

Integral Inequalities for Infimal Convolution and Hamilton-Jacobi Equations

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Dedicated to the memory of Jean Jacques Moreau.

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Let $f, g : \mathbb{R}^N \rightarrow (-\infty, \infty]$ be Borel measurable, bounded below and such that $\inf f + \inf g \geq 0$. We prove that with $m_{f,g} := (\inf f - \inf g)/2$, the inequality $\|(f - m_{f,g})^{-1}\|_\phi + \|(g + m_{f,g})^{-1}\|_\phi \leq 4\|(f \square g)^{-1}\|_\phi$ holds in every Orlicz space L_ϕ , where $f \square g$ denotes the infimal convolution of f and g and where $\|\cdot\|_\phi$ is the Luxemburg norm (i.e., the L^p norm when $L_\phi = L^p$).

Although no genuine reverse inequality can hold in any generality, we also prove that such reverse inequalities do exist in the form $\|(f \square g)^{-1}\|_\phi \leq 2^{N-1}(\|\tilde{f} - m_{f,g}\|_\phi + \|\tilde{g} + m_{f,g}\|_\phi)$, where $\tilde{f}$ and $\tilde{g}$ are suitable transforms of f and g introduced in the paper and reminiscent of, yet very different from, nondecreasing rearrangement.

Similar inequalities are proved for other extremal operations and applications are given to the long-time behavior of the solutions of the Hamilton-Jacobi and related equations.

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1. Introduction

If $f, g : \mathbb{R}^N \rightarrow (-\infty, \infty]$, the infimal convolution $f \square g : \mathbb{R}^N \rightarrow [-\infty, \infty]$, first introduced by Fenchel [9] and Moreau [24], [25], [26], is defined by the formula

$$(f \square g)(x) := \inf_{y \in \mathbb{R}^N} (f(x - y) + g(y)).$$

Since then, this operation and its extension to general vector spaces have found an ever growing variety of applications, including convex functions [13], [29], extension of Lipschitz functions [12], solutions of the Hamilton-Jacobi equations [2], [20], [31] and much more (even a proof of the Hahn-Banach theorem [11]). In fact, there are by now several thousands publications using infimal convolution in

areas as diverse as image processing, economics and finance, information theory, probabilities and statistics, etc. For a glimpse into some of these problems, see the excellent recent survey by Lucet [21].

In this paper, we investigate the mathematical properties of infimal convolution in a new direction, by exploring the existence of integral inequalities involving f, g and $f \square g$. The remark that $f \square g = 0$ whenever $f \geq 0$ and $g \geq 0$ are integrable could cast serious doubts on the value of this program, but they are quickly dispelled by the rebuttal that no similar triviality arises from the integrability of f^{-1} and g^{-1} . Here and everywhere else, $f^{-1} := 1/f$, $g^{-1} := 1/g$, etc. This notation will not be used to denote any set-theoretic inverse.

Omitting technicalities to which we shall return shortly, the first batch of inequalities will relate the (Luxemburg) norm $\|(f \square g)^{-1}\|_\phi$ in any Orlicz space L_ϕ , to the norms $\|(f - z)^{-1}\|_\phi$ and $\|(g + z)^{-1}\|_\phi$ for a suitable constant z independent of ϕ , to be defined in due time. The only restrictions are that f and g must be Borel measurable, bounded below and that $f \square g \geq 0$. The proofs depend crucially upon (a slightly weaker form of) the Brunn-Minkowski inequality.

The setting of Orlicz spaces instead of just the classical L^p spaces introduces only mild additional technicalities, is more natural in many respects and, as we shall see in the examples of Section 7, is useful in some applications. It does not even require any knowledge of Orlicz spaces beyond the definitions of Young functions and of the Luxemburg norm, which will both be reviewed.

This being said, a simple special case asserts that if $f, g \geq 0$ are Borel measurable and $\inf f = \inf g$ (see Theorem 3.5 for a full and much more general statement)

$$\|f^{-1}\|_p + \|g^{-1}\|_p \leq 4\|(f \square g)^{-1}\|_p, \quad (1)$$

for every $1 \leq p \leq \infty$, where $\|\cdot\|_p$ is the norm of $L^p := L^p(\mathbb{R}^N)$. The constant 4 is best possible among all constants independent of p , as is readily seen when $f = g = 1$ and $p = \infty$.

The Borel measurability requirement has to do with the measurability of $f \square g$, without which (1) cannot make sense. Curiously, we were unable to find a discussion of the measurability properties of the infimal convolution in the classical literature, but the evidence points to the fact that f and g Lebesgue measurable does not suffice for the measurability of $f \square g$. Indeed, as is well-known, the strict epigraph of $f \square g$ is the (vector, also called Minkowski) sum of the strict epigraphs of f and g and Sierpiński [30] showed, almost a century ago, that the sum of two Lebesgue measurable sets need not be Lebesgue measurable. In contrast, the sum of two Borel sets is always Lebesgue measurable (but not always a Borel set). See Section 2 for further details.

A peculiar feature of (1) and of more general similar inequalities is that only the left-hand side is unchanged by modifications of f and g on null sets, as long as Borel measurability and $\inf f = \inf g > -\infty$ are preserved.

Most of the paper is actually devoted to perhaps more important -and definitely

more delicate- reverse inequalities which, in a simpler world, would read

$$\|(f \square g)^{-1}\|_p \leq C(\|f^{-1}\|_p + \|g^{-1}\|_p), \quad (2)$$

with $C > 0$ independent of f and g in some suitable class of nonnegative functions. Unfortunately, the main obstacle to (2) is that no remotely general converse of the Brunn-Minkowski inequality holds in any form, even for convex sets. Such a converse is actually trivially true for Euclidean balls, but a direct application of this remark only yields (2) for a narrow subclass of radially symmetric functions.

To take advantage of the converse of the Brunn-Minkowski inequality for balls in a much broader setting, we introduce a new function transform, strongly reminiscent of, yet very different from, nonincreasing rearrangement. The difference is that the upper level sets are rounded before being rearranged, the rounding being performed by using the concept of enclosing ball (see Section 4).

To each function $f : \mathbb{R}^N \rightarrow [-\infty, \infty]$ (no measurability needed), the aforementioned transform associates a measurable radially symmetric function $\hat{f}$, which in turn produces another measurable radially symmetric function $\check{f} := -(-\hat{f})$. In the special case when $f, g \geq 0$ are Borel measurable and $\inf f = \inf g$ (see Theorem 6.1 for a full and much more general statement), the reverse inequality in L^p reads (compare with (1) under the same assumptions)

$$\|(f \square g)^{-1}\|_p \leq 2^{N-1}(\|\check{f}^{-1}\|_p + \|\check{g}^{-1}\|_p). \quad (3)$$

Such an inequality breaks down completely if $\check{f}$ and $\check{g}$ are replaced with f and g , respectively, even if both functions are radially symmetric. For example, if $N = 1, f(x) = x^2 + 1$ and $g(x) = x^2 + 1$ when $x \notin \mathbb{Q}, g(x) = 1$ if $x \in \mathbb{Q}$, then f and g are Borel measurable and $\inf f = \inf g = 1$. But $(f \square g)^{-1} = 1/2$ is in no L^p space with $p < \infty$, whereas f^{-1} and $g^{-1} = f^{-1}$ a.e. are in all of them. (In this example, it turns out that $\check{f} = f$ but $\check{g} = 1$.) This example also shows that, unlike in (1), neither side of (3) is independent of modifications of f or g on null sets that do not affect Borel measurability or $\inf f = \inf g$.

When not trivial (i.e., $\check{f} = f$), the explicit calculation of $\check{f}$ is generally not possible. Nevertheless, the inequality (3) is useful because some simple and general conditions about f and g ensure the finiteness of the right-hand side (Lemma 6.3). There is certainly more to be discovered in that regard.

The proofs of the inequalities involve two other classical extremal operations

$$(f \bar{\wedge} g)(x) := \sup_y \min\{f(x-y), g(y)\}$$

and $(f \vee g)(x) := \inf_y \max\{f(x-y), g(y)\}.$

Either of these operations fully determines the other, but both notations will be useful. In general, $f \vee g = -(-f) \bar{\wedge} (-g)$ and, for nonnegative functions,

$f \vee g = (f^{-1} \bar{\wedge} g^{-1})^{-1}$ will be important. We also prove inequalities similar to (1) and (3) for the operations $\bar{\wedge}$ and $\vee$ (both being often referred to as “level sum” operations in the literature). In fact, a good part of the work will consist in proving integral inequalities for $\bar{\wedge}$, from which those for $\square$ and $\vee$ will be derived.

In the last section, the inequalities are used to obtain L^p (and other) estimates for the inverses of solutions of the Hamilton-Jacobi equations and variants thereof.

Throughout the paper, μ_N denotes the N -dimensional Lebesgue measure and, without a qualifier, measurability always means Lebesgue measurability.

2. Background

The purpose of this short section is to review the basic properties of the operations mentioned in the Introduction, to set the notation used in future sections and to settle basic measurability issues.

Recall that if X and Y are subsets of $\mathbb{R}^N$, their sum $X + Y$ is defined by

$$X + Y := \begin{cases} \{x + y : x \in X, y \in Y\} & \text{if } X \neq \emptyset \text{ and } Y \neq \emptyset, \\ \emptyset & \text{if } X = \emptyset \text{ or } Y = \emptyset. \end{cases}$$

The following key lemma is well-known. The “proof” below merely makes the connection with the deep property behind it.

Lemma 2.1. *If X and Y are Borel subsets of $\mathbb{R}^N$, their sum $X + Y$ is measurable¹.*

Proof. In Euclidean space (any dimension), the continuous image of a Borel set is Lebesgue measurable; see Federer [8, p. 69]. Since $X \times Y$ is a Borel subset of $\mathbb{R}^N \times \mathbb{R}^N$ and the addition is continuous on $\mathbb{R}^N$, the result follows. $\square$

Given two functions $f, g : \mathbb{R}^N \rightarrow [-\infty, \infty]$ and $\xi \in \mathbb{R}$, call F_ξ^+ and G_ξ^+ the upper level sets

$$F_\xi^+ := \{x \in \mathbb{R}^N : f(x) > \xi\}, \quad G_\xi^+ := \{x \in \mathbb{R}^N : g(x) > \xi\} \quad (4)$$

and call W_ξ^+ the corresponding upper level set of $f \bar{\wedge} g$:

$$W_\xi^+ := \{x \in \mathbb{R}^N : (f \bar{\wedge} g)(x) > \xi\}. \quad (5)$$

It is a standard elementary property that

$$W_\xi^+ = F_\xi^+ + G_\xi^+. \quad (6)$$

By Lemma 2.1 and since $f \vee g = -(-f) \bar{\wedge} (-g)$, it follows at once from (6) that:

¹Even a Suslin set, but not necessarily a Borel set.

Lemma 2.2. *If $f, g : \mathbb{R}^N \rightarrow [-\infty, \infty]$ are Borel measurable, then $f \bar{\wedge} g$ and $f \vee g$ are measurable.*

If $f : \mathbb{R}^N \rightarrow [-\infty, \infty]$ we set

$$M_f := \sup f, \quad m_f := \inf f. \quad (7)$$

The next relations are elementary, but important

$$\begin{aligned} M_{f \bar{\wedge} g} &= \min\{M_f, M_g\}, & m_{f \bar{\wedge} g} &\geq \min\{m_f, m_g\}, \\ M_{f \vee g} &\leq \max\{M_f, M_g\}, & m_{f \vee g} &= \max\{m_f, m_g\}. \end{aligned} \quad (8)$$

We now turn to infimal convolution. Given a function $f : \mathbb{R}^N \rightarrow (-\infty, \infty]$, we denote by $E_f := \{(x, \xi) \in \mathbb{R}^N \times \mathbb{R} : f(x) < \xi\}$ the strict epigraph of f . It is also a simple well-known property that if $g : \mathbb{R}^N \rightarrow (-\infty, \infty]$ is another function, then

$$E_{f \square g} = E_f + E_g.$$

Since a function is Borel measurable (measurable) if and only if its strict epigraph is a Borel set (measurable), it follows from Lemma 2.1 that

Lemma 2.3. *If $f, g : \mathbb{R}^N \rightarrow (-\infty, \infty]$ are Borel measurable, then $f \square g$ is measurable.*

For future use, we also note that if $z \in \mathbb{R}$,

$$f \square g = (f - z) \square (g + z). \quad (9)$$

3. First integral inequalities

The Brunn-Minkowski inequality (see e.g. Gardner's survey [10]) asserts that if X, Y are nonempty measurable subsets of $\mathbb{R}^N$ and if $X + Y$ is measurable, then $\mu_N(X + Y)^{1/N} \geq \mu_N(X)^{1/N} + \mu_N(Y)^{1/N}$. Obviously, it fails if X or Y is empty and the other has positive measure. We shall only need the less sharp form

$$\mu_N(X + Y) \geq \mu_N(X) + \mu_N(Y), \quad (10)$$

if X, Y and $X + Y$ are measurable and $X \neq \emptyset, Y \neq \emptyset$.

Lemma 3.1. *If $f, g : \mathbb{R}^N \rightarrow [0, \infty]$ are measurable and $M_f = M_g$ (possibly ∞ ; see (7)) and if $f \bar{\wedge} g$ is measurable, then*

$$\int_{\mathbb{R}^N} f + \int_{\mathbb{R}^N} g \leq \int_{\mathbb{R}^N} f \bar{\wedge} g. \quad (11)$$

Proof. Set $M := M_f = M_g \leq \infty$. If $M = 0$, then $f = g = f \bar{\wedge} g = 0$ and (11) is trivial. In what follows, $M > 0$.

Since $f \geq 0$, it is well-known (see (4)) that $\int_{\mathbb{R}^N} f = \int_0^\infty \mu_N(F_\xi^+) d\xi$. By using $F_\xi^+ = \emptyset$ when $\xi \geq M$, this reads $\int_{\mathbb{R}^N} f = \int_0^M \mu_N(F_\xi^+) d\xi$. Likewise, $\int_{\mathbb{R}^N} g = \int_0^M \mu_N(G_\xi^+) d\xi$, so that $\int_{\mathbb{R}^N} f + \int_{\mathbb{R}^N} g = \int_0^M (\mu_N(F_\xi^+) + \mu_N(G_\xi^+)) d\xi$.

By (5) and (6) and since $f \bar{\wedge} g$ is measurable by hypothesis, $F_\xi^+ + G_\xi^+ = W_\xi^+$ is measurable for every ξ . If $\xi < M$, then $F_\xi^+ \neq \emptyset$ and $G_\xi^+ \neq \emptyset$ by definition of M and so, by (10), $\mu_N(F_\xi^+) + \mu_N(G_\xi^+) \leq \mu_N(F_\xi^+ + G_\xi^+) = \mu_N(W_\xi^+)$. Therefore, $\int_{\mathbb{R}^N} f + \int_{\mathbb{R}^N} g \leq \int_0^M \mu_N(W_\xi^+) d\xi \leq \int_0^\infty \mu_N(W_\xi^+) d\xi$ (by (8), the second inequality is even an equality). Now, $\int_0^\infty \mu_N(W_\xi^+) d\xi = \int_{\mathbb{R}^N} f \bar{\wedge} g$ since $f \bar{\wedge} g \geq 0$ and the proof is complete. $\square$

Of course, (11) does not follow from a pointwise inequality. The condition $M_f = M_g$ cannot be dropped. For example, if $f > 0$ and $g = 0$, then $f \bar{\wedge} 0 = 0$ and (11) fails.

Lemma 3.1 is just the stepping stone for much more general inequalities. Recall that in the theory of Orlicz spaces, a nonconstant function $\phi : [0, \infty] \rightarrow [0, \infty]$ is called a *Young function* if $\phi(0) = 0$ and ϕ is nondecreasing, convex and left continuous ([27]; see also [1], [18] for a simplified treatment limited to N -functions). In particular, $\phi(\infty) = \infty$.

Remark 3.2. If ϕ is a Young function and $h : \mathbb{R}^N \rightarrow [0, \infty]$ is measurable, the monotonicity of ϕ shows at once that $\phi(h)$ is measurable.

If ϕ is a Young function, the corresponding Orlicz space L_ϕ consists of all the measurable functions h on $\mathbb{R}^N$ such that $\int_{\mathbb{R}^N} \phi(\lambda|h|) < \infty$ for some $\lambda > 0$ (this makes sense by Remark 3.2). It is a (complete) normed space for the Luxemburg norm $\|\cdot\|_\phi$ defined by

$$\|h\|_\phi := \inf \left\{ r > 0 : \int_{\mathbb{R}^N} \phi(r^{-1}|h|) \leq 1 \right\}. \quad (12)$$

Since the right-hand side of (12) is finite if and only if $h \in L_\phi$, it will always be understood that $\|h\|_\phi = \infty$ when h is measurable and $h \notin L_\phi$. Thus, $h \in L_\phi$ is equivalent to $\|h\|_\phi < \infty$. Furthermore, it is readily checked that

$$\|h\|_\phi \leq \|k\|_\phi \text{ if } |h| \leq |k| \quad (13)$$

and, by the left-continuity of ϕ and monotone convergence², that if $h \in L_\phi$,

$$\int_{\mathbb{R}^N} \phi(\|h\|_\phi^{-1}|h|) \leq 1. \quad (14)$$

If $\phi(\tau) := \tau^p$ for some $1 \leq p < \infty$, then $\|h\|_\phi = \|h\|_p$. On the other hand, $\|h\|_\phi = \|h\|_\infty$ when ϕ is the indicator function of $[0, 1]$ ($\phi = 0$ in $[0, 1]$ and ∞ outside).

²This is of course a well-known inequality.

Lemma 3.3. *If ϕ is a Young function and if $h : \mathbb{R}^N \rightarrow [0, \infty]$, then (see (7)) $M_{\phi(h)} = \phi(M_h)$.*

Proof. It is plain that $h \leq M_h$ implies $\phi(h) \leq \phi(M_h)$, so that $M_{\phi(h)} \leq \phi(M_h)$. It only remains to show that $\phi(M_h) \leq M_{\phi(h)}$, which is trivial if $M_h = 0$. We henceforth assume $M_h > 0$.

By the monotonicity of ϕ and $\phi(0) = 0$, there is $\tau_1 \in [0, \infty]$ such that $\phi = \infty$ on $(\tau_1, \infty]$ and that $\phi < \infty$ on $[0, \tau_1)$. Specifically, $\tau_1 = \sup\{\tau \geq 0 : \phi(\tau) < \infty\}$. If $\tau_1 \in (0, \infty)$, then $\phi(\tau_1)$ may be finite or infinite. We split the proof into three cases.

(i) $M_h > \tau_1$. If so, $\tau_1 < \infty$ and then $\phi(M_h) = \infty$. The set $\{x \in \mathbb{R}^N : h(x) > \tau_1\}$ is not empty and $\phi(h(x)) = \infty$ for every x in that set. Thus, $\{x \in \mathbb{R}^N : \phi(h(x)) = \infty\} \neq \emptyset$, so that $M_{\phi(h)} = \infty = \phi(M_h)$.

(ii) $M_h = \tau_1 = \infty$. Then, $\phi(M_h) = \phi(\infty) = \infty$. If $\tau > 0$ is finite, $\emptyset \neq \{x \in \mathbb{R}^N : h(x) > \tau\} \subset \{x \in \mathbb{R}^N : \phi(h(x)) \geq \phi(\tau)\}$ (by the monotonicity of ϕ). As a result, $M_{\phi(h)} \geq \phi(\tau)$. By letting $\tau \rightarrow \infty = \tau_1$ and since $\lim_{\tau \rightarrow \infty} \phi(\tau) = \phi(\infty) = \infty$ by the left continuity of ϕ , it follows that $M_{\phi(h)} = \infty = \phi(M_h)$.

(iii) $0 < M_h \leq \tau_1$. If $M_h = \infty$, then $\tau_1 = \infty$ and (ii) above applies. Assume now $M_h < \infty$. For every $\varepsilon > 0$, $S := \{x \in \mathbb{R}^N : h(x) > M_h - \varepsilon\} \neq \emptyset$. If ε is small enough, then $M_h - \varepsilon > 0$ and $S \subset \{x \in \mathbb{R}^N : \phi(h(x)) \geq \phi(M_h - \varepsilon)\}$ by the monotonicity of ϕ . Hence, $M_{\phi(h)} \geq \phi(M_h - \varepsilon)$. Since ϕ is left continuous, $M_{\phi(h)} \geq \phi(M_h)$. $\square$

From Lemma 3.1 and Lemma 3.3, we obtain:

Lemma 3.4. *Let $\phi : [0, \infty] \rightarrow [0, \infty]$ be a Young function. If $f, g : \mathbb{R}^N \rightarrow [0, \infty]$ are Borel measurable and if $M_f = M_g$ (possibly ∞), then $f \bar{\wedge} g$ is measurable and*

$$\max\{\|f\|_\phi, \|g\|_\phi\} \leq \|f \bar{\wedge} g\|_\phi. \quad (15)$$

Proof. By Lemma 2.2, $f \bar{\wedge} g$ is measurable and so, by Remark 3.2, $\phi(f)$, $\phi(g)$ and $\phi(f \bar{\wedge} g)$ are measurable. Since $\phi(\min\{f(y), g(x-y)\}) = \min\{\phi(f(y)), \phi(g(x-y))\}$ by the monotonicity of ϕ , we infer that $\sup_y \phi(\min\{f(y), g(x-y)\}) = (\phi(f) \bar{\wedge} \phi(g))(x)$. By Lemma 3.3 with $h(y) := \min\{f(y), g(x-y)\}$, the left-hand side is $\phi((f \bar{\wedge} g)(x))$, so that $\phi(f \bar{\wedge} g) = \phi(f) \bar{\wedge} \phi(g)$. In particular, $\phi(f) \bar{\wedge} \phi(g)$ is measurable.

Since $M_f = M_g$, then $M_{\phi(f)} = M_{\phi(g)}$, once again by Lemma 3.3. Thus, from the above and from Lemma 3.1 with f and g replaced with $\phi(f)$ and $\phi(g)$, respectively,

$$\int_{\mathbb{R}^N} \phi(f) + \int_{\mathbb{R}^N} \phi(g) \leq \int_{\mathbb{R}^N} \phi(f \bar{\wedge} g). \quad (16)$$

If $r > 0$, then $r^{-1}(f \bar{\wedge} g) = r^{-1}f \bar{\wedge} r^{-1}g$ and $M_{r^{-1}f} = M_{r^{-1}g}$. Thus, (16) for $r^{-1}f$ and $r^{-1}g$ yields $\int_{\mathbb{R}^N} \phi(r^{-1}f) \leq \int_{\mathbb{R}^N} \phi(r^{-1}(f \bar{\wedge} g))$, so that $\|f\|_\phi \leq \|f \bar{\wedge} g\|_\phi$ by

(12). Likewise, $\|g\|_\phi \leq \|f \bar{\wedge} g\|_\phi$ and (15) follows. $\square$

Of course, when $L_\phi = L^1$, Lemma 3.1 yields the stronger $\|f\|_1 + \|g\|_1 \leq \|f \bar{\wedge} g\|_1$ but (15) is optimal when ϕ is arbitrary (let $f = g = 1$ and $L_\phi = L_\infty$).

We are now in a position to prove our first main integral inequality for infimal convolution. Recall once more the notation (7).

Theorem 3.5. *Suppose that $f, g : \mathbb{R}^N \rightarrow (-\infty, \infty]$ are Borel measurable, that $m_f, m_g \in \mathbb{R}$ and that $m_f + m_g \geq 0$. Set*

$$m_{f,g} = (m_f - m_g)/2. \quad (17)$$

Then, $f - m_{f,g}, g + m_{f,g}$ and $f \square g$ are measurable and nonnegative and

$$\|(f - m_{f,g})^{-1}\|_\phi + \|(g + m_{f,g})^{-1}\|_\phi \leq 4\|(f \square g)^{-1}\|_\phi, \quad (18)$$

for every Young function ϕ .

Proof. The measurability of $f \square g$ was established in Lemma 2.3. Next, $\inf(f - m_{f,g}) = \inf(g + m_{f,g}) = (m_f + m_g)/2 \geq 0$, whence $f \square g = (f - m_{f,g}) \square (g + m_{f,g}) \geq 0$ by (9). This also implies $(f - m_{f,g})^{-1} \geq 0, (g + m_{f,g})^{-1} \geq 0$ and $\sup(f - m_{f,g})^{-1} = 2(m_f + m_g)^{-1} = \sup(g + m_{f,g})^{-1}$. Therefore, the inequality (15) is applicable in the form

$$\|(f - m_{f,g})^{-1}\|_\phi + \|(g + m_{f,g})^{-1}\|_\phi \leq 2\|(f - m_{f,g})^{-1} \bar{\wedge} (g + m_{f,g})^{-1}\|_\phi$$

and (18) follows from $0 \leq h^{-1} \bar{\wedge} k^{-1} = (h \vee k)^{-1} \leq 2(h \square k)^{-1}$ when h and k are nonnegative, from $f \square g = (f - m_{f,g}) \square (g + m_{f,g}) \geq 0$ and from (13). $\square$

It was noted in the Introduction that the constant 4 in (18) is already best possible when $L_\phi = L^\infty$.

Remark 3.6. Theorem 3.5 gives a simple necessary condition for the existence of solutions of infimal convolution equations (see [23], [21] and the references therein): Suppose that $h \geq 0$ is measurable and that g is Borel measurable and bounded below. If $h^{-1} \in L_\phi$ for some Orlicz space L_ϕ and $\|(g - m_g + m_h/2)^{-1}\|_\phi > 4\|h^{-1}\|_\phi$ (in particular, if $(g - m_g + m_h/2)^{-1} \notin L_\phi$), the equation $f \square g = h$ has no Borel measurable solution f . Indeed, if f exists, then $m_f = m_h - m_g \in \mathbb{R}$ and (18) cannot hold.

If $z \neq 0$ is a constant, there is no simple pointwise relationship between $(f - z) \vee (g + z)$ and $f \vee g$. As a result, the method of proof of Theorem 3.5 does not yield a variant of (14) or (18) with $f \square g$ replaced with $f \vee g$. However, if $f, g \geq 0$, then $0 \leq f \vee g \leq f \square g$ and such a variant can be obtained as a straightforward corollary of Theorem 3.5:

Corollary 3.7. *Suppose that $f, g : \mathbb{R}^N \rightarrow [-\infty, \infty]$ are Borel measurable, that $g \geq 0$ and that $f \not\equiv \infty$ and $g \not\equiv \infty$, so that $0 \leq m_{f_+} + m_g < \infty$, where $f_+ := \max\{f, 0\}$. Then, $f_+ - m_{f_+,g}, g + m_{f_+,g}$ (see (17)) and $f \vee g$ are measurable and nonnegative and*

$$\|(f_+ - m_{f_+,g})^{-1}\|_\phi + \|(g + m_{f_+,g})^{-1}\|_\phi \leq 4\|(f \vee g)^{-1}\|_\phi, \quad (19)$$

for every Young function ϕ .

Proof. Since $g \geq 0$, it follows that $0 \leq f \vee g = f_+ \vee g \leq f_+ \square g$. By Lemma 2.2, $f \vee g = f_+ \vee g$ is measurable. Therefore, the corollary follows from (13) and from Theorem 3.5 for f_+ and g . $\square$

The constant 4 is also best possible in (19) (among constants independent of ϕ): If $f = 1$ and $g = \ell > 1$ is constant, the inequality for the L^∞ norm is $4(\ell + 1)^{-1} \leq 4\ell^{-1}$. By letting $\ell \rightarrow \infty$, it follows that, in the right-hand side, 4 cannot be lowered.

4. The radial transforms $\hat{f}$ and $\check{f}$

The proof of Lemma 3.1 shows that the existence of a converse of the inequality (11), that is,

$$\int_{\mathbb{R}^N} f \bar{\wedge} g \leq C \left(\int_{\mathbb{R}^N} f + \int_{\mathbb{R}^N} g \right),$$

with $C > 0$ independent of f and g would require $\mu_N(F_\xi^+ + G_\xi^+) \leq C(\mu_N(F_\xi^+) + \mu_N(G_\xi^+))$ for every $\xi > 0$. However, as pointed out in the Introduction, no converse of the Brunn-Minkowski inequality or its weaker form (10) holds in any generality.

The transforms defined in this section will enable us (in the next section) to take advantage of the fact that such a converse trivially exists when X and Y are Euclidean balls. The thought that this case is so special that it cannot have any broad value would result in a serious oversight.

By a classical theorem of Jung [8, p. 200], [17], every nonempty bounded subset X of $\mathbb{R}^N$ is contained in a *unique* closed ball $\overline{B}_X$ with minimal diameter among all closed balls containing X , called the *enclosing ball* of X . If X is unbounded, no closed ball contains X and we set $\overline{B}_X := \mathbb{R}^N$. Lastly, if $X = \emptyset$, every singleton $\{x\}$ satisfies the “minimal diameter” requirement, whence uniqueness, but not existence, is lost. For definiteness, we arbitrarily set $\overline{B}_\emptyset := \{0\}$. Evidently, $X \subset \overline{B}_X$ in all cases. Jung’s theorem also provides the estimate $\text{diam}(X) \leq \text{diam}(\overline{B}_X) \leq \sqrt{2N/(N+1)} \text{diam}(X)$, but we shall only make use of the (trivial) first one.

Remark 4.1. An easily overlooked aspect of enclosing balls is that $X \subset Y$ implies $\mu_N(\overline{B}_X) \leq \mu_N(\overline{B}_Y)$ but *not* $\overline{B}_X \subset \overline{B}_Y$, unless $N = 1$.

The following property of enclosing balls will be important.

Lemma 4.2. *If X_n is a nondecreasing sequence of subsets of $\mathbb{R}^N$, then $\mu_N(\overline{B}_{\cup X_n}) = \lim \mu_N(\overline{B}_{X_n}) = \sup \mu_N(\overline{B}_{X_n})$.*

Proof. Set $X := \cup X_n$. If X is unbounded, then $\overline{B}_X = \mathbb{R}^N$ and so $\mu_N(\overline{B}_X) = \infty$. Since $X := \cup X_n$ and $X_n \subset X_{n+1}$, the diameter of X_n and, hence, that of $\overline{B}_{X_n}$, tends to ∞ . Accordingly, $\lim \mu_N(\overline{B}_{X_n}) = \infty$.

Suppose now that X is bounded, so that $\overline{B}_X$ is a ball. Since $X_n \subset X$ implies $\mu_N(\overline{B}_{X_n}) \leq \mu_N(\overline{B}_X)$ and since $\mu_N(\overline{B}_{X_n})$ is nondecreasing, it is plain that $\lim \mu_N(\overline{B}_{X_n}) \leq \mu_N(\overline{B}_X)$. To prove the converse, call $r_n \geq 0$ the radius of $\overline{B}_{X_n}$. The sequence r_n is nondecreasing and bounded above (by the radius of $\overline{B}_X$) and so it has a limit $r \geq r_n$ for every n . As a result, $\lim \mu_N(\overline{B}_{X_n})$ is the measure of any ball with radius r .

Next, call x_n the center of $\overline{B}_{X_n}$. By a simple contradiction argument, the sequence x_n is bounded (since x_n might not be in $\overline{B}_X$ -see Remark 4.1- this is not totally trivial). After extracting a subsequence, assume that $x_n \rightarrow x \in \mathbb{R}^N$. Every $y \in X$ is in X_n for n large enough. Since $\overline{B}_{X_n} \subset \overline{B}(x_n, r)$, it follows that $y \in \overline{B}(x, r)$. Thus, $X \subset \overline{B}(x, r)$, whence $\mu_N(\overline{B}(x, r)) \geq \mu_N(\overline{B}_X)$ by definition of $\overline{B}_X$. Since $r = \lim r_n$ amounts to $\mu_N(\overline{B}(x, r)) = \lim \mu_N(\overline{B}_{X_n})$, it follows that $\lim \mu_N(\overline{B}_{X_n}) \geq \mu_N(\overline{B}_X)$. $\square$

Given any function $f : \mathbb{R}^N \rightarrow [-\infty, \infty]$, we now proceed to constructing a measurable radially symmetric function $\hat{f} : \mathbb{R}^N \rightarrow [-\infty, \infty]$ whose upper level sets $\hat{F}_\xi^+$ have measure equal to $\mu_N(\overline{B}_{F_\xi^+})$ for every ξ . The construction follows that of the nonincreasing rearrangement of f .

Lemma 4.3. *The function $\mu_N(\overline{B}_{F_\xi^+})$ is nonincreasing and right-continuous on $[-\infty, \infty]$.*

Proof. If $\xi < \eta$, then $F_\eta^+ \subset F_\xi^+$, so that $\mu_N(\overline{B}_{F_\eta^+}) \leq \mu_N(\overline{B}_{F_\xi^+})$. For the right continuity, let $\xi_n \searrow \xi$, so that $F_\xi^+ = \cup F_{\xi_n}^+$ and then, by Lemma 4.2, $\mu_N(\overline{B}_{F_\xi^+}) = \lim \mu_N(\overline{B}_{F_{\xi_n}^+})$. $\square$

Call $\rho_f^+(\xi)$ the radius of $\overline{B}_{F_\xi^+}$. Since $\rho_f^+(\xi)$ is proportional to $\mu_N(\overline{B}_{F_\xi^+})^{1/N}$, it follows from Lemma 4.3 that ρ_f^+ is nonincreasing and right-continuous. Therefore,

$$\gamma_f^+(t) := \inf\{\xi : \rho_f^+(\xi) \leq t\}, \quad (20)$$

is a nonincreasing and right-continuous function on $[0, \infty)$ and

$$\{t \geq 0 : \gamma_f^+(t) > \xi\} = [0, \rho_f^+(\xi)). \quad (21)$$

Indeed, when $f \geq 0$ and $\rho_f^+(\xi)$ is replaced with $\mu_N(F_\xi^+)$, γ_f^+ becomes the non-increasing rearrangement of f and these properties follow uniquely from the

monotonicity and right-continuity of $\mu_N(F_\xi^+)$; see for instance [33, pp. 26–27]. We also point out that in most modern expositions, the nonincreasing rearrangement of a function f is defined to be that of $|f|$. This has not always been the case (see Day [7] or Luxemburg [22]) and the monotonicity and right-continuity properties of nonincreasing rearrangements are independent of whether f or $|f|$ is used in their definition.

We now set

$$\hat{f}(x) := \gamma_f^+(|x|). \quad (22)$$

Some basic properties of $\hat{f}$ are summarized in the next theorem.

Theorem 4.4. *Given $f : \mathbb{R}^N \rightarrow [-\infty, \infty]$, the function $\hat{f}$ has the following properties:*

- (i) $\hat{f}$ is measurable and $\hat{f} = f$ if and only if f is a right-continuous nonincreasing function of $|x|$.
- (ii) $\mu_N(\hat{F}_\xi^+) = \mu_N(\overline{B}_{F_\xi^+})$ for every $\xi \in [-\infty, \infty]$, where $\hat{F}_\xi^+$ denotes the upper ξ -level set of $\hat{f}$.
- (iii) $M_{\hat{f}} \leq M_f$ and $m_{\hat{f}} \geq m_f$ (in particular, $f \geq 0 \Rightarrow \hat{f} \geq 0$). Furthermore, $M_{\hat{f}} = \text{ess sup } \hat{f}$ and $m_{\hat{f}} = \text{ess inf } \hat{f}$.
- (iv) $(f + z)^\wedge = \hat{f} + z$ for $z \in \mathbb{R}$ and $(f(\cdot))^\wedge = \hat{f}(\cdot)$ for $c \in \mathbb{R} \setminus \{0\}$.
- (v) $(cf)^\wedge = c\hat{f}$ for every $c \geq 0$.
- (vi) If $h : \mathbb{R}^N \rightarrow [-\infty, \infty]$ and $h \leq f$, then $\hat{h} \leq \hat{f}$.
- (vii) If f is bounded below on bounded subsets and $\lim_{|x| \rightarrow \infty} f(x) = -\infty$ and if $h : \mathbb{R}^N \rightarrow [-\infty, \infty]$ satisfies $h(x) \leq f(x)$ for $|x|$ large enough, then $\hat{h}(x) \leq \hat{f}(x)$ for $|x|$ large enough. Furthermore, if f is a right-continuous strictly decreasing function of $|x|$ and if $h(x) \leq f(x)$ when $x \notin B$ for some open ball B centered at the origin, then $\hat{h}(x) \leq \hat{f}(x)$ ($= f(x)$ by (i)) for every $x \notin B$.

Proof. (i) The measurability of $\hat{f}$ follows at once from the monotonicity of γ_f^+ and the necessity of the given conditions for $\hat{f} = f$ is obvious. Conversely, if $f(x) = \gamma(|x|)$ with $\gamma : [0, \infty) \rightarrow [-\infty, \infty]$ nonincreasing, the upper level sets of f are balls centered at the origin (possibly $\mathbb{R}^N$) and ρ_f^+ is the distribution function of γ , so that $\gamma_f^+ = \gamma^*$, the nonincreasing rearrangement of γ . Since $\gamma^* = \gamma$ except perhaps at the countably many points of discontinuity of γ , it follows that $\hat{f} = f$ a.e. If γ is right-continuous, then $\gamma^* = \gamma$ and $\hat{f} = f$.

(ii) Just notice that, by (21) and (22), $\hat{F}_\xi^+$ is the open ball with center 0 and radius $\rho_f^+(\xi)$.

(iii) With no loss of generality, assume $M_f < \infty$. Since γ_f^+ is nonincreasing, $\max \gamma_f^+ = \gamma_f^+(0) = \inf\{\xi : \rho_f^+(\xi) = 0\}$. If $\xi \geq M_f$, then $F_\xi^+ = \emptyset$, whence $\overline{B}_{F_\xi^+} = \{0\}$ and so $\rho_f^+(\xi) = 0$. Thus, $\max \gamma_f^+ = \inf\{\xi : \mu_N(F_\xi^+) = 0\} \leq M_f$. On the

other hand, by (22), $\max \gamma_f^+ = M_{\hat{f}}$. This shows that $M_{\hat{f}} \leq M_f$. Furthermore, $M_{\hat{f}} = \text{ess sup } \hat{f}$ by (22) and the right-continuity of γ_f^+ .

That $m_{\hat{f}} \geq m_f$ is obvious if $m_f = -\infty$, or if $m_f = \infty$ (for then $f = \hat{f} = \infty$). If $m_f \in \mathbb{R}$ and $\xi < m_f$, then $\rho_f^+(\xi) = \infty$. Thus, $\gamma_f^+ \geq m_f$ by (20) and so $\hat{f} \geq m_f$, whence $m_{\hat{f}} \geq m_f$. That $m_{\hat{f}} = \text{ess inf } \hat{f}$ follows from the monotonicity of γ_f^+ .

(iv) Since $\{x \in \mathbb{R}^N : f(x) + z > \xi\} = F_{\xi-z}^+$, it follows that $\rho_{f+z}^+(\xi) = \rho_f^+(\xi - z)$, which in turn yields $\gamma_{f+z}^+ = \gamma_f^+ + z$, i.e., $(f + z)^\wedge = \hat{f} + z$. The proofs that $f(c \cdot)^\wedge = \hat{f}(c \cdot)$ if $c \in \mathbb{R} \setminus \{0\}$ is equally straightforward.

(v) Since this is trivial when $c = 0$, assume $c > 0$. Then, $\rho_{cf}^+(\xi) = \rho_f^+(\xi/c)$, whence $\gamma_{cf}^+ = c\gamma_f^+$, i.e. $(cf)^\wedge = c\hat{f}$.

(vi) If $h \leq f$, then (with a self-explanatory notation) $H_\xi^+ \subset F_\xi^+$ and so $\mu_N(\overline{B}_{H_\xi^+}) \leq \mu_N(\overline{B}_{F_\xi^+})$. Hence, $\rho_h^+(\xi) \leq \rho_f^+(\xi)$ and, by (20), $\gamma_h^+ \leq \gamma_f^+$, so that $\hat{h} \leq \hat{f}$.

(vii) Choose an open ball B centered at the origin such that $h(x) \leq f(x)$ when $x \notin B$. Since f is bounded below on bounded subsets, $\inf_B f$ is finite and, if $\xi_0 < \inf_B f$, then $B \subset F_\xi^+$ for every $\xi \leq \xi_0$. By (vi), $\hat{h}$ is increased when h is increased. Thus, if it can be shown that $\hat{h} \leq \hat{f}$ after increasing h , this inequality also holds before h is increased. In particular, we may increase h on B so that $\xi_0 < \inf_B h$ and then $B \subset H_\xi^+$ for every $\xi \leq \xi_0$. Thus, $B \subset H_\xi^+ \cap F_\xi^+$ for every $\xi \leq \xi_0$. On the other hand, $\{x \notin B : h(x) > \xi\} \subset \{x \notin B : f(x) > \xi\}$ since $h \leq f$ on $\mathbb{R}^N \setminus B$. Altogether, if $\xi \leq \xi_0$, then $H_\xi^+ \subset F_\xi^+$ and so $\rho_h^+(\xi) \leq \rho_f^+(\xi)$. As a result,

$$\begin{aligned} \gamma_h^+(t) &:= \inf\{\xi : \rho_h^+(\xi) \leq t\} \leq \inf\{\xi \leq \xi_0 : \rho_h^+(\xi) \leq t\} \\ &\leq \inf\{\xi \leq \xi_0 : \rho_f^+(\xi) \leq t\}. \end{aligned} \quad (23)$$

Since $\lim_{|x| \rightarrow \infty} f(x) = -\infty$, the level set $F_{\xi_0}^+$ is bounded, whence $\rho_f^+(\xi_0) < \infty$. Choose any $t \geq \rho_f^+(\xi_0)$. If $\xi > \xi_0$, then $\rho_f^+(\xi) \leq \rho_f^+(\xi_0) \leq t$ by the monotonicity of ρ_f^+ , so that $\gamma_f^+(t) := \inf\{\xi : \rho_f^+(\xi) \leq t\} = \inf\{\xi \leq \xi_0 : \rho_f^+(\xi) \leq t\}$. By (23), $\gamma_h^+(t) \leq \gamma_f^+(t)$. Since this is true for every $t \geq \rho_f^+(\xi_0)$, it follows that $\hat{h}(x) \leq \hat{f}(x)$ when $|x| \geq \rho_f^+(\xi_0)$.

To complete the proof, assume in addition that $f(x)$ is a strictly decreasing function of $|x|$. We show that $\hat{h}(x) \leq \hat{f}(x)$ when $x \notin B$. If $B = \emptyset$, the result follows from (v). From now on, assume $B \neq \emptyset$ (hence $B \neq \{0\}$ as well since B is open).

By the monotonicity of f in $|x|$, $\inf_{tB} f < \inf_B f$ if $t > 1$. It follows that $F_{\xi_0}^+ \subset tB$ if $t > 1$ and ξ_0 above is close enough to $\inf_B f$. Thus, $\overline{B}_{F_{\xi_0}^+} \subset t\overline{B}$,

so that $\rho_f^+(\xi_0) \leq t\rho$ where ρ is the radius of B . From the above, $\hat{h}(x) \leq \hat{f}(x)$ when $|x| \geq t\rho$ and, hence, when $|x| \geq \rho$ by first letting $t \rightarrow 1$ (which gives only $|x| > \rho$) and next using the right-continuity of $\hat{f}$ and $\hat{h}$ with respect to $|x|$. Since ρ is the radius of B and B is centered at the origin, this means that $\hat{h}(x) \leq \hat{f}(x)$ when $x \notin B$. $\square$

Even when $f \geq 0$, it is not true that $\hat{f} = 0$ implies $f = 0$. The following characterization is important for the proof of the reverse inequalities (specifically, of Theorem 5.6 later).

Lemma 4.5. *If $f : \mathbb{R}^N \rightarrow [0, \infty]$ and $\hat{f} = 0$, then either $f = 0$ or there are $x_0 \in \mathbb{R}^N$ and $0 < z \leq \infty$ such that $f(x) = 0$ if $x \neq x_0$ and $f(x_0) = z$.*

Proof. By (20) and (22) and since ρ_f^+ is nonincreasing, $\hat{f} = 0$ means $\rho_f^+(0) = 0$, i.e., that the enclosing ball $\overline{B}_{F_0^+}$ has radius 0, which happens only when $F_0^+ = \emptyset$ or when $F_0^+ = \{x_0\}$ is a singleton. If $F_0^+ = \emptyset$, then $f = 0$ since $f \geq 0$. Suppose now that $F_0^+ = \{x_0\}$. This means that x_0 is the only point where f is positive, so that $f(x_0) = z > 0$ (possibly ∞) and that $f(x) \leq 0$ when $x \neq x_0$. Since $f \geq 0$, it follows that $f(x) = 0$ when $x \neq x_0$. $\square$

Remark 4.6. By the argument of the above proof, the inequality $M_{\hat{f}} \leq M_f$ in Theorem 4.4 (iii) can be made precise: $M_{\hat{f}} < M_f$ if and only if some upper level set of f is a singleton. Put differently, $M_{\hat{f}} = M_f$ if and only if there is a sequence of *distinct* points x_n such that $f(x_n) \rightarrow M_f$.

We shall also need the transform defined by

$$\check{f} := -(-f)^{\hat{}}, \quad (24)$$

directly given by the formula

$$\check{f}(x) = \gamma_f^-(|x|), \quad (25)$$

where

$$\gamma_f^-(t) := \sup\{\xi : \rho_f^-(\xi) \leq t\} \quad (26)$$

and $\rho_f^-(\xi)$ is the radius of $\overline{B}_{F_\xi^-}$ with, of course, $F_\xi^- = \{x \in \mathbb{R}^N : f(x) < \xi\}$.

Theorem 4.7. *Given $f : \mathbb{R}^N \rightarrow [-\infty, \infty]$, the function $\check{f}$ has the following properties:*

- (i) $\check{f}$ is measurable and $\check{f} = f$ if and only if f is a right-continuous nondecreasing function of $|x|$.
- (ii) $\mu_N(\check{F}_\xi^-) = \mu_N(\overline{B}_{F_\xi^-})$ for every $\xi \in [-\infty, \infty]$, where $\check{F}_\xi^-$ denotes the lower ξ -level set of $\check{f}$.
- (iii) $M_{\check{f}} \leq M_f$ and $m_{\check{f}} \geq m_f$ (in particular, $f \geq 0 \Rightarrow \check{f} \geq 0$). Furthermore, $M_{\check{f}} = \text{ess sup } \check{f}$ and $m_{\check{f}} = \text{ess inf } \check{f}$.

- (iv) $(f + z)^\check{} = \check{f} + z$ for every $z \in \mathbb{R}$ and $(f(c \cdot))^\check{} = \check{f}(c \cdot)$ for $c \in \mathbb{R} \setminus \{0\}$.
- (v) $(cf)^\check{} = c\check{f}$ for every $c \geq 0$ (and $(cf)^\check{} = c\check{f}$ for every $c < 0$).
- (vi) If $h : \mathbb{R}^N \rightarrow [-\infty, \infty]$ and $h \leq f$, then $\check{h} \leq \check{f}$.
- (vii) If $h : \mathbb{R}^N \rightarrow [-\infty, \infty]$ is bounded above on bounded subsets, $\lim_{|x| \rightarrow \infty} h(x) = \infty$ and $h(x) \leq f(x)$ for $|x|$ large enough, then $\check{h}(x) \leq \check{f}(x)$ for $|x|$ large enough. Furthermore, if h is a right-continuous strictly increasing function of $|x|$ and if $h(x) \leq f(x)$ when $x \notin B$ for some open ball B centered at the origin, then $\check{h}(x)$ ($= h(x)$ by (i)) $\leq \check{f}(x)$ for every $x \notin B$.
- (viii) If $f \geq 0$, then $(f^{-1})^\check{} = (\check{f})^{-1}$.
- (ix) $(f_+)^\check{} = \check{f}_+$.

Proof. Parts (i) to (vii) follow at once from (24) and from the corresponding properties in Theorem 4.4.

(viii) First, note that if $f \geq 0$ and $\xi < 0$, then $\rho_f^+(\xi) = \infty$ and $\rho_f^-(\xi) = 0$. Hence, by (20) and (26), $\gamma_f^+(t) := \inf\{\xi \geq 0 : \rho_f^+(\xi) \leq t\}$ and $\gamma_f^-(t) := \sup\{\xi \geq 0 : \rho_f^-(\xi) \leq t\} \geq 0$. Upon replacing f with f^{-1} in the former formula, we get (since passing from f to f^{-1} changes upper level sets into lower ones) $\gamma_{f^{-1}}^+(t) := \inf\{\xi \geq 0 : \rho_{f^{-1}}^+(\xi) \leq t\}$. On the other hand, since $\inf(S^{-1}) = (\sup S)^{-1}$ for every subset $S \subset [0, \infty]$, we have $(\gamma_f^-(t))^{-1} = \inf\{\xi^{-1} \geq 0 : \rho_f^-(\xi) \leq t\} = \inf\{\xi \geq 0 : \rho_f^-(\xi^{-1}) \leq t\}$. This shows that $\gamma_{f^{-1}}^+ = (\gamma_f^-)^{-1}$ and the result follows from (22) and (25).

(ix) The lower level sets $\{x \in \mathbb{R}^N : f_+(x) < \xi\}$ are empty if $\xi \leq 0$ and coincide with F_ξ^- if $\xi > 0$. Thus, $\rho_{f_+}^-(\xi) = 0$ if $\xi \leq 0$ and $\rho_{f_+}^-(\xi) = \rho_f^-(\xi)$ if $\xi > 0$. Therefore, by (26),

$$\begin{aligned} \gamma_{f_+}^-(t) &= \max\{\sup\{\xi \leq 0 : \rho_{f_+}^-(\xi) \leq t\}, \sup\{\xi > 0 : \rho_{f_+}^-(\xi) \leq t\}\} \\ &= \max\{0, \sup\{\xi > 0 : \rho_f^-(\xi) \leq t\}\}. \end{aligned} \quad (27)$$

Suppose first that $\gamma_f^-(t) = \sup\{\xi : \rho_f^-(\xi) \leq t\} \leq 0$. Accordingly, $\{\xi > 0 : \rho_f^-(\xi) \leq t\} = \emptyset$ and so $\sup\{\xi > 0 : \rho_f^-(\xi) \leq t\} = -\infty$. By (27), $\gamma_{f_+}^-(t) = \max\{0, -\infty\} = 0 = (\gamma_f^-)_+(t)$. Suppose next that $\gamma_f^-(t) = \sup\{\xi : \rho_f^-(\xi) \leq t\} > 0$, so that $\gamma_f^-(t) = \sup\{\xi > 0 : \rho_f^-(\xi) \leq t\}$. By (27), $\gamma_{f_+}^-(t) = \max\{0, \gamma_f^-(t)\} = (\gamma_f^-)_+(t)$. This shows that $\gamma_{f_+}^- = (\gamma_f^-)_+$, whence $(f_+)^\check{} = \check{f}_+$ by (25). $\square$

Remark 4.8. By (24) and Remark 4.6, $m_{\check{f}} = m_f$ if and only if there is a sequence of *distinct* points x_n such that $f(x_n) \rightarrow m_f$.

5. Reverse inequalities for $f \bar{\wedge} g$

We begin with the (trivial) converse of the Brunn-Minkowski inequality for Euclidean balls.

Lemma 5.1. *If B_1 and B_2 are Euclidean balls in $\mathbb{R}^N$, then $\mu_N(B_1 + B_2) \leq 2^{N-1}(\mu_N(B_1) + \mu_N(B_2))$.*

Proof. Call r_i the radius of $B_i, i = 1, 2$. It is readily checked that $B_1 + B_2$ is a ball with radius $r_1 + r_2$ and the inequality simply follows from $(r_1 + r_2)^N \leq 2^{N-1}(r_1^N + r_2^N)$. $\square$

In the next lemma, $M_f = M_g$ is *not* needed (compare with Lemma 3.1).

Lemma 5.2. *If $f, g : \mathbb{R}^N \rightarrow [0, \infty]$ are Borel measurable, then*

$$\int_{\mathbb{R}^N} f \bar{\wedge} g \leq 2^{N-1} \left(\int_{\mathbb{R}^N} \hat{f} + \int_{\mathbb{R}^N} \hat{g} \right).$$

Proof. By (5) and (6) and since $f \bar{\wedge} g \geq 0$ is measurable (Lemma 2.2), it follows that $\int_{\mathbb{R}^N} f \bar{\wedge} g = \int_0^\infty \mu_N(F_\xi^+ + G_\xi^+) d\xi$. Next, $F_\xi^+ + G_\xi^+ \subset \overline{B}_{F_\xi^+} + \overline{B}_{G_\xi^+}$ and so $\mu_N(F_\xi^+ + G_\xi^+) \leq \mu_N(\overline{B}_{F_\xi^+} + \overline{B}_{G_\xi^+}) \leq 2^{N-1}(\mu_N(\overline{B}_{F_\xi^+}) + \mu_N(\overline{B}_{G_\xi^+}))$, where Lemma 5.1 was used. This yields $\int_{\mathbb{R}^N} f \bar{\wedge} g \leq 2^{N-1}(\int_0^\infty \mu_N(\overline{B}_{F_\xi^+}) d\xi + \int_0^\infty \mu_N(\overline{B}_{G_\xi^+}) d\xi)$. By Theorem 4.4 (ii), the right-hand side is $2^{N-1}(\int_0^\infty \mu_N(\hat{F}_\xi^+) d\xi + \int_0^\infty \mu_N(\hat{G}_\xi^+) d\xi)$, which in turn equals $2^{N-1}(\int_{\mathbb{R}^N} \hat{f} + \int_{\mathbb{R}^N} \hat{g})$ because $\hat{f}, \hat{g} \geq 0$ by Theorem 4.4 (iii). $\square$

No variant of Lemma 5.2 is true if $\hat{f}$ (or $\hat{g}$) is replaced with f (or g):

Example 5.3. With $N = 1$, let $0 < f \leq 1$ be integrable with $f(n) = 1$ for every $n \in \mathbb{Z}$ and let $g = \chi_{(-1,1)} (= \hat{g})$. Then, $f \bar{\wedge} g = 1$, whence $\int_{\mathbb{R}} f \bar{\wedge} g = \infty$ but $f, g \in L^1$.

If $N = 1, f, g \geq 0$ are even and right-continuous and nonincreasing on $[0, \infty)$ and $M_f = M_g$, then, $\int_{\mathbb{R}} f \bar{\wedge} g = \int_{\mathbb{R}} f + \int_{\mathbb{R}} g$ by Theorem 4.4 (i) and Lemmas 3.1 and 5.2. Is there a different proof? (If $f = g$, this follows from $(f \bar{\wedge} f)(x) = f(x/2)$.)

To go further, we need a simple property of Young functions.

Lemma 5.4. *If ϕ is a Young function and $f : \mathbb{R}^N \rightarrow [0, \infty]$, then $(\phi(f))^\wedge = \phi(\hat{f})$.*

Proof. For brevity, we only give the proof in the more important case when $0 < \phi < \infty$ on $(0, \infty)$, so that ϕ is continuous on $[0, \infty]$ and has an inverse ψ . The general case involves extra technicalities that lengthen the exposition. Recall that $\rho_f^+(\xi)$ denotes the radius of $\overline{B}_{F_\xi^+}$.

Since $\phi(f) \geq 0$, it follows that $\rho_{\phi(f)}^+(\xi) = \infty$ if $\xi < 0$. If $\xi \geq 0$, then $\{x \in \mathbb{R}^N : \phi(f(x)) > \xi\} = \{x \in \mathbb{R}^N : f(x) > \psi(\xi)\} = F_{\psi(\xi)}^+$, so that $\rho_{\phi(f)}^+(\xi) = \rho_f^+(\psi(\xi))$. Thus, by (20) and (22), $(\phi(f))^\wedge(x) = \inf\{\xi \geq 0 : \rho_f^+(\psi(\xi)) \leq |x|\} = \inf\{\phi(\eta) :$

$\rho_f^+(\eta) \leq |x|\} = \phi(\inf\{\eta \geq 0 : \rho_f^+(\eta) \leq |x|\})$. Now, $\inf\{\eta \geq 0 : \rho_f^+(\eta) \leq |x|\} = \hat{f}(x)$ by (20) and (22) because $f \geq 0$ implies $\rho_f^+(\eta) = \infty$ if $\eta < 0$. $\square$

Lemma 5.5. *If ϕ is a Young function and $f : \mathbb{R}^N \rightarrow [0, \infty]$ is measurable, then $\|f\|_\phi \leq \|\hat{f}\|_\phi$.*

Proof. Since $f \geq 0$ implies $\hat{f} \geq 0$, it follows from Theorem 4.4 (ii) and from $\mu_N(F_\xi^+) \leq \mu_N(\overline{B}_{F_\xi^+})$ that $\int_{\mathbb{R}^N} f = \int_0^\infty \mu_N(F_\xi^+) d\xi \leq \int_0^\infty \mu_N(\hat{F}_\xi^+) d\xi = \int_{\mathbb{R}^N} \hat{f}$. Upon replacing f by $\phi(f)$ in this inequality, it follows from Lemma 5.4 that $\int_{\mathbb{R}^N} \phi(f) \leq \int_{\mathbb{R}^N} \phi(\hat{f})$. Now, replace f with $r^{-1}f$ where $r > 0$ and use Theorem 4.4 (v) to get $\int_{\mathbb{R}^N} \phi(r^{-1}f) \leq \int_{\mathbb{R}^N} \phi(r^{-1}\hat{f})$ for every $r > 0$. By (12), this implies $\|f\|_\phi \leq \|\hat{f}\|_\phi$. $\square$

Theorem 5.6. *If $f, g : \mathbb{R}^N \rightarrow [0, \infty]$ are Borel measurable and ϕ is a Young function, then $\phi(f), \phi(g)$ and $\phi(f \bar{\wedge} g)$ are measurable and nonnegative and*

$$\|f \bar{\wedge} g\|_\phi \leq 2^{N-1}(\|\hat{f}\|_\phi + \|\hat{g}\|_\phi). \quad (28)$$

Proof. For the measurability of $\phi(f), \phi(g)$ and $\phi(f \bar{\wedge} g)$, see Lemma 3.4 and Remark 3.2. That all three are nonnegative is trivial. In the proof of Lemma 3.4, we already established that $\phi(f \bar{\wedge} g) = \phi(f) \bar{\wedge} \phi(g)$. Therefore, by Lemma 5.2 for $\phi(f)$ and $\phi(g)$ and by Lemma 5.4, we get

$$\int_{\mathbb{R}^N} \phi(f \bar{\wedge} g) \leq 2^{N-1} \left(\int_{\mathbb{R}^N} \phi(\hat{f}) + \int_{\mathbb{R}^N} \phi(\hat{g}) \right). \quad (29)$$

Suppose first that $\|\hat{f}\|_\phi > 0$ and $\|\hat{g}\|_\phi > 0$. Since (28) is trivial otherwise, we may and shall assume $\|\hat{f}\|_\phi < \infty$ and $\|\hat{g}\|_\phi < \infty$. If so, $0 < r := \|\hat{f}\|_\phi + \|\hat{g}\|_\phi < \infty$ and the inequality (29) for $r^{-1}f$ and $r^{-1}g$ is (use Theorem 4.4 (v))

$$2^{1-N} \int_{\mathbb{R}^N} \phi(r^{-1}(f \bar{\wedge} g)) \leq \int_{\mathbb{R}^N} \phi(r^{-1}\hat{f}) + \int_{\mathbb{R}^N} \phi(r^{-1}\hat{g}).$$

With $\lambda := r^{-1}\|\hat{f}\|_\phi \in (0, 1)$, so that $r^{-1}\hat{f} = \lambda\|\hat{f}\|_\phi^{-1}\hat{f}$ and that $r^{-1}\hat{g} = (1 - \lambda)\|\hat{g}\|_\phi^{-1}\hat{g}$, this reads

$$2^{1-N} \int_{\mathbb{R}^N} \phi(r^{-1}(f \bar{\wedge} g)) \leq \int_{\mathbb{R}^N} \phi\left(\lambda\|\hat{f}\|_\phi^{-1}\hat{f}\right) + \int_{\mathbb{R}^N} \phi\left((1 - \lambda)\|\hat{g}\|_\phi^{-1}\hat{g}\right). \quad (30)$$

Since ϕ is convex and $\phi(0) = 0$, then $\phi(\mu\tau) \leq \mu\phi(\tau)$ when $\tau \geq 0$ and $0 \leq \mu \leq 1$. The choices $\mu = 2^{1-N}$ in the left-hand side of (30) and, next, $\mu = \lambda$ and $\mu = 1 - \lambda$ in its right-hand side, yield

$$\int_{\mathbb{R}^N} \phi\left(2^{1-N}r^{-1}(f \bar{\wedge} g)\right) \leq \lambda \int_{\mathbb{R}^N} \phi\left(\|\hat{f}\|_\phi^{-1}\hat{f}\right) + (1 - \lambda) \int_{\mathbb{R}^N} \phi\left(\|\hat{g}\|_\phi^{-1}\hat{g}\right).$$

By (14), it follows that $\int_{\mathbb{R}^N} \phi(2^{1-N} r^{-1}(f \bar{\wedge} g)) \leq 1$ and so, by (12), $\|f \bar{\wedge} g\|_\phi \leq 2^{N-1} r = 2^{N-1}(\|\hat{f}\|_\phi + \|\hat{g}\|_\phi)$, as claimed in (28).

To complete the proof, suppose now that $\|\hat{f}\|_\phi = 0$ or $\|\hat{g}\|_\phi = 0$. By symmetry, we may and shall assume that $\|\hat{f}\|_\phi = 0$, whence $\hat{f} = 0$ a.e. Since $\hat{f}(x)$ is a nonincreasing and right-continuous function of $|x|$, it follows that $\hat{f} = 0$. Thus, by Lemma 4.5, either $f = 0$ or there are $x_0 \in \mathbb{R}^N$ and $0 < z \leq \infty$ such that $f(x_0) = z$ and $f(x) = 0$ when $x \neq x_0$.

If $f = 0$, then $f \bar{\wedge} g = 0$ since $g \geq 0$ and (28) is trivial. If $f(x_0) = z > 0$ and $f(x) = 0$ when $x \neq x_0$, a straightforward calculation shows that $(f \bar{\wedge} g)(x) = \min\{g(x - x_0), z\}$. In particular, $0 \leq (f \bar{\wedge} g) \leq g(\cdot - x_0)$, whence $\|f \bar{\wedge} g\|_\phi \leq \|g(\cdot - x_0)\|_\phi = \|g\|_\phi \leq \|\hat{g}\|_\phi$, the latter by Lemma 5.5. This proves (28). $\square$

Since $(f \bar{\wedge} f)(x) \geq f(x/2)$, equality holds in (28) if $L_\phi = L^1$ and $0 \leq g = f = \hat{f} \in L^1$. Thus, 2^{N-1} is best possible among all the constants independent of ϕ . On the other hand, even when $L_\phi = L^\infty$, (28) is trivial only under additional assumptions, namely, $M_f = \text{ess sup } f$, $M_g = \text{ess sup } g$ and $\min\{M_f, M_g\} = \text{ess sup}(f \bar{\wedge} g)$ (which is *not* implied by the former two, see Example 5.7 below). If so,

$$\|f \bar{\wedge} g\|_\infty = \min\{\|f\|_\infty, \|g\|_\infty\}, \quad (31)$$

which is much better than (28) since $\|f\|_\infty \leq \|\hat{f}\|_\infty$ and $\|g\|_\infty \leq \|\hat{g}\|_\infty$ by Lemma 5.5. However, (31) is false without the extra assumptions mentioned above and then (28) is no longer trivial in L^∞ . How badly (31) may fail is shown in:

Example 5.7. Let $\mathcal{C} \subset [0, 1]$ be the Cantor set. It is not hard to see that $(x - \mathcal{C}) \cap \mathcal{C} \neq \emptyset$ for every $x \in [0, 2]$ (notice that $(x - \mathcal{C}_k) \cap \mathcal{C}_k \neq \emptyset$ for every $k \in \mathbb{N}$, where $\mathcal{C}_1 = [0, 1]$ and $\mathcal{C}_{k+1} \subset \mathcal{C}_k$ is obtained by removing the open middle thirds of the intervals in $\mathcal{C}_k$). As a result, if $f = \infty$ on $\mathcal{C}$ and 0 outside, then f is Borel measurable, the right-hand side of (31) with $g = f$ is 0, but since $f \bar{\wedge} f = \infty$ on $[0, 2]$, its left-hand side is ∞ .

6. Reverse inequalities for $f \square g$ and $f \vee g$

With the help of Theorem 5.6, it is a simple matter to prove a converse of Theorem 3.5.

Theorem 6.1. *Suppose that $f, g : \mathbb{R}^N \rightarrow (-\infty, \infty]$ are Borel measurable, that $m_f, m_g \in \mathbb{R}$ and that $m_f + m_g \geq 0$. Then $f \square g \geq 0$, $\check{f} - m_{f,g} \geq 0$, $\check{g} + m_{f,g} \geq 0$ and*

$$\|(f \square g)^{-1}\|_\phi \leq 2^{N-1}(\|(\check{f} - m_{f,g})^{-1}\|_\phi + \|(\check{g} + m_{f,g})^{-1}\|_\phi), \quad (32)$$

for every Young function ϕ .

Proof. Use $f - m_{f,g} \geq 0$, $g + m_{f,g} \geq 0$ along with $f \square g = (f - m_{f,g}) \square (g + m_{f,g}) \geq (f - m_{f,g}) \vee (g + m_{f,g}) \geq 0$ to get $\check{f} - m_{f,g} \geq 0$ and $\check{g} + m_{f,g} \geq 0$ (by Theorem

4.7 (iii) and (iv)) as well as $0 \leq (f \square g)^{-1} \leq ((f - m_{f,g}) \vee (g + m_{f,g}))^{-1} = (f - m_{f,g})^{-1} \wedge (g + m_{f,g})^{-1}$. This yields

$$\begin{aligned} \|(f \square g)^{-1}\|_\phi &\leq \|(f - m_{f,g})^{-1} \wedge (g + m_{f,g})^{-1}\|_\phi \leq \\ &2^{N-1}(\|((f - m_{f,g})^{-1})^\wedge\|_\phi + \|((g + m_{f,g})^{-1})^\wedge\|_\phi), \end{aligned}$$

where Theorem 5.6 was used for the second inequality.

Next, $((f - m_{f,g})^{-1})^\wedge = (\check{f} - m_{f,g})^{-1}$ and $((g + m_{f,g})^{-1})^\wedge = (\check{g} + m_{f,g})^{-1}$ by Theorem 4.7 (viii) and (iv) (in that order), which proves (32). $\square$

The comments after Theorem 5.6 may be repeated: Without extra assumptions, (32) is not trivial even when $L_\phi = L^\infty$. The proof of Theorem 6.1 shows that, more generally, if $z \in \mathbb{R}$ and $-m_g \leq z \leq m_f$ (whence $m_f + m_g \geq 0$),

$$\|(f \square g)^{-1}\|_\phi \leq 2^{N-1}(\|(\check{f} - z)^{-1}\|_\phi + \|(\check{g} + z)^{-1}\|_\phi). \quad (33)$$

However, (32) is optimal (to prove $(f \square g)^{-1} \in L_\phi$) among all the inequalities (33). Indeed, if the right-hand side of (33) is finite for some z as above, the right-hand side of (32) is also finite³. To see this, assume first $m_f + m_g = 0$. Then, $z = m_f = -m_g = m_{f,g}$ is the only possible choice in (33) and (32) is optimal by default. Suppose now that $m_f + m_g > 0$ and note that, by Theorem 4.7 (iii), $\check{f} - m_{f,g}$ and $\check{g} + m_{f,g}$ are both bounded below by $(m_f + m_g)/2 > 0$. Thus, $(\check{f} - m_{f,g})^{-1} \in L^\infty$ and so, if $\check{f} - z \geq 0$ and $(\check{f} - z)^{-1} \in L_\phi$ for some z , then $(\check{f} - m_{f,g})^{-1} = [1 - (z - m_{f,g})(\check{f} - m_{f,g})^{-1}](\check{f} - z)^{-1} \in L_\phi$ since $1 - (z - m_{f,g})(\check{f} - m_{f,g})^{-1} \in L^\infty$. Likewise, if $\check{g} + z \geq 0$ and $(\check{g} + z)^{-1} \in L_\phi$, then $(\check{g} + m_{f,g})^{-1} \in L_\phi$.

It is not clear whether 2^{N-1} is best possible in (32), among all the constants independent of ϕ . The remark that $(f \square f)(x) \leq 2f(x/2)$ and the choice $g = f = \check{f} \geq 0$ with $f^{-1} \in L^1 = L_\phi$ only shows that the best constant is at least 2^{N-2} .

There is also a converse of Corollary 3.7:

Corollary 6.2. *Suppose that $f, g : \mathbb{R}^N \rightarrow [-\infty, \infty]$ are Borel measurable, that $g \geq 0$ and that $f \not\equiv \infty, g \not\equiv \infty$. Then, $f \vee g \geq 0$ and*

$$\|(f \vee g)^{-1}\|_\phi \leq 2^N (\|(\check{f}_+ - m_{f_+,g})^{-1}\|_\phi + \|(\check{g} + m_{f_+,g})^{-1}\|_\phi). \quad (34)$$

for every Young function ϕ .

Proof. Since g is nonnegative, $f \vee g = f_+ \vee g \geq (f_+ \square g)/2 = (f_+/2) \square (g/2) \geq 0$. Also, $0 \leq m_{f_+}, m_g < \infty$ since $f \not\equiv \infty$ and $g \not\equiv \infty$. Thus, it suffices to use Theorem 6.1 with $f_+/2$ and $g/2$ along with Theorem 4.7 (v) and (ix). $\square$

³While mostly true, the converse may fail when $m_f + m_g > 0$ and $z = m_f = m_{\check{f}}$ or $z = -m_g = -m_{\check{g}}$.

In many cases, Theorem 6.1 and Corollary 6.2 can be used to prove that $(f \square g)^{-1} \in L_\phi$ or $(f \vee g)^{-1} \in L_\phi$ without any calculation of $\check{f}$ or $\check{g}$, because there are easily verifiable sufficient conditions for the finiteness of the right-hand sides of (32) and (34). The simplest one is given in the following lemma.

Lemma 6.3. *Given $f : \mathbb{R}^N \rightarrow (-\infty, \infty]$ with $m_f > -\infty$ (i.e., f bounded below), suppose that there are constants $c > 0$ and $\alpha > 0$ such that $f(x) \geq h(x) := c|x|^\alpha$ for $|x|$ large enough. Then, for every $z < m_f$,*

$$\|(\check{f} - z)^{-1}\|_\phi \leq 2\|h^{-1}\|_{\phi, \mathbb{R}^N \setminus B} + (m_f - z)^{-1}\|\chi_B\|_\phi, \quad (35)$$

for every open ball B centered at the origin such that⁴ $f \geq h \geq 2z$ outside B and every Young function ϕ . If ϕ is invertible with inverse ψ , this also reads

$$\|(\check{f} - z)^{-1}\|_\phi \leq 2\|h^{-1}\|_{\phi, \mathbb{R}^N \setminus B} + (m_f - z)^{-1}[\psi(\mu_N(B)^{-1})]^{-1}. \quad (36)$$

In particular, if $h^{-1} \in L_\phi(\mathbb{R}^N \setminus B)$, then $(\check{f} - z)^{-1} \in L_\phi$. (Case in point: Since h is explicitly known, $h^{-1} \in L_\phi(\mathbb{R}^N \setminus B)$ can often be checked by a calculation.)

Proof. Let B be as in the theorem. By the “furthermore” part of Theorem 4.7 (vii), $\check{f} \geq h \geq 2z$ outside B . Thus, $h \leq 2(\check{f} - z)$, whence $(\check{f} - z)^{-1} \leq 2h^{-1}$, outside B . Meanwhile, by Theorem 4.7 (iii), $(\check{f} - z)^{-1} \leq (m_{\check{f}} - z)^{-1} < (m_f - z)^{-1} < \infty$ in B (and everywhere else) and (35) follows. To get (36) when ϕ is invertible with inverse ψ , just notice that $\|\chi_B\|_\phi = [\psi(\mu_N(B)^{-1})]^{-1}$ by (12). $\square$

From the proof of Lemma 6.3, $|x|^\alpha$ can be replaced with a function $h(x)$ satisfying general conditions. For example, if $L_\phi = L^p$ with $p \geq 1$, Lemma 6.3 yields $(\check{f} - z)^{-1} \in L^p$ if $\alpha > N/p$ but, if $f(x) \geq c|x|^\alpha(\log|x|)^\beta$ for large $|x|$, the choice of any continuous strictly increasing function h of $|x|$ that coincides with $c|x|^\alpha(\log|x|)^\beta$ for $|x| \geq 1$ (say) shows that $(\check{f} - z)^{-1} \in L^p$ in the limiting case $\alpha = N/p$ if $\beta > p^{-1}$.

Remark 6.4. The proof of Lemma 6.3 also shows that, more generally, (35) is true with m_f replaced with $m_{\check{f}}$ and that this requires only $m_{\check{f}} > -\infty$. This may occasionally be useful, but rarely (see Remark 4.8).

It is more delicate to extend Lemma 6.3 when $z = m_f = m_{\check{f}}$. The extra difficulty is that $(\check{f} - m_f)^{-1} \notin L_{loc}^\infty$ (because $m_{\check{f}} = \text{ess inf } \check{f}$; see Theorem 4.7 (iii)), so that local integrability becomes an issue. This requires further investigation. We only mention without proof (and will not use later) that if m_f is a *unique* and *nondegenerate* minimum of f (plus a mild technical condition) then $(\check{f} - m_f)^{-1} \in L_{loc}^p$ if $N \geq 3$ and $1 \leq p < N/2$.

A direct application of Lemma 6.3 yields the following sample result.

⁴Since $\lim_{|x| \rightarrow \infty} h(x) = \infty$, the existence of B is not an issue.

Theorem 6.5. *Given $f, g : \mathbb{R}^N \rightarrow (-\infty, \infty]$, suppose that there are constants $c > 0$ and $\alpha > 0$ such that $f(x) \geq c|x|^\alpha$ and $g(x) \geq c|x|^\alpha$ for $|x|$ large enough. Then:*

- (i) *If $m_f, m_g \in \mathbb{R}$ and $m_f + m_g > 0$, then $(f \square g)^{-1} \in L^p$ if $p \geq 1$ and $p > N/\alpha$.*
- (ii) *If $g \geq 0$ and $m_{f_+} + m_g > 0$, then $(f \vee g)^{-1} \in L^p$ if $p \geq 1$ and $p > N/\alpha$.*

Proof. (i) Just notice that $m_{f,g} < m_f (\leq m_{\check{f}})$ and $-m_{f,g} < m_g (\leq m_{\check{g}})$ since $m_f + m_g > 0$ and use Lemma 6.3 with $L_\phi = L^p$ and Theorem 6.1.

(ii) If $m_{f_+} = \infty$ or $m_g = \infty$, then $f \vee g = \infty$ and the result is trivial. From now on, $m_{f_+}, m_g < \infty$, whence $m_{f_+}, m_g \in \mathbb{R}$ (because $f_+, g \geq 0$). By Theorem 4.7 (ix), $\check{f}_+ = (f_+)^{\check{}}$, whereas $m_{f_+,g} < m_{f_+}$ and $-m_{f_+,g} < m_g$ since $m_{f_+} + m_g > 0$. Now, use Lemma 6.3 and Corollary 6.2 with $L_\phi = L^p$. $\square$

Lemma 6.3 and its aforementioned variants yield generalizations of Theorem 6.5 to all Orlicz spaces. The proof of Theorem 6.5 does not use the estimate (35) but the next theorem, relevant to the results in the next section, does. As explained after the proof, there is a good reason to confine attention to L^p spaces.

Theorem 6.6. *Suppose that $f, g : \mathbb{R}^N \rightarrow (-\infty, \infty]$ are Borel measurable, that $m_f \geq 0, m_g > -\infty$ and that $m_g > 0$ if $m_f = 0$. Given $1 \leq p \leq \infty$, suppose also that for some $\alpha > N/p$, there are constants $c > 0$ and $\alpha > 0$ such that $f(x) \geq c|x|^\alpha$ for $|x|$ large enough. For $t > 0$, set*

$$f_t(x) := tf(t^{-1}x),$$

so that $m_{f_t} = tm_f$. Lastly, assume $(\check{g} - z)^{-1} \in L^p$ when $z < m_g$ (e.g., if $g(x) \geq c|x|^\alpha$ for $|x|$ large enough by Lemma 6.3, but this is not necessary).

Then, $(f_t \square g)^{-1} \in L^p$ when $tm_f + m_g > 0$ (i.e., $t > 0$ if $m_f = 0$ and $t > -m_f^{-1}m_g$ if $m_f > 0$) and:

- (i) *If $m_f = 0$, then $\|(f_t \square g)^{-1}\|_p = O(t^{N/p})$ as $t \rightarrow \infty$.*
- (ii) *If $m_f > 0$ and $p > N$, then $\lim_{t \rightarrow \infty} \|(f_t \square g)^{-1}\|_p = 0$.*
- (iii) *If $m_f > 0$ and $p \geq N$, then $\|(f_t \square g)^{-1}\|_p = O(t^{-1+N/p})$ as $t \rightarrow \infty$.*

Proof. As in Lemma 6.3, set $h(x) := c|x|^\alpha$ and, in (35), let B be a ball centered at the origin such that $\check{f} \geq h$ and $h \geq m_f + |m_g| \geq 2m_{f,g} = m_f - m_g$ outside B . Evidently, $(\check{f})_t \geq h_t$ ($:= th(t^{-1}\cdot)$) outside tB and, by Theorem 4.7 (iv) and (v), $(f_t)^{\check{}} = (\check{f})_t$. Thus, $(f_t)^{\check{}} \geq h_t$ outside tB . Furthermore, if $t \geq 1$, then $h_t \geq tm_f + |m_g| \geq 2m_{f_t,g} = tm_f - m_g$ outside tB .

Since $h_t(x) = t^{1-\alpha}c|x|^\alpha$, it follows from the above that the estimate (35) can be used with f, h and B replaced with f_t, h_t and tB , respectively, and with $z = m_{f_t,g} = (tm_f - m_g)/2$, provided that $t \geq 1$ and that $tm_f + m_g > 0$ (so that $m_{f_t,g} < m_{f_t} = tm_f$), which holds for large t . Accordingly,

$$\begin{aligned} & \|(\check{f}_t - m_{f_t,g})^{-1}\|_p \\ & \leq 2t^{-1+N/p} \|h^{-1}\|_{p, \mathbb{R}^N \setminus B} + 2(tm_f + m_g)^{-1} t^{N/p} \mu_N(B)^{1/p}, \end{aligned} \quad (37)$$

where $\|\chi_{tB}\|_p = t^{N/p} \mu_N(B)^{1/p}$ was used. Since $\|h^{-1}\|_{p, \mathbb{R}^N \setminus B} < \infty$ by the choice $\alpha > N/p$, it follows that $(\check{f}_t - m_{f_t, g})^{-1} \in L^p$. Also, $(\check{g} + m_{f_t, g})^{-1} \in L^p$ since $-m_{f_t, g} = (m_g - tm_f)/2 < m_g$ when $tm_f + m_g > 0$ and since it is assumed that $(\check{g} - z)^{-1} \in L^p$ when $z < m_g$. Thus, by (32), $(f_t \square g)^{-1} \in L^p$.

The estimates (i), (ii) and (iii) follow from (32) and (37) and from the remarks that (a) if $m_f = 0$, then $(m_g > 0$ and) $(\check{g} + m_{f_t, g})^{-1} = (\check{g} - m_g/2)^{-1}$ is independent of t and (b) if $m_f > 0$, then $\lim_{t \rightarrow \infty} \|(\check{g} + m_{f_t, g})^{-1}\|_p = 0$ by dominated convergence if $p < \infty$ and by $\check{g} + m_{f_t, g} \geq (tm_f + m_g)/2$ if $p = \infty$. $\square$

Similar estimates hold when f is replaced with f_t in Corollary 6.2 and estimates can also be worked out in other spaces L_ϕ , but the technicalities depend on ϕ . For instance, while Theorem 6.6 remains true if L^p is replaced with $L^1 + L^p$, the proof is substantially more demanding (recall that $L^1 + L^p = L_\phi$ with $\phi(\tau) = \tau^p$ in $[0, 1]$ and $\phi(\tau) = p\tau + 1 - p$ in $(1, \infty)$ if $1 \leq p < \infty$ and $L^1 + L^\infty = L_\phi$ with $\phi(\tau) = 0$ in $[0, 1]$ and $\phi(\tau) = t - 1$ in $(1, \infty)$; see e.g. [15])). Choices of h other than $h(x) = c|x|^\alpha$ often lead to challenging calculations.

7. Application to the Hamilton-Jacobi equations

We shall now apply the results of the previous sections to the Hamilton-Jacobi equations in their simplest form (see Subsection 8.1 for a variant)

$$\begin{cases} u_t + H(\nabla u) = 0 & \text{on } (0, \infty) \times \mathbb{R}^N, \\ u(0, \cdot) = g & \text{on } \mathbb{R}^N, \end{cases} \quad (38)$$

where the Hamiltonian H and the initial value g are given functions on $\mathbb{R}^N$.

Roughly speaking, when the Hamiltonian H (initial condition g) is convex, the Hopf-Lax formula (Hopf formula) provides a solution of (38). In both cases, various additional conditions are required of H and g and, as always, what constitutes a solution is somewhat flexible. While the more recent work focuses on viscosity solutions, other definitions exist as well.

Throughout this section, we assume that $g, H : \mathbb{R}^N \rightarrow (-\infty, \infty]$, that $m_g \in \mathbb{R}$ (hence $g \not\equiv \infty$), $H \not\equiv \infty$ and that g is Borel measurable. Further assumptions will be introduced when needed. It is once and for all understood that $t > 0$.

We denote by $(tH)^*$ the Legendre-Fenchel conjugate of tH , that is,

$$(tH)^*(x) := \sup_{y \in \mathbb{R}^N} (x \cdot y - tH(y)) = tH^*(t^{-1}x)$$

Since $(tH)^*$ is always lsc, it is Borel measurable.

7.1. Solutions by the Hopf-Lax formula

In this subsection, H is convex and $H^{**}(0) \in \mathbb{R}$. The Hopf-Lax formula (Hopf [14], Lax [19])

$$u(t, \cdot) = (tH)^* \square g, \quad (39)$$

is known to give a solution of (38) under various conditions about g . That g is real-valued and continuous is a common assumption; see Bardi and Faggian [4] and the references therein. The case when g is lsc and not everywhere finite was considered by Imbert [16] and Strömberg [32]. Chen and Su [6] show that (39) is a solution when g is real-valued, a.e. continuous and satisfies a condition weaker than upper semicontinuity. Undoubtedly, other options can be found in the literature.

The inequality (18) in Theorem 3.5 can be used with $f = (tH)^*$ if $m_{(tH)^*} + m_g \geq 0$. That $m_g > -\infty$ was assumed earlier, whereas $m_{(tH)^*} = -tH^{**}(0) \in \mathbb{R}$. As a result, the condition $m_{(tH)^*} + m_g \geq 0$ is simply

$$-tH^{**}(0) + m_g \geq 0, \quad (40)$$

so that $m_{(tH)^*,g}$ (see (17)) is given by

$$m_{(tH)^*,g} = -(tH^{**}(0) + m_g)/2.$$

Thus, assuming (40), the corresponding inequality (18)

$$\begin{aligned} & \| (2(tH)^* + tH^{**}(0) + m_g)^{-1} \|_\phi + \| (2g - tH^{**}(0) - m_g)^{-1} \|_\phi \\ & \leq 2 \| u(t, \cdot)^{-1} \|_\phi \end{aligned} \quad (41)$$

and the reverse inequality (32) of Theorem 6.1

$$\begin{aligned} & \| u(t, \cdot)^{-1} \|_\phi \\ & \leq 2^{N-1} \left(\| (2((tH)^*)^\vee + tH^{**}(0) + m_g)^{-1} \|_\phi + \| (2\check{g} - tH^{**}(0) - m_g)^{-1} \|_\phi \right), \end{aligned} \quad (42)$$

hold for every Young function ϕ .

Given $\alpha > 1$, call $\alpha' := \alpha/(\alpha - 1) > 1$ the Hölder conjugate of α . It is easily checked and certainly folklore that if there is a constant $d > 0$ such that $H(x) \leq d|x|^{\alpha'}$ for $|x|$ large enough, then $H^*(x) \geq c|x|^\alpha$ for $|x|$ large enough. Consistent with Theorem 6.5, it follows that if also $g(x) \geq c|x|^\alpha$ for $|x|$ large enough, then $u(t, \cdot)^{-1} \in L^p$ for every $p \geq 1, p > N/\alpha$.

Furthermore, since $(tH)^* = tH^*(t^{-1}\cdot)$, Theorem 6.6 gives estimates for $\|u(t, \cdot)^{-1}\|_p$ as $t \rightarrow \infty$ if $H^{**}(0) < 0$ or if $H^{**}(0) = 0$ and $m_g > 0$. (If $H^{**}(0) > 0$, then (42) breaks down when $t > m_g/H^{**}(0)$.) The accuracy (or possible lack thereof) of these estimates can be evaluated by using the inequality (41) with $\|\cdot\|_\phi = \|\cdot\|_p$.

We now look at two classical examples in more detail. In both cases, $H^{**}(0) = H(0) = 0$ will make the inequalities simpler, but confines the discussion to $m_g \geq 0$.

Example 7.1. Suppose that $H(x) := |x|^2/2$, so that $(tH)^*(x) = |x|^2/2t$ and (40) boils down to $m_g \geq 0$ since $H^{**} = H$. If $1 \leq p < \infty$, a quick calculation shows that $\|(2(tH)^* + m_g)^{-1}\|_p = A(p, m_g)t^{N/2p}$ where $A(p, m_g) > 0$ is a constant

which is finite if and only if $m_g > 0$ and $p > N/2$. Accordingly, from (41), $u(t, \cdot)^{-1} \notin L^p$ for any $t > 0$ if $m_g = 0$ or if $N \geq 2$ and $1 \leq p \leq N/2$, even if $u(0, \cdot)^{-1} = g^{-1} \in L^p$.

Assume now $m_g > 0$ and $p > N/2$. By (41) with $H^{**}(0) = 0$ and since $0 \leq g \leq 2g - m_g \leq 2g$,

$$\|u(t, \cdot)^{-1}\|_p \geq c(t^{N/2p} + \|g^{-1}\|_p), \quad (43)$$

for some constant $c > 0$ depending only upon N, p and m_g . Thus, once again, $u(t, \cdot)^{-1} \notin L^p$ if $g^{-1} \notin L^p$.

Conversely, since $((tH)^*)^\check{} = (tH)^*$ by Theorem 4.7 (i), it follows from (42) and from $0 \leq \check{g} \leq 2\check{g} - m_g \leq 2\check{g}$ (since $m_{\check{g}} \geq m_g$; see Theorem 4.7 (iii)) that $u(t, \cdot)^{-1} \in L^p$ for every $t > 0$ if $\check{g}^{-1} \in L^p$ and that $\|u(t, \cdot)^{-1}\|_p \leq C(t^{N/2p} + \|\check{g}^{-1}\|_p)$, where $C > 0$ depends only upon N, p and m_g . In particular, $\|u(t, \cdot)^{-1}\|_p = O(t^{N/2p})$ as $t \rightarrow \infty$ (which is sharp because of (43)). Note that the general estimate of Theorem 6.6 gives only the less precise $O(t^{N/p})$.

Even though $u(t, \cdot)^{-1} \notin L^p$ if $m_g = 0$, this does not preclude $u(t, \cdot)^{-1} \in L_\phi$ for Orlicz spaces outside the L^p scale. For instance, if $L_\phi = L^1 + L^p$, a simple calculation⁵ shows that $(2(tH)^* + m_g)^{-1} \in L^1 + L^p$ if $p > N/2$, with the restriction $N \geq 3$ if $m_g = 0$. If so, by (42), $u(t, \cdot)^{-1} \in L^1 + L^p$ if $\check{g}^{-1} \in L^1 + L^p$ and, by another calculation, $\|u(t, \cdot)^{-1}\|_{L^1 + L^p} = O(t)$ as $t \rightarrow \infty$ (optimal by (41)). In particular, this holds if $\check{g}^{-1} \in L^q$ with $1 \leq q \leq p$ and $p > N/2$. This complements the L^p discussion above, even when $m_g > 0$.

Example 7.2. Suppose that $H(x) = |x|$, so that $(tH)^*$ is the indicator function of the closed ball $\overline{B}(0, t)$. Thus, the formula (39) is simply $u(t, x) = \inf_{|y| \leq t} g(x - y)$.

Once again, $H^{**} = H$, so that (40) amounts to $m_g \geq 0$ and, if ϕ is a Young function, then (with $\omega_N := \mu_N(B(0, 1))$)

$$\|(2(tH)^* + m_g)^{-1}\|_\phi = \inf \{r > 0 : \phi(r^{-1}m_g^{-1}) \omega_N t^N \leq 1\}, \quad (44)$$

which is ∞ if $m_g = 0$. Thus, by (41), $u(t, \cdot)^{-1} \notin L_\phi$ for any $t > 0$ and any ϕ if $m_g = 0$, even if $u(0, \cdot)^{-1} = g^{-1} \in L_\phi$. (Since $u(t, x)^{-1} = \sup_{|y| \leq t} g(x - y)^{-1}$, this also follows from the remark that for every $\varepsilon > 0$, $u(t, \cdot)^{-1} \geq \varepsilon^{-1}$ on some ball of radius t .) Assuming from now on that $m_g > 0$, it follows from (44) that if ϕ has an inverse ψ on $[0, \infty]$, then $\|(2(tH)^* + m_g)^{-1}\|_\phi = m_g^{-1} \psi(\omega_N^{-1} t^{-N})^{-1}$ (where, as usual, $\psi^{-1} = 1/\psi$, not ϕ). In the simple case when $\phi(\tau) = \tau^p$ with $p \geq 1$, this yields

$$\|(2(tH)^* + m_g)^{-1}\|_p = m_g^{-1} \omega_N^{1/p} t^{N/p}. \quad (45)$$

Once again, $((tH)^*)^\check{} = (tH)^*$ by Theorem 4.7 (i). Thus, if $\check{g}^{-1} \in L^p$ (equivalent to $(2\check{g} - m_g)^{-1} \in L^p$), it follows from (42) and (45) that $u(t, \cdot)^{-1} \in L^p$ and that $\|u(t, \cdot)^{-1}\|_p = O(t^{N/p})$ as $t \rightarrow \infty$ (which is sharp because of (41)). This is the general estimate in Theorem 6.6 (i) which, in this example, is therefore optimal.

⁵The formula for ϕ was given at the end of the previous section.

In Example 7.2, $g^{-1} \in L^p$ is not enough to get $u(t, \cdot)^{-1} \in L^p$: If $N = 1$ and if $g = f^{-1/p}$ with $p < \infty$ and f from Example 5.3, then $m_g = 1$, $g^{-1} = f^{1/p} \in L^p$, but $u(t, x) = \inf_{|y| \leq t} f^{-1/p}(x - y) = 1$ if $t \geq 1$, so that $u(t, \cdot)^{-1} = 1 \notin L^p$.

8. Solutions by the Hopf formula

The Hopf formula (Hopf [14])

$$u(t, \cdot) = (tH + g^*)^*$$

gives a solution of (38) when g is convex and various other technical assumptions are satisfied. See for instance Bardi and Evans [3]. In Penot and Volle [28], g and H can be extended real-valued. Below, we assume that g is lsc. Since $u(t, \cdot)$ is convex, there is no measurability issue.

First, $-g^*(0) = \inf g = m_g$ (finite, as assumed above) by definition of g^* . Next, if $h, k : \mathbb{R}^N \rightarrow (-\infty, \infty]$ and $h \not\equiv \infty, k \not\equiv \infty$ (so that h^* and k^* are proper), it is well-known and easily checked that $(h+k)^* \leq h^* \square k^*$. As a result, $u(t, \cdot) = (tH + g^*)^* \leq (tH)^* \square g^{**} = (tH)^* \square g$. On the other hand, by using “inf sup $\geq$ sup inf”, we get $\inf_x (tH + g^*)^*(x) \geq \sup_y \inf_x (x \cdot y - (tH + g^*)(y)) = -tH(0) - g^*(0) = -tH(0) + m_g$. Indeed, $\inf_x (x \cdot y - (tH + g^*)(y)) = -\infty$ if $y \neq 0$ because tH and g^* are proper. This shows that if $-tH(0) + m_g \geq 0$, then $0 \leq u(t, \cdot) \leq (tH)^* \square g$. Furthermore, since $(tH)^* = (tH_C)^*$ where $H_C = H^{**}$ is the closed convex hull of H , it follows that $m_{(tH)^*} = -tH_C(0) \geq -tH(0)$. To ensure that $m_{(tH)^*} \in \mathbb{R}$, i.e., that $H_C(0) > -\infty$, it must be assumed that H is bounded below by an affine function. If so, it follows from Theorem 3.5 with $f = (tH)^*$ that if $-tH(0) + m_g \geq 0$ (hence $-tH_C(0) + m_g \geq 0$), then

$$\begin{aligned} & \| (2(tH)^* + tH_C(0) + m_g)^{-1} \|_\phi + \| (2g - tH_C(0) - m_g)^{-1} \|_\phi \\ & \leq 2 \| u(t, \cdot)^{-1} \|_\phi, \end{aligned} \tag{46}$$

for every Young function ϕ .

Since the inequality (46) depends only upon H_C , it remains true when H is replaced with any proper closed convex function $K \leq H$ (so that $K = K_C$) under the same assumption $-tH(0) + m_g \geq 0$ as above (still needed to ensure $u \geq 0$) because this substitution decreases the left-hand side. This is less accurate, but often more convenient for practical evaluation.

As an illustration of this point, it follows from the discussion in Example 7.1 that $u(t, \cdot)^{-1} \notin L^p$ if $H(x) \geq |x|^2/2$ and either $0 < p \leq N/2$ or $m_g = 0$. Alternatively, from Example 7.2, $u(t, \cdot)^{-1} \notin L_\phi$ for any ϕ if $H(x) \geq |x|$ and $m_g = 0$.

In the opposite direction, if $L \geq H$ is any lsc convex function, then $(tH + g^*) \leq (tL + g^*)$ and, since both tL and g^* are proper and lsc, $(tL + g^*)^* = (tL)^* \square g$ as soon as the relative interiors of $\text{dom } L$ and $\text{dom } g^*$ have nonempty intersection ([29, p. 145]). If so, $(tL)^* \square g \leq u(t, \cdot)$, so that the reverse inequalities of Theorem

6.1 can be used with $f = (tL)^*$ if $-tL(0) + m_g \geq 0$. Since $0 \leq -tL(0) + m_g = m_{(tL)^* \square_g} \leq u(t, \cdot)$, it follows that

$$\begin{aligned} & \|u(t, \cdot)^{-1}\|_\phi \\ & \leq 2^{N-1} \left(\|(2((tL)^*)^\check{+} + tL(0) + m_g)^{-1}\|_\phi + \|(2\check{g} - tL(0) - m_g)^{-1}\|_\phi \right), \end{aligned} \quad (47)$$

for every Young function ϕ . Unlike (46), the inequality (47) does not require H to be bounded below by an affine function.

For instance, by Example 7.1, $\|u(t, \cdot)^{-1}\|_p = O(t^{N/2p})$ as $t \rightarrow \infty$ if $H(x) \leq L(x) := |x|^2/2, m_g > 0$ and $\check{g}^{-1} \in L^p, p > N/2$. Also, by Example 7.2, $\|u(t, \cdot)^{-1}\|_p = O(t^{N/p})$ as $t \rightarrow \infty$ if $H(x) \leq L(x) := |x|, m_g > 0$ and $\check{g}^{-1} \in L^p, p \geq 1$.

8.1. Explicit solutions of related problems

Various explicit formulas for the solution of

$$\begin{cases} u_t + H(u, \nabla u) = 0 & \text{on } (0, \infty) \times \mathbb{R}^N, \\ u(0, \cdot) = g & \text{on } \mathbb{R}^N, \end{cases} \quad (48)$$

when the Hamiltonian depends upon u , have been obtained under suitable (but restrictive) conditions. The result most directly relevant to this paper can be found in the work of Barron et al. [5], complemented and generalized in [2]. It is shown in [2, Theorem 6.11] that if H is continuous on $\mathbb{R}^{N+1}$, with $H(s, x)$ nondecreasing in $s \in \mathbb{R}$, convex and positively homogeneous of degree 1 in $x \in \mathbb{R}^N$, and if $g : \mathbb{R}^N \rightarrow (-\infty, \infty]$ is lsc, then (for $t > 0$)

$$u(t, \cdot) := h_{[t]} \vee g, \quad (49)$$

is the minimal lsc supersolution of (48), where⁶ $h_{[t]}(x) := h(t^{-1}x)$ and

$$h(x) := \inf\{s \in \mathbb{R} : H(s, \cdot)^*(x) \leq 0\}. \quad (50)$$

It is shown in [5] that h is quasiconvex and lsc and that $m_h = -\infty$.

From now on, we assume $g \geq 0$, so that $u(t, \cdot) = h_{[t]+} \vee g (= h_{[t]} \vee g)$. Note that $h_{[t]+} = h_{+[t]}$. Since $m_h = -\infty$ implies $m_{h_{[t]}} = -\infty$, it follows that $m_{h_{[t]+}} = 0$, so that $m_{h_{[t]+}, g} = -m_g/2 \leq 0$.

Corollary 3.7 and Corollary 6.2 with $f = h_{[t]}$ can be used to evaluate $\|u(t, \cdot)^{-1}\|_\phi$. Specifically,

$$\|(2h_{[t]+} + m_g)^{-1}\|_\phi + \|(2g - m_g)^{-1}\|_\phi \leq 2\|u(t, \cdot)^{-1}\|_\phi \quad (51)$$

and

$$\|u(t, \cdot)^{-1}\|_\phi \leq 2^{N+1} \left(\|(2(h_{[t]+})^\check{+} + m_g)^{-1}\|_\phi + \|(2\check{g} - m_g)^{-1}\|_\phi \right), \quad (52)$$

for every Young function ϕ .

⁶This differs from h_t previously defined by $th(t^{-1}\cdot)$.

Example 8.1. Let $H(s, x) = (s_+)^{\alpha}|x|$ with $\alpha > 0$. When $g \geq 0$, the (non-negative) solution (49) of (48) actually solves $u_t + u^{\alpha}|\nabla u| = 0$. By (50) and a straightforward calculation, $h(0) = -\infty$ and $h(x) = |x|^{1/\alpha}$ if $x \neq 0$.

If $m_g = 0$, then $u(t, \cdot)^{-1} \notin L^p$ for any $p \geq 1$ by (51) since $h_{[t]+}^{-1} = t^{1/\alpha}|\cdot|^{-1/\alpha} \notin L^p$.

However, it is readily checked that $h_{[t]+}^{-1} \in L^1 + L^p$ if $1 < N\alpha < p$. If so, it follows from (52) with $m_g = 0$ that $u(t, \cdot) \in L^1 + L^p$ if $\check{g}^{-1} \in L^1 + L^p$ (hence $g^{-1} \in L^1 + L^p$ by (51)) and the calculation of the estimates (51) and (52) is trivial since $m_g = 0$ and since $h_{[t]+} = t^{-1/\alpha}|\cdot|^{1/\alpha} = (h_{[t]+})^{\check{}}$ by Theorem 4.7 (i).

Thus, $\|(h_{[t]+})^{-1}\|_{L^1+L^p} = \|((h_{[t]+})^{\check{}})^{-1}\|_{L^1+L^p} = Ct^{1/\alpha}$ with $C := \| |\cdot|^{-1/\alpha} \|_{L^1+L^p}$, which yields $\|u(t, \cdot)^{-1}\|_{L^1+L^p} = O(t^{1/\alpha})$ as $t \rightarrow \infty$ (optimal).

If $m_g > 0$, then $(2h_{[t]+} + m_g)^{-1} \in L^p$ if and only if $p > N\alpha$. If so and if $\check{g}^{-1} \in L^p$ (equivalent to $(2\check{g} - m_g)^{-1} \in L^p$), it follows from (52) that $u(t, \cdot)^{-1} \in L^p$ with $\|u(t, \cdot)^{-1}\|_p = O(t^{N/p})$ as $t \rightarrow \infty$ (optimal by (51)).

Example 7.2 is recovered when $\alpha = 0$ in Example 8.1. If so, $h(x) = -\infty$ on the closed unit ball and $h(x) = \infty$ outside, so that h_+ is the indicator function of the closed unit ball. Even though the formulas (39) and (49) look different, they both provide the same solution $u(t, x) = \inf_{|y| \leq t} g(x - y)$.

Example 8.2. Let $H(s, x) = e^s|x|$, so that, by (50), $h(0) = -\infty$ and $h(x) = \ln|x|$ if $x \neq 0$. Thus, $h_+(x) = \ln_+|x|$ is continuous, radially symmetric and nondecreasing in $|x|$, so that, once again, $(h_{[t]+})^{\check{}} = h_{[t]+}$ by Theorem 4.7 (i). As always in this subsection, $g \geq 0$.

Since $h_{[t]+} = 0$ on the ball with center 0 and radius t , it follows from (51) that $u(t, \cdot)^{-1} \notin L_{\phi}$ for any $t > 0$ and any Young function ϕ if $m_g = 0$. In addition, the slow growth of $\ln|x|$ reveals that $u(t, \cdot)^{-1} \notin L^p$, $1 \leq p < \infty$, even if $m_g > 0$. However, if $m_g > 0$, then $(2h_{[t]+} + m_g)^{-1} \in L_{\phi}$ if ϕ vanishes fast enough at the origin. Aside from L^{∞} , one of the simplest examples is given by the Young function $\phi(\tau) := e^{\tau-\tau^2}$. (The growth of ϕ at infinity could be damped considerably to enlarge the space L_{ϕ} .) Thus, if $m_g > 0$ and $\check{g}^{-1} \in L_{\phi}$ (equivalent to $(2\check{g} - m_g)^{-1} \in L_{\phi}$), it follows from (52) that $u(t, \cdot)^{-1} \in L_{\phi}$ for every $t > 0$. We did not attempt to estimate $\|u(t, \cdot)^{-1}\|_{\phi}$ as $t \rightarrow \infty$.

References

- [1] R. A. Adams, J. F. Fournier: Sobolev Spaces, 2nd Ed., Academic Press, New York (2003).
- [2] O. Alvarez, E. N. Barron, H. Ishii: Hopf-Lax formulas for semicontinuous data, Indiana Univ. Math. J. 48 (1999) 993–1035.
- [3] M. Bardi, L. C. Evans: On Hopf's formulas for solutions of Hamilton Jacobi equations, Nonlinear Anal. 8 (1984) 1373–1381.
- [4] M. Bardi, S. Faggian: Hopf-type estimates and formulas for nonconvex-nonconcave Hamilton-Jacobi equations, SIAM J. Math. Anal. 29 (1998) 1067–1086.

- [5] E. N. Barron, R. Jensen, W. Liu: Hopf Lax-type formula for $u_t + H(u, Du) = 0$, *J. Differ. Equations* 126 (1996) 49–61.
- [6] G.-Q. Chen, B. Su: Discontinuous solutions for Hamilton-Jacobi equations: uniqueness and regularity, *Discrete Contin. Dyn. Syst.* 9 (2003) 167–192.
- [7] P. W. Day: Rearrangement inequalities, *Can. J. Math.* 24 (1972) 930–943.
- [8] H. Federer: *Geometric Measure Theory*, Springer, Berlin (1996).
- [9] W. Fenchel: *Convex Cones, Sets and Functions*, Princeton University, Princeton (1953).
- [10] R. J. Gardner: The Brunn-Minkowski inequality, *Bull. Amer. Math. Soc.* 39 (2002) 355–405.
- [11] T. Glavosits, Á. Száz: The infimal convolution can be used to easily prove the classical Hahn-Banach theorem, *Rostocker. Math. Kolloq.* 65 (2010) 71–83.
- [12] J.-B. Hiriart-Urruty: Extension of Lipschitz functions, *J. Math. Anal. Appl.* 77 (1980) 539–554.
- [13] J.-B. Hiriart-Urruty, C. Lemaréchal: *Convex Analysis and Minimization Algorithms*, Vols. I and II, Springer, Berlin (1996).
- [14] E. Hopf: Generalized solutions of non linear equations of first-order, *J. Math. Mech.* 14 (1965) 951–973.
- [15] H. Hudzik: Intersections and algebraic sums of Musielak-Orlicz spaces, *Port. Math.* 40 (1981) 287–296.
- [16] C. Imbert: Convex analysis techniques for Hopf-Lax formulae in Hamilton-Jacobi equations, *J. Nonlinear Convex Anal.* 2 (2001) 333–343.
- [17] H. W. E. Jung: Über die kleinste Kugel, die eine räumliche Figur einschliesst, *J. Reine Angew. Math.* 123 (1901) 241–257.
- [18] M. A. Krasnosel'skii, Ya. B. Rutickii: *Convex Functions and Orlicz Spaces*, P. Noordhoff, Groningen (1961).
- [19] P. D. Lax: Hyperbolic systems of conservation laws II, *Commun. Pure Appl. Math.* 10 (1957) 537–566.
- [20] P.-L. Lions: *Generalized Solutions of Hamilton-Jacobi Equations*, Pitman, London (1982).
- [21] Y. Lucet: What shape is your conjugate? A survey of computational convex analysis and its applications, *SIAM Rev.* 52 (2010) 505–542.
- [22] W. A. J. Luxemburg: Rearrangement invariant Banach function spaces, *Proc. Symp. Analysis*, Queens Univ. 10 (1967) 83–144.
- [23] M.-L. Mazure: Equations de convolution et formes quadratiques, *Ann. Mat. Pura Appl.*, IV. Ser. 158 (1991) 75–97.
- [24] J. J. Moreau: *Fonctions à Valeurs dans $[-\infty, \infty]$; Notions Algébriques*, Université de Montpellier, Montpellier (1963).
- [25] J. J. Moreau: Inf-convolution des fonctions numériques sur un espace vectoriel, *C. R. Acad. Sci., Paris* 256 (1963) 5047–5049.

- [26] J. J. Moreau: Inf-convolution, sous-additivité, convexité des fonctions numériques, *J. Math. Pures Appl.*, IX. Sér. 49 (1970) 109–154.
- [27] R. O’Neil: Fractional integration in Orlicz spaces, *Trans. Amer. Math. Soc.* 115 (1965) 300–328.
- [28] J.-P. Penot, M. Volle: Explicit solutions to Hamilton-Jacobi equations under mild continuity and convexity assumptions, *J. Nonlinear Convex Anal.* 1 (2000) 177–199.
- [29] R. T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970).
- [30] W. Sierpiński: Sur la question de la mesurabilité de la base de M. Hamel, *Fundam. Math.* 1 (1920) 105–111.
- [31] T. Strömberg: The operation of infimal convolution, *Diss. Math.* 352 (1996) 1–58.
- [32] T. Strömberg: The Hopf-Lax formula gives the unique viscosity solution, *Differ. Integral Equ.* 15 (2002) 47–52.
- [33] W. P. Ziemer: *Weakly Differentiable Functions*, Springer, New York (1989).