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Control Sweeping Processes

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Dedicated to the memory of Jean Jacques Moreau.

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A control system with internal and external controls is considered in a finite dimensional space. The internal control is a multivalued function of time with convex closed possibly unbounded values, while the external control is represented by a measurable function of time. The internal control acts on the system by normal cones at the points of control values entering the right-hand side of the system. The external control enters the right-hand side of the system as a perturbation. The admissible internal controls are taken from a relatively compact set in the corresponding metric space, whereas the constraint for the external control is a multivalued mapping with closed nonconvex values. Along with the original system we consider an extended system in which the internal control is taken from the closure of the set of admissible controls of the original system, and the external control constraint is the convexified constraint of the original system. Minimization problems for integral functionals over the solutions of the original and extended systems with a nonconvex and the convexified in the exterior control variable integrand are considered. The integrand depends on the phase variable, the internal and external controls. Existence of solutions and relationships between solution sets of the original and extended systems and solutions of optimization problems are investigated.

Keywords: Sweeping process, evolution inclusion, perturbation, normal cone, subdifferential

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1. Statement of the problem

Let $T = [0, 1]$ be the interval of the real line $\mathbb{R}$ with the Lebesgue measure, $\overline{\mathbb{R}} = (-\infty, +\infty]$. Let E be an n-dimensional Euclidean space with the norm $\|\cdot\|$ and the scalar product $\langle\cdot,\cdot\rangle$, Y be an m-dimensional space with the norm $\|\cdot\|_Y$. Let $c(E)$ be the family of all nonempty convex closed sets from E, $d(x, A)$ be the distance form a point x to a set $A \subset E$ and $\{y_i;\ i \geq 1\} \subset E$ be a countable dense set.

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For elements $A, B \in c(E)$ define the distance between them:

$$\rho(A,B) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|d(y_i,A) - d(y_i,B)|}{1 + |d(y_i,A) - d(y_i,B)|}. \tag{1}$$

The function $\rho(\cdot,\cdot)$ defined by (1) is a metric on the space $c(E)$. In the sequel we assume that the space $c(E)$ is endowed with the metric $\rho(\cdot,\cdot)$.

By $C(T, c(E))$ we denote the space of all continuous mappings from T to $c(E)$ with the topology of uniform convergence on T.

Let H be a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_H$. A function $\varphi : H \to \overline{\mathbb{R}}$ is called proper if its effective domain $\operatorname{dom}\varphi = \{x \in H; \varphi(x) < +\infty\}$ is nonempty. The class of all proper, convex and lower semicontinuous functions $\varphi : H \to \overline{\mathbb{R}}$ is denoted by $\Gamma_0(H)$. For a function $\varphi \in \Gamma_0(H)$ and a point $x \in H$, the set

$$\partial\varphi(x) = \{v \in H; \langle v, y - x\rangle_H \le \varphi(y) - \varphi(x),\ \forall y \in H\}$$

is called the subdifferential of the function φ at the point x. It is known that $\partial\varphi(x)$ is a maximal monotone operator, $\operatorname{dom}\partial\varphi = \{x \in H; \partial\varphi(x) \ne \emptyset\} \subset \operatorname{dom}\varphi$, and $\overline{\operatorname{dom}\partial\varphi} = \overline{\operatorname{dom}\varphi}$, where the bar stands for the closure in H.

For a convex closed set $C \subset H$ denote by I_C the indicator function of the set C, i.e. $I_C(x) = 0$ if $x \in C$, $I_C(x) = +\infty$ if $x \notin C$. Then, $I_C \in \Gamma_0(H)$, $\operatorname{dom} I_C = \operatorname{dom}\partial I_C = C$, and the subdifferential $\partial I_C(x)$ at a point $x \in C$ is called the normal cone [2] to the set C at the point x and is denoted by $\mathcal{N}_C(x)$.

Let $G \subset C(T, c(E))$ be a relatively compact set, $\overline{G}$ be its closure and $x_0 : \overline{G} \to E$ be a continuous function, $x_0(C(\cdot)) \in C(0)$, $C(\cdot) \in \overline{G}$.

Consider the control system

$$-\dot{x}(t) \in \mathcal{N}_{C(t)}(x(t)) + F(t, x(t)) + B(t, x(t))u(t), \tag{2}$$

$$x(0) = x_0(C(\cdot)), \tag{3}$$

with the constraints

$$u(t) \in U(t, x(t)), \tag{4}$$

$$C(\cdot) \in G. \tag{5}$$

Along with the constraints (4), (5) consider the constraints

$$u(t) \in \overline{\operatorname{co}}\, U(t, x(t)), \tag{6}$$

$$C(\cdot) \in \overline{G}, \tag{7}$$

where the symbol $\overline{\operatorname{co}}$ denotes the closed convex hull. Here, $F : T \times Q \to E$ is a nonlinear mapping, $B : T \times Q \to \mathcal{L}(Y, E)$, where $\mathcal{L}(Y, E)$ is the space of continuous linear operators (the space of $n \times m$-matrices) from Y to E, $U : T \times Q \to Y$ is a multivalued mapping with closed nonconvex values, $Q = \overline{\operatorname{co}}\,\{\bigcup C(t);\ t \in T,\ C(\cdot) \in \overline{G}\}$.

By a solution of system (2), (3) with constraints (4), (5) we mean a triple $(C(\cdot), x(\cdot), u(\cdot))$, $x(\cdot) \in W^{1,2}(T, E)$, $C(\cdot) \in C(T, c(E))$, $u(\cdot) \in L^2(T, Y)$, such that $x(t) \in \operatorname{dom} \mathcal{N}_{C(t)}$ a.e., $x(0) = x_0(C(\cdot))$, inclusions (2), (4) hold a.e. and (5) takes place. Solutions of system (2), (3) with constraints (6), (7) are defined similarly.

In the sequel elements $C(\cdot), x(\cdot), u(\cdot)$ of the spaces $C(T, c(E))$, $C(T, E)$, $L^2(T, Y)$, if no ambiguity can arise, will be denoted for convenience as C, x and u.

We denote by $\mathcal{R}_{G,U}$ and $\mathcal{R}_{\overline{G},\overline{\text{co}}\, U}$ the sets of all solutions of system (2), (3) with the constraints (4), (5) and (6), (7), respectively. For a fixed $C \in \overline{G}$ the set of all solutions (x, u) of system (2), (3) with the constraints (4) and (6) are denoted by $\mathcal{R}_U(C)$ and $\mathcal{R}_{\overline{\text{co}}\, U}(C)$. If (C, x, u) is a solution of one of the considered systems, then C is called an internal control, u an external control, and x a trajectory. From the definition of solutions it follows that $x(t) \in \operatorname{dom} \mathcal{N}_{C(t)}$ a.e. Since $\operatorname{dom} \mathcal{N}_{C(t)} = C(t)$, we have $x(t) \in C(t)$, $t \in T$. Hence, when an internal control is fixed, we control the trajectories with the help of external controls, and we control the family of trajectories generated by an external control with the help of internal controls.

For a scalar function $g : T \times c(Q) \times Q \times Y \to \mathbb{R}$ consider the function

$$g_U(t, C, x, u) = \begin{cases} g(t, C, x, u), & u \in U(t, x), \\ +\infty, & u \notin U(t, x). \end{cases}$$

Here, $c(Q)$ is the family of all nonempty closed convex sets from Q. Let $g_U^{**}(t, C, x, u)$ be the bipolar (the second conjugate [5]) of the function $u \mapsto g_U(t, C, x, u)$.

For a fixed $C \in \overline{G}$ consider the problems

$$J(C, x, u) = \int_T g(t, C(t), x(t), u(t))\, dt \to \inf \qquad P(C)$$

over the set $\mathcal{R}_U(C)$ and

$$J^{**}(C, x, u) = \int_T g_U^{**}(t, C(t), x(t), u(t))\, dt \to \inf \qquad RP(C)$$

over the set $\mathcal{R}_{\overline{\text{co}}\, U}(C)$. Along with Problems $P(C)$ and $RP(C)$ we consider the problems

$$J(C, x, u) \to \inf \qquad (P)$$

over the set $\mathcal{R}_{G,U}$ and

$$J^{**}(C, x, u) \to \inf \qquad (RP)$$

over the set $\mathcal{R}_{\overline{G},\overline{\text{co}}\, U}$.

The main purpose of this work is to prove the existence of solutions of Problems $RP(C)$, $C \in \overline{G}$, and (RP) and to establish relationships between Problems $RP(C)$, $P(C)$ and also between Problems (RP) and (P).

The sweeping process was introduced in the work [10] as a solution of the differential equation

$$-\dot{x}(t) \in \mathcal{N}_{C(t)}(x(t)), \tag{8}$$

where $C(t)$ is a multivalued mapping with closed convex values. After that sweeping processes have attracted a lot of interest. Both single-valued and multivalued perturbations have been added to the right-hand side of (8) and the values of $C(t)$ have been allowed to be nonconvex. In [10] and the subsequent works the mapping $C(t)$ is assumed to have a finite retraction (this term was introduced in [10]). However, there exists a wide class of multivalued mappings $C(t)$ which do not have finite retraction. The mappings whose values are hyperplanes, half-spaces, etc. belong to this class. The most general results on existence of solutions for sweeping processes in a Hilbert space with a convex valued mapping $C(t)$ not having finite retraction, and with a perturbation whose values are nonconvex closed unbounded sets are obtained in [14]. Control sweeping processes of the type (8) in a finite dimensional space were studied in the works [4, 3]. The values of $C(t)$ in [4] are half-spaces and in [3] are polyhedra. The controls are functional parameters with the help of which the half-spaces and polyhedra are described. The main focus in [4, 3] is to derive necessary optimality conditions for a problem of minimization of an integral functional.

In this paper we address the same problems as those considered in [13] for control system (2), (3) in a Hilbert space with constraints (4), (5) and (6), (7) and with internal controls $C(t)$ whose values are convex compact sets. In the present work we extend the class of admissible internal controls which may not have finite retraction and which may admit closed convex unbounded values.

The work consists of five sections. In the first section we formulate the problems we consider. The second section gives main notations, definitions and auxiliary results. In the third section the control sweeping process is reduced to a control system with a functional parameter and some results on control systems depending on a parameter are stated. The fourth section is devoted to the proof of the main results. In the last fifth section we give examples of compact sets of admissible internal controls.

2. Main definitions, notations and auxiliary results

Let Θ be the zero element of the space E, $cb(Y)$ be the family of all nonempty closed bounded sets from Y with the Hausdorff metric $\mathrm{Dist}(\cdot,\cdot)$.

A point x belongs to the lower limit $\mathrm{Li}_{n\to\infty} A_n$ of a sequence of sets $A_n \subset E$, $n \geq 1$, if any neighborhood of the point x intersects with all sets A_n starting from some number [8].

A point x belongs to the upper limit $\mathrm{Ls}_{n\to\infty} A_n$ of a sequence of sets $A_n \subset E$, $n \geq 1$, if any neighborhood of the point x intersects with infinitely many sets A_n [8].

A sequence of sets $A_n \subset E$, $n \geq 1$, is called convergent to a set $A = \mathrm{Lim}_{n\to\infty} A_n$, if $\mathrm{Li}_{n\to\infty} A_n = A = \mathrm{Ls}_{n\to\infty} A_n$ [8]. The convergence of a sequence A_n, $n \geq 1$, in this sense is usually called the Kuratowski convergence.

Let Z and X be topological spaces. By a multivalued mapping $\Gamma : Z \to X$ me mean a mapping whose values are nonempty subsets from X.

A multivalued mapping $\Gamma : Z \to X$ is called Vietoris lower semicontinuous if for any open set $V \subset X$ the set $\Gamma^{-1}(V) = \{z \in Z;\ \Gamma(z) \cap V \neq \emptyset\}$ is open. If Z and X are metric spaces, then a multivalued mapping $\Gamma : Z \to X$ is Vietoris lower semicontinuous if and only if for any $z \in Z$, $x \in \Gamma(z)$, and any sequence $z_n \to z$, there exists a sequence $x_n \in \Gamma(z_n)$, $n \geq 1$, converging to x.

A multivalued mapping $\Gamma : Z \to X$ is called Vietoris upper semicontinuous if for any open set $V \subset X$ the set $\Gamma^{+1}(V) = \{z \in Z;\ \Gamma(z) \subset V\}$ is open. A multivalued mapping $\Gamma : Z \to X$ with closed values is called Vietoris continuous if it is simultaneously Vietoris lower and upper semicontinuous.

A multivalued mapping $\Gamma : T \to X$ is called measurable [7] if the set $\Gamma^{-1}(V) = \{t \in T;\ \Gamma(t) \cap V \neq \emptyset\}$ is measurable for any closed set $V \subset X$.

If X is a Banach space, then by ω-X we denote the space X with the weak topology. We use the same notation for subsets of X with the topology induced by that of the space ω-X.

On the space $L^p(T, E)$, $1 \leq p < \infty$, along with the standard norm we consider the so-called weak norm

$$\|f\|_\omega := \sup_{0 \leq t \leq t' \leq 1} \left\| \int_t^{t'} f(s)\, ds \right\|. \tag{9}$$

The space $L^p(T, E)$ with the norm (9) we will denote by $L^p_\omega(T, E)$. The relation between the topologies of the spaces ω-$L^p(T, E)$ and $L^p_\omega(T, E)$ is established by the following lemma.

Lemma 2.1 ([6]). *Let $l(\cdot) \in L^p(T, \mathbb{R}^+)$, $1 \leq p < \infty$, and*

$$S(l) = \{f \in L^p(T, E);\ \|f(t)\| \leq l(t)\ \textit{a.e.}\}.$$

Then, the topologies of the spaces ω-$L^p(T, E)$ and $L^p_\omega(T, E)$ coincide on the set $S(l)$.

Theorem 2.2 ([11]). *The following statements are true:*

(*1*) *a sequence $A_n \in c(E)$, $n \geq 1$, converges to $A \in c(E)$ in the space $c(E)$ if and only if the sequence A_n, $n \geq 1$, converges to A in the Kuratowski sense;*

(*2*) *a set $\mathcal{H} \subset c(E)$ is relatively compact if and only if*

$$\sup\{d(\Theta, A);\ A \in \mathcal{H}\} < \infty. \tag{10}$$

Corollary 2.3. *A set $\mathcal{H} \subset c(E)$ is compact if and only if inequality* (10) *holds and for any $i \geq 1$ the set*

$$\{r_i \in \mathbb{R};\ r_i = d(y_i, A),\ A \in \mathcal{H}\}$$

is closed.

Recall that $\{y_i;\ i \geq 1\}$ is a countable dense set in E. Since on every equicontinuous family of mappings from E to $\mathbb{R}$ the topology of pointwise convergence on E coincides with the topology of pointwise convergence on a countable dense set, we have that $A_n \in c(E)$, $n \geq 1$, converges to A in $c(E)$ if and only if $d(x, A_n) \to d(x, A)$, $x \in E$. Statement (*2*) of Theorem 2.2 and Corollary 2.3 essentially are a reformulation of the classical Arzela-Ascoli theorem.

Note that in a finite dimensional space E the convergence of a sequence $A_n \in c(E)$, $n \geq 1$, to $A \in c(E)$ in the Kuratowski sense coincides with the Mosco convergence which in turn is equivalent to the Mosco convergence of the sequence I_{A_n}, $n \geq 1$, of the indicator functions of the sets A_n, $n \geq 1$, to the indicator function I_A of the set A [1].

Denote by $C(T, E)$ and $C(T, c(E))$ the spaces of all continuous mappings from T to E and $c(E)$ with the topology of uniform convergence on T. Note that according to the definition of the metric (1) a mapping $C : T \to c(E)$ is continuous if and only if for any y_i, $i \geq 1$, the function $t \to d(y_i, C(t))$ is continuous.

Theorem 2.4 ([11]). *A set $G \subset C(T, c(E))$ is relatively compact if and only if:*

(*1*) *for any $i \geq 1$ the family $\mathcal{H}_i$ of scalar functions*

$$\mathcal{H}_i = \{t \to d(y_i, C(t));\ C(\cdot) \in G\} \tag{11}$$

is equicontinuous;

(*2*)

$$\sup\{d(\Theta, C(0));\ C(\cdot) \in G\} < \infty. \tag{12}$$

Corollary 2.5. *A set $G \subset C(T, c(E))$ is compact if and only if it is relatively compact and for any $i \geq 1$, $t \in T$ the set*

$$\mathcal{H}_i(t) = \{r_i(t) \in \mathbb{R};\ r_i(t) = d(y_i, C(t));\ C(\cdot) \in G\} \tag{13}$$

is closed.

Theorem 2.4 and Corollary 2.5 are analogues of the classical Arzela-Ascoli theorem.

If we take an arbitrary element $C \in C(T, c(E))$, then the sweeping process may not have solutions. Therefore, we need to characterize the sets of admissible internal controls which are not only relatively compact in the space $C(T, c(E))$, but whose elements guarantees the existence of solutions for the sweeping process. For this we need several definitions.

Let $W^{1,2}(T)$ be the space of absolutely continuous functions $a : T \to \mathbb{R}$ whose derivatives $\dot{a}(\cdot)$ belongs to the space $L^2(T)$ with the norm

$$\|a(\cdot)\|_W = |a(0)| + \left(\int_T |\dot{a}(s)|^2 ds\right)^{\frac{1}{2}}.$$

Definition 2.6. A family G of mappings $C : T \to \ c(E)$ is called $r-$equicontinuous if there exists a set $\mathcal{A} \subset W^{1,2}(T)$ such that for any $C \in G$ and any $r \geq 0$ there exists an element $a_r \in \mathcal{A}$, denoted by $a_r(C)$, such that

$$|d(x, C(t)) - d(x, C(s))| \leq |a_r(C)(t) - a_r(C)(s)| \tag{14}$$

for all $x \in E$, $\|x\| \leq r$, $s, t \in T$ and the set $\{a_r(C);\ C \in G\}$ is bounded in $W^{1,2}(T)$ for any $r \geq 0$.

Definition 2.7. A family G of mappings $C : T \to \ c(E)$ is called weakly $r-$equicontinuous if there exists a set $\mathcal{A} \subset W^{1,2}(T)$ such that for any $C \in G$ and any $r \geq 0$ there exists an element $a_r(C) \in \mathcal{A}$ such that for any $s, t \in T$, $x \in C(s)$, $\|x\| \leq r$, there exists an element $y \in C(t)$ satisfying the inequality

$$\|x - y\| \leq |a_r(C)(t) - a_r(C)(s)| \tag{15}$$

and the set $\{a_r(C);\ C \in G\}$ is bounded in $W^{1,2}(T)$ for any $r \geq 0$.

It is evident that if a family G of mappings from T to $c(E)$ is $r-$equicontinuous, then $G \subset C(T, c(E))$.

Lemma 2.8. *If a family G of mappings $C : T \to \ c(E)$ is $r-$equicontinuous, then it is weakly $r-$equicontinuous. If a family G is weakly $r-$equicontinuous and inequality* (12) *holds, then it is $r-$equicontinuous.*

Proof. If a family G of mappings from T to $c(E)$ is $r-$equicontinuous, $C \in G$, and $t, s \in T$, $x \in C(s)$, $\|x\| \leq r$, then inequality (15) follows from inequality (14). Hence, the family G is weakly $r-$equicontinuous.

To prove the inverse, let G be weakly $r-$equicontinuous and inequality (12) holds. Take $C_* \in G$. According to (12) we have

$$d(\Theta, C_*(0)) \leq \sup\{d(\Theta, C(0));\ C \in G\} \leq l.$$

Consequently, there exists a point $z \in C_*(0)$ such that $\|z\| \leq l$ and a point $y(t) \in C_*(t)$ such that

$$\|z - y(t)\| \leq |a_l(C_*)(0) - a_l(C_*)(t)|, \quad t \in T.$$

Hence,

$$\|y(t)\| \leq l + \int_0^t |\dot{a}_l(C_*)(\tau)| d\tau. \tag{16}$$

Since the set $\{a_l(C);\ C \in G\}$ is bounded in $W^{1,2}(T)$, the set $\{\dot{a}_l(C);\ C \in G\}$ is bounded in $L^2(T)$. Hence, from (16) it follows that there exists a number $M > 0$ independent of C_* such that

$$\|y(t)\| \leq M, \quad t \in T.$$

Since $C_* \in G$ is arbitrary, we have

$$d(\Theta, C(t)) \leq M, \quad t \in T,\ C \in G. \tag{17}$$

Let

$$a_r^*(C)(t) = a_{2r+M}(C)(t), \quad r \geq 0,\ C \in G,$$

and

$$\mathcal{A}^* = \{a_r^*(C);\ C \in G,\ r \geq 0\}.$$

Then, $\mathcal{A}^* \subset \mathcal{A}$.

Let $x \in E$, $\|x\| \leq r$, $s \in T$, $C \in G$ be arbitrary. Then, there exists a point $y \in C(s)$ such that

$$\|x - y\| = d(x, C(s)) \leq d(\Theta, C(s)) + r. \tag{18}$$

From (17), (18) it follows that

$$\|y\| \leq 2r + M.$$

From this inequality and (15) it follows that for any $t \in T$ there is a point $z(t) \in C(t)$ satisfying the inequality

$$\begin{aligned}\|y - z(t)\| &\leq |a_{2r+M}(C)(t) - a_{2r+M}(C)(s)| \\ &= |a_r^*(C)(t) - a_r^*(C)(s)|.\end{aligned} \tag{19}$$

Using (18), (19) we obtain

$$\begin{aligned}d(x, C(t)) &\leq \|x - y\| + \|y - z(t)\| \\ &\leq d(x, C(s)) + |a_r^*(C)(t) - a_r^*(C)(s)|.\end{aligned} \tag{20}$$

Now, inequality (14) follows from (20) and the symmetry between s and t. The lemma follows. □

Corollary 2.9. *A weakly $r-$equicontinuous family G of mappings from T to $c(E)$ satisfying inequality* (12) *is a relatively compact subset of the space $C(T, c(E))$.*

The corollary follows from Lemma 2.1 and Theorem 2.4, since for an $r-$equicontinuous family G the family of scalar functions $\mathcal{H}_i$, $i \geq 1$, defined by equality (11) is equicontinuous.

Lemma 2.10. *The closure of an $r-$equicontinuous family G of mappings from T to $c(E)$ in the space $C(T, c(E))$ is $r-$equicontinuous.*

Proof. Let $\overline{\mathcal{A}}$ be the closure of a set $\mathcal{A}$ in the weak topology of the space $W^{1,2}(T)$. Let $C_n \in G$, $n \geq 1$, be a sequence converging to C in the space $C(T, c(E))$ and $x \in E$, $\|x\| \leq r$, $r \geq 0$. Since the set $\{a_r(C_n);\ n \geq 1\}$ is bounded in the space $W^{1,2}(T)$ and the space $W^{1,2}(T)$ is separable and reflexive, there exists a subsequence $a_r(C_{n_k})$, $k \geq 1$, of the sequence $a_r(C_n)$, $n \geq 1$, converging weakly in $W^{1,2}(T)$ to some function which we denote by $a_r(C) \in \overline{\mathcal{A}}$. The convergence of the sequence $a_r(C_{n_k})$, $k \geq 1$, in the weak topology of the space $W^{1,2}(T)$ to $a_r(C)$ implies the convergence of $a_r(C_{n_k})$, $k \geq 1$, to $a_r(C)$ in the space $C(T, \mathbb{R})$. Passing to the limit as $k \to \infty$ in the inequality

$$|d(x, C_{n_k}(t)) - d(x, C_{n_k}(s)) \leq |a_r(C_{n_k})(t) - a_r(C_{n_k})(s)|$$

we obtain the inequality

$$|d(x, C(t)) - d(x, C(s)) \leq |a_r(C)(t) - a_r(C)(s)|.$$

Since $a_r(C) \in \overline{\mathcal{A}}$, when taking the closure of an $r-$equicontinuous family $G \subset C(T, c(E))$ instead of the family $\mathcal{A} \subset W^{1,2}(T)$ we can take the family $\overline{\mathcal{A}} \subset W^{1,2}(T)$. The lemma follows. □

Corollary 2.11. *Let G be (weakly) $r-$equicontinuous family G of mappings from T to $c(E)$ satisfying inequality* (12). *Then, the closure of the family G in the space $C(T, c(E))$ is compact.*

The corollary follows from Theorem 2.4, Corollary 2.9 and Lemma 2.8.

3. Control sweeping process as a system with functional parameter

In this section we reduce system (2), (3) to a system with a functional parameter in which the internal control plays the role of the parameter. This allows us to use results of the works [15, 12] on control systems depending on a parameter.

We make the following assumptions:

Hypothesis $H_S(G)$. The set $G \subset C(T, c(E))$ is $r-$equicontinuous and satisfies inequality (12).

Denote by Q the set

$$Q = \overline{\text{co}}\,\{x \in E;\ x \in C(t),\ t \in T,\ C \in G\}.$$

It is evident that

$$\overline{\text{co}}\,\{x \in E;\ x \in C(t),\ t \in T,\ C \in \overline{G}\} = Q,$$

where $\overline{G}$ denotes the closure in the space $C(T, c(E))$.

Hypothesis $H_S(F)$. The mapping $F : T \times Q \to E$ has the properties:

(1) the function $t \to F(t, x)$ is measurable;
(2) the function $x \to F(t, x)$ is continuous for a.e. $t \in T$;
(3) there exists a function $\omega \in L^2(T, \mathbb{R}^+)$ such that

$$\langle F(t, x) - F(t, y), x - y \rangle + \omega(t)\|x - y\|^2 \geq 0$$

a.e., $x, y \in Q$;
(4) for every $0 < \eta \leq 1$ there exists a function $L_\eta : [0, +\infty) \to [0, +\infty)$, $L_{\eta_1}(r_1) \leq L_{\eta_2}(r_2)$, $\eta_1 \leq \eta_2$, $r_1 \leq r_2$, $r_1, r_2 \in [0, \infty)$, such that

$$\|F(t, x)\| \leq \eta \| \left(\mathcal{N}_{C(t)}(x)\right)^0 \| + L_\eta(\|x\|),$$

$C \in \overline{G}$, $x \in Q$, where $(\mathcal{N}_{C(t)}(x))^0$ is the element of minimal norm of the set $\mathcal{N}_{C(t)}(x)$.

Hypothesis $H_S(B)$. The matrix $B(t, x)$ with the elements $b_{ij}(t, x)$, $1 \leq i \leq n$, $1 \leq j \leq m$ has the properties:

(1) for any i, j the functions $t \to b_{ij}(t, x)$, $x \in E$, are measurable;
(2) there exists a function $l(\cdot) \in L^2(T, \mathbb{R}^+)$ such that for any i, j

$$|b_{ij}(t, x) - b_{ij}(t, y)| \leq l(t)\|x - y\| \quad \text{a.e., } x, y \in E;$$

(3) for any i, j

$$|b_{ij}(t, x)| \leq c + d\|x\| \quad \text{a.e., } x \in E, \ c, d > 0.$$

Hypothesis $H_S(U)$. The mapping $U : T \times Q \to cb(Y)$ has the properties:

(1) the mapping $t \to U(t, x)$, $x \in Q$, is measurable;
(2) there exists a function $\kappa \in L^2(T, \mathbb{R}^+)$ such that

$$\text{Dist}(U(t, x), U(t, y)) \leq \kappa(t)\|x - y\| \quad \text{a.e., } x, y \in Q;$$

(3)
$$\|U(t, x)\|_Y = \sup\{\|u\|_Y; \ u \in U(t, x)\} \leq m \quad \text{a.e., } x, y \in Q. \tag{21}$$

Let
$$S_Y(m) = \{u \in Y; \ \|u\|_Y \leq m\}$$
and $c(Q)$ be the family of all nonempty convex closed subsets from Q with the metrics (1).

Hypothesis $H_S(g)$. The function $g : T \times c(Q) \times Q \times S_Y(m) \to \mathbb{R}$ has the following properties:

(1) the function $t \to g(t, C, x, u)$ is measurable;
(2) the function $(C, x, u) \to g(t, C, x, u)$ is continuous;

(3)

$$\|g(t, C, x, u)\| \le a(t)(1 + \|x\|) + b(t)d(\Theta, C) \tag{22}$$

a.e., $C \in c(Q)$, $x \in Q$, $u \in S_Y(m)$, $a(\cdot), b(\cdot) \in L^1(T, \mathbb{R}^+)$.

We reduce control system (2), (3) with constraints (4), (6) for the internal and external controls to a control system depending on a functional parameter $C(\cdot)$ from the compact metric space $\overline{G}$. Consider the family of functions $\varphi^t(C) \in \Gamma_0(E)$, $t \in T$, $C \in \overline{G}$, defined by the rule

$$\varphi^t(C) = I_{C(t)}, \tag{23}$$

where $I_{C(t)}$ is the indicator function of the set $C(t)$.

If a sequence $C_n \in \overline{G}$, $n \ge 1$, converges to C, then the sequence $C_n(0)$, $n \ge 1$, converges to $C(0)$ in the space $c(E)$. Since the convergence of a sequence of sets in the space $c(E)$ coincides with the Kuratowski convergence, the mapping $\Gamma : \overline{G} \to c(E)$, $\Gamma(C) = C(0)$, $C \in \overline{G}$ is Vietoris lower semicontinuous. Then, from the classical Michael theorem on continuous selectors [9] it follows that for any $C_0 \in \overline{G}$ and any $y_0 \in C_0(0)$ there exists a continuous function $x_0 : \overline{G} \to E$ such that $x_0(C_0) = y_0$ and $x_0(C_0) \in C(0)$.

Taking into account (23) we may rewrite control system (2), (3) with constraints (4), (6) in the form of a control system depending on a parameter $C \in \overline{G}$:

$$-\dot{x}(C)(t) \in \partial\varphi^t(C)(x(C)(t)) + F(t, x(C)(t)) + B(t, x(C)(t))u(C)(t), \tag{24}$$

$$x(C)(0) = x_0(C) \in \operatorname{dom}\varphi^0(C), \tag{25}$$

with the constraints (4) and (6) in which $u(t)$ and $x(t)$ are replaced with $u(C)(t)$ and $x(C)(t)$, respectively.

For every $C \in \overline{G}$ consider the function $g(C) : T \times Q \times S_Y(m) \to \mathbb{R}$,

$$g(C)(t, x, u) = g(t, C(t), x, u). \tag{26}$$

Let $g_U^{**}(C)(t, x, u)$ be the bipolar of the function $u \mapsto g_{_U}(C)(t, x, u)$,

$$g_{_U}(C)(t, x, u) = \begin{cases} g(C)(t, x, u), & u \in U(t, x), \\ +\infty, & u \notin U(t, x). \end{cases}$$

Consider the functionals

$$J(C)(x, u) = \int_T g(C)(t, x(t), u(t))\, dt, \tag{27}$$

$$J^{**}(C)(x, u) = \int_T g_U^{**}(C)(t, x(t), u(t))\, dt. \tag{28}$$

We note that the functional in (28) is denoted by $J^{**}(C)$ just for a notational convenience. In particular, $J^{**}(C)$ is not the bipolar of $J(C)$.

For a fixed $C \in \overline{G}$ by a solution of system (24), (25) with constraint (4) we mean a pair $(x(C), u(C))$, $x(C) \in W^{1,2}(T, E)$, $x(C)(0) = x_0(C)$, $u(C) \in L^2(T, Y)$ such that $x(C)(t) \in \operatorname{dom} \partial \varphi^t(C)$ a.e. and inclusions (24), (4) hold. A solution of system (24), (25) with constraint (6) is defined similarly.

It is clear that for a fixed $C \in \overline{G}$ solutions $\mathcal{R}_U(C)$ and $\mathcal{R}_{\overline{\text{co}}\,U}(C)$ of control system (2), (3) coincide with solutions of control system (24), (25) with constraints (4) and (6). Likewise, Problems $P(C)$ and $RP(C)$, $C \in \overline{G}$, coincide with the minimization problems for functionals (27) and (28) over solutions of system (24), (25) with constraints (4) and (6).

In order to use results of the work [15] we need to show the validity of Hypotheses $H(\varphi)$ and $H(g)$ for the functions $\varphi^t(C)$ and $g(C)$ defined by equalities (23) and (26) and the validity of Hypotheses $H(F)$, $H(B)$ and $H(U)$ in [15] for the mappings $F(t,x)$, $B(t,x)$ and $U(t,x)$ of our work.

From Hypothesis $H_S(G)$ and Corollary 2.11 it follows that the set $\overline{G}$ is a compact metric space. If a sequence $C_n \in \overline{G}$, $n \geq 1$, converges to C, then for every $t \in T$ the sequence of the sets $C_n(t) \in c(E)$, $n \geq 1$, converges to the set $C(t) \in c(E)$ in the Kuratowski sense. Hence, the sequence of the sets $C_n(t) \in c(E)$, $n \geq 1$, Mosco converges to the set $C(t) \in c(E)$. Whence, for every $t \in T$ the sequence of the functions $\varphi^t(C_n) = I_{C_n(t)}$, $n \geq 1$, Mosco converges to the function $\varphi^t(C) = I_{C(t)}$. Therefore, Hypothesis $H(\varphi)$ 1) in the work [15] holds for the functions $\varphi^t(C)$ defined by equality (23).

Since for any $r_0 > 0$ the set

$$\{x \in E;\ \varphi^0(C) < r_0\} = C_0 \neq \emptyset, \quad C \in \overline{G},$$

Hypothesis $H(\varphi)$ 2) in [15] holds.

From Hypothesis $H_S(G)$ and Lemmas 2.1, 2.8 it follows that the family $\overline{G}$ is weakly $r-$equicontinuous. Consequently, according to Definition 2.2 for any $r \geq 0$, $C \in \overline{G}$ there exists a function $a_r(C) \in W^{1,2}(T)$ such that for any $s, t \in T$, $x \in C(s)$, $\|x\| \leq r$ there exists $y \in C(t)$ satisfying inequality (15) and the set $\{\dot{a}_r(C);\ C \in \overline{G}\}$ is bounded in the space $L^2(T, \mathbb{R})$ for any $r \geq 0$. This means that for every $s, t \in T$, $s \leq t$, $x \in \operatorname{dom} \varphi^s(C)$ with $\|x\| \leq r$ there exists $y \in \operatorname{dom} \varphi^t(C)$ satisfying the inequality

$$\|x - y\| \leq |a_r(C)(t) - a_r(C)(s)| \left(1 + |\varphi^s(C)(x)|^{1/2}\right).$$

Taking any function $b(\cdot) \in L^1(T)$ and letting $b_r(C)(t) = b(t)$, $r \geq 0$, $C \in \overline{G}$, we obtain

$$\varphi^t(C)(y) - \varphi^s(C)(x) \leq |b_r(C)(t) - b_r(C)(s)| \left(1 + |\varphi^s(C)(x)|\right), \quad s \leq t.$$

Therefore, Hypotheses $H(\varphi)$ 3), 4) from [15] hold.

Since for any $r > 0$ the set

$$\{x \in E;\ \|x\| \le r,\ \varphi^t(C) \le r,\ C \in \overline{G}\}, \quad t \in T$$

is relatively compact, Hypothesis $H(\varphi)5)$ in [15] holds. Hence, for the functions $\varphi^t(C)$, $t \in T$, $C \in \overline{G}$, defined by equality (23) all Hypotheses $H(\varphi)$ in [15] hold.

As for Hypotheses $H(F)$, $H(B)$, $H(U)$, they evidently follow from Hypotheses $H_S(F)$, $H_S(B)$, $H_S(U)$ since the mappings $F(t,x)$, $B(t,x)$, $U(t,x)$ do not depend on C.

We now show that the function $g(C)(t,x,u)$ defined by equality (26) satisfies all Hypotheses $H(g)$ in the work [15]. Since any element $C \in \overline{G}$ is a continuous mapping from T to $c(E)$, from Hypotheses $H_S(g)(1),(2)$ it follows that for any $C \in \overline{G}$ the function $t \to g(C)(t,x,u)$, $x \in Q$, $u \in S_Y(m)$ is measurable. Consequently, Hypotheses $H(g)$ 1) in [15] holds. From Hypotheses $H_S(g)(2)$ it follows that the function $(C,x,u) \to g(C)(t,x,u)$ is continuous on $\overline{G} \times Q \times S_Y(m)$ for a.e. t. Hence, for any compact $K \subset Q$ the function $(C,x,u) \to g(C)(t,x,u)$ is uniformly continuous on the compact $\overline{G} \times K \times S_Y(m)$. This means that Hypothesis $H(g)$ 2) in [15] holds for the function $g(C)(t,x,u)$. Since the set $\overline{G}$ is $r-$equicontinuous we have

$$d(\Theta, C(t)) \le d(\Theta, C(0)) + |a_0(C)(t) - a_0(C)(0)|, \quad C \in \overline{G}.$$

From this inequality, inequality (12) and the boundedness of the set $\{a_0(C);\ C \in G\}$ in the space $W^{1,2}(T)$ it follows that there exists a constant $N > 0$ such that

$$d(\Theta, C(t)) \le N, \quad t \in T,\ C \in \overline{G}. \tag{29}$$

From inequalities (22), (29) we infer that

$$|g(C)(t,x,u)| \le a(t)(1 + \|x\|) + b(t)N, \tag{30}$$

a.e., $x \in Q$, $C \in \overline{G}$, $u \in S_Y(m)$. Hence, the function $g(C)(t,x,u)$ satisfies Hypothesis $H(g)$ 3) in the work [15]. Therefore, the function $g(C)(t,x,u)$ satisfies all Hypotheses $H(g)$ in [15].

Now we can use the main results of the work [15] reformulated for our systems. In the sequel we assume that the hypotheses stated at the beginning of this section hold.

Theorem 3.1. *For the mapping $C \to \mathcal{R}_U(C)$ ($\mathcal{R}_{\overline{\text{co}}\,U}(C)$), $C \in \overline{G}$, the following statements are true:*

1) *for any $C \in \overline{G}$ the set $\mathcal{R}_U(C)$ ($\mathcal{R}_{\overline{\text{co}}\,U}(C)$) is not empty and there exists a continuous function $s : \overline{G} \to C(T,E) \times L^2(T,Y)$ such that*

$$s(C) \in \mathcal{R}_U(C)\ (\mathcal{R}_{\overline{\text{co}}\,U}(C)), \quad C \in \overline{G}; \tag{31}$$

2) *if $D \in \overline{G}$ is a closed set and $s_D : D \to C(T,E) \times L^2(T,Y)$ is a continuous function, $s_D(C) \in \mathcal{R}_U(C)$ $(\mathcal{R}_{\overline{\mathrm{co}}\,U}(C))$, $C \in D$, then there exists a continuous function $s : \overline{G} \to C(T,E) \times L^2(T,Y)$ such that inclusion* (31) *holds and $s(C) = s_D(C)$, $C \in D$;*
3) *the mapping $C \to \mathcal{R}_U(C)(\mathcal{R}_{\overline{\mathrm{co}}\,U}(C))$ is Vietoris lower semicontinuous from $\overline{G}$ to $C(T,E) \times L^2(T,Y)$ with closed values in $C(T,E) \times L^2(T,Y)$;*
4) *the mapping $C \to \mathcal{R}_{\overline{\mathrm{co}}\,U}(C)$ is Vietoris continuous from $\overline{G}$ to $C(T,E) \times L^2_\omega(T,Y)$ with compact values in $C(T,E) \times L^2_\omega(T,Y)$.*

Theorem 3.2. *For any $C_* \in \overline{G}$, $(x_*, u_*) \in \mathcal{R}_{\overline{\mathrm{co}}\,U}(C_*)$ there exists a sequence $(x_n, u_n) \in \mathcal{R}_U(C_*)$, $n \geq 1$, such that*

$$(x_n, u_n) \to (x_*, u_*) \quad \text{in } C(T,E) \times L^2_\omega(T,Y), \tag{32}$$

$$\lim_{n\to\infty} \sup_{0 \leq t \leq t' \leq 1} \left| \int_t^{t'} \Big(g_U^{**}(\tau, C_*(\tau), x_*(\tau), u_*(\tau)) - g(\tau, C_*(\tau), x_n(\tau), u_n(\tau)) \Big) \, d\tau \right| = 0. \tag{33}$$

Taking into account Lemma 2.1, the definition of the set $S_Y(m)$ and equality (26), Theorems 3.1, 3.2 are reformulations of Theorems 4.4, 5.1 from [15] applied to our problems.

Theorem 3.3. *For any $C_* \in \overline{G}$ Problem $RP(C)$ has a solution and*

$$\min_{(x,u)\in\mathcal{R}_{\overline{\mathrm{co}}U}(C_*)} J^{**}(C_*, x, u) = \inf_{(x,u)\in\mathcal{R}_U(C_*)} J(C_*, x, u).$$

For any solution $(x_*, u_*) \in \mathcal{R}_{\overline{\mathrm{co}}U}(C_*)$ of Problem $RP(C_*)$ there exists a minimizing sequence $(x_n, u_n) \in \mathcal{R}_U(C_*)$, $n \geq 1$, of Problem $P(C_*)$ converging to (x_*, u_*) in the space $C(T,E) \times L^2_\omega(T,Y)$ and such that equality (33) *holds.*

Conversely, if $(x_n, u_n) \in \mathcal{R}_U(C_*)$, $n \geq 1$, is a minimizing sequence of Problem $P(C_*)$, then there exists a subsequence (x_{n_k}, u_{n_k}), $k \geq 1$, of the sequence (x_n, u_n), and a solution (x_*, u_*) of Problem $RP(C_*)$ such that the sequence (x_{n_k}, u_{n_k}), $k \geq 1$, converges to (x_*, u_*) in the space $C(T,E) \times L^2_\omega(T,Y)$ and equality (33) *holds for the sequence (x_{n_k}, u_{n_k}), $k \geq 1$.*

Set

$$m_U(C) = \inf\{J(C, x, u);\ (x,u) \in \mathcal{R}_U(C)\}, \quad C \in \overline{G}, \tag{34}$$

$$m_{\overline{\mathrm{co}}U}(C) = \min\{J^{**}(C, x, u);\ (x,u) \in \mathcal{R}_{\overline{\mathrm{co}}\,U}(C)\}, \quad C \in \overline{G}. \tag{35}$$

According to Theorem 3.3 for any $C \in \overline{G}$ we have

$$m_{\overline{\mathrm{co}}U}(C) = m_U(C). \tag{36}$$

Let

$$\mathcal{R}^m_{\overline{\text{co}}\,U}(C) = \{(x,u) \in \mathcal{R}_{\overline{\text{co}}\,U}(C);\ m_{\overline{\text{co}}\,U}(C) = J^{**}(C,x,u)\}. \tag{37}$$

Theorem 3.4. *The function $C \to m_{\overline{\text{co}}\,U}(C)$, $C \in \overline{G}$, is continuous and the multivalued mapping $C \to \mathcal{R}^m_{\overline{\text{co}}\,U}(C)$ is Vietoris upper semicontinuous from $\overline{G}$ to $C(T,E) \times L^2_\omega(T,Y)$ with compact values in $C(T,E) \times L^2_\omega(T,Y)$.*

In view of Lemma 2.1, Theorems 3.3, 3.4 are reformulations of Theorems 5.2, 5.3 from [15].

4. Main results

In this section we give the main results of the present work.

Theorem 4.1. *The set $\mathcal{R}_{\overline{G},\overline{\text{co}}\,U}$ is compact in the space $C(T,c(E)) \times C(T,E) \times L^2_\omega(T,Y)$. For any element $(C_*,x_*,u_*) \in \mathcal{R}_{\overline{G},\overline{\text{co}}\,U}$ there exists a sequence $(C_n,x_n,u_n) \in \mathcal{R}_{G,U}$, $n \geq 1$, such that*

$$C_n \to C_* \quad \text{in } C(T,c(E)), \tag{38}$$

$$(x_n,u_n) \to (x_*,u_*) \quad \text{in } C(T,E) \times L^2_\omega(T,Y), \tag{39}$$

$$\lim_{n\to\infty} \sup_{0\le t\le t'\le 1} \left| \int_t^{t'} \big(g^{**}_U(\tau,C_*(\tau),x_*(\tau),u_*(\tau)) - g(\tau,C_n(\tau),x_n(\tau),u_n(\tau))\big)\,d\tau \right| = 0. \tag{40}$$

Proof. From the statement *4*) of Theorem 3.1 and the compactness of the set $\overline{G}$ it follows that the set $\mathcal{R}_{\overline{\text{co}}\,U}(\overline{G}) = \{\bigcup \mathcal{R}_{\overline{\text{co}}\,U}(C);\ C \in \overline{G},\}$ is compact in $C(T,E) \times L^2_\omega(T,Y)$. Since $\mathcal{R}_{\overline{G},\overline{\text{co}}\,U} \subset \overline{G} \times \mathcal{R}_{\overline{\text{co}}\,U}(\overline{G})$, to prove the compactness of the set $\mathcal{R}_{\overline{G},\overline{\text{co}}\,U}$ it is enough to prove the closedness of this set in the space $C(T,c(E)) \times C(T,E) \times L^2_\omega(T,Y)$.

Let a sequence $(C_n,x_n,u_n) \in \mathcal{R}_{\overline{G},\overline{\text{co}}\,U}$, $n \geq 1$, converge to (C,x,u) in the space $C(T,c(E)) \times C(T,E) \times L^2_\omega(T,Y)$. Then $C \in \overline{G}$. Since $(x_n,u_n) \in \mathcal{R}_{\overline{\text{co}}\,U}(C_n)$, $n \geq 1$, from the statement *4*) of Theorem 3.1 it follows that $(x,u) \in \mathcal{R}_{\overline{\text{co}}\,U}(C)$. Consequently, $(C,x,u) \in \mathcal{R}_{\overline{G},\overline{\text{co}}\,U}$. The compactness of the set $\mathcal{R}_{\overline{G},\overline{\text{co}}\,U}$ is proven.

Let $(C_*,x_*,u_*) \in \mathcal{R}_{\overline{G},\overline{\text{co}}\,U}$. Then, using Theorem 3.2 we infer that there exists a sequence $(y_n,v_n) \in \mathcal{R}_U(C_*)$, $n \geq 1$, such that

$$(y_n,v_n) \to (x_*,u_*) \quad \text{in } C(T,E) \times L^2_\omega(T,Y), \tag{41}$$

$$\lim_{n\to\infty} \sup_{0\le t\le t'\le 1} \left| \int_t^{t'} \Big(g^{**}_U(\tau,C_*(\tau),x_*(\tau),u_*(\tau)) - g(\tau,C_*(\tau),y_n(\tau),v_n(\tau))\Big)\,d\tau \right| = 0. \tag{42}$$

For $C_* \in \overline{G}$ there exists a sequence $C_m \in G$, $m \geq 1$, such that

$$C_m \to C_*, \quad m \to \infty, \text{ in } C(T, c(E)). \tag{43}$$

Fix $n \geq 1$. Then, according to the statement *2*) of Theorem 3.1 there exists a sequence $(z_m^n, w_m^n) \in \mathcal{R}_U(C_m)$, $m \geq 1$, such that

$$(z_m^n, w_m^n) \to (y_n, v_n), \quad m \to \infty, \text{ in } C(T, E) \times L^2(T, Y). \tag{44}$$

Without loss of generality, we may assume that $w_m^n(t) \to v_n(t)$, $m \to \infty$, a.e. Using Hypothesis $H_S(g)$, (43), (44) we obtain

$$g(t, C_m(t), z_m^n(t), w_m^n(t)) \to g(t, C_*(t), y_n(t), v_n(t)), \quad m \to \infty, \text{ a.e.} \tag{45}$$

Then, from (26), (30) it follows that the convergence in (45) takes place in the space $L^1(T, \mathbb{R})$. Using (44), (45) we infer that for any $n \geq 1$ there exists a number m_n such that

$$\|z_{m_n}^n(t) - y_n\|_{C(T,E)} \leq \frac{1}{n}, \tag{46}$$

$$\|w_{m_n}^n(t) - v_n\|_{L^2(T,Y)} \leq \frac{1}{n}, \tag{47}$$

$$\int_T |g(t, C_*(t), y_n(t), v_n(t)) - g(t, C_{m_n}(t), z_{m_n}^n(t), w_{m_n}^n(t))|\, dt \leq \frac{1}{n}. \tag{48}$$

Since $(z_{m_n}^n, w_{m_n}^n) \in \mathcal{R}_U(C_{m_n})$, $n \geq 1$, we have

$$(C_{m_n}, z_{m_n}^n, w_{m_n}^n) \in \mathcal{R}_{G,U}, \quad n \geq 1. \tag{49}$$

Set

$$x_n = z_{m_n}^n, \quad u_n = w_{m_n}^n, \quad C_n = C_{m_n}, \quad n \geq 1. \tag{50}$$

Using relations (41)–(43), (46)–(49) we deduce that the sequence (C_n, x_n, u_n), $n \geq 1$, defined by equalities (50) satisfies relations (38)–(40). The theorem follows. $\square$

Corollary 4.2. *There holds the equality*

$$\mathcal{R}_{\overline{G}, \overline{\mathrm{co}}\, U} = \overline{\mathcal{R}_{G,U}},$$

where the bar stands for the closure in the space $C(T, c(E)) \times C(T, E) \times L^2_\omega(T, Y)$.

Theorem 4.3. *Problem* (RP) *has a solution and*

$$\min_{(C,x,u) \in \mathcal{R}_{\overline{G}, \overline{\mathrm{co}} U}} J^{**}(C, x, u) = \inf_{(C,x,u) \in \mathcal{R}_{G,U}} J(C, x, u). \tag{51}$$

For any solution $(C_*, x_*, u_*) \in \mathcal{R}_{\overline{G}, \overline{\mathrm{co}} U}$ *of Problem* (RP) *there exists a minimizing sequence* $(C_n, x_n, u_n) \in \mathcal{R}_{G,U}$, $n \geq 1$, *of Problem* (P) *converging to* (C_*, x_*, u_*) *in the space* $C(T, c(E)) \times C(T, E) \times L^2_\omega(T, Y)$ *and equality* (40) *holds.*

Conversely, if $(C_n, x_n, u_n) \in \mathcal{R}_{G,U}$*,* $n \geq 1$*, is a minimizing sequence of Problem* (P)*, then there exists a subsequence* $(C_{n_k}, x_{n_k}, u_{n_k})$*,* $k \geq 1$*, of the sequence* (C_n, x_n, u_n)*,* $n \geq 1$*, and a solution* $(C_*, x_*, u_*) \in \mathcal{R}_{\overline{G},\overline{\mathrm{co}}U}$ *of Problem* (RP) *such that the sequence* $(C_{n_k}, x_{n_k}, u_{n_k})$*,* $k \geq 1$*, converges to* (C_*, x_*, u_*) *in the space* $C(T, c(E)) \times C(T, E) \times L^2_\omega(T, Y)$ *and equality* (40) *holds for this sequence.*

Proof. From the compactness of the set $\overline{G}$, (36) and Theorem 3.4 it follows that there exists $C_* \in \overline{G}$ such that

$$m_{\overline{\mathrm{co}}\,U}(C_*) = \min_{C \in \overline{G}} m_{\overline{\mathrm{co}}\,U}(C) = \inf_{C \in G} m_U(C). \tag{52}$$

Using Theorem 3.3 we obtain that there exist $(x_*, u_*) \in \mathcal{R}_{\overline{\mathrm{co}}U}(C_*)$ such that

$$J^{**}(C_*, x_*, u_*) = \min_{C \in \overline{G}} m_{\overline{\mathrm{co}}\,U}(C). \tag{53}$$

From inequalities (52), (53), (34), (35) and Theorem 3.3 it directly follows that Problem (RP) has a solution and equality (51) holds.

Let $(C_*, x_*, u_*) \in \mathcal{R}_{\overline{G},\overline{\mathrm{co}}U}$ be a solution of Problem (RP). According to Theorem 4.1 there exists a sequence $(C_n, x_n, u_n) \in \mathcal{R}_{G,U}$, $n \geq 1$, for which relations (38)–(40) hold. Since

$$\begin{aligned} &|J^{**}(C_*, x_*, u_*) - J(C_n, x_n, u_n)| \\ &\leq \sup_{0 \leq t \leq t' \leq 1} \left| \int_t^{t'} \Big(g_U^{**}(\tau, C_*(\tau), x_*(\tau), u_*(\tau)) \right. \\ &\qquad \left. - g(\tau, C_n(\tau), x_n(\tau), u_n(\tau)) \Big) \, d\tau \right|, \end{aligned} \tag{54}$$

from (54), (40) it follows that $(C_n, x_n, u_n) \in \mathcal{R}_{G,U}$, $n \geq 1$, is a minimizing sequence for Problem (P), and the first part of the theorem follows.

Now, we prove the second part of the theorem. Let $(C_n, x_n, u_n) \in \mathcal{R}_{G,U}$, $n \geq 1$, be a minimizing sequence for Problem (P). Then,

$$\min_{(C,x,u) \in \mathcal{R}_{\overline{G},\overline{\mathrm{co}}U}} J^{**}(C, x, u) = \lim_{n \to \infty} J(C_n, x_n, u_n). \tag{55}$$

From Corollary 4.2, Theorem 4.1 and (26), (30) we infer that there exists a subsequence $(C_{n_k}, x_{n_k}, u_{n_k})$, $k \geq 1$, of the sequence (C_n, x_n, u_n), $n \geq 1$, and $(C_*, x_*, u_*) \in \mathcal{R}_{\overline{G},\overline{\mathrm{co}}U}$, $\lambda \in L^1(T)$ such that

$$(C_{n_k}, x_{n_k}, u_{n_k}) \to (C_*, x_*, u_*) \tag{56}$$

in the space $C(T, c(E)) \times C(T, E) \times L^2_\omega(T, Y)$,

$$g(t, C_{n_k}(t), x_{n_k}(t), u_{n_k}(t)) \to \lambda(t) \tag{57}$$

in the space ω-$L^1(T,\mathbb{R})$.

As was shown above the function $g(C)(t,x,u)$ defined by equality (26) satisfies all Hypotheses $H(g)$ in the work [15]. Since

$$(x_*, u_*) \in \mathcal{R}_{\overline{\text{co}}U}(C_*), \tag{58}$$

using Lemmas 5.1, 5.2 in [15], (56) and reasoning, e.g., as in the proof of Theorem 5.3 in [15] we conclude that

$$g_U^{**}(t, C_*(t), x_*(t), u_*(t)) \le \lambda(t) \quad \text{a.e.} \tag{59}$$

From (55), (58), (59) it follows that

$$\begin{aligned} &\min_{(C,x,u)\in\mathcal{R}_{\overline{G},\overline{\text{co}}U}} J^{**}(C,x,u) = \int_T g_U^{**}(t, C_*(t), x_*(t), u_*(t)\, dt \\ &\le \int_T \lambda(t)\, dt = \lim_{n\to\infty} \int_T g(t, C_{n_k}(t), x_{n_k}(t), u_{n_k}(t))\, dt. \end{aligned} \tag{60}$$

From (59), (60) we infer that

$$g_U^{**}(t, C_*(t), x_*(t), u_*(t)) = \lambda(t) \quad \text{a.e.} \tag{61}$$

Using (57), (61), (26), (30) and Lemma 2.1 we obtain

$$g(t, C_{n_k}(t), x_{n_k}(t), u_{n_k}(t)) \to g_U^{**}(t, C_*(t), x_*(t), u_*(t)) \quad \text{in } L^1_\omega(T,\mathbb{R}). \tag{62}$$

From the definition of the norm (9) in the space $L^1_\omega(T,\mathbb{R})$, (60), (62) it follows that $(C_*, x_*, u_*) \in \mathcal{R}_{\overline{G},\overline{\text{co}}U}$ is a solution of Problem (RP) and for the sequence $(C_{n_k}, x_{n_k}, u_{n_k})$, $k \ge 1$, relations (56), (40) hold. The theorem follows. $\square$

Let

$$\begin{aligned} m_* &= \min\{J^{**}(C,x,u);\ (C,x,u) \in \mathcal{R}_{\overline{G},\overline{\text{co}}\,U}\}, \\ \mathcal{R}^{m_*}_{\overline{G},\overline{\text{co}}\,U} &= \{(C,x,u) \in \mathcal{R}_{\overline{G},\overline{\text{co}}\,U};\ m_* = J^{**}(C,x,u)\}. \end{aligned}$$

Theorem 4.4. *The set $\mathcal{R}^{m_*}_{\overline{G},\overline{\text{co}}\,U}$ is compact in the space $C(T,c(E)) \times C(T,E) \times L^2_\omega(T,Y)$.*

Proof. Since

$$m_* = \min\{m_{\overline{\text{co}}\,U}(C)\ \ C \in \overline{G}\},$$

from Theorem 3.4 it follows that the set

$$\mathcal{D} = \{C \in \overline{G};\ m_* = m_{\overline{\text{co}}\,U}(C)\}$$

is compact in $C(T,c(E))$. Since the set $\mathcal{R}^{m_*}_{\overline{G},\overline{\text{co}}\,U}$ coincides with the graph of the multivalued mapping $C \to \mathcal{R}^m_{\overline{\text{co}}\,U}(C)$ defined by equality (37), the statement of the theorem follows from Theorem 3.4. $\square$

Remark 4.5. An example of a function $g(t, C, x, u)$ satisfying Hypotheses $H_S(g)$ is the function

$$g(t, C, x, u) = g_1(t, x, u) + g_2(t, C),$$

where $g_1 : T \times Q \times S_Y(m) \to \mathbb{R}$ is a Caratheodory function satisfying the inequality

$$|g_1(t, x, u)| \le a(t)(1 + \|x\|) \quad \text{a.e.},$$

$u \in S_Y(m)$, $a \in L^1(T, \mathbb{R}^+)$, and the function $g_2(t, C)$ has the form:

$$d(y(t), C), \quad C \in c(Q),$$

where $y : T \to Y$ is a fixed continuous function.

Remark 4.6. For control system (2), (3) we consider constraints (4), (5) and (6), (7) which are conceptually different in the sense that we pass from constraint (5) to constraint (7) by taking the closure, but to pass from constraint (4) to constraint (6) we take the closed convex hull. If for a continuous function $x : T \to Q$ we denote

$$S(x) = \{u \in L^2(T, Y);\ u(t) \in U(t, x(t)) \ \text{a.e.}\},$$

then under our assumptions on the multivalued mapping $U(t, x)$ the equality

$$\overline{S}(x) = \{u \in L^2(T, Y);\ u(t) \in \overline{\text{co}}\, U(t, x(t)) \ \text{a.e.}\}$$

holds for the closure $\overline{S}(x)$ of the set $S(x)$ in the space ω-$L^2(T, Y)$ and thus, according to Lemma 2.1, in the space $L^2_\omega(T, Y)$. Therefore, for control system (2), (3) instead of control constraints (4), (5) and (6), (7) we may consider the constraints

$$C(\cdot) \in G, \quad u(\cdot) \in S(x), \tag{63}$$

$$C(\cdot) \in \overline{G}, \quad u(\cdot) \in \overline{S}(x). \tag{64}$$

Since now the passage to constraint (64) is made by taking the closure of constraint (63) the above mentioned conceptual difference disappears.

5. Examples

In this section we give examples of sets $G \subset C(T, c(E))$ which are r-equicontinuous and compact. Let $V \subset W^{1,2}(T, E)$ and $B \subset W^{1,2}(T)$ be sets of functions $v(t)$ and $b(t)$ with the properties:

1)
$$\|v(t)\| = 1, \quad v \in V; \tag{65}$$

2) the set V is bounded in $W^{1,2}(T, E)$;
3) the set B is bounded in $W^{1,2}(T)$ and, hence,

$$|b(0)| \le N, \quad b \in B,\ N > 0. \tag{66}$$

Consider the family G of mappings $C : T \to c(E)$ defined by the equality

$$C(t) = \{x \in E;\ \langle v(t), x\rangle - b(t) = 0\}, \tag{67}$$

$v \in V$, $b \in B$, $t \in T$.

Lemma 5.1. *The family G of mappings $C : T \to c(E)$ defined by equality (67) is r-equicontinuous and a relatively compact subset of the space $C(T, c(E))$. If, in addition, the sets V and B are closed in $W^{1,2}(T, E)$ and $W^{1,2}(T)$ equipped with the weak topology (i.e. in the spaces ω-$W^{1,2}(T, E)$ and ω-$W^{1,2}(T)$) respectively, then the family G is r-equicontinuous and a compact subset in $C(T, c(E))$.*

Proof. For convenience, the functions v and b appearing in the definition of the mapping C given by the formula (67) will be denoted by $v(C)$ and $b(C)$, respectively. From the well-known equality

$$d(x, C(t)) = |\langle v(C)(t), x\rangle - b(C)(t)|/\|v(C)(t)\|, \tag{68}$$

and (65) it follows that

$$\begin{aligned} &|d(x, C(t)) - d(x, C(s))| \\ &\le \|v(C)(t) - v(C)(s)\|\|x\| + |b(C)(t) - b(C)(s)|. \end{aligned} \tag{69}$$

Consider the functions

$$a_r(C)(t) = \int_0^t \left(\|\dot v(C)(\tau)\|r + |\dot b(C)(\tau)|\right) d\tau, \quad C \in G,\ r \ge 0. \tag{70}$$

Since the sets $\{v(C);\ C \in G\}$ and $\{b(C);\ C \in G\}$ are bounded in $W^{1,2}(T, E)$ and $W^{1,2}(T)$, the sets $\{\dot v(C);\ C \in G\}$, $\{\dot b(C);\ C \in G\}$ are bounded in $L^2(T, E)$ and $L^2(T, \mathbb{R})$, respectively. Therefore,

$$\mathcal{A} = \{a_r(C);\ r \ge 0,\ C \in G\} \subset W^{1,2}(T)$$

and the set $\{a_r(C);\ C \in G\}$ is bounded in $W^{1,2}(T)$ for any $r \ge 0$. From (68)–(70), (66) we obtain

$$|d(x, C(t)) - d(x, C(s))| \le |a_r(C)(t) - a_r(C)(s)|, \tag{71}$$

$\|x\| \le r$, $t, s \in T$, $C \in G$,

$$d(\Theta, C(0)) \le N, \quad C \in G. \tag{72}$$

Now, from (71), (72) and Corollary 2.11 it follows that the family G is $r-$equicontinuous and a relatively compact subset of the space $C(T, c(E))$.

In order to finish the proof of the lemma it is enough to show the closedness of the set G. If the sets V and B are bounded in $W^{1,2}(T, E)$ and $W^{1,2}(T)$ and closed in the topologies of the spaces ω-$W^{1,2}(T,E)$ and ω-$W^{1,2}(T)$, then from the equalities

$$v(t) = v(0) + \int_0^t \dot{v}(\tau)\, d\tau, \qquad b(t) = b(0) + \int_0^t \dot{b}(\tau)\, d\tau$$

it follows that the sets V and B are compact in the spaces $C(T,E)$ and $C(T,\mathbb{R})$, respectively.

Let a sequence $C_n \in G$, $n \geq 1$,

$$C_n(t) = \{x \in E;\ \langle v(C_n)(t), x\rangle - b(C_n)(t) = 0\}, \quad t \in T,$$

converges to C in $C(T, c(E))$. Then, there exists a subsequence $C_{n_k} \in G$, $k \geq 1$, of the sequence $C_n \in G$, $n \geq 1$, such that the sequence $v(C_{n_k})$, $k \geq 1$, converges in $C(T,E)$ to $v \in V$, and the sequence $b(C_{n_k})$, $k \geq 1$, converges in $C(T,\mathbb{R})$ to $b \in B$. Passing to the limit as $k \to \infty$ in the equality

$$d(x, C_{n_k}(t)) = |\langle v(C_{n_k})(t), x\rangle - b(C_{n_k})(t)|, \quad x \in E,$$

we obtain

$$d(x, C(t)) = |\langle v(t), x\rangle - b(t)|, \quad x \in E. \tag{73}$$

Since $C(t) = \{x \in E;\ d(x, C(t)) = 0\}$, from (73) it follows that

$$C(t) = \{x \in E;\ \langle v(t), x\rangle - b(t) = 0\}.$$

Therefore, $C \in G$. The lemma follows. □

Let $\overline{V}$ and $\overline{B}$ be the closures of the sets V and B in the topologies of the spaces ω-$W^{1,2}(T,E)$ and ω-$W^{1,2}(T)$. Since the set $\overline{V}$ is compact in $C(T,E)$, equality (65) holds for any $v \in \overline{V}$. Denote by Γ the family of all mappings $C : T \to c(E)$ defined by formula (67) for $v \in \overline{V}$ and $b \in \overline{B}$.

Corollary 5.2. *The closure $\overline{G}$ of the family Γ of mappings $C : T \to c(E)$ defined by equality* (67) *in the space $C(T, c(E))$ for $v \in V$, $b \in B$ coincides with Γ and is $r-$equicontinuous and a compact set in $C(T, c(E))$.*

Proof. From Lemma 5.1 it follows that $\overline{G} \subset \Gamma$ and is $r-$equicontinuous and a compact set in $C(T, c(E))$. Let $C \in \Gamma$,

$$C(t) = \{x \in E;\ \langle v(t), x\rangle - b(t) = 0\}, \quad v \in \overline{V},\ b \in \overline{B}. \tag{74}$$

Since the spaces $W^{1,2}(T,E)$ and $W^{1,2}(T)$ are separable and reflexive, the sets $\overline{V}$ and $\overline{B}$ are metrizable compacts in the topologies of the spaces ω-$W^{1,2}(T,E)$ and ω-$W^{1,2}(T)$. Let $v_n \in V$, $b_n \in B$, $n \geq 1$, be sequences converging to v and b in the topologies of the spaces ω-$W^{1,2}(T,E)$ and ω-$W^{1,2}(T)$. Then v_n, b_n, $n \geq 1$, converge to v and b in the spaces $C(T,E)$ and $C(T,\mathbb{R})$. Set

$$C_n(t) = \{x \in E;\ \langle v_n(t), x\rangle - b_n(t) = 0\}, \quad n \geq 1.$$

Then, $C_n \in G$, $n \geq 1$, and

$$d(x, C_n(t)) = |\langle v_n(t), x\rangle - b_n(t)|, \quad x \in E,\ n \geq 1. \tag{75}$$

Passing to the limit as $n \to \infty$ in (75) and taking into account (74) we obtain

$$\lim_{n\to\infty} d(x, C_n(t)) = |\langle v(t), x\rangle - b(t)| = d(x, C(t)), \quad x \in E.$$

From this equality it follows that for every $t \in T$ the sequence $C_n(t)$, $n \geq 1$, converges to $C(t)$ in the space $c(E)$. Since the set $\overline{G}$ is compact in $C(T, c(E))$, the sequence C_n, $n \geq 1$, converges to C in the space $C(T, c(E))$. Therefore, $\overline{G} = \Gamma$. □

Consider the family G of mappings $C : T \to c(E)$ defined by the formula

$$C(t) = \{x \in E;\ \langle v(t), x\rangle - b(t) \leq 0\}, \tag{76}$$

$v \in V$, $b \in B$, $t \in T$.

Lemma 5.3. *For the family G of mappings $C : T \to c(E)$ defined by equality* (76) *all the statements of Lemma* 5.1 *hold true.*

Proof. Let $C \in G$, $t \in T$, $x \in C(t)$. If $x \notin C(s)$, $s \in T$, $s \neq t$, in view of (76) we have

$$\langle v(C)(s), x\rangle - b(C)(s) > 0. \tag{77}$$

According to (77)

$$d(x, C(s)) = |\langle v(C)(s), x\rangle - b(C)(s)| = \langle v(C)(s), x\rangle - b(C)(s). \tag{78}$$

From (76), (78) it follows that

$$\begin{aligned} d(x, C(s)) &\leq \langle v(C)(s), x\rangle - b(C)(s) - \langle v(C)(t), x\rangle + b(C)(t) \\ &\leq \|v(C)(t) - v(C)(s)\|\|x\| + |b(C)(t) - b(C)(s)|. \end{aligned} \tag{79}$$

If $x \in C(s)$, then $d(x, C(s)) = 0$. Hence, for any $x \in C(t)$ inequality (79) holds. Using (70), (79) we infer that for any $r \geq 0$, $s, t \in T$, $x \in C(t)$, $\|x\| \leq r$ there exists an element $y \in C(s)$ such that

$$\|x - y\| \leq |a_r(C)(t) - a_r(C)(s)|. \tag{80}$$

Therefore, the family G is weakly $r-$equicontinuous. If $\Theta \in C(0)$, then $d(\Theta, C(0)) = 0$. If $\Theta \notin C(0)$, then from (78) it follows that

$$d(\Theta, C(0)) \leq |b(C)(0)| \leq N, \quad C \in G.$$

From this inequality, (80), Lemma 2.1 and Corollary 2.11 we deduce that G is $r-$equicontinuous and a relatively compact subset of the space $C(T, c(E))$.

Let V and B be closed in the topologies of the spaces ω-$W^{1,2}(T,E)$ and ω-$W^{1,2}(T)$ and $C_n \in G$, $n \geq 1$, be a sequence converging to C in $C(T, c(E))$. As in the proof of Lemma 5.1 there exists a subsequence C_{n_k}, $k \geq 1$, of the sequence C_n, $n \geq 1$, such that the sequences $v(C_{n_k})$, $b(C_{n_k})$, $k \geq 1$, converge in $C(T,E)$ and $C(T,\mathbb{R})$ to $v \in V$ and $b \in B$, consequently. For any $x \in E$, $t \in T$, we have either $x \in C_{n_k}(t)$ or $x \notin C_{n_k}(t)$, $k \geq 1$. In the first case,

$$\langle v(C_{n_k})(t), x\rangle - b(C_{n_k})(t) \leq 0 = d(x, C_{n_k}(t)).$$

In the second case,

$$\langle v(C_{n_k})(t), x\rangle - b(C_{n_k})(t) = d(x, C_{n_k}(t)).$$

Therefore, for any $x \in E$, $t \in T$ the inequality

$$\langle v(C_{n_k})(t), x\rangle - b(C_{n_k})(t) \leq d(x, C_{n_k}(t)) \tag{81}$$

holds. Passing to the limit as $k \to \infty$ in (81) we obtain

$$\langle v(t), x\rangle - b(t) \leq d(x, C) \tag{82}$$

Let

$$C^*(t) = \{x \in E;\ \langle v(t), x\rangle - b(t) \leq 0\}. \tag{83}$$

From (82), (83) it follows that

$$C(t) \subset C^*(t). \tag{84}$$

Suppose that

$$C(t) \neq C^*(t). \tag{85}$$

Then, there exists a point $x^* \in C^*(t)$, $x^* \notin C(t)$. Hence, $d(x^*, C(t)) = \delta > 0$. Since

$$d(x^*, C_{n_k}(t)) \to d(x^*, C(t)) \quad \text{as } k \to \infty,$$

for k large enough we have

$$d(x^*, C_{n_k}(t)) \geq \frac{\delta}{2} > 0.$$

Consequently,

$$0 < \frac{\delta}{2} \leq \langle v(C_{n_k})(t), x^*\rangle - b(C_{n_k})(t) = d(x^*, C_{n_k}(t)).$$

Passing to the limit as $k \to \infty$ in this inequality we obtain

$$0 < \frac{\delta}{2} \leq \langle v(t), x^*\rangle - b(t).$$

From this inequality and (83) it follows that $x^* \notin C^*(t)$. But this contradicts to our assumption that $x^* \in C^*(t)$ and $x^* \notin C(t)$. Therefore, $x^* \in C(t)$ and according to (84), $C^*(t) = C(t)$. Since $v \in V$, $b \in B$, in view of (76), $C \in G$. Hence, the family G is closed and according to Corollary 2.11 is a compact $r-$equicontinuous subset of the space $C(T, c(E))$. The lemma follows. □

Denote by Γ the family of all mappings $C : T \to c(E)$ defined by equality (76) for $v \in \overline{V}$ and $b \in \overline{B}$, where $\overline{V}$ and $\overline{B}$ are the closures of V and B in the topologies of the spaces ω-$W^{1,2}(T,E)$ and ω-$W^{1,2}(T)$.

Corollary 5.4. *The closure $\overline{G}$ of the family G of mappings $C : T \to c(E)$ defined by equality* (76) *in the space $C(T,c(E))$ for $v \in V$ and $b \in B$ coincides with Γ and is $r-$equicontinuous and a compact set in $C(T,c(E))$.*

Proof. From Lemma 5.3 it follows that Γ is $r-$equicontinuous and a compact set in $C(T,c(E))$ and $\overline{G} \subset \Gamma$. Hence, we need to prove the equality $\overline{G} = \Gamma$. Let $C^* \in \Gamma$. Then,

$$C^*(t) = \{x \in E;\ \langle v(C^*)(t), x\rangle - b(C^*)(t) \leq 0\}, \tag{86}$$

$v(C^*) \in \overline{V}$, $b(C^*) \in \overline{B}$. Since the set G is relatively compact there exist sequences $C_n \in G$, $v(C_n) \in V$, $b(C_n) \in B$, $n \geq 1$, such that

$$C_n(t) = \{x \in E;\ \langle v(C_n)(t), x\rangle - b(C_n)(t) \leq 0\}, \quad t \in T, \tag{87}$$

and the sequence $C_n \in G$, $n \geq 1$, converges in $C(T,c(E))$ to $C \in \Gamma$, $v(C_n) \to v(C^*)$, $b(C_n) \to b(C^*)$ in $C(T,E)$ and $C(T,\mathbb{R})$. From (87) it follows that for any $x \in E$ the inequality

$$\langle v(C_n)(t), x\rangle - b(C_n)(t) \leq d(x, C_n(t)), \quad t \in T.$$

holds. Passing to the limit as $n \to \infty$ in this inequality we obtain

$$\langle v(C^*)(t), x\rangle - b(C^*)(t) \leq d(x, C(t)). \tag{88}$$

From (86), (88) it follows that inclusion (84) holds. If we suppose that $C(t) \neq C^*(t)$, reasoning as in the proof of Lemma 5.3 we derive a contradiction. Therefore, $C(t) = C^*(t)$ and thus $\overline{G} = \Gamma$. $\square$

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