w}'_{\varepsilon_n}\right)$. We have $\lim_{n\to+\infty} \tilde{w}'_{\varepsilon_n} - \bar{w}_{\varepsilon_n} = 0$ and, possibly extracting a subsequence, we know that

$$\bar{w}_{\varepsilon_n} \to \bar{w}_* \in D_1(q).$$

Hence

$$\lim_{n\to+\infty} \big\langle n(q'_{\varepsilon_n}), M^{-1}(q'_{\varepsilon_n})\left(n(q'_{\varepsilon_n}) + \tilde{w}'_{\varepsilon_n}\right)\big\rangle = 0 = \big\langle n(q), M^{-1}(q)\,(n(q) + \bar{w}_*)\big\rangle$$

and $\bar{w}_* \in D_{1,\text{adm}}(q) = \{\tilde{w}\}$. It follows that

$$\langle u'_{\varepsilon_n}, \tilde{w}'_{\varepsilon_n}\rangle \to_{n\to+\infty} \langle u^-, \tilde{w}\rangle < 0.$$

Hence, there exists $N' \geq N$ such that

$$\langle u'_{\varepsilon_n}, \tilde{w}'_{\varepsilon_n}\rangle < 0 \quad \forall n \geq N'$$

and we infer that

$$\lambda'_{\varepsilon_n} = \frac{-\langle u'_{\varepsilon_n}, \tilde{w}'_{\varepsilon_n}\rangle}{\big\langle n(q'_{\varepsilon_n}) + \tilde{w}'_{\varepsilon_n}, M^{-1}(q'_{\varepsilon_n})\left(n(q'_{\varepsilon_n}) + \tilde{w}'_{\varepsilon_n}\right)\big\rangle}.$$

It follows that

$$\lim_{n\to+\infty} \lambda'_{\varepsilon_n} = \tilde{\lambda} \quad \text{and} \quad \lim_{n\to+\infty} \mathcal{S}_{min}(q'_{\varepsilon_n}, u'_{\varepsilon_n}) = \mathcal{S}_{min}(q, u^-)$$

which gives a contradiction.

The choice of $\mathcal{S}_{min}$ means that, whenever a contact occurs, the right velocity is always given as the Argmin with respect to the kinetic metric of all the admissible right velocities. It follows that discrete tangential shocks happen for any $(q, u^-) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $g(q) \geq 0$, $u^- \in T(q)$ and $\tilde{u} \neq u^-$. This a quite different discrete contact law than $\mathcal{S}_{JJM}$ but it leads to the same asymptotic behavior which can be summarized as

$$d_H\left(\{u^-, \tilde{u}\}, \mathcal{A}_\varepsilon(q, u^-)\right) \to_{\varepsilon\to 0} 0$$
$$\text{if } \gamma(q) < 0 \text{ or } \gamma(q) = 0 \text{ and (H4) does not hold,}$$

and

$$d_H\left([u^-, \tilde{u}], \mathcal{A}_\varepsilon(q, u^-)\right) \to_{\varepsilon\to 0} 0 \quad \text{if } \gamma(q) = 0 \text{ and (H4) holds.}$$

Finally let us emphasize that the assumption $\dim(\mathrm{Span}(D_1(q'))) = 1$ for all q' in a neighborhood of q plays a minor role in the proofs since we only used it to show that $\mathrm{Argmin}_{w\in D_1(q')} \langle n(q'), M^{-1}(q')(n(q') + w)\rangle$ is reduced to a single point. But this last property is also satisfied when $\dim(\mathrm{Span}(D_1(q'))) \geq 1$ and $D_1(q')$ is strictly convex. More precisely

Lemma 5.1. *Let $q \in \partial K$ such that $M^{-1}(q)C(q) \cap T(q) \neq \{0\}$. Let us assume furthermore that there exists $r_q > 0$ such that $D_1(q')$ is strictly convex for any $q' \in \overline{B}(q, r_q)$ i.e. for any w_1 and w_2 belonging to $D_1(q')$ such that $w_1 \neq w_2$, and for any $\gamma \in (0,1)$, $\gamma w_1 + (1-\gamma)w_2$ belongs to the relative interior of $D_1(q')$, i.e. there exists a open subset $\mathcal{O}$ of $\mathbb{R}^d$ such that*

$$\gamma w_1 + (1-\gamma)w_2 \in \mathcal{O} \cap \mathrm{Span}(D_1(q')) \subset D_1(q').$$

Then for all $q' \in B(q, r_q)$, $\mathrm{Argmin}_{w\in D_1(q')} \langle n(q'), M^{-1}(q')(n(q') + w)\rangle$ is reduced to a single point.

Proof. For all $q' \in \mathbb{R}^d$, $D_1(q')$ is a compact subset of $\mathbb{R}^d$ and so

$$\mathrm{Argmin}_{w\in D_1(q')} \left\langle n(q'), M^{-1}(q')(n(q') + w)\right\rangle \neq \emptyset.$$

With the same computations as in Lemma 4.3 we obtain that, possibly reducing r_q, we have

$$\left\langle n(q'), M^{-1}(q')\hat{w}'\right\rangle \leq -\frac{1}{2}\left\langle n(q), M^{-1}(q)n(q)\right\rangle < 0$$

for all $q' \in B(q, r_q)$ and for all $\hat{w}' \in \mathrm{Argmin}_{w\in D_1(q')} \langle n(q'), M^{-1}(q')(n(q') + w)\rangle$.

Now let $q' \in B(q, r_q)$ and $\hat{w}'_1$, $\hat{w}'_2$ be such that $\hat{w}'_1 \neq \hat{w}'_2$ and

$$\hat{w}'_1, \hat{w}'_2 \in \mathrm{Argmin}_{w \in D_1(q')} \left\langle n(q'), M^{-1}(q') \, (n(q') + w) \right\rangle .$$

We define $\hat{w}' = \frac{1}{2}(\hat{w}'_1 + \hat{w}'_2)$. Reminding that $D_1(q')$ is convex, we have

$$\hat{w}' \in \mathrm{Argmin}_{w \in D_1(q')} \left\langle n(q'), M^{-1}(q') \, (n(q') + w) \right\rangle$$

and there exists an open subset $\mathcal{O}$ of $\mathbb{R}^d$ such that

$$\hat{w}' \in \mathcal{O} \cap \mathrm{Span}\,(D_1(q')) \subset D_1(q').$$

It follows that there exists $r > 0$ such that the open ball $B(\hat{w}', r)$ of center $\hat{w}'$ and radius r in $\mathbb{R}^d$ is included in $\mathcal{O}$ and

$$\hat{w}' \in B(\hat{w}', r) \cap \mathrm{Span}\,(D_1(q')) \subset D_1(q').$$

Moreover $\hat{w}'_1 \neq 0$ and $\mathrm{Span}(\hat{w}'_1, \hat{w}') \subset \mathrm{Span}\,(D_1(q'))$. Hence

$$\hat{w}' + \frac{r}{2\|\hat{w}'_1\|}\hat{w}'_1 \in B(\hat{w}', r) \cap \mathrm{Span}\,(D_1(q')) \subset D_1(q').$$

But

$$\begin{aligned} &\left\langle n(q'), M^{-1}(q') \left(n(q') + \hat{w}' + \frac{r}{2\|\hat{w}'_1\|}\hat{w}'_1 \right) \right\rangle \\ = \; &\left\langle n(q'), M^{-1}(q') \, (n(q') + \hat{w}') \right\rangle + \frac{r}{2\|\hat{w}'_1\|} \left\langle n(q'), M^{-1}(q') \hat{w}'_1 \right\rangle \\ < \; &\left\langle n(q'), M^{-1}(q') \, (n(q') + \hat{w}') \right\rangle = \min_{w \in D_1(q')} \left\langle n(q'), M^{-1}(q') \, (n(q') + w) \right\rangle \end{aligned}$$

which is absurd. □

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