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Properties of Completely Monotone and Bernstein Functions Related to the Shape of Their Measures

Hristo Sendov
Department of Statistical and Actuarial Sciences, Western University,
1151 Richmond Str., London, ON, N6A 5B7 Canada
hssendov@stats.uwo.ca

Shen Shan*
Department of Statistical and Actuarial Sciences, Western University,
1151 Richmond Str., London, ON, N6A 5B7 Canada
sshan2@uwo.ca

Dedicated to the memory of Jean Jacques Moreau.

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We are interested in Bernstein and Lévy measures having certain convexity-type properties. We thoroughly investigate how the shape of these measures results in various properties of the corresponding completely monotone and Bernstein functions. We hope that by understanding these connections one would be able to shed more light on the several conjectures posed in [9].

Keywords: Bernstein function, completely monotone function, Bernstein measure, Lévy measure, harmonically convex function

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1. Introduction

The need for the current investigations emerged from the works [9] and [10]. These works described intriguing parallels between completely monotone functions with harmonically convex measures and Bernstein functions, having Lévy measures with harmonically concave tail. The connections with convex analysis, the Coupon Collector's Problem, Pólay's criterion for characteristic functions, necessitate the need to better understand the structure of measures that either have a harmonically convex cumulative distribution function or have a harmonically concave tail. We hope that this work will contribute to the solution of the

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several open problems formulated in [9].

A non-negative function $f(x) : (0,\infty) \to [0,\infty)$ is called *completely monotone*, if it is infinitely differentiable and

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all } x > 0 \text{ and } n \geq 1.$$

A non-negative function $g(x) : (0,\infty) \to [0,\infty)$ is called *Bernstein*, if it is infinitely differentiable and

$$(-1)^{n-1} g^{(n)}(x) \geq 0 \quad \text{for all } x > 0 \text{ and } n \geq 1.$$

Completely monotone and Bernstein functions are classical objects with numerous applications in areas ranging from complex analysis and monotone operator theory to stochastic Lévy processes. As the well-known Bernstein representation theorem shows that completely monotone functions are the Laplace transforms of measures on $[0,\infty)$.

Theorem 1.1 (Bernstein). *A function $f(x)$ is completely monotone, if and only if it is the Laplace transform of a unique Radon measure μ on $[0,\infty)$, that is,*

$$f(x) = \int_{[0,\infty)} e^{-xt}\, \mu(dt)$$

for all $x > 0$, while measure μ is such that the above integral is finite for all $x > 0$.

The measure μ in the Bernstein representation theorem will be called *Bernstein measure*.

The Bernstein functions are in one-to-one correspondence with the Laplace exponent of the subordinator Lévy processes. The monograph [8] is an excellent introduction to the subject. The Lévy-Khintchine representation theorem characterizes completely the Bernstein functions.

Theorem 1.2 (Lévy-Khintchine). *A function $g(x)$ is Bernstein, if and only if it admits the representation*

$$g(x) = a + bx + \int_{(0,\infty)} (1 - e^{-xt})\, \nu(dt)$$

for some constants $a, b \geq 0$ and a Radon measure ν on $(0,\infty)$ which satisfies the integrability condition

$$\int_{(0,\infty)} (1 \wedge t)\, \nu(dt) < \infty. \tag{1}$$

The triplet (a, b, ν) uniquely determines the Bernstein function $g(x)$, and vice versa.

The measure ν in this representation is usually called the *Lévy measure* of the Bernstein function $g(x)$, and (a, b, ν) is called the *Lévy triplet* of $g(x)$.

Clearly, if $g(x)$ is a Bernstein function, then $g'(x)$ is completely monotone, but there are completely monotone functions that do not have a Bernstein primitive. The following proposition clarifies this implication by characterizing completely monotone functioins that have a Bernstein primitive.

Proposition 1.3 (Proposition 3.4 in [8]). *Suppose $f(x)$ is completely monotone with measure μ. It has a primitive $g(x)$, which is Bernstein, if and only if the measure μ satisfies*

$$\int_{(0,\infty)} \frac{1}{1+t}\,\mu(dt) < \infty. \tag{2}$$

All measures considered in this paper are Radon measures. In particular, they are σ-finite and finite on compact sets, see [13, Definition 19.15]. The cumulative distribution function for measure μ on $[0, \infty)$ is denoted by

$$F_\mu(x) = \mu[0, x],$$

while the tail of measure ν on $(0, \infty)$ is denoted by

$$\bar{\nu}(x) = \nu(x, \infty).$$

These notions are used without further clarification. We say that a measure μ has a convexity property if the function $F_\mu(x)$ has that property. Similarly, we say that the tail of a measure ν has a convexity property if the function $\bar{\nu}(x)$ does so. (See Definition 2.5 for details.)

Recently, interest in Bernstein functions has focused on different subclasses of such functions, see for example [2], [4], and [11]. The subclass of Bernstein functions examined in [9], requires a convexity-type property on the tail of the Lévy measure. More specifically, it requires that the function $x \mapsto x\nu(x, \infty)$ be concave on $(0, \infty)$. A parallel property for Bernstein measures is that the function $x \mapsto x\mu[0, x]$ is convex on $[0, \infty)$. In this work we give several characterizations of such measures.

Pólya criterion, see [3, Theorem 3.3.10] and [5, Theorem 4.3.3], is well-known as it provides a simple sufficient condition for a function to be a characteristic function. We derive an analogous representation for measures with desirable convexity properties. See Definition 2.5 for details of the convexity properties mentioned in Tables 1.1 and 1.2.

The conditions on the measure τ in Table 1.1 depend on each case and the reader should refer to the corresponding theorems for details. The last row in Table 1.1 is trivial. The function $x \to \mu[0, 1/x]$ is concave, non-increasing, and non-negative. It can not be concave, unless it is a constant on $(0, \infty)$, implying $\mu[0, x] = a$.

μ on $[0,\infty)$	Characterization	Reference
harmonically convex	$\mu[0,x] = a + \int_{(0,\infty)} \left(1 - \frac{1}{xs}\right)^{+} \tau(ds)$	Thm. 3.1
concave	$\mu[0,x] = a + bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds)$	Thm. 4.1
convex	$\mu[0,x] = a + bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^{+} \tau(ds)$	Thm. 5.1
harmonically concave	$\mu[0,x] = a$	N/A

Table 1.1: Characterization of measure μ on $[0,\infty)$ with convexity properties

Again, the conditions on the measure τ in Table 1.2 depend on each case and the reader should refer to the corresponding theorems for details. The third row is trivial, since $x \mapsto \nu(x,\infty)$ is non-increasing and non-negative, it can not be concave unless $\nu(x,\infty)$ is constant. Continuity of Radon measure implies $\nu(x,\infty) = 0$.

Applying the representations in Tables 1.1 and 1.2, we characterize completely monotone and Bernstein function with those convexity properties on their measures. The main results are listed in Tables 1.3 and 1.4 with corresponding references for details.

The paper is organized as follows. The second section introduces the important concepts and notions. In the third section, we characterize harmonically convex measures and measures with harmonically concave tail, which leads to the characterization for completely monotone functions with harmonically convex measure and Bernstein functions, whose Lévy measure has harmonically concave tail. In the fourth section, we discuss concave measures and measures with convex tail. Convex measures and measures with harmonically convex tail are considered in Section 5 and Section 6.

2. Background Results

2.1. Harmonic convexity and concavity

Let I be a convex interval of $\mathbb{R}$. A function $h : I \to \mathbb{R}$ is *convex* on I if

$$h(\alpha x + (1-\alpha)y) \leq \alpha h(x) + (1-\alpha)h(y)$$

for $x, y \in I$, and $\alpha \in [0,1]$. The function h is *concave* if $-h$ is convex. If h is twice differentiable on an open interval I, then h is convex on I, if and only if $h''(x) \geq 0$ for all x in I.

ν on $(0,\infty)$	Characterization	Reference
harmonically concave tail	$\nu(x,\infty) = \dfrac{b}{x} + \displaystyle\int_{(0,\infty)} \left(1 \wedge \frac{1}{xs}\right) \tau(ds)$	Thm. 3.2
convex tail	$\nu(x,\infty) = \displaystyle\int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^{+} \tau(ds)$	Thm. 4.2
concave tail	$\nu(x,\infty) = 0$	N/A
harmonically convex tail	$\nu(x,\infty) = \dfrac{b}{x} + \displaystyle\int_{(0,\infty)} \left(\frac{1}{xs} - 1\right)^{+} \tau(ds)$	Thm. 6.1

Table 1.2: Characterization of measure ν on $(0,\infty)$ with convexity properties on its tail

Every convex function is locally Lipschitz continuous on the interior of its domain. The left and the right directional derivatives exist (in the extended sense) for every $x \in I$ and

$$h'_+(x) \le h'_-(y) \le h'_+(y) \le h'_-(z) \tag{3}$$

for all $x < y < z$ in I, see [7, Theorem 24.1]. The function $h'_+(x)$ is right-continuous while $h'_-(x)$ is left continuous. Moreover, for any x, y in the interior of I we have

$$h(y) - h(x) = \int_{(x,y)} h'_+(t)\,dt = \int_{(x,y)} h'_-(t)\,dt, \tag{4}$$

see [7, Corollary 24.2.1] for details.

Definition 2.1. A function $h : (0,\infty) \to \mathbb{R}$ is called *harmonically convex (concave)* if the function $h(1/x)$ is convex (concave) on $(0,\infty)$.

The justification for the terminology comes from the fact that $2/(1/x + 1/y)$ is the harmonic mean of x and y, and h is harmonically convex, if and only if

$$h\left(\frac{2}{1/x + 1/y}\right) \le \frac{h(x) + h(y)}{2}$$

for every $x, y \in (0,\infty)$. Examples of harmonically convex functions can be found in [6]. It is not clear how to tell whether the function is harmonically convex by looking at its graph, see for example [9, Figure 1]. However, once noticing that $2/(1/x + 1/y) \le (x + y)/2$ holds for $x, y > 0$, one can tell that:

a) a non-decreasing convex function on $(0,\infty)$ is harmonically convex; and
b) a non-increasing concave function on $(0,\infty)$ is harmonically concave.

Property on μ	Characterization	Reference
harmonically convex	$f(x) = a + \int_{(0,\infty)} \frac{x}{s} k\left(\frac{x}{s}\right) \tau(ds)$	Prop. 3.5
	$f(x) - xf'(x)$ is completely monotone	Cor. 3.6
concave	$f(x) = a + \frac{b}{x} + \int_{(0,\infty)} r(xs)\, \tau(ds)$	Prop. 4.5
	$f(x) + xf'(x)$ is completely monotone	Cor. 4.6
convex	$f(x) = a + \frac{b}{x} + \int_{(0,\infty)} l(xs)\, \tau(ds)$	Prop. 5.4
	$x(f(x) - \mu(\{0\}))$ is completely monotone	Cor. 5.5
harmonically concave	$f(x) = a$	N/A

Table 1.3: Characterization of completely monotone functions $f(x)$ with convexity properties on their measure μ

The following result, a variant of [6, Lemma 2.2], clarifies the relationship between convexity and harmonic convexity.

Lemma 2.2. *A function $h : (0, \infty) \to R$ is harmonically convex (concave), if and only if $xh(x)$ is convex (concave).*

Alternatively, $h(x)$ is convex on $(0, \infty)$, if and only if $h(x)/x$ is harmonically convex. If h is twice differentiable, then it is harmonically convex, if and only if $2h'(x) + xh''(x) \geq 0$ for all $x > 0$.

The next lemma can be verified directly from the definition of directional derivative. It shows that if a function h is harmonically convex then its left and right directional derivatives exist.

Lemma 2.3. *Suppose the left (reps. right) directional derivative of $\varphi : I \to \mathbb{R}$ exists at $x \neq 0$. Then, the right (reps. left) directional derivative of $h(x) := \varphi(1/x)$ exists and*

$$h'_+(x) = -\varphi'_-\left(\frac{1}{x}\right)\frac{1}{x^2}, \qquad \textit{resp. } h'_-(x) = -\varphi'_+\left(\frac{1}{x}\right)\frac{1}{x^2}.$$

Proposition 2.4. *Formula* (4) *is valid if the function h is harmonically convex on* $(0, \infty)$.

Property on ν	Characterization	Reference
harmonically concave tail	$g(x)=a+bx+\displaystyle\int_{(0,\infty)}\left(1-\frac{x}{s}k\left(\frac{x}{s}\right)\right)\tau(ds)$	Prop. 3.8
	$g(x)-xg'(x)$ is Bernstein	Cor. 3.9
convex tail	$g(x)=a+bx+\displaystyle\int_{(0,\infty)}(1-r(xs))\,\tau(ds)$	Prop. 4.8
	$g(x)+xg'(x)$ is Bernstein	Cor. 4.9
concave tail	$g(x)=a+bx$	N/A
harmonically convex tail	$g(x)=a+bx$	Cor. 6.2 b)

Table 1.4: Characterization of Bernstein functions $g(x)$ with convexity properties on the tail of their Lévy measure ν

Proof. For any $0<x<y$, since the function h has bounded variation (on intervals away from 0), we have

$$\begin{aligned} h(y)-h(x) &= \int_{(x,y)} dh(t) = \int_{(-1/x,-1/y)} dh(-1/s) \\ &= \int_{(-1/x,-1/y)} \big(h(-1/s)\big)'_-\, ds \\ &= \int_{(-1/x,-1/y)} h'_+(-1/s)\frac{1}{s^2}\, ds = \int_{(x,y)} h'_+(t)\, dt. \end{aligned}$$

Change of variable $t=-1/s$ is applied in the second and the fifth equality, see [12, Theorem 11a]. The third equality follows by (4) applied to the convex function $h(-1/s)$. Analogously, Equation (4) holds for the left derivative. $\square$

Definition 2.5. a) Let μ be a measure on $[0,\infty)$. We say that μ is *(harmonically) convex* on $(0,\infty)$ if $F_\mu(x)$ is (harmonically) convex on $(0,\infty)$.

b) Let ν be a measure on $(0,\infty)$. We say that ν has *(harmonically) concave tail* if $\bar{\nu}(x)$ is (harmonically) concave on $(0,\infty)$.

It is easy to see that if a measure μ is convex, then it is harmonically convex, because the cumulative distribution function of μ is non-decreasing. By Lemma 2.2, a measure μ is harmonically convex, if and only if the function $x\mapsto xF_\mu(x)$ is convex.

2.2. Pólya's criterion

The property of a measure being harmonically convex is realted to the notion of a characteristic function as the following trivial modification of the Polya's criterion shows, see [3, Theorem 3.3.10]. (Here, x^+ denotes the function that is zero if $x < 0$ and x otherwise.)

Theorem 2.6 (Polya's criterion). *Suppose a function $\varphi : [0, \infty) \mapsto [0, \infty)$ is convex and continuous, with $\varphi(0) = 1$ and*

$$\lim_{x\to\infty} \varphi(x) = 0.$$

Then, it admits the representation

$$\varphi(x) = \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds), \tag{5}$$

for some probability measure τ on $(0, \infty)$. Furthermore, $\varphi(|x|)$ is the characteristic function of a symmetric distribution.

The theorem fails without the assumption $\lim_{x\to 0+} \varphi(x) = 1$ since $x \mapsto \mathbf{1}_{\{0\}}(x)$ satisfies all other conditions in the theorem, but does not admit such integral representation. Here, $\mathbf{1}_{\{0\}}(x)$ is the indicator function of the set $\{0\}$.

A probability measure μ on $[0, \infty)$ can not be convex unless $\mu(\{0\}) = 1$, however, it can be harmonically convex. Such probability measure are easily characterized by Polya's criterion.

Corollary 2.7. *A probability measure μ on $(0, \infty)$ is harmonically convex, if and only if there is a probability measure τ on $(0, \infty)$ such that*

$$\mu(0, x] = \int_{(0,\infty)} \left(1 - \frac{1}{xs}\right)^+ \tau(ds).$$

Proof. Let $\varphi(x)$ be equal to $\mu(0, 1/x]$, for $x > 0$, and to 1 when $x = 0$. Then, $\varphi(x)$ is non-negative, non-increasing, $\varphi(0) = 1$, $\lim_{x\to 0+} \varphi(x) = 1$, and $\lim_{x\to\infty} \varphi(x) = 0$. Moreover, μ is a harmonically convex measure, if and only if φ is convex. The proof is completed after replacing x by $1/x$ in (5). □

3. Characterization of harmonically convex measures and measures with harmonically concave tail

This section is devoted to the proof of Theorem 3.1 and Theorem 3.2. Following the ideas used in the proof of Polya's criterion, we characterize harmonically convex measures and measures having harmonically concave tail.

Theorem 3.1. *A measure μ on $[0,\infty)$ is harmonically convex, if and only if there is a measure τ on $(0,\infty)$ such that*

$$\mu[0,x] = a + \int_{(0,\infty)} \left(1 - \frac{1}{xs}\right)^+ \tau(ds), \tag{6}$$

where $a \geq 0$ and τ satisfies the integrability condition $\tau(1,\infty) < \infty$.

Proof. First, we show the sufficiency. If (6) holds for any $x > 0$, then

$$\mu[0,1/x] = a + \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds).$$

The integral is well-defined because $\mu[0,1/x] \leq a + \tau(x,\infty) < \infty$, for any $x > 0$. As the function $x \mapsto (1 - x/s)^+$ is convex for any $s > 0$, the integral is also convex. Hence, μ is harmonically convex.

Now, we show the necessity. If μ is harmonically convex, then $\varphi(x) := \mu[0,1/x]$ is convex, non-increasing and non-negative on $(0,\infty)$. The following limits exist

$$a := \lim_{x\to\infty} \varphi(x) = \mu(\{0\}) \geq 0, \qquad \lim_{x\to\infty} \varphi'_+(x) = 0, \qquad \lim_{x\to 0+} \varphi'_+(x) \geq -\infty.$$

Define a Radon measure on $(0,\infty)$ by

$$\mu^*(p,q] = \begin{cases} \varphi'_+(q) - \varphi'_+(p) & \text{if } 0 < p < q < \infty, \\ \varphi'_+(q) - \lim\limits_{x\to 0+} \varphi'_+(x) & \text{if } 0 = p < q < \infty. \end{cases}$$

For all $t > 0$, we have

$$\mu^*(t,\infty) = 0 - \varphi'_+(t) = -\varphi'_+(t).$$

Consider the measure τ defined on $(0,\infty)$ by $\tau(dt) = t\mu^*(dt)$. By (4), for $M > 0$,

$$\varphi(x) = \varphi(x) - \varphi(M) + \varphi(M) = -\int_{(x,M)} \varphi'_+(t)\,dt + \varphi(M).$$

Letting M approaching infinity, and using Fubini's theorem, we obtain

$$\begin{aligned}
\varphi(x) &= a - \int_{(x,\infty)} \varphi'_+(t)\,dt = a + \int_{(x,\infty)} \int_{(t,\infty)} \mu^*(ds)\,dt \\
&= a + \int_{(x,\infty)} \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt = a + \int_{(x,\infty)} \frac{1}{s} \int_{(x,s)} dt\,\tau(ds) \\
&= a + \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds).
\end{aligned}$$

A change of variable leads to (6). The integrability condition on τ follows from

$$\tau(1,\infty) = \int_{(1,\infty)} \tau(ds) \leq \int_{(1,\infty)} \left(2 - \frac{1}{s}\right) \tau(ds) \leq 2\varphi(1/2) < \infty.$$

The proof is completed. □

Theorem 3.2. *A measure ν on $(0,\infty)$ has harmonically concave tail, if and only if there is a measure τ on $(0,\infty)$ such that*

$$\nu(x,\infty) = \frac{b}{x} + \int_{(0,\infty)} \left(1 \wedge \frac{1}{xs}\right) \tau(ds), \tag{7}$$

where $b \geq 0$ and τ satisfies

$$\int_{(0,\infty)} \left(1 \wedge \frac{1}{s}\right) \tau(ds) < \infty. \tag{8}$$

Proof. First, we show the sufficiency. If (7) holds for any $x > 0$, then

$$\nu(1/x,\infty) = bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds).$$

The integral is well-defined for any $x > 0$, since τ satisfies condition (8). As the function $x \mapsto \min(1, x/s)$ is concave, the integral is concave, which implies measure ν has harmonically concave tail.

Now, we show the necessity. If ν has harmonically concave tail, then function $\varphi(x) := \nu(1/x,\infty)$ is non-negative, non-decreasing and concave on $(0,\infty)$. The following limits exist

$$\lim_{x\to 0+} \varphi(x) = 0, \qquad \lim_{x\to 0^+} \varphi'_+(x) \leq \infty, \qquad b := \lim_{x\to\infty} \varphi'_+(x) < \infty.$$

Define a Radon measure μ^* on $(0,\infty)$ by

$$\mu^*(p,q] = \begin{cases} \varphi'_+(p) - \varphi'_+(q) & \text{if } 0 < p < q < \infty, \\ \lim\limits_{x\to 0+} \varphi'_+(x) - \varphi'_+(q) & \text{if } 0 = p < q < \infty. \end{cases}$$

For all $t > 0$, we have

$$\mu^*(t,\infty) = \varphi'_+(t) - b.$$

Consider the measure τ on $(0,\infty)$ defined by $\tau(dt) = t\mu^*(dt)$. By (4), for $\epsilon > 0$,

$$\varphi(x) = \varphi(x) - \varphi(\epsilon) + \varphi(\epsilon) = \int_{(\epsilon,x)} \varphi'_+(t)\,dt + \varphi(\epsilon).$$

Letting ϵ approaching 0, and using Fubini's theorem, we obtain

$$\begin{aligned}
\varphi(x) &= \int_{(0,x)} \varphi'_+(t)\,dt = bx + \int_{(0,x)} (\varphi'_+(t) - b)\,dt \\
&= bx + \int_{(0,x)} \int_{(t,\infty)} \mu^*(ds)\,dt = bx + \int_{(0,x)} \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt \\
&= bx + \int_{(0,x]} \int_{(0,s)} \frac{1}{s}\,dt\,\tau(ds) + \int_{(x,\infty)} \int_{(0,x)} \frac{1}{s}\,dt\,\tau(ds) \\
&= bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds).
\end{aligned}$$

Representation (7) follows, while the integrability condition (8) utilizes the fact that $\varphi(1) = \nu(1,\infty) < \infty$. This concludes the proof. $\square$

Denote by $\delta_0(dx)$, the Dirac delta measure on $[0,\infty)$, having mass 1 at 0 and mass 0 everywhere else.

Corollary 3.3. *a) A measure μ is harmonically convex on $[0,\infty)$, if and only if*

$$\mu(dx) = a\delta_0(dx) + \frac{1}{x^2}\left(\int_{(1/x,\infty)} \frac{1}{s}\,\tau(ds)\right) dx, \tag{9}$$

for some constant $a \geq 0$ and measure τ on $(0,\infty)$ satisfying

$$\tau(1,\infty) < \infty.$$

b) A Bernstein measure μ is harmonically convex, if and only if (9) holds and the measure τ on $(0,\infty)$ satisfies

$$\tau(1,\infty) < \infty \quad \text{and} \quad \int_{(0,1]}\int_{(0,1)} e^{-x/ts}\,dt\,\tau(ds) < \infty \quad \text{for all } x > 0. \tag{10}$$

c) A measure ν has harmonically concave tail on $(0,\infty)$, if and only if

$$\nu(dx) = \frac{1}{x^2}\left(b + \int_{(1/x,\infty)} \frac{1}{s}\,\tau(ds)\right) dx, \tag{11}$$

for some constant $b \geq 0$ and measure τ on $(0,\infty)$ satisfying (8), that is

$$\int_{(0,\infty)} \left(1 \wedge \frac{1}{s}\right) \tau(ds) < \infty.$$

d) A Lévy measure ν has harmonically concave tail, if and only if (11) holds with $b = 0$ and the measure τ on $(0,\infty)$ satisfies

$$\tau(0,1] < \infty \quad \text{and} \quad \int_{(1,\infty)} \frac{\ln(s)}{s}\,\tau(ds) < \infty. \tag{12}$$

Proof. *a)* Formula (9) follows from the following representation

$$\mu[0,x] = a + \int_{(1/x,\infty)}\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt,$$

which is inferred from the proof of Theorem 3.1. Here, measure τ satisfies the integrability condition $\tau(1,\infty) < \infty$.

b) Suppose now μ is a Bernstein measure and it is harmonically convex. Its Laplace transform is well-defined for all $x > 0$. By (9) and Fubini's theorem:

$$\begin{aligned}\infty > \int_{[0,\infty)} e^{-xu}\,\mu(du) &= a + \int_{(0,\infty)} e^{-xu}\,\mu(du)\\ &= a + \int_{(0,\infty)} e^{-xu}\frac{1}{u^2}\int_{(1/u,\infty)} \frac{1}{s}\,\tau(ds)\,du\\ &= a + \int_{(0,\infty)} \frac{1}{s}\int_{(1/s,\infty)} e^{-xu}\frac{1}{u^2}\,du\,\tau(ds)\\ &= a + \int_{(0,\infty)}\int_{(0,1)} e^{-x/ts}\,dt\,\tau(ds),\end{aligned}$$

where in the last equality, we changed the variable $u = 1/ts$. Once we notice that

$$\int_{(1,\infty)}\int_{(0,1)} e^{-x/ts}\,dt\,\tau(ds) \le \tau(1,\infty) < \infty,$$

integrability condition (10) follows.

Conversely, suppose a measure μ on $[0,\infty)$ have the representation (9) with condition (10). By part *a*), measure μ is harmonically convex. Reversing the steps above shows that its Laplace transform is well-defined for all $x > 0$. Therefore, μ is a Bernstein measure.

c) Formula (11) follows immediately from the following representation

$$\nu(x,\infty) = \frac{b}{x} + \int_{(0,1/x)}\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt,$$

inferred from the proof of Theorem 3.2. Here, measure τ satisfies the integrability condition (8).

d) Suppose now ν is a Lévy measure with harmonically concave tail. Then, from (1), (11), and Fubini's theorem, we obtain

$$\begin{aligned}\infty > \int_{(1,\infty)} \nu(dt) &= b + \int_{(1,\infty)}\int_{(1/t,\infty)} \frac{1}{t^2 s}\,\tau(ds)\,dt\\ &= b + \int_{(0,1)}\int_{(1/s,\infty)} \frac{1}{t^2 s}\,dt\,\tau(ds) + \int_{[1,\infty)}\int_{(1,\infty)} \frac{1}{t^2 s}\,dt\,\tau(ds)\\ &= b + \int_{(0,\infty)} \left(1 \wedge \frac{1}{s}\right)\,\tau(ds)\\ &\ge b + \tau(0,1]\end{aligned}$$

and

$$\begin{aligned}\infty > \int_{(0,1]} t\,\nu(dt) &= \int_{(0,1]} \frac{b}{t}\,dt + \int_{(0,1]} \int_{(1/t,\infty)} \frac{1}{ts}\,\tau(ds)\,dt \\ &= \int_{(0,1]} \frac{b}{t}\,dt + \int_{(1,\infty)} \int_{(1/s,1]} \frac{1}{ts}\,dt\,\tau(ds) \\ &= \int_{(0,1]} \frac{b}{t}\,dt + \int_{(1,\infty)} \frac{\ln(s)}{s}\,\tau(ds).\end{aligned}$$

Together with (8), we conclude $b = 0$, as well as

$$\tau(0,1] < \infty \quad \text{and} \quad \int_{(1,\infty)} \frac{\ln(s)}{s}\,\tau(ds) < \infty.$$

Conversely, suppose a measure ν on $(0,\infty)$ have the representation (11) with $b = 0$ and condition (12). By part c), measure ν has harmonically concave tail. It is also a Lévy measure, as (1) is implied by condition (12). □

3.1. Applications to completely monotone and Bernstein functions

As an application of the above representations, we characterize completely monotone functions with harmonically convex measure and Bernstein functions having Lévy measure with harmonically concave tail. We need the well-known *exponential integral function* on $(0,\infty)$, defined by

$$\mathrm{E}_1(x) := \int_{(x,\infty)} \frac{e^{-t}}{t}\,dt,$$

and use it to define the kernel

$$k(x) := -\mathrm{E}_1'(x) - \mathrm{E}_1(x) = \frac{e^{-x}}{x} - \int_{(x,\infty)} \frac{e^{-t}}{t}\,dt. \tag{13}$$

Function $k(x)$ plays a role in the next developments, hence we briefly summarize its properties.

Lemma 3.4. *Function $k(x)$ defined by* (13) *has the following properties.*

(a) *It satisfies the inequality* $0 \le k(x) \le e^{-x}/x$.

(b) *It is completely monotone with convex measure, more precisely*

$$k(x) = \int_{(0,\infty)} e^{-xt} \frac{(t-1)^+}{t}\,dt. \tag{14}$$

(c) *The product $xk(x)$ is completely monotone with harmonically convex measure. Moreover, $xk(x)$ is the derivative of a Bernstein function, explicitly*

$$xk(x) = \int_{(1,\infty)} e^{-xt} \frac{1}{t^2}\,dt = \left(\int_{(1,\infty)} (1 - e^{-xt}) \frac{1}{t^3}\,dt \right)'. \tag{15}$$

(*d*) *The function* $1-xk(x)$ *is Bernstein, whose Lévy measure has harmonically concave tail, more precisely*

$$1 - xk(x) = \int_{(1,\infty)} (1 - e^{-xt}) \frac{1}{t^2}\, dt. \tag{16}$$

Proof. (*a*) The first inequality follows from

$$\mathrm{E}_1(x) = \int_{(x,\infty)} \frac{e^{-t}}{t}\, dt \leq \frac{1}{x} \int_{(x,\infty)} e^{-t}\, dt = \frac{e^{-x}}{x} = -\mathrm{E}_1'(x),$$

while the second inequality is straightforward.

(*b*) Representation (14) is readily verified. The Bernstein measure for $k(x)$ is convex, as its cumulative distribution function is $(x - \ln(x) - 1) \cdot \mathbf{1}_{(x>1)}$ for $x \in [0, \infty)$, which is convex.

(*c*) The Bernstein measure μ of $xk(x)$ is harmonically convex, since

$$x\mu[0, x] = (x - 1) \cdot \mathbf{1}_{(x>1)}$$

is convex on $(0, \infty)$. One can readily verify (15) and see that $xk(x)$ is the derivative of a Bernstein function.

(*d*) The Lévy measure ν of $1 - xk(x)$ has harmonically concave tail, since

$$x\nu(x, \infty) = x \cdot \mathbf{1}_{(x\leq 1)} + \mathbf{1}_{(x>1)}$$

is concave. □

We note that, in general, part *c*) of Lemma 3.4 implies part b), since a completely monotone function $f(x)$ has convex measure with no mass at zero, if and only if $xf(x)$ is completely monotone, see Corollary 5.5, [1, Proposition 2.7.13], or [9, Lemma 5.1].

Proposition 3.5. *Suppose* $f(x)$ *is completely monotone with measure* μ. *Then,* μ *is harmonically convex, if and only if there exists a unique measure* τ *on* $(0, \infty)$ *such that*

$$f(x) = a + \int_{(0,\infty)} \frac{x}{s} k\left(\frac{x}{s}\right) \tau(ds), \tag{17}$$

where $k(x)$ *is defined in* (13), $a \geq 0$ *is constant, and* τ *satisfies* (10).

Proof. For the necessity, let $a := \mu(\{0\}) \geq 0$. By Corollary 3.3 part *b*), for all $x > 0$, we have

$$\begin{aligned} f(x) &= \int_{[0,\infty)} e^{-xu}\, \mu(du) = a + \int_{(0,\infty)} e^{-xu}\, \mu(du) \\ &= a + \int_{(0,\infty)} e^{-xu} \frac{1}{u^2} \int_{(1/u,\infty)} \frac{1}{s}\, \tau(ds)\, du \\ &= a + \int_{(0,\infty)} \frac{1}{s} \int_{(1/s,\infty)} e^{-xu} \frac{1}{u^2}\, du\, \tau(ds), \end{aligned}$$

where τ satisfies conditions (10). The change of variable $u = t/s$ gives

$$f(x) = a + \int_{(0,\infty)} \int_{(1,\infty)} e^{-xt/s} \frac{1}{t^2} \, dt \, \tau(ds).$$

Finally, by (15), we obtain representation (17). The uniqueness of measure τ follows from the uniquness of μ.

For the sufficiency, suppose (17) holds. By Lemma 3.4 part c), we know $xk(x)$ is completely monotone with harmonically convex measure. Then, so is $(x/s)k(x/s)$ for all $s > 0$. (Indeed, if $f(x)$ is completely monotone function with harmonically convex measure μ, then so is the measure of the completely monotone $f(xs)$ for any $s > 0$, as its cumulative distribution function is $F_\mu(x/s)$.) The proof follows from here. □

We point out that the measures μ and τ is Proposition 3.5 are related through Equation (9).

Proposition 3.5 allows us to give a new proof of an intrinsic characterization of completely monotone functions with harmonically convex measure. This was one of the main results in [9], see Lemma 4.1 there, where the proof used the real inversion formula for the Laplace-Stieltjes integrals. We avoid that tool now.

Corollary 3.6. *Suppose $f(x)$ is completely monotone with measure μ. Then, μ is harmonically convex, if and only if $f(x) - xf'(x)$ is completely monotone.*

Proof. Suppose measure μ has mass a at zero in the proof of both directions. If $f(x)$ has harmonically convex measure, by Proposition 3.5, we know $f(x)$ has representation (17). Thus, using [3, Theorem A.5.2] for differentiating under the integral, we have

$$f(x) - xf'(x) = a - \int_{(0,\infty)} \frac{x^2}{s^2} k'\left(\frac{x}{s}\right) \tau(ds) = a + \int_{(0,\infty)} e^{-x/s} \, \tau(ds),$$

where $a \geq 0$ and τ satisfies integrability condition (10). The last integral is convergent since the expression on the left is finite and defines a completely monotone function.

Conversely, if $f(x) - xf'(x)$ is completely monotone, we have

$$f(x) - xf'(x) = a' + \int_{(0,\infty)} e^{-xt} \, \eta(dt),$$

for $a' \geq 0$ and measure η on $(0, \infty)$. Therefore, we obtain

$$f''(x) = \frac{(f(x) - xf'(x))'}{-x} = \frac{1}{x} \int_{(0,\infty)} e^{-xt} t \, \eta(dt).$$

Integrating both sides twice, and using Fubini's theorem, gives us

$$\begin{aligned}
f(z) - a &= \int_{(z,\infty)} \int_{(y,\infty)} f''(x)\, dx\, dy \\
&= \int_{(z,\infty)} \int_{(y,\infty)} \int_{(0,\infty)} \frac{e^{-xt}t}{x}\, \eta(dt)\, dx\, dy \\
&= \int_{(0,\infty)} \left(\int_{(z,\infty)} \int_{(z,x)} \frac{e^{-xt}}{x}\, dy\, dx \right) t\, \eta(dt) \\
&= \int_{(0,\infty)} \left(\int_{(z,\infty)} e^{-xt}\, dx - z \int_{(z,\infty)} \frac{e^{-xt}}{x}\, dx \right) t\, \eta(dt) \\
&= \int_{(0,\infty)} \left(\frac{e^{-zt}}{t} - z\mathrm{E}_1(zt) \right) t\, \eta(dt) \\
&= \int_{(0,\infty)} ztk(zt)\, \eta(dt).
\end{aligned}$$

Let $F_\eta(x) = \eta(0, x]$ and noting that $-F_\eta(1/s)$ is a non-decreasing function, define measure τ on $(0, \infty)$ by $\tau(ds) := d(-F_\eta(1/s))$. Thus, replacing the argument z by x, and changing variable by $t = 1/s$, we have

$$f(x) = a + \int_{(0,\infty)} \frac{x}{s} k\left(\frac{x}{s}\right)\, \tau(ds).$$

Now, if we can show that τ satisfies (10), then, by Proposition 3.5, the Bernstein measure μ of $f(x)$ is harmonically convex. First,

$$\tau(1, \infty) = \int_{(1,\infty)} d(-F_\eta(1/s)) = F_\eta(1) - \lim_{s\to\infty} F_\eta(1/s) = F_\eta(1) < \infty.$$

For the second inequality,

$$\begin{aligned}
\int_{(0,1]} \int_{(0,1)} e^{-x/ts}\, dt\, \tau(ds) &= \int_{(0,1]} \int_{(0,1)} e^{-x/ts}\, dt\, d(-F_\eta(1/s)) \\
&= \int_{(1,\infty)} \int_{[1,\infty)} \frac{e^{-xuv}}{v^2}\, dF_\eta(u)\, dv,
\end{aligned}$$

where in the last equality, we changed the variables by $v = 1/t$ and $u = 1/s$, and also change the order of integration using Fubini's theorem. Continuing, we have

$$\begin{aligned}
\int_{(1,\infty)} \int_{[1,\infty)} \frac{e^{-xuv}}{v^2}\, dF_\eta(u)\, dv &\le \left(\int_{(1,\infty)} \frac{1}{v^2}\, dv \right) \left(\int_{(0,\infty)} e^{-xu}\, \eta(du) \right) \\
&= f(x) - xf'(x) - a' < \infty.
\end{aligned}$$

Therefore, measure τ satisfies (10). The proof is completed. $\square$

Combining Corollary 3.3 part b), Proposition 3.5, and Proposition 1.3, we arrive at the following corollary.

Corollary 3.7. *Suppose $f(x)$ is completely monotone with harmonically convex measure μ. It has a Bernstein primitive, if and only if the measure τ in* (17) *satisfies*

$$\int_{(0,\infty)} \left(1 - \frac{\ln(s+1)}{s}\right) \tau(ds) < \infty. \tag{18}$$

Proof. We only have to verify the integrability condition (18) on τ is equivalent to (2) on μ. Using (9), we have

$$\begin{aligned} \int_{(0,\infty)} \frac{1}{1+t}\,\mu(dt) &= \int_{(0,\infty)} \frac{1}{(1+t)t^2} \left(\int_{(1/t,\infty)} \frac{1}{s}\,\tau(ds)\right) dt \\ &= \int_{(0,\infty)} \left(\int_{(1/s,\infty)} \frac{1}{t^2(1+t)}\,dt\right) \frac{1}{s}\,\tau(ds) \\ &= \int_{(0,\infty)} \left(1 - \frac{\ln(s+1)}{s}\right) \tau(ds), \end{aligned}$$

where in the second equality, we changed the order of integration using Fubini's theorem. ☐

The next proposition characterizes Bernstein functions, whose Lévy measure has harmonically concave tail.

Proposition 3.8. *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then, ν has harmonically concave tail, if and only if there exists a unique triplet (a, b, τ) such that*

$$g(x) = a + bx + \int_{(0,\infty)} \left(1 - \frac{x}{s} k\left(\frac{x}{s}\right)\right) \tau(ds), \tag{19}$$

where $k(x)$ is defined in (13), *$a, b \geq 0$ are constants, and τ is a measure on $(0,\infty)$, satisfying* (12).

Proof. We show necessity first. If ν has harmonically concave tail, by Corollary 3.3 part d), for all $x > 0$, we have

$$\begin{aligned} g(x) &= a + bx + \int_{(0,\infty)} (1 - e^{-xu})\,\nu(du) \\ &= a + bx + \int_{(0,\infty)} (1 - e^{-xu}) \frac{1}{u^2} \int_{(1/u,\infty)} \frac{1}{s}\,\tau(ds)\,du \\ &= a + bx + \int_{(0,\infty)} \frac{1}{s} \int_{(1/s,\infty)} (1 - e^{-xu}) \frac{1}{u^2}\,du\,\tau(ds), \end{aligned}$$

where τ satisfies (12). The change of variable $u = t/s$ gives

$$g(x) = a + bx + \int_{(0,\infty)} \int_{(1,\infty)} (1 - e^{-xt/s}) \frac{1}{t^2}\, dt\, \tau(ds).$$

Apply now (16) to get representation (19). The uniqueness of the triplet (a, b, τ) follows from the uniqueness of the Lévy triplet (a, b, ν).

For the sufficiency, suppose (19) holds. By part *d*) of Lemma 3.4 we know $1 - xk(x)$ is a Bernstein function with Lévy measure having harmonically concave tail. Then, so is $1 - (x/s)k(x/s)$ for all $s > 0$. The proof follows from here. $\square$

This proposition allows us to give a new proof of an intrinsic characterization of Bernstein functions, whose Lévy measure has harmonically concave tail. This was one of the main results in [9], see Lemma 6.1 there, where the proof used the real inversion formula for the Laplace-Stieltjes integrals.

Corollary 3.9. *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then, ν has harmonically concave tail, if and only if $g(x) - xg'(x)$ is also Bernstein.*

Proof. Suppose $g(x)$ has Lévy triplet (a, b, ν). If ν has harmonically concave tail, by Proposition 3.8, we know $g(x)$ has representation (19). Thus, using [3, Theorem A.5.2] for differentiating under the integral, we have

$$g(x) - xg'(x) = a + \int_{(0,\infty)} \left(1 + \frac{x^2}{s^2} k'\left(\frac{x}{s}\right)\right) \tau(ds) = a + \int_{(0,\infty)} (1 - e^{-x/s})\, \tau(ds),$$

where $a \geq 0$ and τ satisfies integrability condition (12). The integral is convergent since the left-hand side is finite and shows that $g(x) - xg'(x)$ is a Bernstein function.

Conversely, suppose $g(x) - xg'(x)$ is a Bernstein function with Lévy-Khintchine representation:

$$g(x) - xg'(x) = a' + b'x + \int_{(0,\infty)} (1 - e^{-xt})\, \eta(dt),$$

where $a', b' \geq 0$, and η satisfies (1). Notice that

$$b' = \lim_{x\to\infty} \frac{g(x) - xg'(x)}{x} = \lim_{x\to\infty} \frac{g(x)}{x} - \lim_{x\to\infty} g'(x) = b - b = 0.$$

Therefore, we obtain

$$-g''(x) = \frac{(g(x) - xg'(x))'}{x} = \frac{1}{x} \int_{(0,\infty)} e^{-xt} t\, \eta(dt).$$

Integrating twice both sides of this equality and using Fubini's theorem, we obtain

$$\begin{aligned}
g(z) - a - bz &= \int_{(0,z)} \int_{(y,\infty)} -g''(x)\, dx\, dy \\
&= \int_{(0,z)} \int_{(y,\infty)} \int_{(0,\infty)} \frac{e^{-xt} t}{x}\, \eta(dt)\, dx\, dy \\
&= \int_{(0,\infty)} \left(\int_{(0,z]} \int_{(0,x)} \frac{e^{-xt}}{x}\, dy\, dx + \int_{(z,\infty)} \int_{(0,z)} \frac{e^{-xt}}{x}\, dy\, dx \right) t\, \eta(dt) \\
&= \int_{(0,\infty)} \left(\frac{1 - e^{-zt}}{t} + z \mathrm{E}_1(zt) \right) t\, \eta(dt) \\
&= \int_{(0,\infty)} (1 - ztk(zt))\, \eta(dt).
\end{aligned}$$

The penultimate equality gives the following identity, needed later

$$g(z) - a - bz = \int_{(0,\infty)} (1 - e^{-zt})\, \eta(dt) + z \int_{(0,\infty)} t \mathrm{E}_1(zt)\, \eta(dt). \tag{20}$$

Utilizing the fact that $\bar{\eta}(1/s)$ is non-decreasing, define measure τ on $(0,\infty)$ by $\tau(ds) = d\bar{\eta}(1/s) = d(\eta(1/s,\infty))$. Thus, replacing the argument z by x, and changing variable $t = 1/s$, we have

$$g(x) = a + bx + \int_{(0,\infty)} \left(1 - \frac{x}{s} k\left(\frac{x}{s}\right)\right)\, \tau(ds).$$

We will be done if we show that τ satisfies (12), since that implies the Lévy measure ν has harmonically concave tail, by Proposition 3.8. Notice that η is a Lévy measure, and

$$\tau(0,1] = \int_{(0,1]} d\bar{\eta}(1/s) = \bar{\eta}(1) - \lim_{s \to 0^+} \bar{\eta}(1/s) = \bar{\eta}(1) < \infty.$$

Next, we need to verify the finiteness of the following integral

$$\begin{aligned}
\int_{(1,\infty)} \frac{\ln(s)}{s}\, \tau(ds) &= \int_{(1,\infty)} \frac{\ln(s)}{s}\, d\bar{\eta}(1/s) = \int_{(0,1)} -t \ln(t)\, \eta(dt) \\
&\le e^z \int_{(0,1)} t \mathrm{E}_1(zt)\, \eta(dt) \le e^z \int_{(0,\infty)} t \mathrm{E}_1(zt)\, \eta(dt) < \infty,
\end{aligned}$$

where for the first inequality we used the fact that $0 \le -\ln(t) \le e^z \mathrm{E}_1(zt)$, for every $t \in (0,1)$ and $z > 0$. (Indeed, consider $q(t) := e^{-z} \ln(t) + \mathrm{E}_1(zt)$ and note that $q(1) = \mathrm{E}_1(z) \ge 0$ and $q'(t) = e^{-z}(1 - e^{z(1-t)})/t \le 0$.) The finiteness of the last integral follows from (20), since all other terms in that equality are finite. $\square$

4. Characterization of concave measures and measures with convex tail

In this section we characterize concave measures and measures with convex tail. That is accomplished in Theorems 4.1 and 4.2. The development parallels that of Section 3.

Theorem 4.1. *A measure μ on $[0,\infty)$ is concave, if and only if there is a measure τ on $(0,\infty)$ such that*

$$\mu[0,x] = a + bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds), \tag{21}$$

where $a, b \geq 0$ and τ satisfies the integrability condition

$$\int_{(0,\infty)} \left(1 \wedge \frac{1}{s}\right) \tau(ds) < \infty. \tag{22}$$

Proof. First, we show the sufficiency. If (21) and (22) hold, then the integral in (21) is convergent for all $x > 0$. Since the function $x \mapsto \min(1, x/s)$ is concave for all $s > 0$, the integral is concave, which implies that μ is concave measure.

Now, we show necessity. Since $F(x) := \mu[0,x]$ is non-decreasing and concave, the following limits of right derivatives exist in $[0,\infty]$:

$$b := \lim_{x\to\infty} F'_+(x) < \infty, \qquad \lim_{x\to 0+} F'_+(x) \leq \infty.$$

It is also clear that $a := \lim_{x\to 0+} F(x) = F(0) = \mu(\{0\}) \geq 0$ is finite. Define the Radon measure μ^* on $(0,\infty)$ by

$$\mu^*(p,q] = \begin{cases} F'_+(p) - F'_+(q) & \text{if } 0 < p < q < \infty, \\ \lim\limits_{x\to 0+} F'_+(x) - F'_+(q) & \text{if } 0 = p < q < \infty, \end{cases}$$

and note that for all $t > 0$, we have

$$\mu^*(t,\infty) = F'_+(t) - \lim_{x\to\infty} F'_+(x) = F'_+(t) - b.$$

Consider the measure τ defined on $(0,\infty)$ by $\tau(dt) = t\mu^*(dt)$. By (4), for $\epsilon > 0$,

$$F(x) = F(x) - F(\epsilon) + F(\epsilon) = \int_{(\epsilon,x)} F'_+(t)\, dt + F(\epsilon).$$

Letting ϵ approaching 0, and using Fubini's theorem, we obtain

$$\begin{aligned}
F(x) &= \int_{(0,x)} F'_+(t)\,dt + F(0) = a + bx + \int_{(0,x)} (F'_+(t) - b)\,dt \\
&= a + bx + \int_{(0,x)} \int_{(t,\infty)} \mu^*(ds)\,dt \\
&= a + bx + \int_{(0,x]} \int_{(0,s)} \frac{1}{s}\,dt\,\tau(ds) + \int_{(x,\infty)} \int_{(0,x)} \frac{1}{s}\,dt\,\tau(ds) \\
&= a + bx + \int_{(0,\infty)} \left(1 \wedge \frac{x}{s}\right) \tau(ds).
\end{aligned}$$

Representation (21) follows, while $F(1) < \infty$ implies the integrability condition (22). This completes the proof. $\square$

Theorem 4.2. *A measure ν on $(0,\infty)$ has convex tail, if and only if there is a measure τ on $(0,\infty)$ such that*

$$\nu(x,\infty) = \int_{(0,\infty)} \left(1 - \frac{x}{s}\right)^+ \tau(ds), \tag{23}$$

where τ satisfies the integrability condition $\tau(1,\infty) < \infty$.

Proof. We show sufficiency first. If (23) holds for measure ν with $\tau(1,\infty) < \infty$, then $\nu(x,\infty)$ is well defined and since the function $x \mapsto (1 - x/s)^+$ is convex for all $s > 0$, the integral is convex, implying that ν has a convex tail.

Now, we show the necessity. If measure ν has convex tail, then $\bar{\nu}(x) = \nu(x,\infty)$ is non-increasing, convex, and non-negative. The following limits exist in $[-\infty, 0]$:

$$\lim_{x\to\infty} \bar{\nu}(x) = 0, \qquad \lim_{x\to\infty} \bar{\nu}'_+(x) = 0, \qquad \lim_{x\to 0+} \bar{\nu}'_+(x) \geq -\infty.$$

Define the Radon measure μ^* on $(0,\infty)$ by

$$\mu^*(p,q] = \begin{cases} \bar{\nu}'_+(q) - \bar{\nu}'_+(p) & 0 < p < q < \infty, \\ \bar{\nu}'_+(q) - \lim\limits_{x\to 0+} \bar{\nu}'_+(x) & 0 = p < q < \infty, \end{cases}$$

and note that for all $t > 0$, we have

$$\mu^*(t,\infty) = \lim_{x\to\infty} \bar{\nu}'_+(x) - \bar{\nu}'_+(t) = -\bar{\nu}'_+(t).$$

Consider the measure τ defined on $(0,\infty)$ by $\tau(dt) = t\mu^*(dt)$. By (4), for $M > 0$,

$$\nu(x,\infty) = \bar{\nu}(x) - \bar{\nu}(M) + \bar{\nu}(M) = -\int_{(x,M)} \bar{\nu}'_+(t)\,dt + \bar{\nu}(M).$$

Letting M approaching infinity, and using Fubini's theorem, we obtain

$$\begin{aligned}\nu(x,\infty) &= -\int_{(x,\infty)} \bar{\nu}'_+(t)\,dt = \int_{(x,\infty)}\int_{(t,\infty)} \mu^*(ds)\,dt \\ &= \int_{(x,\infty)}\int_{(x,s)} dt\,\mu^*(ds) = \int_{(x,\infty)} \frac{s-x}{s}\,\tau(ds) \\ &= \int_{(0,\infty)} \left(1-\frac{x}{s}\right)^+ \tau(ds).\end{aligned}$$

The integrability condition $\tau(1,\infty) < \infty$ follows from the existence of $\bar{\nu}(x)$. Indeed,

$$\tau(1,\infty) = \int_{(1,\infty)} \tau(ds) \le \int_{(1,\infty)} \left(2-\frac{1}{s}\right) \tau(ds) \le 2\bar{\nu}(1/2) < \infty.$$

The proof is completed. $\square$

Corollary 4.3. *a) A measure μ is concave on $[0,\infty)$, if and only if*

$$\mu(dx) = a\delta_0(dx) + \left(b + \int_{(x,\infty)} \frac{1}{s}\,\tau(ds)\right) dx, \tag{24}$$

for some constant $a \ge 0$ and measure τ on $(0,\infty)$ satisfying (22), that is

$$\int_{(0,\infty)} \left(1 \wedge \frac{1}{s}\right) \tau(ds) < \infty.$$

In particular, every concave measure on $[0,\infty)$ is a Bernstein measure.

b) A measure ν has convex tail on $(0,\infty)$, if and only if

$$\nu(dx) = \left(\int_{(x,\infty)} \frac{1}{s}\,\tau(ds)\right) dx, \tag{25}$$

for some measure τ on $(0,\infty)$ satisfying

$$\tau(1,\infty) < \infty.$$

c) A Lévy measure ν has convex tail, if and only if (25) holds and the measure τ on $(0,\infty)$ is another Lévy measure, that is

$$\int_{(0,\infty)} (1 \wedge s)\,\tau(ds) < \infty. \tag{26}$$

Proof. *a)* Formula (24) follows immediately from the following representation

$$\mu[0,x] = a + bx + \int_{(0,x)}\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt,$$

which is inferred from the proof of Theorem 4.1. Here, τ satisfies condition (22). Consider the Laplace transform of such a measure. Using Fubini's theorem,

$$\begin{aligned}\int_{[0,\infty)} e^{-tx}\,\mu(dt) &= a + \int_{(0,\infty)} e^{-xt}\left(b + \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\right) dt\\ &= a + \frac{b}{x} + \int_{(0,\infty)}\int_{(t,\infty)} e^{-xt}\frac{1}{s}\,\tau(ds)\,dt\\ &= a + \frac{b}{x} + \int_{(0,\infty)}\int_{(0,s)} e^{-xt}\frac{1}{s}\,dt\,\tau(ds)\\ &= a + \frac{b}{x} + \frac{1}{x}\int_{(0,\infty)} \frac{1-e^{-xs}}{s}\,\tau(ds).\end{aligned}$$

The last integral is convergent, if (22) holds, as the following inequality shows

$$\frac{1-e^{-xs}}{s} \le \begin{cases} 1/s & \text{if } s \ge 1,\\ 1+x & \text{if } s \in (0,1).\end{cases}$$

Thus, a Bernstein measure is concave, if and only if it admits representation (24) with condition (22).

b) Formula (25) follows immediately from the following representation

$$\nu(x,\infty) = \int_{(x,\infty)}\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt,$$

inferred from the proof of Theorem 4.2. Here, measure $\tau(1,\infty) < \infty$.

c) Suppose now ν is Lévy measure with convex tail. By part *b*) we know that $\tau(1,\infty) < \infty$. Next, from (1) and (25), using Fubini's theorem, we obtain the integrability of the following integrals.

$$\begin{aligned}\infty > \int_{(0,1]} t\,\nu(dt) &= \int_{(0,1]} t\left(\int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\right) dt\\ &= \int_{(0,1]}\int_{(0,s)} \frac{t}{s}\,dt\,\tau(ds) + \int_{(1,\infty)}\int_{(0,1)} \frac{t}{s}\,dt\,\tau(ds)\\ &= \frac{1}{2}\int_{(0,1]} s\,\tau(ds) + \frac{1}{2}\int_{(1,\infty)} \frac{1}{s}\,\tau(ds),\end{aligned}$$

implies that

$$\int_{(0,1]} s\,\tau(ds) < \infty.$$

This shows that condition (26) holds.

Conversely, suppose ν has representation (25) while τ satisfies (26). By part *b*), ν has convex tail. Reversing the steps above shows that

$$\int_{(0,1]} t\,\nu(dt) < \infty.$$

Combining it with

$$\begin{aligned}\int_{(1,\infty)} \nu(dt) &= \int_{(1,\infty)} \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\,dt = \int_{(1,\infty)} \int_{(1,s)} \frac{1}{s}\,dt\,\tau(ds)\\ &= \int_{(1,\infty)} \left(1-\frac{1}{s}\right)\tau(ds) \le \int_{(1,\infty)} 1\,\tau(ds) < \infty.\end{aligned}$$

shows that ν is a Lévy measure. □

4.1. Applications to completely monotone and Bernstein functions

As an application of the above representations, we characterize completely monotone functions with concave measure and Bernstein functions having Lévy measure with convex tail. We first define the following kernel

$$r(x) := \frac{1-e^{-x}}{x}, \quad x > 0. \tag{27}$$

Below we list several properties of $r(x)$ that will be needed.

Lemma 4.4. *Function $r(x)$ defined by (27) has the following properties.*

(a) *It satisfies the inequalities $0 \le r(x) \le \min(1, 1/x)$.*

(b) *It is completely monotone with concave measure, more precisely*

$$r(x) = \int_{(0,1)} e^{-xt}\,dt. \tag{28}$$

(c) *The function $1 - r(x)$ is Bernstein, whose Lévy measure has convex tail, more precisely,*

$$1 - r(x) = \int_{(0,1)} (1 - e^{-xt})\,dt. \tag{29}$$

Proof. *a*) We only need to show that $r(x) \le 1/x$ and $r(x) \le 1$. The first inequality is clear by (27). The second inequality is equivalent to $1 - e^{-x} \le x$ for $x > 0$.

b) The Bernstein measure μ for $r(x)$ is concave as $\mu[0,x] = \min(x, 1)$.

c) The Lévy measure ν for $1 - r(x)$ is convex, as $\bar{\nu}(x) = (1 - x)^+$. □

Proposition 4.5. *Suppose $f(x)$ is completely monotone with measure μ. Then, μ is concave, if and only if there exists a unique measure τ on $(0,\infty)$ such that*

$$f(x) = a + \frac{b}{x} + \int_{(0,\infty)} r(xs)\,\tau(ds), \tag{30}$$

where $r(x)$ is defined in (27), $a, b \geq 0$ are constants and measure τ satisfies (22).

Proof. For the necessity, let $a := \mu(\{0\}) \geq 0$. By Corollary 4.3 part a), for all $x > 0$, we have

$$\begin{aligned} \int_{[0,\infty)} e^{-xt}\,\mu(dt) &= a + \int_{(0,\infty)} e^{-xt}\left(b + \int_{(t,\infty)} \frac{1}{s}\,\tau(ds)\right) dt \qquad (31)\\ &= a + \frac{b}{x} + \int_{(0,\infty)} \int_{(0,s)} e^{-xt}\frac{1}{s}\,dt\,\tau(ds) \\ &= a + \frac{b}{x} + \int_{(0,\infty)} \frac{1-e^{-xs}}{xs}\,\tau(ds) \\ &= a + \frac{b}{x} + \int_{(0,\infty)} r(xs)\,\tau(ds), \end{aligned}$$

where τ satisfies conditions (22). The uniqueness of measure τ follows the uniqueness of measure μ.

For the sufficiency, suppose (30) holds. By Lemma 4.4 part b), completely monotone function $r(x)$ has concave measure. Then, so is the measure of $r(xs)$ for all $s > 0$. Since $1/x$ is also completely monotone with concave measure, the proof follows. ☐

The proposition provides with an intrinsic characterization of completely monotone functions with concave measure.

Corollary 4.6. *Suppose $f(x)$ is completely monotone with measure μ. The following conditions are equivalent:*

(*1*) *μ is concave;*
(*2*) *$xf(x)$ is Bernstein function;*
(*3*) *$f(x) + xf'(x)$ is completely monotone function.*

Proof. If $f(x)$ has concave measure, by Proposition 4.5, we have

$$xf(x) = ax + b + \int_{(0,\infty)} (1-e^{-xs})\frac{1}{s}\,\tau(ds),$$

where $a, b \geq 0$ and τ satisfies (22). This shows that $xf(x)$ is Bernstein with Lévy measure $\tau(ds)/s$.

Conversely, if $xf(x)$ is Bernstein with triplet (a', b', η), we know

$$f(x) = \frac{a'}{x} + b' + \int_{(0,\infty)} \frac{1 - e^{-xs}}{x} \eta(ds) = \frac{a'}{x} + b' + \int_{(0,\infty)} r(xs)\, s\, \eta(ds),$$

where $a', b' \geq 0$ and η satisfies (1). Define measure τ on $(0, \infty)$ by $\tau(ds) = s\eta(ds)$. The integrability condition on η implies that τ satisfies (22). By Proposition 4.5, completely monotone function $f(x)$ has concave measure.

This shows the equivalence between (*1*) and (*2*). The equivalence between (*2*) and (*3*) is routine. □

Combining Corollary 4.3 part *a*), Proposition 4.5 and Proposition 1.3, we have the following corollary.

Corollary 4.7. *Suppose $f(x)$ is completely monotone with concave measure. It has a Bernstein primitive, if and only if* (30) *holds with $b = 0$ and τ satisfies*

$$\int_{(0,\infty)} \frac{\ln(s+1)}{s} \tau(ds) < \infty. \tag{32}$$

Proof. Observe that condition (32) implies condition (22), since

$$\frac{1}{2}\left(1 \wedge \frac{1}{s}\right) \leq \frac{\log(1+s)}{s} \quad \text{for } s > 0.$$

Let μ be the Bernstein measure of $f(x)$. Using Fubini's theorem, we have

$$\begin{aligned}
\int_{(0,\infty)} \frac{1}{1+t} \mu(dt) &= \int_{(0,\infty)} \frac{1}{1+t} \left(b + \int_{(t,\infty)} \frac{1}{s} \tau(ds)\right) dt \\
&= \int_{(0,\infty)} \frac{b}{1+t} dt + \int_{(0,\infty)} \int_{(0,s)} \frac{1}{(1+t)s} dt\, \tau(ds) \\
&= \int_{(0,\infty)} \frac{b}{1+t} dt + \int_{(0,\infty)} \frac{\ln(s+1)}{s} \tau(ds).
\end{aligned}$$

This shows that, condition (2) is equivalent to (32) with $b = 0$ in (30) □

Next proposition characterizes Bernstein functions, whose Lévy measure has convex tail.

Proposition 4.8. *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then, ν has convex tail, if and only if there exists a unique triplet (a, b, τ) such that*

$$g(x) = a + bx + \int_{(0,\infty)} (1 - r(xs))\, \tau(ds), \tag{33}$$

where $r(x)$ is defined in (27)*, $a, b \geq 0$ are constants, and τ is a Lévy measure.*

Proof. We show necessity first. If ν has convex tail, by Corollary 4.3 part c), for all $x > 0$, we have

$$\begin{aligned} g(x) &= a + bx + \int_{(0,\infty)} (1 - e^{-xt})\, \nu(dt) \\ &= a + bx + \int_{(0,\infty)} (1 - e^{-xt}) \int_{(t,\infty)} \frac{1}{s}\, \tau(ds)\, dt \\ &= a + bx + \int_{(0,\infty)} \int_{(0,s)} \frac{1 - e^{-xt}}{s}\, dt\, \tau(ds) \\ &= a + bx + \int_{(0,\infty)} \left(1 - \frac{1 - e^{-xs}}{xs}\right) \tau(ds) \\ &= a + bx + \int_{(0,\infty)} (1 - r(xs))\, \tau(ds), \end{aligned}$$

where τ is a Lévy measure. The uniqueness of the triplet (a, b, τ) follows from the uniqueness of the triplet (a, b, ν).

For the sufficiency, suppose (33) holds. By Lemma 4.4 part c) we have that $1 - r(x)$ is a Bernstein function having Lévy measure with convex tail. It is easy to see that $1 - r(xs)$ is a Bernstein function having Lévy measure with convex tail, for all $s > 0$. The proof follows. $\square$

Next proposition provides with an intrinsic characterization of Bernstein functions, whose Lévy measure has convex tail. Comparing it with Corollary 4.6, shows the parallel between completely monotone functions with concave measure and Bernstein functions, whose Lévy measure has convex tail.

Corollary 4.9. *Suppose $g(x)$ is Bernstein with Lévy measure ν. Then, ν has convex tail, if and only if $g(x) + xg'(x)$ is Bernstein.*

Proof. Suppose $g(x)$ has Lévy triplet (a, b, ν). If ν has convex tail, then using representation (33), we have that

$$\begin{aligned} g(x) + xg'(x) &= a + 2bx + \int_{(0,\infty)} (1 - r(xs) - xsr'(xs))\, \tau(ds) \\ &= a + 2bx + \int_{(0,\infty)} (1 - e^{-xs})\, \tau(ds) \end{aligned}$$

is a Bernstein function, since τ is a Lévy measure by Proposition 4.8.

Conversely, if $g(x) + xg'(x)$ is Bernstein with Lévy triplet (a', b', η), we know

$$g(x) + xg'(x) = a' + b'x + \int_{(0,\infty)} (1 - e^{-xt})\, \eta(dt).$$

Note that $g(x) + xg'(x) = (xg(x))'$ and $\lim_{x\to 0} xg(x) = 0$. Hence, we have

$$\begin{aligned} zg(z) &= \int_{(0,z)} \big(a' + b'x + \int_{(0,\infty)} (1 - e^{-xt})\, \eta(dt)\big)\, dx \\ &= a'z + \frac{b'}{2}z^2 + \int_{(0,\infty)} \int_{(0,z)} (1 - e^{-xt})\, dx\, \eta(dt) \\ &= a'z + \frac{b'}{2}z^2 + \int_{(0,\infty)} \left(z - \frac{1 - e^{-zt}}{t}\right) \eta(dt), \end{aligned}$$

and dividing both sides by z gives

$$g(x) = a' + \frac{b'}{2}z + \int_{(0,\infty)} (1 - r(xt))\, \eta(dt),$$

where $a', b' \geq 0$ and η is a Lévy measure. Proposition 4.8 shows, that ν has convex tail. □

5. Characterization of convex measures

In this section we characterize convex measure on $[0, \infty)$ in Theorem 5.1 and investigate the properties of the related completely monotone functions.

Theorem 5.1. *A measure μ on $[0, \infty)$ is convex, if and only if there is a measure τ on $(0, \infty)$ such that*

$$\mu[0, x] = a + bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^+ \tau(ds), \tag{34}$$

where $a, b \geq 0$ and τ satisfies the integrability condition

$$\int_{(0,1]} \frac{1}{s}\, \tau(ds) < \infty. \tag{35}$$

Proof. First, we show the sufficiency. Condition (35) implies the convergence of the integral in (34), for any $x > 0$. Since function $x \mapsto (x/s - 1)^+$ is convex for all $s > 0$, the integral is convex, which implies that μ is a convex measure.

Now, we show necessity. If μ is a convex measure, function $F(x) := \mu[0, x]$ is non-decreasing and convex, hence the following limits exist:

$$a := \lim_{x\to 0+} F(x) = \mu(\{0\}) < \infty, \qquad b := \lim_{x\to 0+} F'_+(x) < \infty.$$

Define the Radon measure μ^* on $(0, \infty)$ by

$$\mu^*(p, q] = \begin{cases} F'_+(q) - F'_+(p) & \text{if } 0 < p < q < \infty, \\ F'_+(q) - \lim\limits_{x\to 0+} F'_+(x) & \text{if } 0 = p < q < \infty, \end{cases}$$

and note that for any $t > 0$, we have

$$\mu^*(0,t] = F'_+(t) - \lim_{x\to 0+} F'_+(x) = F'_+(t) - b.$$

Consider the measure τ defined on $(0,\infty)$ by $\tau(dt) = t\mu^*(dt)$. By (4), for $\epsilon > 0$,

$$F(x) = F(x) - F(\epsilon) + F(\epsilon) = \int_{(\epsilon,x)} F'_+(t)\,dt + F(\epsilon).$$

Letting ϵ approaching 0, and using Fubini's theorem, we obtain

$$\begin{aligned} F(x) &= a + \int_{(0,x)} F'_+(t)\,dt = a + bx + \int_{(0,x)} (F'_+(t) - b)\,dt \\ &= a + bx + \int_{(0,x)} \int_{(0,t]} \mu^*(ds)\,dt = a + bx + \int_{(0,x)} \int_{[s,x)} \frac{1}{s}\,dt\,\tau(ds) \\ &= a + bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^+ \tau(ds). \end{aligned}$$

The integrability condition (35) follows from

$$\int_{(0,1]} \frac{1}{s}\,\tau(ds) \le \int_{(0,1]} \left(\frac{2}{s} - 1\right) \tau(ds) \le \int_{(0,\infty)} \left(\frac{2}{s} - 1\right)^+ \tau(ds) \le F(2) < \infty.$$

This closes the proof. □

Corollary 5.2. *a) A measure μ is convex on $[0,\infty)$, if and only if*

$$\mu(dx) = a\delta_0(dx) + \left(b + \int_{(0,x]} \frac{1}{s}\,\tau(ds)\right) dx, \tag{36}$$

for some constant $a, b \ge 0$ and measure τ on $(0,\infty)$ satisfying (35), that is

$$\int_{(0,1]} \frac{1}{s}\,\tau(ds) < \infty.$$

b) A Bernstein measure μ on is convex, if and only if (36) holds and the measure τ on $(0,\infty)$ satisfies

$$\int_{(0,\infty)} \frac{e^{-xs}}{s}\,\tau(ds) < \infty, \quad \textit{for all } x > 0. \tag{37}$$

Proof. *a)* Formula (36) follows immediately from the following representation

$$\mu[0,x] = a + bx + \int_{(0,x)} \int_{(0,t]} \frac{1}{s}\,\tau(ds)\,dt,$$

inferred from the proof of Theorem 5.1. Here, τ satisfies condition (35).

b) Suppose now Bernstein measure μ is convex. By (36), we have

$$\begin{aligned}\infty > \int_{[0,\infty)} e^{-xt}\,\mu(dt) &= a + \int_{(0,\infty)} e^{-xt}\left(b + \int_{(0,t]} \frac{1}{s}\,\tau(ds)\right) dt \\ &= a + \frac{b}{x} + \int_{(0,\infty)} \frac{1}{s} \int_{[s,\infty)} e^{-xt}\,dt\,\tau(ds) \\ &= a + \frac{b}{x} + \frac{1}{x}\int_{(0,\infty)} \frac{e^{-xs}}{s}\,\tau(ds).\end{aligned}$$

That is, τ satisfies (37).

Conversely, if a measure μ on $[0,\infty)$ has representation (36) with condition (37), then it is convex by part *a*). Its Laplace transform is well-defined for all $x > 0$ by by tracing the steps above backwards. That is, μ is a Bernstein measure. □

5.1. Applications to completely monotone and Bernstein functions

As an application of the above representation, we characterize completely monotone functions with convex measures. Define the kernel

$$l(x) := \frac{e^{-x}}{x}, \quad x > 0. \tag{38}$$

The necessary properties of function $l(x)$ are listed below.

Lemma 5.3. *Function $l(x)$ defined by (38) has the following properties.*

(*a*) *It satisfies the inequalities $0 \le l(x) \le 1/x$.*
(*b*) *It is completely monotone with convex measure, more precisely*

$$l(x) = \int_{(1,\infty)} e^{-xt}\,dt. \tag{39}$$

Proof. Part *a*) is trivial. For part *b*), note that the cumulative distribution function of the measure of $l(x)$ is $x \mapsto (x-1)^+$. □

We use the kernel $l(x)$ to construct the following representation.

Proposition 5.4. *Suppose $f(x)$ is completely monotone with measure μ. Then, μ is convex, if and only if there exists a unique measure τ on $(0,\infty)$ such that*

$$f(x) = a + \frac{b}{x} + \int_{(0,\infty)} l(xs)\,\tau(ds), \tag{40}$$

where $l(x)$ is defined in (38), $a, b \ge 0$ are constants, and τ satisfies (37).

Proof. For the necessity, let $a := \mu(\{0\}) \geq 0$. By Corollary 5.2 part b), for all $x > 0$, we have

$$\begin{aligned} f(x) &= \int_{[0,\infty)} e^{-xt}\,\mu(dt) = a + \int_{(0,\infty)} e^{-xt} \left(b + \int_{(0,t]} \frac{1}{s}\,\tau(ds) \right) dt \\ &= a + \frac{b}{x} + \int_{(0,\infty)} \int_{[s,\infty)} e^{-xt} \frac{1}{s}\, dt\, \tau(ds) = a + \frac{b}{x} + \int_{(0,\infty)} \frac{e^{-xs}}{xs}\, \tau(ds) \\ &= a + \frac{b}{x} + \int_{(0,\infty)} l(xs)\, \tau(ds), \end{aligned}$$

where τ satisfies conditions (37). The uniqueness of τ follows from the uniqueness of μ.

For the sufficiency, suppose (40) holds. By Lemma 5.3 part b) we know $l(x)$ is completely monotone with convex measure, then so is $l(sx)$ for all $s > 0$. Since $1/x$ is also completely monotone with convex measure, the proof follows. $\square$

Proposition 5.4 allows us to give an alternative proof of an intrinsic characterization of completely monotone functions with convex measure, see [9, Lemma 5.1] for a different proof, as well as [1, Proposition 2.7.13].

Corollary 5.5. *Suppose $f(x)$ is completely monotone with measure μ. Then, μ is convex, if and only if $xf(x) - x\mu(\{0\})$ is completely monotone.*

Proof. If μ is convex, then $f(x)$ has representation (40), with $a = \lim_{x\to\infty} f(x) = \mu(\{0\})$. Thus,

$$xf(x) = x\mu(\{0\}) + b + \int_{(0,\infty)} \frac{e^{-xs}}{s}\, \tau(ds),$$

where $b \geq 0$ and τ satisfies integrability condition (37). Hence, $xf(x) - x\mu(\{0\})$ is completely monotone with Bernstein measure $\tau(ds)/s$.

Conversely, if $xf(x) - x\mu(\{0\})$ is completely monotone, then

$$xf(x) = x\mu(\{0\}) + a' + \int_{(0,\infty)} e^{-xs}\, \eta(ds),$$

for some constant $a' \geq 0$ and Bernstein measure η. Letting $\tau(ds) = s\eta(ds)$ on $(0, \infty)$, we obtain

$$f(x) = \mu(\{0\}) + \frac{a'}{x} + \int_{(0,\infty)} l(xs)\, \tau(ds),$$

where τ satisfies (37). Proposition 5.4, completes the proof. $\square$

Combining Corollary 5.2 part b) with Proposition 1.3, we have the following corollary.

Corollary 5.6. *Suppose $f(x)$ is completely monotone with convex measure μ. It can not have a Bernstein primitive, unless $f(x)$ is a constant.*

Proof. We only have to verify the integrability condition (2) can not be satisfies unless μ vanishes on $(0,\infty)$. By (36), we have

$$\begin{aligned}\int_{(0,\infty)} \frac{1}{1+t}\,\mu(dt) &= \int_{(0,\infty)} \frac{1}{1+t}\left(b+\int_{(0,t]}\frac{1}{s}\,\tau(ds)\right)dt\\ &= \int_{(0,\infty)} \frac{b}{1+t}\,dt + \int_{(0,\infty)}\frac{1}{s}\int_{[s,\infty)}\frac{1}{1+t}\,dt\,\tau(ds).\end{aligned}$$

However, $1/(1+t)$ is not integrable on $[s,\infty)$ for any $s>0$. So, $f(x)$ can not have a Bernstein primitive, unless $b=0$ and τ vanishes on $(0,\infty)$. That implies that μ vanishes on $(0,\infty)$, or that $f(x)$ is a constant. □

In [10, Theorem 3.1, Lemma 4.1] the following hereditary properties are shown.

- If a completely monotone function $f(x)$ has (harmonically) convex measure, then $-f'(x)$ also has (harmonically) convex measure; and
- If a Bernstein function $g(x)$ has Lévy measure with harmonically concave tail, then $g'(x)$ has harmonically convex measure.

Alternatively, Corollary 5.6 says that the derivative of a Bernstein function $g(x)$, cannot have convex measure, unless $g(x)$ is affine. Curiously, we have a partial converse of the first bullet.

Proposition 5.7. *Suppose $f(x)$ is completely monotone. If $-f'(x)$ has convex measure, then $f(x)$ has harmonically convex measure.*

Proof. Note that the measure of $-f'(x)$ has no mass at 0. Then, by Corollary 5.5, $-xf'(x)$ is completely monotone, hence so is the sum $f(x)-xf'(x)$. Corollary 3.6 now shows that $f(x)$ has harmonically convex measure. □

6. Characterization of measures with harmonically convex tail

This section is devoted to the proof of Theorem 6.1, which characterize measure with harmonically convex tail. Then, Corollary 6.2 shows that there are no Bernstein functions, whose measure has harmonically convex tail.

Theorem 6.1. *A measure ν on $(0,\infty)$ has harmonically convex tail, if and only if there is a measure τ on $(0,\infty)$ such that*

$$\nu(x,\infty) = \frac{b}{x} + \int_{(0,\infty)} \left(\frac{1}{xs}-1\right)^{+} \tau(ds), \tag{41}$$

where $b\ge 0$ and τ satisfies the integrability condition

$$\int_{(0,1]} \frac{1}{s}\,\tau(ds) < \infty. \tag{42}$$

Proof. First, we show the sufficiency. If (41) and (42) hold, then

$$\nu(1/x,\infty) = bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^+ \tau(ds),$$

with the integral being convergent for all $x > 0$. Since the function $x \mapsto (x/s - 1)^+$ is convex, the integral on the right hand side of the above equation is convex, which implies that ν has harmonically convex tail.

Now, we show necessity. If ν has harmonically convex tail, then function $\varphi(x) := \nu(1/x,\infty)$ is non-negative, non-decreasing and convex. The following limits exist and are non-negative:

$$\lim_{x\to 0+} \varphi(x) = 0, \qquad b := \lim_{x\to 0+} \varphi'_+(x) < \infty.$$

Define the Radon measure μ^* on $(0,\infty)$ by

$$\mu^*(p,q] = \begin{cases} \varphi'_+(q) - \varphi'_+(p) & \text{if } 0 < p < q < \infty, \\ \varphi'_+(q) - \lim\limits_{x\to 0+} \varphi'_+(x) & \text{if } 0 = p < q < \infty, \end{cases}$$

and note that for any $t > 0$, we have

$$\mu^*(0,t] = \varphi'_+(t) - b.$$

Consider the measure τ defined on $(0,\infty)$ by $\tau(dt) = t\mu^*(dt)$. By (4), for $\epsilon > 0$,

$$\varphi(x) = \varphi(x) - \varphi(\epsilon) + \varphi(\epsilon) = \int_{(\epsilon,x)} \varphi'_+(t)\,dt + \varphi(\epsilon).$$

Letting ϵ approaching 0, and using Fubini's theorem, we obtain

$$\begin{aligned}\varphi(x) &= \int_{(0,x)} \varphi'_+(t)\,dt = bx + \int_{(0,x)} (\varphi'_+(t) - b)\,dt \\ &= bx + \int_{(0,x)} \int_{(0,t]} \mu^*(ds)\,dt = bx + \int_{(0,x)} \int_{[s,x)} \frac{1}{s}\,dt\,\tau(ds) \\ &= bx + \int_{(0,\infty)} \left(\frac{x}{s} - 1\right)^+ \tau(ds).\end{aligned}$$

Representation (41) follows. The integrability condition (42) holds, because

$$\int_{(0,1]} \frac{1}{s}\,\tau(ds) \le \int_{(0,1]} \left(\frac{2}{s} - 1\right) \tau(ds) \le \int_{(0,\infty)} \left(\frac{2}{s} - 1\right)^+ \tau(ds) \le \varphi(2) < \infty.$$

The proof is completed. □

Corollary 6.2. *a) A measure ν on $(0,\infty)$ is convex, if and only if*

$$\nu(dx) = \frac{1}{x^2}\left(b + \int_{(0,1/x]} \frac{1}{s}\,\tau(ds)\right) dx, \tag{43}$$

for some constant $b \geq 0$ and measure τ satisfying (42).

b) A Lévy measure ν can not have harmonically convex tail, unless ν vanishes on $(0,\infty)$.

Proof. *a*) Formula (43) follows immediately from the following representation

$$\nu(x,\infty) = \frac{b}{x} + \int_{(0,1/x)} \int_{(0,t]} \frac{1}{s}\,\tau(ds)\,dt,$$

inferred from the proof of Theorem 6.1, and τ satisfies (42).

b) If Lévy measure ν has harmonically convex tail, representation (43) holds. We verify that integrability condition (1) does not hold for ν:

$$\begin{aligned}
\int_{(0,1]} t\,\nu(dt) &= \int_{(0,1]} \frac{1}{t}\left(b + \int_{(0,1/t]} \frac{1}{s}\,\tau(ds)\right) dt \\
&= \int_{(0,1]} \frac{b}{t}\,dt + \int_{(0,1]} \frac{1}{t} \int_{(0,1/t]} \frac{1}{s}\,\tau(ds)\,dt \\
&= \int_{(0,1]} \frac{b}{t}\,dt + \int_{(1,\infty)} \frac{1}{s} \int_{(0,1/s]} \frac{1}{t}\,dt\,\tau(dt) + \int_{(0,1]} \frac{1}{s} \int_{(0,1]} \frac{1}{t}\,dt\,\tau(dt).
\end{aligned}$$

However, $1/t$ is not integrable over either $(0,1]$ or $(0,1/s]$ for any $s > 1$. Hence, the above expression is finite, only if $b = 0$ and τ vanishes on $(0,\infty)$. That implies ν vanishes on $(0,\infty)$. □

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