Journal of Convex Analysis
Volume 23 (2016), No. 3, 921–946

Sweeping Processes and Rate Independence

Vincenzo Recupero*
Dipartimento di Scienze Matematiche, Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
vincenzo.recupero@polito.it

Dedicated to the memory of Jean Jacques Moreau.

Received: March 25, 2015
Revised manuscript received: April 13, 2015
Accepted: April 14, 2015

We introduce a new reparametrization technique for convex-valued functions of bounded variation. By means of this technique we are able to reduce discontinuous BV sweeping processes to the Lipschitz continuous case by using only tools from measure theory. In particular, from the regular case we deduce existence, continuous dependence, and convergence of the catching-up algorithm.

Keywords: Sweeping processes, differential inclusions, convex sets, Hausdorff distance, rate independence, functions of bounded variation

2010 Mathematics Subject Classification: 34G25, 34A60, 47J20, 74C05

1. Introduction

Jean Jacques Moreau introduced the *sweeping processes* in a series of articles [26, 27, 28, 29, 32] which culminated in the celebrated paper [33], originating a research that is still active. A sweeping process is an evolution differential inclusion, and its most simple formulation can be described as follows. Let $\mathcal{H}$ be a real Hilbert space and, for every positive time t, let $\mathcal{C}(t)$ be a given nonempty, closed and convex subset of $\mathcal{H}$ such that the mapping $t \longmapsto \mathcal{C}(t)$ is locally Lipschitz continuous when the family of closed subsets of $\mathcal{H}$ is endowed with the Hausdorff metric. One has to find a locally Lipschitz continuous function $y : [0, \infty[\longrightarrow \mathcal{H}$ such that

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [0, \infty[\,, \tag{1}$$

$$y'(t) + N_{\mathcal{C}(t)}(y(t)) \ni 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, \infty[, \tag{2}$$

$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0), \tag{3}$$

y_0 being a prescribed point in $\mathcal{H}$. Here $\mathcal{L}^1$ is the Lebesgue measure, $\mathrm{Proj}_{\mathcal{C}(0)}$ denotes the projection operator on $\mathcal{C}(0)$, and $N_{\mathcal{C}(t)}$ is the exterior normal cone

*The author is a member of GNAMPA-INdAM.

ISSN 0944-6532 / $ 2.50

to $\mathcal{C}(t)$ at $y(t)$ (all the precise definitions will be given in Section 2). When the interior of $\mathcal{C}(t)$ is nonempty, the process defined by (1)–(3) has a nice and useful geometrical-mechanical interpretation which we recall from [33]: "the moving point $y(t)$ remains at rest as long as it happens to lie in the interior of $\mathcal{C}(t)$; when caught up with the boundary of the moving set, it can only proceed in an inward normal direction, as if pushed by this boundary, so as to go on belonging to $\mathcal{C}(t)$". Moreau was originally motivated by elastoplaticity and friction dynamics (cf. [29, 32, 34]), but now sweeping processes have found applications to nonsmooth mechanics (see [25]), to economics ([14, 11]), to electrical circuits (cf., e.g., [1, 6]), to crowd motion modeling ([23]), and to other fields (see, e.g., the references in the recent paper [2]).

Differential inclusion (2) can be seen as a time dependent gradient flow

$$y'(t) + \partial\phi^t(y(t)) \ni 0,$$

indeed $N_{\mathcal{C}(t)} = \partial\phi^t$, the subdifferential of the indicator function $\phi^t = I_{\mathcal{C}(t)}$, defined by $I_{\mathcal{C}(t)}(x) = 0$ if $x \in \mathcal{C}(t)$, $I_{\mathcal{C}(t)}(x) = \infty$ otherwise. Problem (1)–(3) was solved by Moreau by approximating ϕ^t by what is now called Moreau-Yosida regularization:

$$\phi^t_\lambda(x) := \min_{y\in\mathcal{H}} \left(\frac{1}{2\lambda}\|y-x\|^2 + \phi^t(y)\right), \quad x \in \mathcal{H}, \tag{4}$$

where $\|\cdot\|$ is the norm of $\mathcal{H}$ and $\lambda > 0$. The resulting equation $y'_\lambda(t) + \partial\phi^t_\lambda(y_\lambda(t)) = 0$ admits a unique continuously differentiable solution satisfying the initial condition (3), and in [26] it is proved that if λ approaches 0, then y_λ converges to a function $y =: \mathsf{S}(\mathcal{C})$ which uniquely solves (1)–(3).

If $\mathcal{C}(t)$ is not regular enough in time, for instance if the mapping $t \longmapsto \mathcal{C}(t)$ is of bounded variation, then one expects that the solution of any reasonable generalized sweeping processes is of bounded variation, hence in general its ($\mathscr{L}^1$-a.e.) pointwise derivative does not coincide with the distributional one, and an alternative formulation has to be found. In [33] Moreau introduced the following generalization of (2)–(3). If $t \longmapsto \mathcal{C}(t)$ is right continuous and with local bounded variation, one has to find a right continuous local BV function $y : [0,\infty[\longrightarrow \mathcal{H}$ such that there exist a positive measure μ and a μ-locally integrable function $w : [0,\infty[\longrightarrow \mathcal{H}$ satisfying

$$\mathrm{D}y = w\mu, \tag{5}$$
$$y(t) \in \mathcal{C}(t) \quad \forall t \in [0,\infty[\,, \tag{6}$$
$$w(t) + N_{\mathcal{C}(t)}(y(t)) \ni 0 \quad \text{for } \mu\text{-a.e. } t \in [0,\infty[, \tag{7}$$
$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0), \tag{8}$$

where $\mathrm{D}y$ denotes the differential measure of y, that is its distributional derivative.

Problem (5)–(8) is solved in [33] by means of the so called "catching-up algorithm". It consists in choosing a suitable sequence of subdivisions $0 = t_1^n < \cdots < t_m^n < \cdots$, with $t_m \to \infty$ as $m \to \infty$, and in defining the functions y_n by

$$y_0^n := \mathrm{Proj}_{\mathcal{C}(0)}(y_0), \qquad y_j^n := \mathrm{Proj}_{\mathcal{C}(t_j^n)}(y_{j-1}^n),$$
$$y_n(t) := y_j^n \quad \text{if } t \in [t_{j-1}^n, t_j^n[.$$

In [33] it is proved that y_n converges to a function $y =: \mathsf{S}_c(\mathcal{C})$ of local bounded variation which is exactly the solution of the sweeping process "in the sense of differential measures", i.e. such that (5)–(8) hold (the subscript $_c$ in S_c stands for "catching-up"). Then it is proved that for regular $\mathcal{C}(t)$, the solution y solves (5)–(8), so that S_c is actually an extension of S.

Tha aim of the present paper is to solve the BV sweeping processes (5)–(8) by reducing it to the Lispchitz continuous case (1)–(3). This method is based on the observation that sweeping processes are *rate independent*. This means that if S is the solution operator of the sweeping process, associating with a Lipschitz $\mathcal{C}$ the solution y of (1)–(3), then we have

$$\mathsf{S}(\mathcal{C} \circ \gamma) = \mathsf{S}(\mathcal{C}) \circ \gamma \tag{9}$$

for every Lipschitz continuous increasing reparametrization γ of time. For the theory of general rate independent operators we refer, e.g., to [15, 9, 17, 24, 43], and we remark that property (9) was already observed by J. J. Moreau in [33, Proposition 2i], even if he did not use the term "rate independence".

In particular we can reparametrize the rectifiable curve $\mathcal{C}(t)$ by the arc length $\ell_{\mathcal{C}}(t) := \mathrm{V}(\mathcal{C}, [0,t])$, the variation of $\mathcal{C}$ on $[0,t]$, so we find a Lipschitz continuous $\widetilde{\mathcal{C}}$ such that $\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$. This reparametrization method was used by Moreau in [26, 29] in order to reduce to the Lipschitz continuous case the sweeping process driven by an absolutely continuous moving set $\mathcal{C}(t)$. Now, whatever generalized BV-solution of problem (1)–(3) one may define, rate independence suggests that its solution may be $y = \mathsf{S}(\widetilde{\mathcal{C}}) \circ \ell$. We used this idea in [41] to deal with the case of a continuous BV moving set $\mathcal{C}(t)$ and, together with some measure theory tools, we reduced the associated sweeping process to the classical Lipschitz case.

It is natural to try to extend this procedure to the case where $\mathcal{C}$ is a discontinuous function of bounded variation. When we reparametrize a discontinuous BV mapping $\mathcal{C}$ by the arc length $\ell_{\mathcal{C}}$ we find a Lipschitz continuous function $\widetilde{\mathcal{C}}$ which is defined on the image $\ell_{\mathcal{C}}([0,\infty[)$ and such that $\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$. At this point we need to extend the definition of $\widetilde{\mathcal{C}}$ to the whole $[0,\infty[$, in other words we have to fill in the jumps of $\mathcal{C}$. But it is not clear what kind of curve we have to choose to connect two endpoints of a jump of $\mathcal{C}$. In order to see which problems arise, let us consider the special case $\mathcal{C}(t) := u(t) - \mathcal{Z}$, where $\mathcal{Z}$ is a fixed nonempty closed convex set, and $u : [0,\infty[\longrightarrow \mathcal{H}$ is a given function of bounded variation (cf. [29, p. 294]). In this case we can restate problem (1)–(3) without any reference to the Hausdorff distance, and the solution operator S can be rewritten as an operator $\mathsf{P} : u \longmapsto y$ associating with u the solution $y = \mathsf{S}(u - \mathcal{Z})$. The operator P is usually called "play operator" and is one the main building blocks of the theory of hysteresis operators (see, e.g., the vectorial case in [15, 16, 17]).

Since we are now in a Hilbert setting, there is a canonical way to fill in the jumps of u: it consists in choosing segments, which represent the shortest trajectory. This natural possibility was observed in [19] and was analyzed in [39], where we defined $\widetilde{u}_s$ to be the arc length reparametrization of u obtained by filling in the jumps with segments, and we proved that the operator P_s defined by $\mathsf{P}_s(u) := \mathsf{P}(\widetilde{u}_s) \circ \ell_u$ produces an extension of P having good continuity properties (the subscript $_s$ in $\widetilde{u}_s$ and P_s stands for "segments"). However in [39] a counterexample shows that, if $\mathcal{C}(t) = u(t) - \mathcal{Z}$ with u of bounded variation, then the extension $\mathsf{P}_s(u)$ is different from the solution $\mathsf{P}_c(u) := \mathsf{S}_c(\mathcal{C})$ in the sense of differential measures. They agree if $\mathcal{H}$ is the real line (cf. [38]), but when $\mathcal{H} = \mathbb{R}^n$, with $n \geq 2$, they agree if and only if $\mathcal{Z}$ is a non obtuse polyhedron (see [20]). In the continuous BV case there are no jumps and P_s and P_c coincide (cf. [36, 41]). This procedure is generalized in [35, 37, 39] for a large class of rate independent operators: in the one dimensional case it unifies several extension results presented in [43]; in the multidimensional case the extension by "filling in the jumps" can be interpreted as an alternative model for some discontinuous phenomena and it works for instance for the so called Mroz model (cf. [18, 7, 8]).

If we come back to the general framework of sweeping processes, a segment $g_s(t) := (1-t)a + tb$, $t \in [0, 1]$, joining two points $a, b \in \mathcal{H}$, corresponds to the convex-valued curve of the form $\mathcal{G}_s(t) := (1-t)\mathcal{A} + t\mathcal{B}$, connecting two closed convex sets $\mathcal{A}$ and $\mathcal{B}$. Thus from the previous discussion it follows that this kind of curves are not the proper choice to fill in the jumps of $\mathcal{C}(t)$ if we want to solve problem (5)–(8), even if they are geodesics. In order to get the solution in the sense of differential measure, we need to find another geodesic $\mathcal{G}(t)$ connecting $\mathcal{A}$ and $\mathcal{B}$ with the property that any "initial" point $a \in \mathcal{A}$ is swept by $\mathcal{G}(t)$ to its projection on $\mathcal{B}$, in other words for any initial datum $a \in \mathcal{A}$, the final point of trajectory of the solution of the sweeping process driven by $\mathcal{G}(t)$ must be $\mathrm{Proj}_{\mathcal{B}}(a)$. In this way we would recover the catching-up algorithm by means of a continuous sweeping process and the reduction to the Lipschitz case would work. The proper curve is given by $\mathcal{G}(t) := (\mathcal{A} + D_{t\rho}) \cap (\mathcal{B} + D_{(1-t)\rho})$, where ρ is the Hausdorff distance between $\mathcal{A}$ and $\mathcal{B}$, and D_r is the closed ball with radius $r \geq 0$ and center 0.

The paper is organized as follows. In the next section we present some preliminaries and in Section 3 we state the results on Lipschitz continuous sweeping processes. The sweeping processes driven by the geodesic $\mathcal{G}(t)$ is studied in Section 4, while in Section 5 we define the reparametrization $\widetilde{\mathcal{C}}(t)$ of a BV convex-valued function $\mathcal{C}(t)$ by the arc length $\ell_{\mathcal{C}}$, where the jumps are filled in by the geodesics of the form $\mathcal{G}(t)$. In the final section we reduce the BV sweeping process (5)–(8) to the Lipschitz continuous case by means of the formula $\mathcal{C} = \widetilde{\mathcal{C}} \circ \ell_{\mathcal{C}}$. This formula also allows to recover the continuous dependence on the data and the convergence of the catching up algorithm.

2. Preliminaries

In this section we recall the main definitions and tools needed in the paper. The set of integers greater than or equal to 1 will be denoted by $\mathbb{N}$. Given an interval I of the real line $\mathbb{R}$, we say that a function $f : I \longrightarrow \mathbb{R}$ is *increasing* if $(f(t) - f(s))(t - s) \geq 0$ for every $t, s \in I$. If $\mathscr{B}(I)$ indicates the family of Borel sets in I, $\mu : \mathscr{B}(I) \longrightarrow [0, \infty]$ is a measure, $p \in [1, \infty]$, and E is a Banach space, then the space of E-valued functions which are p-integrable with respect to μ will be denoted by $L^p(I, \mu; E)$ or simply by $L^p(\mu; E)$. As usual the set of locally p-integrable functions is denoted by $L^p_{\mathrm{loc}}(I, \mu; E)$. We do not identify two functions which are equal μ-almost everywhere (μ-a.e.). For the theory of integration of vector valued functions we refer, e.g., to [22, Chapter VI]. When $\mu = \mathcal{L}^1$, the one dimensional Lebesgue measure, we write $L^p(I; E) := L^p(I, \mu; E)$.

2.1. Functions with values in a metric space

In this section we assume that

$$(X, d) \text{ is a complete metric space,} \tag{10}$$

where we admit that d is an extended metric, i.e. X is a set and

$$d : X \times X \longrightarrow [0, \infty]$$

satisfies the usual axioms of a distance, but may take on the value ∞. The notion of completeness remains unchanged and the topology induced by d is defined in the usual way by means of the open balls $B_r(x_0) := \{x \in X \ : \ d(x, x_0) < r\}$ for $r > 0$ and $x_0 \in X$, so that it satisfies the first axiom of countability. The general topological notions of interior, closure and boundary of a subset $Y \subseteq X$ will be respectively denoted by $\mathrm{int}(Y)$, $\mathrm{cl}(Y)$ and ∂Y. The underlying metric d will always be clear from the context. If $x \in X$ and $Y \subseteq X$, we also set $d(x, Y) := \inf_{y \in Y} d(x, y)$.

If $I \subseteq \mathbb{R}$ is an interval and $f \in X^I$, the space of X-valued functions defined on I, then $\mathrm{Cont}(f)$ denotes the continuity set of f, and $\mathrm{Discont}(f) := I \setminus \mathrm{Cont}(f)$. For $S \subseteq I$ we write $\mathrm{Lip}(f, S) := \sup\{d(f(s), f(t))/|t - s| \ : \ s, t \in S,\ s \neq t\}$, $\mathrm{Lip}(f) := \mathrm{Lip}(f, I)$, the Lipschitz constant of f, and $\mathrm{Lip}(I; X) := \{f \in X^I \ : \ \mathrm{Lip}(f) < \infty\}$, the set of X-valued Lipschitz continuous functions on I. As usual $\mathrm{Lip}_{\mathrm{loc}}(I; X) := \{f \in X^I \ : \ \mathrm{Lip}(f, J) < \infty \ \forall J \text{ compact in } I\}$. We recall now the notion of BV function with values in a metric space (see, e.g., [4, 44]).

Definition 2.1. Given an interval $I \subseteq \mathbb{R}$, a function $u \in X^I$, and a subinterval $J \subseteq I$, the *(pointwise) variation of* u *on* J is defined by

$$\mathrm{V}(u, J) := \sup\left\{\sum_{j=1}^{m} d(u(t_{j-1}), u(t_j)) \ : \ m \in \mathbb{N},\ t_j \in J \ \forall j,\ t_0 < \cdots < t_m\right\}.$$

If $\mathrm{V}(u,I) < \infty$ we say that u *is of bounded variation on* I and we set $BV(I;X) := \{u \in X^I \ :\ \mathrm{V}(u,I) < \infty\}$ and $BV_{\mathrm{loc}}(I;X) := \{u \in X^I \ :\ \mathrm{V}(u,J) < \infty\ \forall J$ *compact* in $I\}$.

It is well known that the completeness of X implies that every $u \in BV(I;X)$ admits one sided limits $u(t-), u(t+)$ at every point $t \in I$, with the convention that $u(\inf I-) := u(\inf I)$ if $\inf I \in I$, and that $u(\sup I+) := u(\sup I)$ if $\sup I \in I$. Moreover there exist $u(-\infty) := \lim_{t\to-\infty} u(t)$ and $u(\infty) := \lim_{t\to\infty} u(t)$. We set

$$BV^l(I;X) := \{u \in BV(I;X)\ :\ u(t-) = u(t)\ \forall t \in I\},$$
$$BV^r(I;X) := \{u \in BV(I;X)\ :\ u(t) = u(t+)\ \forall t \in I\}.$$

Accordingly we set $BV^l_{\mathrm{loc}}(I;X) := \{u \in BV_{\mathrm{loc}}(I;X)\ :\ u(t-) = u(t)\ \forall t \in I\}$ and $BV^r_{\mathrm{loc}} := \{u \in BV_{\mathrm{loc}}(I;X)\ :\ u(t) = u(t+)\ \forall t \in I\}$. Clearly we have $\mathrm{Lip}_{\mathrm{loc}}(I;X) \subseteq BV_{\mathrm{loc}}(I;X)$.

In the next definition we introduce some natural metrics in $BV(I;X)$.

Definition 2.2. For every $u, v \in BV(I;X)$ and every interval $J \subseteq I$ we set

$$d_{\infty,J}(u,v) := \sup_{t\in J} d(u(t),v(t)), \tag{11}$$

$$d_{us,J}(u,v) := d_{\infty,J}(u,v) + |\,\mathrm{V}(u,J) - \mathrm{V}(v,J)|. \tag{12}$$

We set $d_\infty := d_{\infty,I}$ and $d_{us} := d_{us,I}$, which are called respectively *uniform metric on* $BV(I;X)$ and *uniform strict metric on* $BV(I;X)$. We say that $u_n \to u$ *uniformly strictly on* J if $d_{us,J}(u_n,u) \to 0$ as $n \to \infty$. We say that $u_n \to u$ *locally uniformly strictly on* I if $d_{us,J}(u_n,u) \to 0$ as $n \to \infty$ for every interval J compact in I.

Observe that the metric d_{us} is the metric that fits with the convergence of the catching up algorithm (cf. formula (29)). Of course d_∞ is the distance inducing the topology of uniform convergence. Now we recall the notion of geodesic.

Definition 2.3. Assume that $x, y \in X$ and $d(x,y) < \infty$. A function $g \in \mathrm{Lip}([0,1]\,;X)$ is called *a geodesic connecting* x *to* y if $g(0) = x$, $g(1) = y$, and $\mathrm{V}(g,[0,1]) = d(x,y)$.

2.2. Hilbert spaces

Throughout the remainder of the paper we assume that

$$\begin{cases} \mathcal{H} \text{ is a real Hilbert space with inner product } (x,y) \longmapsto \langle x,y\rangle \\ \|x\| := \langle x,x\rangle^{1/2}, \end{cases} \tag{13}$$

and we endow $\mathcal{H}$ with the natural metric defined by $d(x,y) := \|x-y\|$, $x,y \in \mathcal{H}$. We also use the notation

$$D_r := \{x \in \mathcal{H}\ :\ \|x\| \le r\}, \quad r \ge 0.$$

We set

$$\mathscr{C}_{\mathcal{H}} := \{\mathcal{K} \subseteq \mathcal{H} \ : \ \mathcal{K} \text{ nonempty, closed and convex}\}.$$

If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $x \in \mathcal{H}$, then $\mathrm{Proj}_{\mathcal{K}}(x)$ is the projection on $\mathcal{K}$, i.e. $y = \mathrm{Proj}_{\mathcal{K}}(x)$ is the unique point such that $d(x, \mathcal{K}) = \|x - y\|$, and it is also characterized by the fact that $y \in \mathcal{K}$ and satisfies the variational inequality

$$\langle x - y, v - y \rangle \leq 0 \quad \forall v \in \mathcal{K}.$$

If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $x \in \mathcal{K}$, then $N_{\mathcal{K}}(x)$ denotes the *(exterior) normal cone of* $\mathcal{K}$ *at* x:

$$N_{\mathcal{K}}(x) := \{u \in \mathcal{H} \ : \ \langle u, v - x \rangle \leq 0 \ \forall v \in \mathcal{K}\} = \mathrm{Proj}_{\mathcal{K}}^{-1}(x) - x. \tag{14}$$

It is well-known that if $\mathrm{int}(\mathcal{K}) \neq \varnothing$ and $x \in \partial\mathcal{K}$, then $N_{\mathcal{K}}(x) \neq \{0\}$ (see, e.g., [17, Proposition 2.11]). We endow the set $\mathscr{C}_{\mathcal{H}}$ with the Hausdorff distance. Here is the definition.

Definition 2.4. The *Hausdorff distance* $\mathrm{d}_{\mathscr{H}} : \mathscr{C}_{\mathcal{H}} \times \mathscr{C}_{\mathcal{H}} \longrightarrow [0, \infty]$ is defined by

$$\mathrm{d}_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) := \max\left\{\sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(b, \mathcal{A})\right\}, \quad \mathcal{A}, \mathcal{B} \subseteq \mathscr{C}_{\mathcal{H}}.$$

For the proof of the following proposition see, e.g., [10, Theorem II-14, Section II.3.14, p. 47] and [3, Lemma 3.71, Section 3.16, p. 110].

Proposition 2.5. *The distance* $\mathrm{d}_{\mathscr{H}}$ *makes* $\mathscr{C}_{\mathcal{H}}$ *a complete metric space. If* $\mathcal{A}, \mathcal{B} \in \mathscr{C}_{\mathcal{H}}$ *then*

$$\mathrm{d}_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) = \inf\{\rho > 0 \ : \ \mathcal{A} \subseteq \mathcal{B} + D_\rho, \ \mathcal{B} \subseteq \mathcal{A} + D_\rho\},$$

where $\inf \varnothing := \infty$.

Now we recall the notion of subdifferential (cf. [5, Chapter 2]). If $\phi : \mathcal{H} \longrightarrow [0, \infty]$ is convex and lower semicontinuous and $D(\phi) := \{x \in \mathcal{H} \ : \ \phi(x) \neq \infty\} \neq \varnothing$, then for $x \in \mathcal{H}$ the *subdifferential of* ϕ *at* x is defined by $\partial\phi(x) := \{y \in \mathcal{H} \ : \ \langle y, v - x \rangle + \phi(x) \leq \phi(v) \ \forall v \in \mathcal{H}\}$. The *domain* of $\partial\phi$ is defined by $D(\partial\phi) := \{x \in \mathcal{H} \ : \ \partial\phi(x) \neq \varnothing\}$. If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $I_{\mathcal{K}}$, the *indicator function of* $\mathcal{K}$, is defined by $I_{\mathcal{K}}(x) = 0$ if $x \in \mathcal{K}$ and $I_{\mathcal{K}}(x) = \infty$ if $x \notin \mathcal{K}$, then $\partial I_{\mathcal{K}}(x) = N_{\mathcal{K}}(x)$ for every $x \in D(I_{\mathcal{K}}) = D(\partial I_{\mathcal{K}}) = \mathcal{K}$.

2.3. Differential measures

Let $I \subseteq \mathbb{R}$ be an interval and let E be a Banach space with norm $\|\cdot\|_E$ and let $\mathscr{B}_c(I)$ be the family of precompact Borel subsets of I. We recall that a *(Borel) vector measure* on I is a map $\mu : \mathscr{B}_c(I) \longrightarrow E$ such that $\mu(\bigcup_{n=1}^\infty B_n) = \sum_{n=1}^\infty \mu(B_n)$ whenever (B_n) is a sequence of mutually disjoint sets in $\mathscr{B}_c(I)$

and $\bigcup_{n=1}^{\infty} B_n \in \mathscr{B}_c(I)$. The *total variation of* μ is the positive measure $|\mu| : \mathscr{B}(I) \longrightarrow [0,\infty]$ defined by

$$|\mu|(B)$$
$$:= \sup\left\{\sum_{n=1}^{\infty} \|\mu(B_n)\|_E \ : \ B = \bigcup_{n=1}^{\infty} B_n,\ B_n \in \mathscr{B}_c(I),\ B_h \cap B_k = \varnothing \text{ if } h \neq k\right\}.$$

The vector measure μ is said to be *with bounded variation* if $|\mu|(I) < \infty$. Finally we say that μ is *with local bounded variation* if $|\mu|(J) < \infty$ for every interval J compact in I (see, e.g., [12, Chapter I, Section 3.]).

Let E_j, $j = 1,2,3$, be Banach spaces with norms $\|\cdot\|_{E_j}$ and let $E_1 \times E_2 \longrightarrow E_3 : (x_1,x_2) \longmapsto x_1 \bullet x_2$ be a bilinear form such that $\|x_1 \bullet x_2\|_{E_3} \leq \|x_1\|_{E_1}\|x_2\|_{E_2}$ for every $x_j \in E_j$, $j = 1,2$. Assume that $\mu : \mathscr{B}_c(I) \longrightarrow E_2$ is a vector measure with local bounded variation and let $f : I \longrightarrow E_1$ be a *step map with respect to* μ, i.e. there exist $f_1,\ldots,f_m \in E_1$ and $A_1,\ldots,A_m \in \mathscr{B}_c(I)$ mutually disjoint such that $|\mu|(A_j) < \infty$ for every j and

$$f = \sum_{j=1}^{m} \mathbb{1}_{A_j} f_j,$$

where $\mathbb{1}_S$ is the characteristic function of a set S, i.e. $\mathbb{1}_S(x) := 1$ if $x \in S$ and $\mathbb{1}_S(x) := 0$ if $x \notin S$. For such f we define

$$\int_I f \bullet \mathrm{d}\mu := \sum_{j=1}^{m} f_j \bullet \mu(A_j) \in E_3.$$

If $St(|\mu|;E_1)$ is the set of E_1-valued maps with respect to μ, then the map $St(|\mu|;E_1) \longrightarrow E_3 : f \longmapsto \int_I f \bullet \mathrm{d}\mu$ is linear and continuous when $St(|\mu|;E_1)$ is endowed with the L^1-semimetric $\|f-g\|_{L^1(|\mu|;E_1)} := \int_I \|f-g\|_{E_1}\,\mathrm{d}\,|\mu|$. Therefore it admits a unique continuous extension $\mathsf{I}_\mu : L^1(|\mu|;E_1) \longrightarrow E_3$ and we set

$$\int_I f \bullet \mathrm{d}\mu := \mathsf{I}_\mu(f), \quad f \in L^1(|\mu|;E_1).$$

As usual $\int_S f \bullet \mathrm{d}\mu := \int_I \mathbb{1}_S f \bullet \mathrm{d}\mu$ for every $S \in \mathscr{B}(I)$. We will use the previous integral in two particular cases, namely when

a) $E_1 = \mathbb{R}$, $E_2 = E_3 = \mathcal{H}$, $\lambda \bullet x := \lambda x$ ($\int_I f \bullet \mathrm{d}\mu = \int_I f\,\mathrm{d}\mu$, integral of a real function with respect to a vector measure);

b) $E_1 = E_2 = \mathcal{H}$, $E_3 = \mathbb{R}$, $x_1 \bullet x_2 := \langle x_1, x_2\rangle$ ($\int_I f \bullet \mathrm{d}\mu = \int_I \langle f, \mathrm{d}\mu\rangle$, integral of a vector function with respect to a vector measure).

The following proposition (cf. [12, Theorem 1, section III.17.2, p. 358]) provides a connection between functions with bounded variation and vector measures.

Theorem 2.6. *If $f \in BV_{\mathrm{loc}}(I;\mathcal{H})$ then there exists a unique vector measure of local bounded variation $\mu_f : \mathscr{B}_c(I) \longrightarrow \mathcal{H}$ such that for every $c, d \in I$ with $-\infty < c < d < \infty$ we have*

$$\mu_f(]c,d[) = f(d-) - f(c+), \qquad \mu_f([c,d]) = f(d+) - f(c-),$$
$$\mu_f([c,d[) = f(d-) - f(c-), \qquad \mu_f(]c,d]) = f(d+) - f(c+).$$

Moreover if $f_+ : I \longrightarrow \mathcal{H}$ is defined by $f_+(t) := f(t+)$ if $t \in \mathrm{int}(I)$, $f_+(t) := f(t)$ if $t \in \partial I$, then $\mu_f = \mu_{f_+}$. Vice versa if $\mu : \mathscr{B}_c(I) \longrightarrow \mathcal{H}$ is a vector measure with local bounded variation, if $a \in I$, and if $f_\mu : I \longrightarrow \mathcal{H}$ is defined by $f_\mu(t) := \mu([a,t])$ if $t \geq a$, $f_\mu(t) := -\mu([t,a])$ if $t < a$, then $f_\mu \in BV_{\mathrm{loc}}(I;\mathcal{H})$ and $\mu_{f_\mu} = \mu$.

The measure μ_f is called *Lebesgue-Stieltjes measure* or *differential measure* of f. Like in the scalar case if I is right-open and $f \in BV^r_{\mathrm{loc}}(I;\mathcal{H})$ then $\mu_f = \mathrm{D}f$, the distributional derivative of f, i.e.

$$-\int_I \varphi'(t) f(t)\,\mathrm{d}t = \int_I \varphi\,\mathrm{d}\mathrm{D}f \quad \forall \varphi \in C^1_c(\mathrm{int}(I))$$

(cf. [39, Section 2]). Let us also recall that if $f \in \mathrm{Lip}_{\mathrm{loc}}(I;\mathcal{H})$ then there exists the derivative $f'(t)$ for $\mathcal{L}^1$-a.e. in $t \in I$ and $\mathrm{D}f = f'\mathcal{L}^1$ (cf. [5, Appendix]). Observe that the functional $g \longmapsto \int_I \langle f, \mathrm{d}\mathrm{D}g\rangle$ is linear in the integrator function g. In [39, Section A.4] we show that the integral with respect to $\mathrm{D}f$ coincides with the Young integral with respect to f (cf. [19, Section 3]). Therefore from [19, Corollary 3.15] we infer the following

Proposition 2.7. *If $f \in BV^r_{\mathrm{loc}}(I;\mathcal{H})$, $a, b \in I$, and $a < b < \sup I$, then*

$$\int_{[a,b]} \langle f, \mathrm{d}\mathrm{D}f\rangle = \frac{1}{2}\left(\|f(b)\|^2 - \|f(a)\|^2\right) + \frac{1}{2}\sum_{t \in [a,b]} \|f(t) - f(t-)\|^2.$$

In [39, Lemma 6.4 and Theorem 6.1] it is also proved that

Proposition 2.8. *Assume that $I, J \subseteq \mathbb{R}$ are right-open intervals, and that $h : I \longrightarrow J$ is increasing and right continuous.*

(*i*) $\mathrm{D}h(h^{-1}(B)) = \mathcal{L}^1(B)$ *for every $B \in \mathscr{B}(h(\mathrm{Cont}(h)))$.*

(*ii*) *If $f \in \mathrm{Lip}_{\mathrm{loc}}(J;\mathcal{H})$ and $g : I \longrightarrow \mathcal{H}$ is defined by*

$$g(t) := \begin{cases} f'(h(t)) & \text{if } t \in \mathrm{Cont}(h) \\ \dfrac{f(h(t+)) - f(h(t-))}{h(t+) - h(t-)} & \text{if } t \in \mathrm{Discont}(h), \end{cases}$$

then $f \circ h \in BV_{\mathrm{loc}}(I;\mathcal{H})$ and $\mathrm{D}(f \circ h) = g\,\mathrm{D}h$. This result holds even if f' is replaced by any of its $\mathcal{L}^1$-representives.

3. Review of sweeping processes

In this section we briefly review the main existence result for Lipschitz continuous sweeping processes and we recall the definition of "catching-up" algorithm. We aim to emphasize that the existence of solution of problem (1)–(3) can also be proved by means of a regularization method with no use of discretization procedures.

Problem 3.1. *Let $y_0 \in \mathcal{H}$ and $\mathcal{C} \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathcal{C}_{\mathcal{H}})$ be given. One has to find a function $y \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ such that* (1)–(3) *hold. Such function y is called* solution of the sweeping processes with driving term $\mathcal{C}$ and initial condition y_0.

The uniqueness of solutions of Problem 3.1 follows from the "monotonicity" of the normal cone to a set $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$, that is

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq 0 \quad \forall x_j \in \mathcal{K},\ \forall v_j \in N_{\mathcal{K}}(x_j),\ j = 1, 2.$$

Indeed if y_1, y_2 are two solutions, then from (2) we get $0 \geq \langle y_1'(t) - y_2'(t), y_1(t) - y_2(t)\rangle = (1/2)(\mathrm{d}/\mathrm{d}t)\|y_1(t)-y_2(t)\|^2$ for $\mathcal{L}^1$-a.e. $t \geq 0$, hence $t \longmapsto \|y_1(t)-y_2(t)\|^2$ is decreasing, and from (3) it follows that Problem 3.1 has at most one solution. Concerning existence, we would like to emphasize that the solution of sweeping processes can be obtained as a limit of continuously differentiable approximating functions. As mentioned in the Introduction, this method was used by Moreau in [29] by replacing the indicator function $\phi^t = I_{\mathcal{C}(t)}$ of $\mathcal{C}(t)$ by the Moreau-Yosida approximation (4). It turns out (cf. e.g. [5, Section 2.8, p. 46]) that the normal cone $N_{\mathcal{C}(t)}(x) = \partial I_{\mathcal{C}(t)}$ is replaced by its Yosida approximation

$$(N_{\mathcal{C}(t)})_\lambda(x) = \partial(I_{\mathcal{C}(t)})_\lambda = \frac{1}{\lambda}\left(x - \mathrm{Proj}_{\mathcal{C}(t)}(x)\right), \quad x \in \mathcal{H}, \tag{14}$$

where $\lambda > 0$. Therefore the regularized problem consists in searching for a function $y_\lambda \in C^1([0,\infty[\,;\mathcal{H})$ such that

$$y_\lambda'(t) + \frac{1}{\lambda}\left(y_\lambda(t) - \mathrm{Proj}_{\mathcal{C}(t)}(y_\lambda(t))\right) = 0 \quad \forall t \geq 0, \tag{15}$$

$$y_\lambda(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0). \tag{16}$$

The map defined by (14) is clearly Lipschitz continuous in the variable $x \in \mathcal{H}$, and Moreau proved (cf. [33, formula 2.17]) that it is also continuous in time, therefore standard results in ordinary differential equations imply that there is a unique solution to (15)–(16). By proving further time regularity properties of $t \longmapsto (N_{\mathcal{C}(t)})_\lambda(y_\lambda(t))$, he was able to deduce enough a priori estimates allowing to prove the following existence theorem for Problem 3.1.

Theorem 3.2. *Assume that $\mathcal{C} \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ and $y_0 \in \mathcal{H}$. Then there exists a unique $\mathsf{S}(\mathcal{C}) := \mathsf{S}(\mathcal{C}, y_0) := y \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ such that*

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [0,\infty[\,, \tag{17}$$

$$y'(t) + N_{\mathcal{C}(t)}(y(t)) \ni 0 \quad \textit{for } \mathcal{L}^1\textit{-a.e. } t \in [0,\infty[, \tag{18}$$

$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0). \tag{19}$$

Moreover

$$\mathrm{Lip}(y,[0,t]) \le \mathrm{Lip}(\mathcal{C},[0,t]) \quad \forall t > 0. \tag{20}$$

If $y_\lambda \in C^1([0,\infty[\,;\mathcal{H})$ *is the unique function satisfying* (15)–(16)*, then* $y_\lambda \to y$ *locally uniformly in* $[0,\infty[$ *as* $\lambda \searrow 0$.

Thanks to Theorem 3.2 we can define the solution operator

$$\mathsf{S} = \mathsf{S}(\cdot, y_0) : \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}}) \longrightarrow \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}) \tag{21}$$

assigning to every $\mathcal{C} \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ the unique function $\mathsf{S}(\mathcal{C}) := \mathsf{S}(\mathcal{C}, y_0) := y$ satisfying (17)–(19). We will often omit the dependence on the initial datum y_0 if no confusion may arise. This operator has the semigroup property

$$\mathsf{S}(\mathcal{C}, y_0)(t) = \mathsf{S}(\mathcal{C}(\cdot + s), \mathsf{S}(\mathcal{C}, y_0)(s))(t-s) \quad \forall t, s,\ 0 \le s < t. \tag{22}$$

Definition 3.3. If $y_0 \in \mathcal{H}$ and $\mathcal{C} \in \mathrm{Lip}([0,T]\,;\mathscr{C}_{\mathcal{H}})$ for some $T > 0$, then we say that y is a *solution of the sweeping process with driving term* $\mathcal{C}$ *and initial condition* y_0 if it is the restriction to $[0,T]$ of the solution of the sweeping process with initial condition y_0 and driving term $\mathcal{C}_T \in \mathrm{Lip}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ defined by $\mathcal{C}_T(t) := \mathcal{C}(t)$ if $t \in [0,T]$ and $\mathcal{C}_T(t) := \mathcal{C}(T)$ if $t > T$.

From (17)–(20) one can derive the following continuous dependence result (see [21, Theorem 4]).

Theorem 3.4. *Assume that* $\mathcal{C}_1, \mathcal{C}_2 \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$*,* $y_{0,1}, y_{0,2} \in \mathcal{H}$*, and set* $y_k := \mathsf{S}(\mathcal{C}_k, y_{0,k})$*,* $k = 1,2$*. Then*

$$\begin{aligned} &\|y_1(t) - y_2(t)\| \\ &\le \|y_{0,1} - y_{0,2}\| + 2\left(\mathrm{Lip}(\mathcal{C}_1,[0,t]) + \mathrm{Lip}(\mathcal{C}_2,[0,t])\right) \int_0^t \mathrm{d}_{\mathcal{H}}(\mathcal{C}_1(s), \mathcal{C}_2(s))\,\mathrm{d}s \end{aligned}$$

for every $t \ge 0$.

For non regular data $\mathcal{C} \in BV^r_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$, the following notion of generalized solution is introduced in [33].

Problem 3.5. *Given* $\mathcal{C} \in BV^r_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ *and* $y_0 \in \mathcal{H}$ *one has to find a function* $y \in BV^r_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ *such that there exist a measure* $\mu : \mathscr{B}([0,\infty[) \longrightarrow [0,\infty]$ *and a function* $w \in L^1_{\mathrm{loc}}(\mu;\mathcal{H})$ *satisfying*

$$\mathrm{D}y = w\mu \tag{23}$$

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [0,\infty[\,, \tag{24}$$

$$w(t) + N_{\mathcal{C}(t)}(y(t)) \ni 0 \quad \text{for } \mu\text{-a.e. } t \in [0,\infty[, \tag{25}$$

$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0). \tag{26}$$

In [33] Moreau proved that Problem 3.5 has solutions by means of the so called "catching-up algorithm". It consists in choosing a sequence of subdivisions (t^n_m) such that

$$0 = t^n_0 < \cdots < t^n_m < \cdots, \qquad \lim_{m\to\infty} t^n_m = \infty \quad \forall n \in \mathbb{N}. \tag{27}$$

Then we define $\mathcal{C}_n := [0,\infty[\longrightarrow \mathscr{C}_{\mathcal{H}}$ by

$$\mathcal{C}_n(t) := \mathcal{C}(t^n_{j-1}) \quad \text{if } t \in \left[t^n_{j-1}, t^n_j\right[. \tag{28}$$

If $\mathcal{C} \in BV^{r}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ then it is possible to choose the sequence (t^n_m) such that (cf. [33, formulas (1.11), (1.12)]) $\mathcal{C}_n \to \mathcal{C}$ locally uniformly strictly on $[0,\infty[$, i.e.

$$d_{us,[c,d]}(\mathcal{C}_n, \mathcal{C}) \to 0 \quad \text{as } n \to \infty,\ \forall c, d \geq 0,\ c < d. \tag{29}$$

Then one defines the local step function $y_n \in BV^{r}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ by

$$y^n_0 := y_0, \qquad y^n_j := \mathrm{Proj}_{\mathcal{C}(t^n_j)}(y^n_{j-1}) \tag{30}$$

$$y_n(t) := y^n_j \quad \text{if } t \in [t^n_{j-1}, t^n_j[. \tag{31}$$

In [33] it is proved that there exists $y \in BV^{r}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ such that $y_n \to y$ locally uniformly, and y is the unique solution of Problem 3.5. In the next sections we will show a new procedure allowing to reduce Problem 3.5 directly to Problem 3.1. This reduction will be obtained by means of a suitable reparametrization of the driving moving set $\mathcal{C}(t)$.

4. A particular sweeping process

In this section we find the solution of a particular sweeping process driven by a curve $\mathcal{G} : [0,1] \longrightarrow \mathscr{C}_{\mathcal{H}}$ having the property that $\mathsf{S}(\mathcal{G}, y_0)(1) = \mathrm{Proj}_{\mathcal{G}(1)}(y_0)$ for any initial datum $y_0 \in \mathcal{G}(0)$. Before we need some simple technical lemmas.

Lemma 4.1. *Assume that $\rho > 0$, $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$, and $\mathcal{K}_\rho := \mathcal{K} + D_\rho$. Then*

(i) $\mathcal{K}_\rho = \{x \in \mathcal{H} \ :\ \|x - \mathrm{Proj}_{\mathcal{K}}(x)\| \leq \rho\} \in \mathscr{C}_{\mathcal{H}}$.
(ii) $\partial\mathcal{K}_\rho = \{x \in \mathcal{H} \ :\ \|x - \mathrm{Proj}_{\mathcal{K}}(x)\| = \rho\}$.
(iii) *If $y \in \partial\mathcal{K}_\rho$ and $x = \mathrm{Proj}_{\mathcal{K}}(y)$, then $N_{\mathcal{K}_\rho}(y) = \{\lambda(y - x) \ :\ \lambda \geq 0\}$.*

Proof. The proofs of (i) and (ii) are straightforward. In order to show (iii), observe that $\mathrm{int}(\mathcal{K}_\rho) \neq \varnothing$, therefore $N_{\mathcal{K}_\rho}(y) \neq \{0\}$. Take $n \in N_{\mathcal{K}_\rho}(y)$ with $\|n\| = 1$. Thus, since $\|y - x\| = \rho$ and $x + \rho n \in \mathcal{K}_\rho$, we have $\langle n, x + \rho n - y\rangle \leq 0$, which implies that $\langle n - (y-x)/\rho, y - x\rangle = \langle n, y - x\rangle - \rho \geq \langle n, \rho n\rangle - \rho = 0$. On the other hand by the Cauchy-Schwarz inequality we have $\langle n - (y-x)/\rho, y - x\rangle = \langle n, y - x\rangle - \rho \leq \|n\|\,\|y - x\| - \rho = 0$. Thus we have shown that $n = (y - x)/\rho$ which implies that every vector of the normal cone $N_{\mathcal{K}_\rho}(y)$ is necessarily of the form $\lambda(y - x)$ for some $\lambda \geq 0$. ☐

Lemma 4.2. *Assume that $\mathcal{Z}, \mathcal{W} \in \mathscr{C}_{\mathcal{H}}$ and $\mathcal{Z} \cap \mathcal{W} \neq \varnothing$. Then $\partial(\mathcal{Z} \cap \mathcal{W}) \subseteq \partial\mathcal{Z} \cup \partial\mathcal{W}$ and*

$$x \in \partial\mathcal{Z} \cap \operatorname{int}(\mathcal{W}) \quad \Longrightarrow \quad x \in \partial(\mathcal{Z} \cap \mathcal{W}), \quad N_{\mathcal{Z}\cap\mathcal{W}}(x) = N_{\mathcal{Z}}(x).$$

Proof. The inclusions $\partial(\mathcal{Z} \cap \mathcal{W}) \subseteq \partial\mathcal{Z} \cup \partial\mathcal{W}$ and $\partial\mathcal{Z} \cap \operatorname{int}(\mathcal{W}) \subseteq \partial(\mathcal{Z} \cap \mathcal{W})$ obviously hold in any topological space, thus we address the statement about the normal cones. It is easily seen that $N_{\mathcal{Z}}(x) \subseteq N_{\mathcal{Z}\cap\mathcal{W}}(x)$, so let us consider $n \in N_{\mathcal{Z}\cap\mathcal{W}}(x)$, i.e. $\langle n, v - x\rangle \leq 0$ for every $v \in \mathcal{Z} \cap \mathcal{W}$, and take $z \in \mathcal{Z} \setminus \mathcal{W}$. Then the segment joining z and x is contained in $\mathcal{Z}$ and, since $x \in \operatorname{int}(\mathcal{W})$, there exists a point on this segment, say $w_\lambda := (1-\lambda)z + \lambda x$ with $\lambda \in \,]0,1[$ small enough, such that $w_\lambda \in \mathcal{W}$. Therefore $\langle n, z - x\rangle = \langle n, w_\lambda + \lambda(z-x) - x\rangle = \langle n, w_\lambda - x\rangle + \lambda\langle n, z-x\rangle \leq \lambda\langle n, z-x\rangle$ which yields $(1-\lambda)\langle n, z-x\rangle \leq 0$. Therefore $N_{\mathcal{Z}\cap\mathcal{W}}(x) \subseteq N_{\mathcal{Z}}(x)$ and we are done. $\square$

Lemma 4.3. *If $\mathcal{Z}, \mathcal{W} \in \mathscr{C}_{\mathcal{H}}$ then $\mathcal{Z} \cap (\mathcal{W} + D_\rho) \subseteq ((\mathcal{Z} + D_\rho) \cap \mathcal{W}) + D_\rho$.*

Proof. Take $x \in \mathcal{Z} \cap (\mathcal{W} + D_\rho)$. Thus $x \in \mathcal{Z}$ and there exists $w \in \mathcal{W}$ such that $d(x,w) \leq \rho$. It follows that $w \in (\mathcal{Z} + D_\rho) \cap \mathcal{W}$ and $d(x,w) \leq \rho$, that is $x \in ((\mathcal{Z} + D_\rho) \cap \mathcal{W}) + D_\rho$. $\square$

Now we can introduce the curve $\mathcal{G}$. Let $\mathcal{A}, \mathcal{B} \in \mathscr{C}_{\mathcal{H}}$ be such that $\rho := d_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) < \infty$ and let us define, for every $t \in [0,1]$, the following closed convex subset of $\mathcal{H}$:

$$\mathcal{G}(t) := \mathcal{G}_{(\mathcal{A},\mathcal{B})}(t) := (\mathcal{A} + D_{t\rho}) \cap (\mathcal{B} + D_{(1-t)\rho}), \quad t \in [0,1]. \tag{32}$$

In the remainder of this section we simply write $\mathcal{G}$ rather than $\mathcal{G}_{(\mathcal{A},\mathcal{B})}$. Observe that $\mathcal{G}(t) \neq \varnothing$ for every $t \in [0,1]$, indeed, if $a \in \mathcal{A}$ and $b = \operatorname{Proj}_{\mathcal{B}}(a)$, then $\|b-a\| \leq \rho$, hence $x_t = a + t(b-a) = b + (1-t)(a-b) \in \mathcal{G}(t)$ for every $t \in [0,1]$. The $\mathscr{C}_{\mathcal{H}}$-valued curve $\mathcal{G}$ can be found in [42] where $\mathcal{A}$ and $\mathcal{B}$ are assumed to be compact sets of an arbitrary metric space, and it is shown a proof that $\mathcal{G}$ is a geodesic. For the sake of completeness we prefer to provide a proof in a slightly different form which makes use of convexity.

Proposition 4.4. *Let $\mathcal{A}, \mathcal{B} \in \mathscr{C}_{\mathcal{H}}$ be such that $\rho := d_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) < \infty$ and let $\mathcal{G} : [0,1] \longrightarrow \mathscr{C}_{\mathcal{H}}$ be defined by (32). Then*

$$d_{\mathscr{H}}(\mathcal{G}(s), \mathcal{G}(t)) = d_{\mathscr{H}}(\mathcal{A}, \mathcal{B})|t-s| \quad \forall s,t \in [0,1], \tag{33}$$

i.e. $\mathcal{G} \in \operatorname{Lip}([0,1]\,;\mathscr{C}_{\mathcal{H}})$ with $\operatorname{Lip}(\mathcal{G}) = d_{\mathscr{H}}(\mathcal{A}, \mathcal{B})$, and $\mathcal{G}$ is a geodesic connecting $\mathcal{A}$ to $\mathcal{B}$.

Proof. Let us suppose that $s < t$. Since $\rho = d_{\mathscr{H}}(\mathcal{A}, \mathcal{B})$ we have that $\mathcal{A} \subseteq \mathcal{B} + D_\rho$

and $\mathcal{B} \subseteq \mathcal{A} + D_\rho$, hence $\mathcal{G}(0) = \mathcal{A}$ and $\mathcal{G}(1) = \mathcal{B}$. Using Lemma 4.3 we get

$$\begin{aligned}
\mathcal{G}(s) &= (\mathcal{A} + D_{s\rho}) \cap (\mathcal{B} + D_{(1-s)\rho}) \\
&= (\mathcal{A} + D_{s\rho}) \cap (\mathcal{B} + D_{(1-t)\rho} + D_{(t-s)\rho}) \\
&\subseteq \big[(\mathcal{A} + D_{s\rho} + D_{(t-s)\rho}) \cap (\mathcal{B} + D_{(1-t)\rho})\big] + D_{(t-s)\rho} \\
&= \big[(\mathcal{A} + D_{t\rho}) \cap (\mathcal{B} + D_{(1-t)\rho})\big] + D_{(t-s)\rho} \\
&= \mathcal{G}(t) + D_{(t-s)\rho}
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathcal{G}(t) &= (\mathcal{A} + D_{t\rho}) \cap (\mathcal{B} + D_{(1-t)\rho}) \\
&= (\mathcal{A} + D_{s\rho} + D_{(t-s)\rho}) \cap (\mathcal{B} + D_{(1-t)\rho}) \\
&\subseteq \big[(\mathcal{A} + D_{s\rho}) \cap (\mathcal{B} + D_{(1-s)\rho})\big] + D_{(t-s)\rho} \\
&= \mathcal{G}(s) + D_{(t-s)\rho}.
\end{aligned}$$

Therefore

$$d_{\mathscr{H}}(\mathcal{G}(s), \mathcal{G}(t)) \le (t-s)\rho = (t-s) d_{\mathscr{H}}(\mathcal{A}, \mathcal{B}). \tag{34}$$

In order to prove the opposite inequality, let us observe that by the triangle inequality and by (34) we have

$$\begin{aligned}
d_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) &\le d_{\mathscr{H}}(\mathcal{A}, \mathcal{G}(s)) + d_{\mathscr{H}}(\mathcal{G}(s), \mathcal{G}(t)) + d_{\mathscr{H}}(\mathcal{G}(t), \mathcal{B}) \\
&\le s d_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) + d_{\mathscr{H}}(\mathcal{G}(s), \mathcal{G}(t)) + (1-t) d_{\mathscr{H}}(\mathcal{A}, \mathcal{B})
\end{aligned}$$

hence $(t-s) d_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) \le d_{\mathscr{H}}(\mathcal{G}(s), \mathcal{G}(t))$. ☐

Lemma 4.5. *Let $\mathcal{A}, \mathcal{B} \in \mathscr{C}_{\mathcal{H}}$ be such that $\rho := d_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) < \infty$ and let $\mathcal{G}_{(\mathcal{A},\mathcal{B})} = \mathcal{G} : [0,1] \longrightarrow \mathscr{C}_{\mathcal{H}}$ be defined by (32). If $y_0 \in \mathcal{A}$, then let $t_0 \in [0,1]$ be the unique number such that*

$$\|y_0 - \mathrm{Proj}_{\mathcal{B}}(y_0)\| = (1 - t_0)\rho \tag{35}$$

and define $y \in \mathrm{Lip}([0,1]\,;\mathcal{H})$ by

$$y(t) := \begin{cases} y_0 & \text{if } t \in [0, t_0[\\ y_0 + \dfrac{t - t_0}{1 - t_0}(\mathrm{Proj}_{\mathcal{B}}(y_0) - y_0) & \text{if } t \in [t_0, 1[\\ \mathrm{Proj}_{\mathcal{B}}(y_0) & \text{if } t = 1 \end{cases}. \tag{36}$$

Then

$$y(t) \in \mathcal{G}_{(\mathcal{A},\mathcal{B})}(t) \quad \forall t \in [0,1], \tag{37}$$

$$y'(t) + N_{\mathcal{G}_{(\mathcal{A},\mathcal{B})}(t)}(y(t)) \ni 0 \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in [0,1], \tag{38}$$

$$y(0) = \mathrm{Proj}_{\mathcal{G}_{(\mathcal{A},\mathcal{B})}(0)}(y_0) = y_0, \tag{39}$$

i.e. y is the unique solution of the sweeping process driven by $\mathcal{G}_{(\mathcal{A},\mathcal{B})}$ with initial condition $y_0 \in \mathcal{A}$.

Proof. We use the notation $\mathcal{K}_\rho := \mathcal{K} + D_\rho$ for $\mathcal{K} \in \mathscr{C}_\mathcal{H}$ and $\rho \geq 0$. Observe that $t_0 \in [0,1]$ is the unique number satisfying (35) because $0 \leq d(y_0, \mathcal{B}) \leq \rho$. If $y_0 \in \mathcal{B}$ we have that $t_0 = 1$, $y(t) = y_0 = \mathrm{Proj}_\mathcal{B}(y_0) \in \mathcal{G}(t)$ and $y'(t) = 0$ for every $t \in [0,1]$, and there is nothing to prove. Therefore we assume that $y_0 \notin \mathcal{B}$, i.e. $t_0 < 1$, and from Lemma 4.1-(*ii*) and formula (35) we infer that

$$y_0 \in \partial \mathcal{B}_{(1-t)\rho} \iff t = t_0, \tag{40}$$

that is t_0 is the unique time when $\partial\mathcal{B}_{(1-t)\rho}$ meets y_0. If $t \in \,]t_0, 1]$ we have that

$$\|y(t) - \mathrm{Proj}_\mathcal{B}(y_0)\| = \frac{1-t}{1-t_0}\|y_0 - \mathrm{Proj}_\mathcal{B}(y_0)\| = \frac{1-t}{1-t_0}(1-t_0)\rho = (1-t)\rho,$$

therefore

$$y(t) \in \partial\mathcal{B}_{(1-t)\rho} \quad \forall t \in [t_0, 1]\,. \tag{41}$$

On the other hand thanks to Lemma 4.1-(*i*) we have

$$d(y_0, \partial\mathcal{A}_{t\rho}) \geq t\rho \quad \forall t \in [0,1]\,, \tag{42}$$

hence

$$y(t) = y_0 + \frac{t-t_0}{1-t_0}(\mathrm{Proj}_\mathcal{B}(y_0) - y_0) \in \mathrm{int}(\mathcal{A}_{t\rho}) \quad \forall t \in \,]t_0, 1]\,. \tag{43}$$

Now observe that, as

$$\partial\mathcal{G}(t) \subseteq \partial\mathcal{A}_{t\rho} \cup \partial\mathcal{B}_{(1-t)\rho} \quad \forall t \in [0,1]\,, \tag{44}$$

we have that $y_0 \in \mathrm{int}(\mathcal{G}(t))$ for every $t \in \,]0, t_0[$, and from (41), (43), and Lemma 4.2 we infer that

$$y(t) \in \partial\mathcal{G}(t) \quad \forall t \in [t_0, 1[$$

and

$$N_{\mathcal{G}(t)}(y(t)) = N_{\mathcal{B}_{(1-t)\rho}}(y(t)) \quad \forall t \in [t_0, 1[\,. \tag{45}$$

Therefore, since

$$y'(t) = \begin{cases} 0 & \text{if } t \in \,]0, t_0[\\ \dfrac{1}{1-t_0}(\mathrm{Proj}_\mathcal{B}(y_0) - y_0) & \text{if } t \in \,]t_0, 1[, \end{cases}$$

we infer from (45) and from Lemma 4.1-(*iii*) that $y'(t) \in N_{\mathcal{G}(t)}(y(t))$ for every $t \in \,]0,1[\, \smallsetminus \{t_0\}$. □

We conclude this section with a uniform bound for the distance between two geodesics of form $\mathcal{G}_{(\mathcal{A},\mathcal{B})}$. Before we need the following

Lemma 4.6. *If $\mathcal{A}, \mathcal{B} \in \mathscr{C}_\mathcal{H}$ and $\rho := d_\mathscr{H}(\mathcal{A}, \mathcal{B}) < \infty$, then $d_\mathscr{H}(\mathcal{A} + D_{t\rho}, \mathcal{B} + D_{(1-t)\rho}) = \max\{2t\rho, 2(1-t)\rho\}$.*

Proof. The conclusion follows from the fact that $\rho := d_{\mathscr{H}}(\mathcal{A},\mathcal{B})$ is the minimal number such that the following inclusions hold:

$$\mathcal{A} + D_{t\rho} \subseteq \mathcal{B} + D_\rho + D_{t\rho} = \mathcal{B} + D_{(1-t)\rho} + D_{2t\rho},$$
$$\mathcal{B} + D_{(1-t)\rho} \subseteq \mathcal{A} + D_\rho + D_{(1-t)\rho} = \mathcal{A} + D_{t\rho} + D_{2(1-t)\rho}. \qquad \square$$

Proposition 4.7. *Assume that $\mathcal{A},\mathcal{B} \in \mathscr{C}_{\mathcal{H}}$, $0 < d_{\mathscr{H}}(\mathcal{A},\mathcal{B}) < \infty$, and $\varepsilon \in]0, d_{\mathscr{H}}(\mathcal{A},\mathcal{B})/8[$. If $\mathcal{A}^\varepsilon,\mathcal{B}^\varepsilon \in \mathscr{C}_{\mathcal{H}}$, $d_{\mathscr{H}}(\mathcal{A}^\varepsilon,\mathcal{A}) < \varepsilon$, and $d_{\mathscr{H}}(\mathcal{B}^\varepsilon,\mathcal{B}) < \varepsilon$, then*

$$d_{\mathscr{H}}\left(\mathcal{G}_{(\mathcal{A}^\varepsilon,\mathcal{B}^\varepsilon)}(t), \mathcal{G}_{(\mathcal{A},\mathcal{B})}(t)\right) < 3\varepsilon \quad \forall t \in [0,1].$$

Proof. For simplicity we set $\mathcal{G} := \mathcal{G}_{(\mathcal{A},\mathcal{B})}$ and $\mathcal{G}_\varepsilon := \mathcal{G}_{(\mathcal{A}^\varepsilon,\mathcal{B}^\varepsilon)}$. We also set

$$\rho := d_{\mathscr{H}}(\mathcal{A},\mathcal{B}), \qquad \rho_\varepsilon := d_{\mathscr{H}}(\mathcal{A}^\varepsilon,\mathcal{B}^\varepsilon),$$

and

$$\rho_t := d_{\mathscr{H}}(\mathcal{A} + D_{t\rho}, \mathcal{B} + D_{(1-t)\rho}), \qquad \rho_{\varepsilon,t} := d_{\mathscr{H}}(\mathcal{A}^\varepsilon + D_{t\rho_\varepsilon}, \mathcal{B}^\varepsilon + D_{(1-t)\rho_\varepsilon})$$

for $t \in [0,1]$. From Lemma 4.6 we infer that

$$\rho \le \rho_t \le 2\rho, \qquad \rho_\varepsilon \le \rho_{\varepsilon,t} \le 2\rho_\varepsilon \tag{46}$$

for every $t \in [0,1]$, and from the triangle inequality for $d_{\mathscr{H}}$ we get that $|\rho_\varepsilon - \rho| \le 2\varepsilon$, which easily yields

$$d_{\mathscr{H}}(D_{t\rho_\varepsilon}, D_{t\rho}) \le t|\rho_\varepsilon - \rho| \le 2t\varepsilon \le 2\varepsilon \quad \forall t \in [0,1] \tag{47}$$

so that $3\varepsilon < \rho - 5\varepsilon$. Then from (47), (46), and (33) (with $\mathcal{A}$, $\mathcal{B}$, ρ, s, t replaced by $\mathcal{A} + D_\varepsilon$, $\mathcal{B} + D_{(1-t)\rho}$, ρ_t, 0 and $3\varepsilon/\rho_t$), we have, since $3\varepsilon < \rho - 5\varepsilon < \rho - 3\varepsilon$,

$$\begin{aligned}
\mathcal{G}_\varepsilon(t) &= (\mathcal{A}^\varepsilon + D_{t\rho_\varepsilon}) \cap (\mathcal{B}^\varepsilon + D_{(1-t)\rho_\varepsilon}) \\
&\subseteq (\mathcal{A} + D_{t\rho} + D_{3\varepsilon}) \cap (\mathcal{B} + D_{(1-t)\rho} + D_{3\varepsilon}) \\
&\subseteq (\mathcal{A} + D_{t\rho} + D_{3\varepsilon}) \cap (\mathcal{B} + D_{(1-t)\rho} + D_{\rho-3\varepsilon}) \\
&\subseteq (\mathcal{A} + D_{t\rho} + D_{3\varepsilon}) \cap (\mathcal{B} + D_{(1-t)\rho} + D_{\rho_t-3\varepsilon}) \\
&= (\mathcal{A} + D_{t\rho} + D_{(3\varepsilon/\rho_t)\rho_t}) \cap (\mathcal{B} + D_{(1-t)\rho} + D_{(1-3\varepsilon/\rho_t)\rho_t}) \\
&\subseteq \left[(\mathcal{A} + D_{t\rho}) \cap (\mathcal{B} + D_{(1-t)\rho})\right] + \rho_t D_{3\varepsilon/\rho_t} = \mathcal{G}(t) + D_{3\varepsilon}.
\end{aligned} \tag{48}$$

On the other hand, using the same arguments, since $3\varepsilon < \rho - 5\varepsilon < \rho_\varepsilon - 3\varepsilon$ by (47), we find

$$\begin{aligned}
\mathcal{G}(t) &= (\mathcal{A} + D_{t\rho}) \cap (\mathcal{B} + D_{(1-t)\rho}) \\
&\subseteq (\mathcal{A}^\varepsilon + D_{t\rho_\varepsilon} + D_{3\varepsilon}) \cap (\mathcal{B}^\varepsilon + D_{(1-t)\rho_\varepsilon} + D_{3\varepsilon}) \\
&\subseteq (\mathcal{A}^\varepsilon + D_{t\rho_\varepsilon} + D_{3\varepsilon}) \cap (\mathcal{B}^\varepsilon + D_{(1-t)\rho_\varepsilon} + D_{\rho-5\varepsilon}) \\
&\subseteq (\mathcal{A}^\varepsilon + D_{t\rho_\varepsilon} + D_{3\varepsilon}) \cap (\mathcal{B}^\varepsilon + D_{(1-t)\rho_\varepsilon} + D_{\rho_\varepsilon-3\varepsilon}) \\
&\subseteq (\mathcal{A}^\varepsilon + D_{t\rho_\varepsilon} + D_{3\varepsilon}) \cap (\mathcal{B}^\varepsilon + D_{(1-t)\rho_\varepsilon} + D_{\rho_{\varepsilon,t}-3\varepsilon}) \\
&= (\mathcal{A}^\varepsilon + D_{t\rho_\varepsilon} + D_{(3\varepsilon/\rho_{\varepsilon,t})\rho_{\varepsilon,t}}) \cap (\mathcal{B}^\varepsilon + D_{(1-t)\rho_\varepsilon} + D_{(1-3\varepsilon/\rho_{\varepsilon,t})\rho_{\varepsilon,t}}) \\
&\subseteq \left[(\mathcal{A}^\varepsilon + D_{t\rho_\varepsilon}) \cap (\mathcal{B}^\varepsilon + D_{(1-t)\rho_\varepsilon})\right] + \rho_{\varepsilon,t} D_{3\varepsilon/\rho_{\varepsilon,t}} = \mathcal{G}_\varepsilon(t) + D_{3\varepsilon}.
\end{aligned} \tag{49}$$

We conclude by collecting (48) and (49). $\square$

Corollary 4.8. *Assume that $\mathcal{A}, \mathcal{B}, \mathcal{A}_n, \mathcal{B}_n \in \mathscr{C}_{\mathcal{H}}$ for every $n \in \mathbb{N}$, and $\mathcal{A}_n \to \mathcal{A}$, $\mathcal{B}_n \to \mathcal{B}$ as $n \to \infty$. Then $\mathcal{G}_{(\mathcal{A}_n,\mathcal{B}_n)}$ is uniformly convergent to $\mathcal{G}_{(\mathcal{A},\mathcal{B})}$ as $n \to \infty$.*

Remark 4.9. The results on the intersection of moving convex sets contained in [31] do not apply to the previous situation, indeed in general $\mathcal{G}(t)$ is unbounded and/or with empty interior.

Remark 4.10. The curve $\mathcal{G}$ is not the only curve with the property that $\mathsf{S}(\mathcal{G}, y_0)(1) = \mathrm{Proj}_{\mathcal{B}}(y_0)$ for every $y_0 \in \mathcal{A}$. It is not hard to prove that another example is provided by

$$\mathcal{F}(t) := \begin{cases} (1-2t)\mathcal{A} + 2t\mathcal{B}_\rho & \text{if } t \in [0, 1/2[\\ 2(1-t)\mathcal{B}_\rho + (2t-1)\mathcal{B} & \text{if } t \in [1/2, 1]. \end{cases} \tag{50}$$

This is not a geodesic connecting $\mathcal{A}$ and $\mathcal{B}$, and is not useful in order to prove next Proposition 6.3. However it is a geodesic for the asymmetric distance $\rho = e(\mathcal{A}, \mathcal{B}) := \sup_{a \in \mathcal{A}} d(a, \mathcal{B})$, therefore it suits well for the analysis of the sweeping process driven by a moving convex set with local bounded retraction (cf. [30, 33]).

5. "Catching-up" reparametrization

Now we introduce the reparametrization by the arc length of a BV function with values in a metric space X. The novelty with respect to other approaches is that we fill in the jumps of the function with curves chosen in a prescribed family of geodesics $\mathscr{G}$. We also prove some convergence results on sequences of reparametrizations. Assume that (10) holds and set

$$F := \{(x, y) \in X \times X \;:\; 0 < d(x, y) < \infty\}. \tag{51}$$

Proposition 5.1. *If $u \in BV^{\mathrm{r}}_{\mathrm{loc}}([0, \infty[\,; X)$, let $\ell_u : [0, \infty[\longrightarrow [0, \infty[$ be defined by*

$$\ell_u(t) := \mathrm{V}(u, [0, t]), \quad t \in [0, \infty[\,.$$

(*i*) *We have $\mathrm{Discont}(u) = \mathrm{Discont}(\ell_u)$ and there is a unique $U : \ell_u([0, \infty[) \longrightarrow X$ such that*

$$u = U \circ \ell_u. \tag{52}$$

Moreover $\mathrm{Lip}(U) \leq 1$.

(*ii*) *Let F be given by (51), and let $\mathscr{G} = (g_{(x,y)})_{(x,y) \in F}$ be a family in $\mathrm{Lip}([0, 1]; X)$ such that $g_{(x,y)}(0) = x$ and $g_{(x,y)}(1) = y$. If $\widetilde{u} : [0, \infty[\longrightarrow X$ is defined by*

$$\widetilde{u}(\sigma) := \begin{cases} U(\sigma) & \text{if } \sigma \in \ell_u(\mathrm{Cont}(u)) \\ g_{(u(t-),u(t))}\left(\dfrac{\sigma - \ell_u(t-)}{\ell_u(t) - \ell_u(t-)}\right) & \text{if } \sigma \in [\ell_u(t-), \ell_u(t)], \\ & \quad t \in \mathrm{Discont}(u), \end{cases} \tag{53}$$

and by $\widetilde{u}(\sigma) := u(\infty)$ if $\sup \ell_u([0, \infty[) \leq \sigma < \infty$, then $\widetilde{u} \in \mathrm{Lip}([0, \infty[\,; X)$,

$$u = \widetilde{u} \circ \ell_u, \tag{54}$$

and

$$\begin{aligned}\widetilde{u}([0,\infty[) &= u([a,\infty[) \bigcup \left(\bigcup_{t \in \mathrm{Discont}(u)} g_{(u(t-),u(t))}([0,1]) \right) \\ &= \widetilde{u}(\ell_u([0,\infty[)) \bigcup \left(\bigcup_{t \in \mathrm{Discont}(u)} g_{(\widetilde{u}(\ell_u(t-)),\widetilde{u}(\ell_u(t)))}([0,1]) \right). \end{aligned} \tag{55}$$

Moreover, if $g_{(x,y)}$ is a geodesic connecting x to y for every $(x,y) \in F$, then $\mathrm{Lip}(\widetilde{u}) = \mathrm{Lip}(U) \le 1$ *and* $\mathrm{V}(\widetilde{u}, [0,\infty[) = \mathrm{V}(u, [0,\infty[)$.

Proof. The proof of (i) follows the classical lines as in [13, Section 2.5.16, p. 109]. The statements in (ii) are a straighforward consequence of the fact that $g_{(u(t-),u(t))}$ is a geodesic connecting $u(t-) = U(\ell_u(t-))$ to $u(t) = U(\ell_u(t))$ for every $t \in \mathrm{Discont}(u) = \mathrm{Discont}(\ell_u)$. □

The arc length reparametrization of a continuous function of bounded variation u is a classical argument. If u is not continuous, in [13, Section 2.5.16, p. 109] the jumps are "filled in" with segments belonging to a Banach space where X is embedded. However it seems that such procedure is not useful to our purposes.

Definition 5.2. Under the assumptions of Proposition 5.1, the function $\widetilde{u}$ defined in (53) will be called *arc length reparametrization of u associated with the family $\mathscr{G}$.*

When $X = \mathscr{C}_{\mathcal{H}}$ and the family $\mathscr{G}$ is given by the geodesics (32), then the arc length reparamerization of $\mathcal{C} \in BV^r_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ associated to $(\mathcal{G}_{(\mathcal{A},\mathcal{B})})$ will be called "catching up" reparametrization, since it is essentially a continuous version of the catching up algorithm (30)–(31).

Proposition 5.3. *Let $J \subseteq [0,\infty[$ be an interval. Assume that $u, u_n \in BV^r_{\mathrm{loc}}([0,\infty[\,;X)$, $\mathrm{V}(u,J) < \infty$, $\mathrm{V}(u_n,J) < \infty$ for every $n \in \mathbb{N}$, and $d_{us,J}(u_n,u) \to 0$ as $n \to \infty$. Then $\ell_{u_n} \to \ell_u$ uniformly on J.*

Proof. For simplicity we assume that $0 = \inf J \in J$ (which is what we need), but the proof clearly holds in the general case. Let $\varepsilon > 0$ and let $K \subseteq J$ an interval arbitrarily fixed. Then there are points $\inf K \le t_0 < \cdots < t_m \le \sup K$ such that

$$\mathrm{V}(u,K) < \varepsilon + \sum_{j=1}^{m} \mathrm{d}(u_n(t_{j-1}), u_n(t_j)). \tag{56}$$

Moreover there exists $n_\varepsilon \in \mathbb{N}$ (which does not depend on K) such that

$$\sup_{s\in J} d(u(s), u_n(s)) < \varepsilon/m, \quad \mathrm{V}(u_n,J) < \mathrm{V}(u,J) + \varepsilon \ \forall n \ge n_\varepsilon. \tag{57}$$

It follows that

$$\begin{aligned}
\mathrm{V}(u,K) &< \varepsilon + \sum_{j=1}^{m} d(u(t_{j-1}), u(t_j)) \\
&\le \varepsilon + \sum_{j=1}^{m} \Big(d(u(t_{j-1}), u_n(t_{j-1})) + d(u_n(t_{j-1}), u_n(t_j)) + d(u_n(t_j), u(t_j)) \Big) \\
&\le \varepsilon + \sum_{j=1}^{m} \Big(\varepsilon/m + d(u_n(t_{j-1}), u_n(t_j)) + \varepsilon/m \Big) \le \mathrm{V}(u_n, K) + 3\varepsilon.
\end{aligned}$$

Now fix an arbitrary $t \in J$ and apply the previous inequality with $K = J \setminus [0,t]$ and $K = [0,t]$. We get

$$\mathrm{V}(u, J \setminus [0,t]) < \mathrm{V}(u_n, J \setminus [0,t]) + 3\varepsilon, \tag{58}$$

$$\mathrm{V}(u, [0,t]) < \mathrm{V}(u_n, [0,t]) + 3\varepsilon. \tag{59}$$

Therefore we also have

$$\begin{aligned}
\mathrm{V}(u_n, [0,t]) &= \mathrm{V}(u_n, J) - \mathrm{V}(u_n, J \setminus [0,t]) \\
&< \mathrm{V}(u, J) + \varepsilon + 3\varepsilon - \mathrm{V}(u, J \setminus [0,t]) \\
&= \mathrm{V}(u, [0,t]) + 4\varepsilon,
\end{aligned}$$

i.e.

$$\mathrm{V}(u_n, [0,t]) < \mathrm{V}(u, [0,t]) + 4\varepsilon. \tag{60}$$

Thus from (59)–(60) we get $|\ell_{u_n}(t) - \ell_u(t)| < 4\varepsilon$ whenever $n \ge n_\varepsilon$, and we are done. □

The assumption on $\mathscr{G}$ in the next proposition is an abstract version of the thesis of Proposition 4.7.

Proposition 5.4. *Assume that $\mathscr{G} = (g_{(x,y)})_{(x,y)\in F}$ is a family of geodesics in* $\mathrm{Lip}([0,1]\,;X)$ *connecting x to y for every $(x,y) \in F$, and it satisfies the following property:*

$$\forall \varepsilon > 0\ \exists \delta > 0\ :$$

$$\left.\begin{array}{c} 0 < d(x,y) < \varepsilon, \\ x^\varepsilon, y^\varepsilon \in F, \\ \max\{d(x,x^\varepsilon), d(y,y^\varepsilon)\} < \delta \end{array}\right\} \implies \sup_{\sigma\in[0,1]} d(g_{(x^\varepsilon,y^\varepsilon)}(\sigma), g_{(x,y)}(\sigma)) < \varepsilon.$$

If $u, u_n \in BV^r_{\mathrm{loc}}([0,\infty[\,;X)$ for every $n \in \mathbb{N}$ and $u_n \to u$ locally uniformly strictly on $[0,\infty[$, then $\widetilde{u}_n \to \widetilde{u}$ locally uniformly strictly on $[0,\infty[$.

Proof. First of all let us observe $\mathrm{Lip}(\widetilde{u}_n) \le 1$, thus $(\widetilde{u}_n)$ is equi-Lipschitz continuous on $[0,\infty[$. Now let $c, d \in \mathbb{R}$ be such that $0 \le c < d < \infty$ and let

$a, b \in [0, \infty[$ be such that $c = \ell_u(a)$, $d = \ell_u(b)$. Fix $\varepsilon > 0$ and let $n_\varepsilon \in \mathbb{N}$ be such that $\sup_{t \in [a,b]} d(u_n(t), u(t)) < \min\{\varepsilon, \delta\}$ for all $n \geq n_\varepsilon$. Therefore

$$u_n([a,b]) \subseteq \{x \in X \ : \ d(x, u([a,b])) < \varepsilon\} \quad \forall n \geq n_\varepsilon.$$

If $t \in [a,b]$, the following three cases (a), (b), (c) can occur.

(a) Assume that $t \in \mathrm{Discont}(u_n)$ definitely, and $t \in \mathrm{Cont}(u)$.
Then $\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} u_n(t-) = u(t)$ and we have

$$\begin{aligned} d(g_{(u_n(t-),u_n(t))}(\sigma), u(t)) &\leq d(g_{(u_n(t-),u_n(t))}(\sigma), u_n(t)) + d(u_n(t), u(t)) \\ &= (1-\sigma) d(u_n(t-), u_n(t)) + \varepsilon \\ &\leq d(u_n(t-), u(t)) + d(u(t), u_n(t)) + \varepsilon \\ &= d(u_n(t-), u(t-)) + d(u(t), u_n(t)) + \varepsilon \leq 3\varepsilon. \end{aligned} \tag{61}$$

(b) Assume that $t \in \mathrm{Cont}(u_n)$ frequently.
Then by standard uniform convergence arguments we have that $t \in \mathrm{Cont}(u)$, therefore $d(u_n(t), u(t)) < \varepsilon$ for every $n \in \mathbb{N}$ such that $t \in \mathrm{Cont}(u_n)$. If n_k is a subsequence of n such that $t \in \mathrm{Discont}(u_{n_k})$ for every k, then from (a) we infer that

$$d(g_{(u_{n_k}(t-),u_{n_k}(t))}(\sigma), u(t)) < 3\varepsilon. \tag{62}$$

(c) Assume that $t \in \mathrm{Discont}(u_n)$ definitely, and $t \in \mathrm{Discont}(u)$.
Then $u_n(t-) \to u(t-)$ and $u_n(t) \to u(t)$ as $n \to \infty$, and $d(u_n(t-), u(t-)) < \varepsilon$, $d(u_n(t), u(t)) < \varepsilon$ for every $n \geq n_\varepsilon$. We consider two subcases:
(c.1) If $\varepsilon < d(u(t-), u(t))$ then from the assumption on $\mathscr{G}$ we know that

$$\sup_{\sigma \in [0,1]} d(g_{(u_n(t-),u_n(t))}(\sigma), g_{(u(t-),u(t))}(\sigma)) < \varepsilon. \tag{63}$$

(c.2) If $\varepsilon \geq d(u(t-), u(t))$ then for every $\sigma \in [0,1]$ we have that

$$\begin{aligned} d(g_{(u_n(t-),u_n(t))}(\sigma), u(t)) &\leq d(g_{(u_n(t-),u_n(t))}(\sigma), u_n(t)) + d(u_n(t), u(t)) \\ &= d(u_n(t-), u_n(t))(1-\sigma) + \varepsilon \qquad (64) \\ &\leq (d(u(t-), u(t)) + 2\varepsilon)(1-\sigma) + \varepsilon \leq 4\varepsilon. \end{aligned}$$

Taking into account (55), from (61)–(64) we infer that

$$\widetilde{u}_n([c,d]) \subseteq \{x \in X \ : \ d(x, \widetilde{u}([c,d])) < \varepsilon\} \quad \forall n \geq n_\varepsilon.$$

Thus, as $\widetilde{u}([c,d])$ is compact, we can apply the generalized Arzelá-Ascoli theorem and infer that, up to subsequences, there exists $\widehat{u} \in \mathrm{Lip}([c,d]\,; X)$ such that $\widetilde{u}_n \to \widehat{u}$ uniformly on $[c,d]$. Since $\ell_{u_n} \to \ell_u$ uniformly on $[a,b]$ by Proposition 5.3, taking the limit in the equality $u_n = \widetilde{u}_n \circ \ell_{u_n}$ we get $u = \widehat{u} \circ \ell_u$, thus from (54) we get $\widetilde{u} = \widehat{u}$ on $\ell_u([a,b])$. If $\sigma \notin \ell_u([a,b])$, then $\sigma \in [\ell_u(t-), \ell_u(t)]$ for some $t \in \mathrm{Discont}(u)$. Then, arguing as in (a), (b), (c), we get that $t \in \mathrm{Discont}(u_n)$

definitely, therefore $u_n(t-) \to u(t-)$ and $u_n(t) \to u(t)$ as $n \to \infty$, and the assumption on $\mathscr{G}$ implies that $g_{(u_n(t-),u_n(t))} \to g_{(u(t-),u(t))}$ uniformly on $[0,1]$. Hence, thanks to Lemma 5.3, it follows that

$$\begin{aligned}\widetilde{u}_n(\sigma) &= g_{(u_n(t-),u_n(t))}\left(\frac{\sigma - \ell_{u_n}(t-)}{\ell_{u_n}(t) - \ell_{u_n}(t-)}\right)\\ &\to g_{(u(t-),u(t))}\left(\frac{\sigma - \ell_u(t-)}{\ell_u(t) - \ell_u(t-)}\right) = \widetilde{u}(\sigma)\end{aligned}$$

as $n \to \infty$ and we are done. $\square$

Now from Lemma 5.4 and Proposition 4.7 we infer the following

Corollary 5.5. *Assume that $\mathcal{C}, \mathcal{C}_n \in BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ and let $\widetilde{\mathcal{C}}, \widetilde{\mathcal{C}}_n$ be their reparametrizations by the arc length associated with the family $(\mathcal{G}_{(\mathcal{A},\mathcal{B})})$. If $\mathcal{C}_n \to \mathcal{C}$ locally uniformly strictly on $[0,\infty[$, then $\widetilde{\mathcal{C}}_n \to \widetilde{\mathcal{C}}$ locally uniformly strictly on $[0,\infty[$.*

6. *BV* sweeping processes: reduction to the Lipschitz case

In this section we provide a new proof of the existence and uniqueness of solutions of BV sweeping processes. It consists in reducing to the Lipschitz continuous case by means of the "catching-up" reparametrization technique shown in Section 5. This reduction makes use of measure theory tools only, and shows that BV sweeping processes are somehow a direct consequence of the regular case. Let us set

$$\mathscr{F} := \{(\mathcal{A},\mathcal{B}) \in \mathscr{C}_{\mathcal{H}} \ : \ 0 < \mathrm{d}_{\mathscr{H}}(\mathcal{A},\mathcal{B}) < \infty\}.$$

Theorem 6.1. *Assume that $\mathcal{C} \in BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$, $y_0 \in \mathcal{H}$, and let $(\mathcal{G}_{(\mathcal{A},\mathcal{B})})_{(\mathcal{A},\mathcal{B})\in\mathscr{F}}$ be the family in $\mathrm{Lip}([0,1]\,;\mathscr{C}_{\mathcal{H}})$ defined by (32). Let $\ell_{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ be respectively the arc length and the arc length reparametrization of $\mathcal{C}$ associated with $(\mathcal{G}_{(\mathcal{A},\mathcal{B})})_{\mathscr{F}}$. Then $y := \mathsf{S}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}} \in BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ is the unique function such that there exists a measure $\mu : \mathscr{B}([0,\infty[) \longrightarrow [0,\infty]$ and a function $w \in L^1_{\mathrm{loc}}(\mu;\mathcal{H})$ such that*

$$\mathrm{D}y = w\mu, \tag{65}$$

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [0,\infty[\,, \tag{66}$$

$$w(t) + N_{\mathcal{C}(t)}(y(t)) \ni 0 \quad \textit{for } \mu\textit{-a.e. } t \in [0,\infty[, \tag{67}$$

$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0). \tag{68}$$

We can take $\mu = \mathrm{D}\ell_{\mathcal{C}}$, and w given by (69).

Proof. Let us begin with existence. Since $\widetilde{\mathcal{C}} \in \mathrm{Lip}([0,\infty[\,;\mathcal{H})$, by Theorem 3.2 we have that $\mathsf{S}(\widetilde{\mathcal{C}}) \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$. Let us set

$$\hat{y} := \mathsf{S}(\widetilde{\mathcal{C}}), \qquad y := \mathsf{S}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}},$$

thus conditions (66) and (68) follow immediately from (17) and (19). If $w : [0,\infty[\longrightarrow \mathcal{H}$ is defined by

$$w(t) := \begin{cases} \mathsf{S}(\widetilde{\mathcal{C}})'(\ell_{\mathcal{C}}(t)) & \text{if } t \in \operatorname{Cont}(\ell_{\mathcal{C}}) \\ \dfrac{\mathsf{S}(\widetilde{\mathcal{C}})(\ell_{\mathcal{C}}(t)) - \mathsf{S}(\widetilde{\mathcal{C}})(\ell_{\mathcal{C}}(t-))}{\ell_{\mathcal{C}}(t) - \ell_{\mathcal{C}}(t-)} & \text{if } t \in \operatorname{Discont}(\ell_{\mathcal{C}}), \end{cases} \tag{69}$$

then, since $\ell_{\mathcal{C}}$ is right continuous and increasing, from Proposition 2.8-(*ii*) we infer that $y \in BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ and $\mathrm{D}y = w\,\mathrm{D}\ell_{\mathcal{C}}$, i.e. (65) holds with $\mu = \mathrm{D}\ell_{\mathcal{C}}$. Let us set

$$Z := \{t \in [0,\infty[\ : \ -\hat{y}'(t) \notin N_{\widetilde{\mathcal{C}}(t)}(\hat{y}(t))\}. \tag{70}$$

From formula (18) we deduce that

$$\mathcal{L}^1(Z) = 0,$$

therefore, thanks to Proposition 2.8-(*i*), we have that

$$\begin{aligned} & \mathrm{D}\ell_{\mathcal{C}}(\{t \in \operatorname{Cont}(\ell_{\mathcal{C}}) \ : \ -w(t) \notin N_{\mathcal{C}(t)}(y(t))\}) \\ = \ & \mathrm{D}\ell_{\mathcal{C}}(\{t \in \operatorname{Cont}(\ell_{\mathcal{C}}) \ : \ -\hat{y}'(\ell_{\mathcal{C}}(t)) \notin N_{\widetilde{\mathcal{C}}(\ell_{\mathcal{C}}(t))}(\hat{y}(\ell_{\mathcal{C}}(t))\}) \\ = \ & \mathrm{D}\ell_{\mathcal{C}}(\{t \in \operatorname{Cont}(\ell_{\mathcal{C}}) \ : \ \ell_{\mathcal{C}}(t) \in Z\}) = \mathcal{L}^1(Z) = 0. \end{aligned} \tag{71}$$

Now let us fix $t \in \operatorname{Discont}(\ell_{\mathcal{C}})$ and observe that if $s \in [\ell_{\mathcal{C}}(t-), \ell_{\mathcal{C}}(t)]$ then $\widetilde{\mathcal{C}}(s) = \mathcal{G}_{(\mathcal{C}(t-),\mathcal{C}(t))}(s - \ell_{\mathcal{C}}(t-)/(\ell_{\mathcal{C}}(t) - \ell_{\mathcal{C}}(t-)))$. Hence the semigroup property (22) and Lemma 4.5 imply that $\mathsf{S}(\widetilde{\mathcal{C}})(\ell_{\mathcal{C}}(t)) = \operatorname{Proj}_{\mathcal{C}(t)}(\hat{y}(\ell_{\mathcal{C}}(t-)))$, therefore

$$-w(t) = \frac{\mathsf{S}(\widetilde{\mathcal{C}})(\ell_{\mathcal{C}}(t-)) - \mathsf{S}(\widetilde{\mathcal{C}})(\ell_{\mathcal{C}}(t))}{\ell_{\mathcal{C}}(t) - \ell_{\mathcal{C}}(t-)} = \frac{\hat{y}(\ell_{\mathcal{C}}(t-)) - \operatorname{Proj}_{\mathcal{C}(t)}(\hat{y}(\ell_{\mathcal{C}}(t-)))}{\ell_{\mathcal{C}}(t) - \ell_{\mathcal{C}}(t-)}.$$

It follows that $-w(t) \in N_{\mathcal{C}(t)}(\operatorname{Proj}_{\mathcal{C}(t)}(\hat{y}(\ell_{\mathcal{C}}(t-)))) = N_{\mathcal{C}(t)}(\hat{y}(\ell_{\mathcal{C}}(t)))) = N_{\mathcal{C}(t)}(y(t))$), hence from (70) and (71) we obtain

$$\mathrm{D}\ell_{\mathcal{C}}(\{t \in [0,\infty[\ : \ -w(t) \notin N_{\mathcal{C}(t)}(y(t))\}) = 0, \tag{72}$$

and (67) is proved with $\mu = \mathrm{D}\ell_{\mathcal{C}}$. Concerning uniqueness, assume that y_1 and y_2 are two solutions, and μ_j, w_j are such that $\mathrm{D}y_j = w_j\mu_j$, $j = 1, 2$. Then $\langle w_j(t), v - y_j(t)\rangle \geq 0$ for every $v \in \mathcal{C}(t)$ and for μ_j-a.e. $t \in [0,\infty[$. We can take $v = y_i$, $i \neq j$ and infer that

$$\begin{aligned} \int_{[0,t]} \langle w_j(s), y_i(s) - y_j(s)\rangle \,\mathrm{d}\mu_j(s) &= \int_{[0,t]} \langle y_i(s) - y_j(s), \mathrm{d}(w_j\mu_j)(s)\rangle \\ &= \int_{[0,t]} \langle y_i(s) - y_j(s), \mathrm{d}\mathrm{D}y_j(s)\rangle \geq 0 \end{aligned}$$

hence, subtracting these inequalities for $j = 1, 2$ and using Proposition 2.7 we get

$$0 \geq \int_{[0,t]} \langle y_1 - y_2, \mathrm{d}\mathrm{D}(y_1 - y_2)\rangle \geq \|y_1(t) - y_2(t)\|^2_{\mathcal{H}}/2$$

for every $t \geq 0$ and we are done. $\square$

The previous theorem defines the solution operator $\overline{\mathsf{S}}$ of the BV sweeping process defined in Problem 3.5

$$\overline{\mathsf{S}} = \overline{\mathsf{S}}(\cdot, y_0) : BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}}) \longrightarrow BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H}) \tag{73}$$

and we have the "representation formula"

$$\overline{\mathsf{S}}(\mathcal{C}) = \mathsf{S}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}}.$$

It is possible to check directly that the operator $\overline{\mathsf{S}}$ actually extends the operator S, indeed we have the following

Proposition 6.2. *If $\mathcal{C} \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ then $y := \overline{\mathsf{S}}(\mathcal{C}) = \mathsf{S}(\widetilde{\mathcal{C}}) \circ \ell_{\mathcal{C}} \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty[\,;\mathcal{H})$ is the unique solution of Problem* 3.1, *i.e. is the unique function satisfying* (1)–(3).

The proof can be found in [41, Proposition 3.2] where we deal with the continuous case. From the Lipschitz continuous case we can also deduce the following continuous dependence on the data.

Proposition 6.3. *Assume that $y_0, y_{0,n} \in \mathcal{H}$ and $\mathcal{C}, \mathcal{C}_n \in BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ for every $n \in \mathbb{N}$. If $\|y_{0,n} - y_0\| \to 0$ and $d_{us,[0,t]}(\mathcal{C}_n, \mathcal{C}) \to 0$ for every $t > 0$, then $d_{\infty,[0,t]}(\mathsf{S}(\mathcal{C}_n, y_{0,n}), \mathsf{S}(\mathcal{C}, y_0)) \to 0$ for every $t > 0$.*

Proof. From Theorem 3.4, estimate (20), and Proposition 5.1-(ii), we infer that for every $t \geq 0$

$$\begin{aligned}
&\|\mathsf{S}(\mathcal{C}_n, y_{0,n})(t) - \mathsf{S}(\mathcal{C}, y_0)(t)\| \\
&= \|\mathsf{S}(\widetilde{\mathcal{C}}_n, y_{0,n})(\ell_{\mathcal{C}_n}(t)) - \mathsf{S}(\widetilde{\mathcal{C}}, y_0)(\ell_{\mathcal{C}}(t))\| \\
&\leq \|\mathsf{S}(\widetilde{\mathcal{C}}_n, y_{0,n})(\ell_{\mathcal{C}_n}(t)) - \mathsf{S}(\widetilde{\mathcal{C}}_n, y_{0,n})(\ell_{\mathcal{C}}(t))\| \\
&\qquad + \|\mathsf{S}(\widetilde{\mathcal{C}}_n, y_{0,n})(\ell_{\mathcal{C}}(t)) - \mathsf{S}(\widetilde{\mathcal{C}}, y_0)(\ell_{\mathcal{C}}(t))\| \\
&\leq \mathrm{Lip}(\mathsf{S}(\widetilde{\mathcal{C}}_n, y_{0,n}), [0,t])|\ell_{\mathcal{C}}(t) - \ell_{\mathcal{C}_n}(t)| + \|y_{0,n} - y_0\| + 2(\mathrm{Lip}(\mathsf{S}(\widetilde{\mathcal{C}}, y_0), [0, \ell_{\mathcal{C}}(t)]) \\
&\qquad + \mathrm{Lip}(\mathsf{S}(\widetilde{\mathcal{C}}_n, y_{0,n}), [0, \ell_{\mathcal{C}}(t)])) \int_0^{\ell_{\mathcal{C}}(t)} d_{\mathscr{H}}(\widetilde{\mathcal{C}}(s), \widetilde{\mathcal{C}}_n(s))\,\mathrm{d}s \\
&\leq \|y_{0,n} - y_0\| + |\ell_{\mathcal{C}}(t) - \ell_{\mathcal{C}_n}(t)| + 2\ell_{\mathcal{C}}(t) \sup_{s\in[0,\ell_{\mathcal{C}}(t)]} d_{\mathscr{H}}(\widetilde{\mathcal{C}}(s), \widetilde{\mathcal{C}}_n(s)).
\end{aligned}$$

The conclusion now follows Propositions 5.3 and Corollary 5.5 $\square$

We finally deduce the convergence of the catching-up algorithm.

Corollary 6.4. *Assume that* $\mathcal{C} \in BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ *and let* $\mathcal{C}_n \in BV^{\mathrm{r}}_{\mathrm{loc}}([0,\infty[\,;\mathscr{C}_{\mathcal{H}})$ *be such that* (27)–(29) *hold for every* $n \in \mathbb{N}$. *Then* $y_n = \mathsf{S}(\mathcal{C}_n, y_{0,n})$ *is given by* (30)–(31) *and* $d_{\infty,[0,t]}(\mathsf{S}(\mathcal{C}_n, y_{0,n}), \mathsf{S}(\mathcal{C}, y_0)) \to 0$ *as* $n \to \infty$ *for every* $t > 0$.

Acknowledgements. I would like to thank an anonymous referee for pointing out that the proof of Lemma 4.1 was incomplete. She/he also showed me different proofs of Lemmas 4.1-(*iii*) and 4.2: the first using the "Slater condition", the second exploiting the notion of subdifferential.

References

[1] K. Addy, S. Adly, B. Brogliato, D. Goeleven: A method using the approach of Moreau and Panagiotopoulos for the mathematical formulation of non-regular circuits in electronics, Nonlinear Anal., Hybrid Syst. 1 (2007) 30–43.

[2] S. Adly, T. Haddad, L. Thibault: Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities, Math. Program. 148 (2014) 5–47.

[3] C. D. Aliprantis, K. C. Border: Infinite Dimensional Analysis, 3rd Ed., Springer, Berlin (2006).

[4] L. Ambrosio: Metric space valued functions of bounded variation, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 17 (1990) 439–478.

[5] H. Brézis: Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland Mathematical Studies 5, North-Holland, Amsterdam (1973).

[6] B. Brogliato, L. Thibault: Existence and uniqueness of solutions for non-autonomous complementarity dynamical systems, J. Convex Analysis 17 (2010) 961–990.

[7] M. Brokate, K. Dreßler, P. Krejčí: On the Mróz model, Eur. J. Appl. Math. 7 (1996) 473–497.

[8] M. Brokate, P. Krejčí, D. Rachinskii: Some analytical properties of the multidimensional continuous Mróz model of plasticity, Control Cybern. 27 (1998) 199–215.

[9] M. Brokate, J. Sprekels: Hysteresis and Phase Transitions, Applied Mathematical Sciences 121, Springer, New York (1996).

[10] C. Castaing, M. Valadier: Convex Analysis and Measurable Multifunctions, Springer, Berlin (1977).

[11] B. Cornet: Existence of slow solutions for a class of differential inclusions, J. Math. Anal. Appl. 96 (1983) 130–147.

[12] N. Dinculeanu: Vector Measures, International Series of Monographs in Pure and Applied Mathematics 95, Pergamon Press, Oxford (1967).

[13] H. Federer: Geometric Measure Theory, Springer, Berlin (1969).

[14] C. Henry: An existence theorem for a class of differential equations with multi-valued right-hand side, J. Math. Anal. Appl. 41 (1973) 179–186.

[15] M. A. Krasnosel'skiĭ, A. V. Pokrovskiĭ: Systems with Hysteresis, Springer, Berlin (1989).

[16] P. Krejčí: Vector hysteresis models, Eur. J. Appl. Math. 2 (1991) 281–292.

[17] P. Krejčí: Hysteresis, Convexity and Dissipation in Hyperbolic Equations, Gakuto International Series Mathematical Sciences and Applications 8, Gakkōtosho, Tokyo (1997).

[18] P. Krejčí: personal communication.

[19] P. Krejčí, P. Laurençot: Generalized variational inequalities, J. Convex Analysis 9 (2002) 159–183.

[20] P. Krejčí, V. Recupero: Comparing BV solutions of rate independent processes, J. Convex Analysis 21 (2014) 121–146.

[21] M. Kunze, M. D. P. Monteiro Marques: An introduction to Moreau's sweeping processes, in: Impacts in Mechanical Systems - Analysis and Modelling (Grenoble, 1999), B. Brogliato (ed.), Lecture Notes in Physics 551, Springer, Berlin (2000) 1–60.

[22] S. Lang: Real and Functional Analysis, 3rd Ed., Springer, New York (1993).

[23] B. Maury, J. Venel: A mathematical framework for a crowd motion model, C. R., Math., Acad. Sci. Paris 346 (2008) 1245–1250.

[24] A. Mielke: Evolution in rate-independent systems, in: Handbook of Differential Equations, Evolutionary Equations, Vol. 2, C. Dafermos, E. Feireisl (eds.), Elsevier, Amsterdam (2005) 461–559.

[25] M. D. P. Monteiro Marques: Differential Inclusions in Nonsmooth Mechanical Problems - Shocks and Dry Friction, Birkhäuser, Basel (1993).

[26] J. J. Moreau: Rafle par un convexe variable I, Trav. Sémin. d'Anal. Convexe, Montpellier 1, Exposé 15 (1971) 43 pp.

[27] J. J. Moreau: Rafle par un convexe variable II, Trav. Sémin. d'Anal. Convexe, Montpellier 2, Exposé 3 (1972) 36 pp.

[28] J. J. Moreau: Problème d'évolution associé à un convexe mobile d'un espace hilbertien, C. R. Acad. Sci., Paris, Sér. A 276 (1973) 791–794.

[29] J. J. Moreau: On unilateral constraints, friction and plasticity, in: New Variational Techniques in Mathematical Physics (Bolzano, 1973), G. Capriz, G. Stampacchia (eds.), Cremonese, Rome (1974) 173–322.

[30] J. J. Moreau: Multiapplications à retraction finie, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 1 (1974) 169–203.

[31] J. J. Moreau: Intersection of moving convex sets in a normed space, Math. Scand. 36 (1975) 159–173.

[32] J. J. Moreau: Application of convex analysis to the treatment of elastoplastic systems, in: Applications of Methods of Functional Analysis to Problems in Mechanics, P. Germain, B. Nayroles (eds.), Lecture Notes in Mathematics 503, Springer, Berlin (1976) 56–89.

[33] J. J. Moreau: Evolution problem associated with a moving convex set in a Hilbert space, J. Differ. Equations 26 (1977) 347–374.

[34] J. J. Moreau: An introduction to unilateral dynamics, in: Novel Approaches in Civil Engineering, M. Frémond, F. Maceri (eds.), Springer, Berlin (2002) 1–46.

[35] V. Recupero: On locally isotone rate independent operators, Appl. Math. Lett. 20 (2007) 1156–1160.

[36] V. Recupero: The play operator on the rectifiable curves in a Hilbert space, Math. Methods Appl. Sci. 31 (2008) 1283–1295.

[37] V. Recupero: BV-extension of rate independent operators, Math. Nachr. 282 (2009) 86–98.

[38] V. Recupero: On a class of scalar variational inequalities with measure data, Appl. Anal. 88 (2009) 1739–1753.

[39] V. Recupero: BV solutions of rate independent variational inequalities, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 10 (2011) 269–315.

[40] V. Recupero: Extending vector hysteresis operators, J. Phys.: Conf. Ser. 268 (2011) 12–24.

[41] V. Recupero: A continuity method for sweeping processes, J. Differ. Equations 251 (2011) 2125–2142.

[42] J. Serra: Hausdorff distances and interpolations, in: Mathematical Morphology and its Applications to Image and Signal Processing (Amsterdam, 1998), H. Heijmans, J. Roerdink (eds.), Kluwer, Dordrecht (1998) 107–114.

[43] A. Visintin: Differential Models of Hysteresis, Applied Mathematical Sciences 111, Springer, Berlin (1994).

[44] W. Ziemer: Weakly Differentiable Functions, Springer, New York (1989).