Journal of Convex Analysis
Volume 23 (2016), No. 4, 1185–1204

Locating a Semi-Obnoxious Facility – A Toland-Singer Duality Based Approach

Andrea Wagner
Vienna University of Economics and Business Institute for Statistics and Mathematics, Department of Finance, Accounting and Statistics, Welthandelsplatz 1, 1020 Wien, Austria
andrea.wagner@wu.ac.at

Juan-Enrique Martínez-Legaz*
Unitat de Fonaments de l'Anàlisi Econòmica,
Departament d'Economia i d'Història Econòmica,
Universitat Autónoma de Barcelona
JuanEnrique.Martinez.Legaz@uab.cat

Christiane Tammer
Institute of Mathematics, Martin-Luther-University Halle-Wittenberg,
Theodor-Lieser Str. 5, 06120 Halle/Saale
christiane.tammer@mathematik.uni-halle.de

Dedicated to the memory of Jean Jacques Moreau.

Received: January 5, 2015
Revised manuscript received: March 2, 2015
Accepted: March 7, 2015

We consider the problem of locating a facility amongst a given collection of attraction and repulsion points. The goal is to find a location x in the Euclidean space $\mathbb{R}^n$ for a facility, such that the difference between the weighted sum of distances from x to the attraction points and the weighted sum of distances to the repulsion points is minimized. The corresponding objective function constitutes as a D.C. function. Based on the duality theory by Toland and Singer for the class of D.C. programs, we formulate a dual problem to the given location problem. Taking into account the special structure of the location problem, we present geometrical properties of the model, give conditions for the existence of an optimal solution, obtain duality results, describe the relationship between primal and dual elements, and formulate an algorithm which determines exact solutions for the location problem by reducing this non-convex optimization problem to a finite number of linear programs.

Keywords: Locational analysis, conjugate duality, obnoxious facilities, D.C. optimization, geometric duality, linear vector optimization

*The second author was supported by the MICINN of Spain [grant MTM2011-29064-C03-01], and under the Australian Research Councils Discovery Projects funding scheme [project number DP140103213]. He is affiliated to MOVE (Markets, Organizations and Votes in Economics).

ISSN 0944-6532 / $ 2.50

1. Introduction

Location problems appear in many variants and with different constraints depending on the application in practice. In most location models the criteria for finding an optimal location for a new facility have economic issues. Travel time or travel costs to the given facilities are to be minimized. Especially, in economics, engineering, town planning and landscape development, methods of locational analysis are very useful for decision makers. Location models and corresponding algorithms are well-studied in the operations research literature and have been discussed by many authors from the theoretical as well as the computational point of view.

An overview on these questions can be found amongst others in the books by Love, Morris and Wesolowsky (1988) [16], Drezner (1995) [6], Hamacher (1995) [11] and Drezner and Hamacher (2002) [7].

In location models in town planning and landscape development the goal is to establish a new facility taking into account a certain set of existing facilities. Such facilities could be living areas, hospitals, warehouses, service centers or fire stations. Whereas, for instance, a hospital is considered as a desirable facility, there are other facilities, like airports, railway stations, dump sites, power plants or wireless stations, that are useful and necessary for the community, but are also a source of negative effects such as noise, smoke or electromagnetic energy. Those facilities are called *semi-obnoxious (push-pull or semi-desirable) facilities*. Existing facilities that shall be close to the new facility are called *attraction points*, whereas those facilities that are asked to be far away are called *repulsion points*. The first attempts on location problems with undesirable facilities go back to Goldman and Dearing [10] and Church and Garfinkel [5].

In this study, we consider gauges in $\mathbb{R}^n$ as distances, which are defined by the well known Minkowski functional. Note that the kind of distances to repulsion points may depend on the kind of aversion. For example noises and stench do not need to pass along streets. Instead it seems to be reasonable to use the Euclidean distance for them. However the distance to other facilities should be measured by a distance function given by the course of the roads. Hence, we consider a location problem with mixed distances.

The goal, when locating a semi-desirable facility, is to minimize the weighted sum of distances between the new facility and the attraction points and to maximize the weighted sum of distances to the repulsive ones. The objective function constitutes as a difference of two convex functions, *D.C. function* for short, so that it is possible to formulate the location problem as a *D.C. programming problem* [1, 2, 12, 18, 19, 20, 21, 29, 30] which, in contrast with the classical Fermat-Weber problem, is non-convex.

Altogether, in this paper we study the continuous problem of locating a single semi-obnoxious facility in the Euclidean space $\mathbb{R}^n$ with a D.C. objective function and mixed distances given by polyhedral gauges.

Our approach strictly differs from the methods developed earlier on location problems with semi-desirable facilities. In the literature, a frequent approach for solving such location problems are branch and bound methods e.g. [4, 8, 17, 25, 29, 31].

The main tool in our paper is to apply the *duality theory* by Toland [28] and Singer [26] for D.C. optimization problems. Contrary to the classical duality theory in convex analysis, the duality theory by Toland and Singer does not need any special assumptions, such as constraint qualifications, on the objective or the constraints of (P), except the convexity and lower semicontinuity of the functions involved in the objective.

Our paper is organized as follows: In Section 2 we introduce the location model (P) in detail and in Section 3 we recall some basic definitions and properties that are applied within this study. In Section 4 we formulate a Toland-Singer dual problem (D) to the primal location problem (P). As known for the classical location problem [9], we introduce the terms elementary convex sets and grids w.r.t. attraction and w.r.t. repulsion for both, the primal and the dual problem (P) and (D). Using the special structure of (P) and (D), we give statements concerning the existence and attainment of finite optimal solutions and present geometrical properties, duality statements and discretization results as well as a relation between primal and dual elementary convex sets, based on results from the theory of geometric duality [13].

Based on the results presented in Section 4, we formulate an algorithm in Section 5, which determines exact solutions for the dual pair of optimization problems (P) and (D) by reducing the non-convex optimization problems to a finite number of linear programs.

2. The Model

The goal, when locating a semi-desirable facility, is to minimize the weighted sum of distances to the attracting points and to maximize the weighted sum of distances to the repulsive ones. In this study, distances are measured by mixed gauges.

For a compact convex set $B \subseteq \mathbb{R}^n$, which contains the origin in its interior, the gauge $\gamma_B : \mathbb{R}^n \to \mathbb{R}$ is defined as the well known Minkowski functional [24]

$$\gamma_B(x) := \min \{\lambda \geq 0 | \, x \in \lambda B\} . \tag{1}$$

The gauge distance from $a \in \mathbb{R}^n$ to $x \in \mathbb{R}^n$ is defined by $\gamma_B(x - a)$ [9]. The set B is called the *unit ball* associated with γ_B. Vice versa, the unit ball B is determined by its gauge γ_B, i.e., $B := \{x \in \mathbb{R}^n | \, \gamma_B(x) \leq 1\}$. Since the set B is supposed to be convex, the gauge γ_B is convex, too. If the set B is a convex polytope then the associated gauge γ_B is said to be a *polyhedral gauge* [9].

In order to formulate the location problem, we consider attraction points $\overline{a}^1, \ldots,$ $\overline{a}^{\overline{M}} \in \mathbb{R}^n$, $(\overline{M} \geq 1)$, with weights $\overline{w}_1, \ldots, \overline{w}_{\overline{M}} > 0$, as well as repulsion points

$\underline{a}^1, \dots, \underline{a}^{\underline{M}} \in \mathbb{R}^n$, ($\underline{M} \geq 1$), with weights $\underline{w}_1, \dots, \underline{w}_{\underline{M}} > 0$, and polyhedral unit balls $\overline{B}_1, \dots, \overline{B}_{\overline{M}}$ and $\underline{B}_1, \dots, \underline{B}_{\underline{M}}$, which induce the gauge distances that we associate with $\overline{a}^1, \dots, \overline{a}^{\overline{M}}$ and $\underline{a}^1, \dots, \underline{a}^{\underline{M}}$, respectively. Throughout this paper we study the location problem with obnoxious facilities given by

$$\alpha := \inf_{x \in \mathbb{R}^n} \{g(x) - h(x)\}, \tag{P}$$

with functions $g, h : \mathbb{R}^n \to \mathbb{R}_+$ defined as

$$g(x) := \sum_{m=1}^{\overline{M}} \overline{\phi}_m(x), \qquad \overline{\phi}_m(x) := \overline{w}_m \gamma_{\overline{B}_m}(x - \overline{a}^m), \quad (m = 1, \dots, \overline{M}), \tag{2}$$

$$h(x) := \sum_{m=1}^{\underline{M}} \underline{\phi}_m(x), \qquad \underline{\phi}_m(x) := \underline{w}_m \gamma_{\underline{B}_m}(x - \underline{a}^m), \quad (m = 1, \dots, \underline{M}). \tag{3}$$

Note that the objective function $g - h : \mathbb{R}^n \to \mathbb{R}$, being a difference of two convex functions g and h, is a D.C. function (see Section 3.2).

3. Preliminaries

Before discussing the location problem (P) in detail, we recall some elementary definitions and results concerning dual distance functions and D.C. optimization.

Throughout the paper, we will use standard notions and notations from convex analysis [22, 24]:

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an arbitrary function. As usual we define the effective domain of f by $\operatorname{dom} f := \{x \in \mathbb{R}^n | \, f(x) < +\infty\}$, its epigraph by $\operatorname{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} | \, r \geq f(x)\}$ and the *(Fenchel-) conjugate* function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of f by

$$f^*(y) := \sup_{x \in \operatorname{dom} f} \{\langle y, x \rangle - f(x)\}.$$

The *subdifferential* of f at a point $\bar{x} \in \operatorname{dom} f$ is defined as the set

$$\partial f(\bar{x}) := \{p \in \mathbb{R}^n | \, \forall x \in \mathbb{R}^n : \; f(x) - f(\bar{x}) \geq \langle p, x - \bar{x} \rangle\}.$$

Let $B \subseteq \mathbb{R}^n$ be a convex set. The *normal cone* to B at x is defined by

$$N_B(x) := \begin{cases} \{p \in \mathbb{R}^n | \, \forall y \in B : \; \langle p, y - x \rangle \leq 0\} & \text{if } x \in B, \\ \emptyset & \text{if } x \notin B. \end{cases}$$

The indicator function $\mathbb{I}_B : \mathbb{R}^n \to \{0, +\infty\}$ of B is given by

$$\mathbb{I}_B(x) := \begin{cases} 0 & \text{if } x \in B, \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, the *infimal convolution* of the convex functions $f_1, \dots, f_M : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f(x) := (f_1 \Box \cdots \Box f_M)(x) := \inf \left\{ \sum_{m=1}^{M} f_m(x^m) \,\middle|\, \sum_{m=1}^{M} x^m = x \right\}.$$

3.1. Dual Distance Functions

The polar set

$$B^* := \{ y \in \mathbb{R}^n | \, \forall x \in B : \; \langle x, y \rangle \leq 1 \} \tag{4}$$

defines the dual unit ball of B, and the dual gauge of γ_B is the gauge associated with B^*.

If B is polyhedral then its polar B^* is polyhedral, too (and hence if γ_B is a polyhedral gauge then so is the dual gauge γ_{B^*}). The dual unit ball B^* is also a closed bounded convex set in $\mathbb{R}^n$, which contains the origin in its interior [24]. It follows directly from the definitions in (1) and (4) that

$$\gamma_{B^*}(y) = \max_{x \in B} \langle x, y \rangle \,. \tag{5}$$

The polar set $B^{**} := (B^*)^*$ of the polar set B^* is the set B itself [24], i.e., $\gamma_B = \gamma_{B^{**}}$. Hence, by (5), a gauge can also be written as

$$\gamma_B(x) = \max_{y \in B^*} \langle x, y \rangle \,. \tag{6}$$

3.2. D.C. Functions and D.C. Programming Problems

Throughout this paper we follow the notational convention that $(+\infty) - (+\infty) = (+\infty)$. Consider two convex functions $g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The function $g - h$ is called *D.C.*, and the minimization problem

$$\inf_{x \in \mathbb{R}^n} \{ g(x) - h(x) \}$$

is said to be a *D.C. programming problem*. From the duality theory by Toland [28] and Singer [26] we obtain the following theorem, which provides a dual D.C. optimization problem for a given primal D.C. program, (cf. also [15]).

Theorem 3.1 ([26, 28]). *Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, and let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper closed convex function. Then*

$$\inf_{x \in \operatorname{dom} g} \{ g(x) - h(x) \} = \inf_{y \in \operatorname{dom} h^*} \{ h^*(y) - g^*(y) \},$$

where g^* and h^* are the conjugate functions of g and h.

Note that in Theorem 3.1 the minimization problem on the left hand side is considered to be the primal problem and the one on the right hand side is the corresponding dual problem.

Theorem 3.2 ([15, 30]). *Let $g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper closed convex functions. If $\widehat{x} \in \operatorname{dom} g \cap \operatorname{dom} h$ is a global minimizer of $g - h$ on $\mathbb{R}^n$, then $\partial h(\widehat{x}) \subseteq \partial g(\widehat{x})$. In the same way, if $\widehat{y} \in \operatorname{dom} h^* \cap \operatorname{dom} g^*$ is a global minimizer of $h^* - g^*$ on $\mathbb{R}^n$, then $\partial g^*(\widehat{y}) \subseteq \partial h^*(\widehat{y})$.*

An important duality relationship for D.C. minimization problems is given in the following proposition.

Proposition 3.3 ([15, Proposition 4.7]). *Let g, h be given as in Theorem 3.1. Further, let $\mathcal{X}$ be the set of minimizers of $g - h$ and $\mathcal{Y}$ be the set of minimizers of $h^* - g^*$. Then*

$$\mathcal{X} \supseteq \bigcup_{y \in \mathcal{Y}} \partial g^*(y), \qquad \mathcal{Y} \supseteq \bigcup_{x \in \mathcal{X}} \partial h(x).$$

Remark 3.4. In the case when g and h are polyhedral convex functions, we even have equalities in Proposition 3.3.

Proof. We only prove the first equality, the second one follows analogously.

Let $\widehat{x} \in \mathcal{X}$. Then it holds, by Proposition 3.3, that $\partial h(\widehat{x}) \subseteq \mathcal{Y}$, and by Theorem 3.2 we have $\partial h(\widehat{x}) \subseteq \partial g(\widehat{x})$. Due to the fact that h is a polyhedral function we have $\partial h(\widehat{x}) \neq \emptyset$. For $\widehat{y} \in \partial h(\widehat{x})$ we have, by [14, Corollary 1.4.4], that $\widehat{x} \in \partial g^*(\widehat{y})$, and thus the opposite inclusion

$$\mathcal{X} \subseteq \bigcup_{\widehat{y} \in \mathcal{Y}} \partial g^*(\widehat{y})$$

is satisfied as well. □

3.3. Elementary Convex Sets

For the classical Fermat-Weber-Problem (cf. [33]) the definition of elementary convex sets was introduced in [9]. Since each of the functions g and h (see (2), (3)) has the same structure as the objective function in the classical Fermat-Weber-Problem, we introduce the following generalized notations in order to distinguish between attraction and repulsion:

Definition 3.5. Consider the location problem (P). We say that a non-empty polyhedron $\mathcal{C} \subseteq \mathbb{R}^n$ is an *elementary convex set w.r.t. attraction*, if there exists a tuple $\overline{\pi} = \left\{\overline{y}^1, \ldots, \overline{y}^{\overline{M}}\right\} \in \overline{w}_1 \overline{B}_1^* \times \ldots \times \overline{w}_{\overline{M}} \overline{B}_{\overline{M}}^*$, such that

$$\mathcal{C} = \bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m)\right],$$

and we say that a non-empty polyhedron $\mathcal{C} \subseteq \mathbb{R}^n$ is an *elementary convex set w.r.t. repulsion*, if there exists a tuple $\underline{\pi} = \left\{\underline{y}^1, \ldots, \underline{y}^{\underline{M}}\right\} \in \underline{w}_1 \underline{B}_1^* \times \ldots \times \underline{w}_{\underline{M}} \underline{B}_{\underline{M}}^*$, such that

$$\mathcal{C} = \bigcap_{m=1}^{\underline{M}} \left[\underline{a}^m + N_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m)\right].$$

Moreover, we call the sets

$$\overline{G} := \bigcup_{m=1}^{\overline{M}} \bigcup_{e \in \operatorname{ext}(\overline{B}_m)} \overline{a}^m + \mathbb{R}_+ e \quad \text{and} \quad \underline{G} := \bigcup_{m=1}^{\underline{M}} \bigcup_{e \in \operatorname{ext}(\underline{B}_m)} \underline{a}^m + \mathbb{R}_+ e$$

the *(construction) grids* w.r.t. attraction and w.r.t. repulsion, respectively. A point $x \in \mathbb{R}^n$ is called *grid point* (also *intersection point*) w.r.t. attraction (repulsion), if x is an extreme point of an elementary convex set $\mathcal{C}$ w.r.t. attraction (resp. repulsion). We denote by $\overline{\mathcal{I}}$ and $\underline{\mathcal{I}}$ the sets of all grid points w.r.t. attraction and w.r.t. repulsion, respectively.

Consider a unit ball $B \subseteq \mathbb{R}^n$ and a weight $w > 0$. Then the normal cone to the weighted dual ball wB^* at $y \in \operatorname{bd} wB^*$ is the convex cone $N_{wB^*}(y) = \{\lambda x | \, x \in F(y), \, \lambda \geq 0\}$ generated by the exposed face $F(y) := \{x \in B | \, \langle y, x\rangle = w\}$ of B, see [9, Section 2].

Example 3.6. Consider a location problem with two attraction points, $\overline{a}^1 = (2,2)^T$ and $\overline{a}^2 = (9,4)^T$, and one repulsion point, $\underline{a}^1 = (7,1)^T$, and the corresponding unit balls given by their extreme points:

$$\begin{aligned} \operatorname{ext}(\overline{B}_1) &= \{(1,1),(1,-1),(-1,1),(-1,-1)\}, \\ \operatorname{ext}(\overline{B}_2) &= \{(1,0),(0,1),(-1,0),(0,-1)\}, \\ \operatorname{ext}(\underline{B}_1) &= \{(1,0),(0,1),(1,-1)\}. \end{aligned}$$

The corresponding construction grids $\overline{G}$ and $\underline{G}$ are illustrated in Figure 3.1.

4. Duality Results

In the following we formulate a dual problem (D) to the given problem (P) according to the duality theory by Toland and Singer, see Theorem 3.1. We present a necessary and sufficient condition for the existence and attainment of finite optimal solutions of the dual pair of optimization problems (P) and (D), we develop some geometrical properties and duality statements and we introduce the terms elementary convex sets and grids with respect to attraction and to repulsion for the dual problem (D). Finally, we present discretization results for both problems.

Using some basic calculus rules for conjugate functions [14], the conjugates $\overline{\phi}_m^*, \underline{\phi}_m^* : \mathbb{R}^n \to \mathbb{R}$ of the weighted distance functions $\overline{\phi}_m$ and $\underline{\phi}_m$, see (2) and (3), are determined by

$$\overline{\phi}_m^*(\overline{y}^m) = \langle \overline{y}^m, \overline{a}^m\rangle + \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m), \quad (m = 1, \ldots, \overline{M}), \tag{7}$$

$$\underline{\phi}_m^*(\underline{y}^m) = \left\langle \underline{y}^m, \underline{a}^m\right\rangle + \mathbb{I}_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m), \quad (m = 1, \ldots, \underline{M}), \tag{8}$$

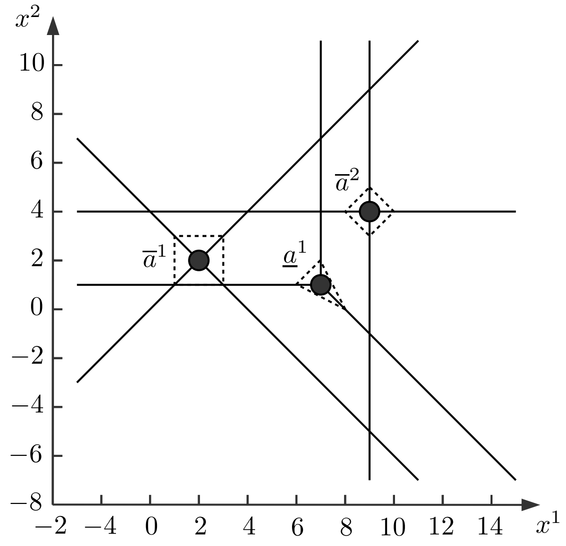


Figure 3.1: Construction grid $\overline{G}$ with respect to the attraction points $\overline{a}^1 = (2,2)^T$ and $\overline{a}^2 = (9,4)^T$ and their assigned unit balls $\overline{B}_1, \overline{B}_2$; construction grid $\underline{G}$ with respect to the repulsion point $\underline{a}^1 = (7,1)^T$ and its assigned unit ball $\underline{B}_1$.

where the indicator functions with respect to the weighted dual unit balls $\overline{w}_m\overline{B}_m^*$ and $\underline{w}_m\underline{B}_m^*$ are induced by the gauges that appear in (2) and (3), and the inner products $\langle \overline{y}^m, \overline{a}^m \rangle$ and $\langle \underline{y}^m, \underline{a}^m \rangle$ are induced by the shiftings $x - \overline{a}^m$ and $x - \underline{a}^m$ in (2) and (3).

Since the unit balls $\overline{B}_1, \ldots, \overline{B}_{\overline{M}}$ and $\underline{B}_1, \ldots, \underline{B}_{\underline{M}}$ are supposed to be closed convex sets, the weighted distances $\overline{\phi}_1, \ldots, \overline{\phi}_{\overline{M}}$ and $\underline{\phi}_1, \ldots, \underline{\phi}_{\underline{M}}$ are convex functions whose domains coincide with $\mathbb{R}^n$. Thus, the conjugates of g and h are given by the infimal convolutions $g^* = \overline{\phi}_1^* \Box \ldots \Box \overline{\phi}_{\overline{M}}^*$ and $h^* = \underline{\phi}_1^* \Box \ldots \Box \underline{\phi}_{\underline{M}}^*$, respectively.

The following minimization problem is the **Toland-Singer dual problem** of (P):

$$\beta := \inf_{y \in \mathbb{R}^n} \{h^*(y) - g^*(y)\}, \tag{D}$$

with conjugates $g^*, h^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ given by

$$g^*(y) = \inf \left\{ \sum_{m=1}^{\overline{M}} \left[\mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m) + \langle \overline{y}^m, \overline{a}^m \rangle \right] \,\middle|\, \sum_{m=1}^{\overline{M}} \overline{y}^m = y \right\}, \tag{9}$$

$$h^*(y) = \inf \left\{ \sum_{m=1}^{\underline{M}} \left[\mathbb{I}_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m) + \langle \underline{y}^m, \underline{a}^m \rangle \right] \,\middle|\, \sum_{m=1}^{\underline{M}} \underline{y}^m = y \right\}. \tag{10}$$

4.1. Existence of Finite Optimal Solutions

The weights of repulsive facilities $\underline{a}^1, \ldots, \underline{a}^{\underline{M}}$ in the location problem (P) may have a strong influence on the optimal objective value. It may even happen that the objective function does not have a lower bound, i.e., $\alpha = -\infty$. In order to avoid this case, we give a necessary and sufficient condition in Theorem 4.2 below for the existence of a finite optimal solution.

Proposition 4.1 ([27, Remark 8.3]). *Let $g, h : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be two arbitrary functions and g^*, h^* be the corresponding conjugates. If g and h satisfy $\alpha := \inf_{x\in\mathbb{R}^n} \{g(x) - h(x)\} > -\infty$ then we have* $\operatorname{dom} g \subseteq \operatorname{dom} h$.
If $\alpha > -\infty$ then also $\beta := \inf_{y\in\mathbb{R}^n} \{h^*(y) - g^*(y)\} > -\infty$ and hence $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*$.

Theorem 4.2 (Finiteness Criterion). (i) *The optimal objective value of the dual pair of optimization problems (P) and (D) is finite, i.e., $\beta := \inf_{y\in\mathbb{R}^n}\{h^*(y) - g^*(y)\} = \inf_{y\in\mathbb{R}^n}\{g(x) - h(x)\} =: \alpha \in \mathbb{R}$, if and only if*

$$\sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^* \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*. \tag{11}$$

(ii) *Further the dual optimization problem (D) attains a minimal solution whenever* $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*$. *Then also the primal problem (P) does. Hence we can substitute infimum by minimum in both problems.*

Proof. (i) The function g^*, as defined in (9), attains a finite objective value at $y \in \mathbb{R}^n$ if and only if there exists a tuple $\left(\overline{y}^1, \dots, \overline{y}^{\overline{M}}\right)$ such that $\sum_{m=1}^{\overline{M}} \overline{y}^m = y$ and $\overline{y}^m \in \overline{w}_m \overline{B}_m^*$ for all $m = 1, \dots, \overline{M}$, i.e.,

$$g^*(y) \in \mathbb{R} \quad \Leftrightarrow \quad y \in \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*.$$

An analogous statement holds for h^*. Hence, the corresponding effective domains of h^* and g^* are

$$\operatorname{dom} h^* = \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^* (\neq \emptyset) \quad \text{and} \quad \operatorname{dom} g^* = \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^* (\neq \emptyset). \tag{12}$$

By Proposition 4.1 we have

$$\beta \in \mathbb{R} \quad \Rightarrow \quad \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^* \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*.$$

Further, if $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*$ then the objective value $h^*(y) - g^*(y)$ is finite for all $y \in \operatorname{dom} h^*$ and hence we also have

$$\beta \in \mathbb{R} \quad \Leftarrow \quad \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^* \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*.$$

(ii) The effective domain of h^* as the weighted sum of a finite number of convex, closed and bounded balls $\underline{B}_1^*, \ldots, \underline{B}_{\underline{M}}^*$, is bounded and closed itself.

The function g^*, as the conjugate of the polyhedral function g, is polyhedral, too, and can be decomposed as the sum of a piece-wise affine function $\widehat{g}$ and the indicator function $\mathbb{I}_{\operatorname{dom} g^*}$, so that $g^* = \widehat{g} + \mathbb{I}_{\operatorname{dom} g^*}$, see [3]. Analogously, h^* can be decomposed as the sum of a piece-wise affine function $\widehat{h}$ and the indicator function $\mathbb{I}_{\operatorname{dom} h^*}$, so that $h^* = \widehat{h} + \mathbb{I}_{\operatorname{dom} h^*}$. Consequently, we know from Weierstrass Theorem that the objective function $h^* - g^* = (\widehat{h} + \mathbb{I}_{\operatorname{dom} h^*}) - (\widehat{g} + \mathbb{I}_{\operatorname{dom} g^*})$ attains a minimum in $\operatorname{dom} h^* \cap \operatorname{dom} g^*$ and thus in $\operatorname{dom} h^*$ when $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*$; hence, in this case the dual optimization problem (D)) has an optimal solution and its optimal value is finite. Then the same can be said about the primal problem (P), and so we can substitute infimum by minimum in both problems. □

By Theorem 4.2 we have a necessary and sufficient condition for the existence and attainment of finite optimal solutions of the dual pair of optimization problems (P) and (D).

Note that in the case that all unit balls $\overline{B}_m$ $(m = 1, \ldots, \overline{M})$ and $\underline{B}_m$ $(m = 1, \ldots, \underline{M})$ coincide, condition (11) simplifies as follows:

$$\beta := \inf_{y \in \mathbb{R}^n} \{h^*(y) - g^*(y)\} \in \mathbb{R} \quad \Leftrightarrow \quad \sum_{m=1}^{\underline{M}} \underline{w}_m \le \sum_{m=1}^{\overline{M}} \overline{w}_m.$$

Hence Theorem 4.2 is a generalization of Theorem 1 in [8], see also [23, Theorem 2.3].

4.2. Geometrical Properties of Primal and Dual Elements

In the following we present geometrical properties, duality results, optimality conditions and discretization results for the pair of dual optimization problems (P) and (D).

Proposition 4.3. *For* $\overline{\phi}_1^*, \ldots, \overline{\phi}_{\overline{M}}^* : \mathbb{R}^n \to \mathbb{R}$ *as given in* (7)*, and* $\underline{\phi}_1^*, \ldots, \underline{\phi}_{\underline{M}}^* : \mathbb{R}^n \to \mathbb{R}$ *as defined in* (8) *we have*

$$\partial \overline{\phi}_m^*(\overline{y}^m) = \overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m), \quad (m = 1, \ldots, \overline{M}),$$
$$\partial \underline{\phi}_m^*(\underline{y}^m) = \underline{a}^m + N_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m), \quad (m = 1, \ldots, \underline{M}).$$

Proof. From (7) we obtain, for all $m = 1, \ldots, \overline{M}$, that

$$\partial \overline{\phi}_m^*(\overline{y}^m) = \partial \left[\langle \overline{y}^m, \overline{a}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m) \right]. \tag{13}$$

Since the linear function $\langle \cdot, \overline{a}^m \rangle$ is convex and continuous and the indicator function $\mathbb{I}_{\overline{w}_m \overline{B}_{m*}}$ is convex, we know that

$$\partial \left[\langle \overline{y}^m, \overline{a}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m) \right] = \partial \langle \overline{y}^m, \overline{a}^m \rangle + \partial \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m). \tag{14}$$

Further, it holds $\partial \langle \overline{y}^m, \overline{a}^m \rangle = \{\overline{a}^m\}$ and

$$\partial \mathbb{I}_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m) = N_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m).$$

Thus, by (13) and (14), we have

$$\partial \overline{\phi}^*_m(\overline{y}^m) = \overline{a}^m + N_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m).$$

The second equality follows analogously, by taking into account (8). $\square$

Proposition 4.4. *The subdifferentials $\partial g^*(y)$ and $\partial h^*(y)$ are non-empty for all $y \in \operatorname{dom} g^*$ and $y \in \operatorname{dom} h^*$, respectively.*

Proof. As in the proof of Theorem 4.2, we have $g^* = \widehat{g} + \mathbb{I}_{\operatorname{dom} g^*}$. By applying the sum rule for subdifferentials we obtain for all $y \in \operatorname{dom} g^*$ that $\partial g^*(y) \supseteq \partial \widehat{g}(y) + \partial \mathbb{I}_{\operatorname{dom} g^*}(y) = \partial \widehat{g}(y) + N_{\operatorname{dom} g^*}(y)$; the latter set is obviously non-empty for all $y \in \operatorname{dom} g^*$. Analogously, we obtain $\partial h^*(y) \neq \emptyset$ for all $y \in \operatorname{dom} h^*$. $\square$

Theorem 4.5. *For each tuple $\overline{\pi} = (\overline{y}^1, \ldots, \overline{y}^{\overline{M}}) \in \overline{w}_1 \overline{B}^*_1 \times \cdots \times \overline{w}_{\overline{M}} \overline{B}^*_{\overline{M}}$ the following statements are equivalent:*

1. $\bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}^+_m}(\overline{y}^m)\right] \neq \emptyset,$
2. *g^* is exact at $y := \sum_{m=1}^{\overline{M}} \overline{y}^m$, i.e., $g^*(y) = \sum_{m=1}^{\overline{M}} \langle \overline{a}^m, \overline{y}^m \rangle$,*
3. $\bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}^+_m}(\overline{y}^m)\right] = \partial g^* \left(\sum_{m=1}^{\overline{M}} \overline{y}^m\right).$

Moreover, for each $y \in \operatorname{dom} g^*$ there exists a tuple $\overline{\pi} \in \overline{w}_1 \overline{B}^*_1 \times \cdots \times \overline{w}_{\overline{M}} \overline{B}^*_{\overline{M}}$, such that these statements hold.

Analogously, for each tuple $\underline{\pi} = (\underline{y}^1, \ldots, \underline{y}^{\underline{M}}) \in \underline{w}_1 \underline{B}^*_1 \times \cdots \times \underline{w}_{\underline{M}} \underline{B}^*_{\underline{M}}$ the following statements are equivalent:

1. $\bigcap_{m=1}^{\underline{M}} \left[\underline{a}^m + N_{\underline{w}_m \underline{B}^*_m}(\underline{y}^m)\right] \neq \emptyset,$
2. *h^* is exact at $y := \sum_{m=1}^{\underline{M}} \underline{y}^m$, i.e., $h^*(y) = \sum_{m=1}^{\underline{M}} \langle \underline{a}^m, \underline{y}^m \rangle$,*
3. $\bigcap_{m=1}^{\underline{M}} \left[\underline{a}^m + N_{\underline{w}_m \underline{B}^*_m}(\underline{y}^m)\right] = \partial h^* \left(\sum_{m=1}^{\underline{M}} \underline{y}^m\right).$

Moreover, for each $y \in \operatorname{dom} h^*$ there exists a tuple $\underline{\pi} \in \underline{w}_1 \underline{B}^*_1 \times \cdots \times \underline{w}_{\underline{M}} \underline{B}^*_{\underline{M}}$ such that these statements hold.

Proof. We will only prove the statements concerning g^*. Those concerning h^* can be proved analogously.

1. $\Rightarrow$ 2. $\Rightarrow$ 3.: By Proposition 4.3 we have

$$\bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m)\right] = \bigcap_{m=1}^{\overline{M}} \partial \overline{\phi}^*_m(\overline{y}^m),$$

and from the definition of g^* in (9) we obtain

$$\partial \left(\overline{\phi}^*_1 \square \ldots \square \overline{\phi}^*_{\overline{M}}\right)(y) = \partial g^*(y).$$

The assertions follow directly.

3. $\Rightarrow$ *1.*: For $(\overline{y}^1, \dots, \overline{y}^{\overline{M}}) \in \overline{w}_1\overline{B}_1^* \times \cdots \times \overline{w}_{\overline{M}}\overline{B}_{\overline{M}}^*$ it obviously follows from (12) that

$$y := \sum_{m=1}^{\overline{M}} \overline{y}^m \in \sum_{m=1}^{\overline{M}} \overline{w}_m\overline{B}_m^* = \operatorname{dom} g^*.$$

The assertion follows since $\partial g^*(y) \neq \emptyset$ for all $y \in \operatorname{dom} g^*$, see Proposition 4.4.

The existence of such a tuple $(\overline{y}^1, \dots, \overline{y}^{\overline{M}}) \in \overline{w}_1\overline{B}_1^* \times \cdots \times \overline{w}_{\overline{M}}\overline{B}_{\overline{M}}^*$ follows directly. $\square$

Proposition 4.6. *For each $x \in \mathbb{R}^n$ the subdifferentials $\partial g(x)$ and $\partial h(x)$ are given by*

$$\partial g(x) = \sum_{m=1}^{\overline{M}} \operatorname*{argmax}_{y \in \overline{w}_m\overline{B}_m^*} \langle x - \overline{a}^m, y\rangle, \qquad \partial h(x) = \sum_{m=1}^{\underline{M}} \operatorname*{argmax}_{y \in \underline{w}_m\underline{B}_m^*} \langle x - \underline{a}^m, y\rangle.$$

Proof. It holds by (2) that

$$\partial g(x) = \partial \sum_{m=1}^{\overline{M}} \overline{\phi}_m(x) = \sum_{m=1}^{\overline{M}} \partial\overline{\phi}_m(x)$$

for all $x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$, then for all $m = 1, \dots, \overline{M}$ we obtain the following equivalences:

$$\begin{aligned}
&\overline{y}^m \in \partial\overline{\phi}_m(x) && \\
\Leftrightarrow\ & x \in \partial\overline{\phi}_m^*(\overline{y}^m) && (15)\\
\Leftrightarrow\ & x \in \overline{a}^m + N_{\overline{w}_m\overline{B}_m^*}(\overline{y}^m) && (16)\\
\Leftrightarrow\ & x - \overline{a}^m \in N_{\overline{w}_m\overline{B}_m^*}(\overline{y}^m) && \\
\Leftrightarrow\ & \forall y \in \overline{w}_m\overline{B}_m^* : \langle x - \overline{a}^m, y - \overline{y}^m\rangle \leq 0 \quad \text{and} \quad \overline{y}^m \in \overline{w}_m\overline{B}_m^* && (17)\\
\Leftrightarrow\ & \forall y \in \overline{w}_m\overline{B}_m^* : \langle x - \overline{a}^m, y\rangle \leq \langle x - \overline{a}^m, \overline{y}^m\rangle \quad \text{and} \quad \overline{y}^m \in \overline{w}_m\overline{B}_m^* && \\
\Leftrightarrow\ & \overline{y}^m \in \operatorname*{argmax}_{y \in \overline{w}_m\overline{B}_m^*} \langle x - \overline{a}^m, y\rangle. &&
\end{aligned}$$

Equivalence (15) holds by Corollary 1.4.4 in [14], and (16) holds by Proposition 4.3. Further, equivalence (17) holds by the definition of the normal cone. The proof of the second equality follows analogously. $\square$

Corollary 4.7. *For $x \in \mathbb{R}^n$ and $\overline{y}^m \in \overline{w}_m\overline{B}_m^*$, $m \in \{1, \dots, \overline{M}\}$, the following equivalence holds:*

$$x \in \overline{a}^m + N_{\overline{w}_m\overline{B}_m^*}(\overline{y}^m) \quad \Leftrightarrow \quad \overline{y}^m \in \operatorname*{argmax}_{y \in \overline{w}_m\overline{B}_m^*} \langle x - \overline{a}^m, y\rangle,$$

and analogously, for $x \in \mathbb{R}^n$ and $\underline{y}^m \in \underline{w}_m\underline{B}_m^*$, $m \in \{1, \dots, \underline{M}\}$, we have

$$x \in \underline{a}^m + N_{\underline{w}_m\underline{B}_m^*}(\underline{y}^m) \quad \Leftrightarrow \quad \underline{y}^m \in \operatorname*{argmax}_{y \in \underline{w}_m\underline{B}_m^*} \langle x - \underline{a}^m, y\rangle.$$

Proof. By Proposition 4.3 and Proposition 4.6 we have

$$\partial\overline{\phi}_m^*(\overline{y}^m) = \overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m), \quad \text{and} \quad \partial\overline{\phi}_m(x) = \operatorname*{argmax}_{y\in\overline{w}_m\overline{B}_m^*} \langle x - \overline{a}^m, y\rangle .$$

The assertion follows. □

From Theorem 4.5 and Definition 3.5 we know that the elementary convex sets for the primal problem (P) coincide with the subdifferentials of the conjugate functions g^* and h^*. Analogously, we define dual elementary convex sets using the subdifferentials of the functions g and h:

Definition 4.8. Consider the dual problem (D), for $x \in \mathbb{R}^n$ we call the sets

$$\partial g(x) = \sum_{m=1}^{\overline{M}} \operatorname*{argmax}_{y\in\overline{w}_m\overline{B}_m^*} \langle x - \overline{a}^m, y\rangle , \quad \text{and} \quad \partial h(x) = \sum_{m=1}^{\underline{M}} \operatorname*{argmax}_{y\in\underline{w}_m\underline{B}_m^*} \langle x - \underline{a}^m, y\rangle .$$

dual elementary convex set w.r.t. attraction and w.r.t. repulsion, respectively.

The corresponding *dual construction grids* are given by

$$\overline{G}_D := \bigcup_{\{x\in\mathbb{R}^n \,|\, \dim \partial g(x)=1\}} \partial g(x), \qquad \underline{G}_D := \bigcup_{\{x\in\mathbb{R}^n \,|\, \dim \partial h(x)=1\}} \partial h(x).$$

A point $y \in \mathbb{R}^n$ is said to be a *dual grid point* (also *dual intersection point*) w.r.t. attraction (repulsion), if y is an extreme point of an elementary convex set w.r.t. attraction (resp. repulsion). We denote by $\overline{\mathcal{I}}_D$ and $\underline{\mathcal{I}}_D$ the sets of all grid points w.r.t. attraction and repulsion, respectively.

Using the following duality statements we are able to determine the set of optimal solutions of (D) out of the set of optimal solutions of (P) and vice versa. The following corollary provides a sufficient optimality condition for dual solutions based on the knowledge about primal optimal solutions:

Corollary 4.9. *Let $x \in \bigcap_{m=1}^{\underline{M}}[\underline{a}^m + N_{\underline{w}_m\underline{B}_m^*}(\underline{y}^m)]$ be an optimal solution of the primal problem (P). Then $\sum_{m=1}^{\underline{M}} \underline{y}^m$ is an optimal solution of the dual problem (D).*

Proof. By Corollary 4.7 and $x \in \bigcap_{m=1}^{\underline{M}}[\underline{a}^m + N_{\underline{w}_m\underline{B}_m}(\underline{y}^m)]$, we have

$$\underline{y}^m \in \operatorname*{argmax}_{y\in\underline{w}_m\underline{B}_m^*} \left\langle x - \underline{a}^m, \underline{y}\right\rangle$$

for all $m = 1, \ldots, \underline{M}$. By Proposition 4.6, we have $\sum_{m=1}^{\underline{M}} \underline{y}^m \in \partial h(x)$ and thus, by Proposition 3.3, $\sum_{m=1}^{\underline{M}} \underline{y}^m$ is an optimal solution of the dual problem (D). □

The following theorem provides a sufficient optimality condition for primal solutions based on the knowledge about dual optimal solutions:

Corollary 4.10. *Let y be an optimal solution of the dual problem (D) such that g^* is exact at $y = \sum_{m=1}^{\overline{M}} \overline{y}^m$ for $\overline{y}^m \in \overline{w}_m \overline{B}^*_m$, $m = 1, \dots, \overline{M}$. Then each $x \in \bigcap_{m=1}^{\overline{M}} [\overline{a}^m + N_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m)]$ is optimal for the primal problem (P).*

Proof. Let y be an optimal solution of the dual problem (D) and assume that g^* is exact at $y = \sum_{m=1}^{\overline{M}} \overline{y}^m$. Then, by Theorem 4.5, we know that $\bigcap_{m=1}^{\overline{M}} [\overline{a}^m + N_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m)] = \partial g^*(y)$; this set is non-empty, as we know from Proposition 4.4. The assertion follows from Proposition 3.3. □

In preparation for an algorithm for solving the location problem (P) with obnoxious facilities, the following discretization result plays a major role.

Theorem 4.11 (Discretization Result). *The set $\underline{\mathcal{I}}_D$ of dual grid points in $\underline{G}_D$ w.r.t. repulsion is a finite dominating set for the optimal points of the dual problem (D), i.e., $\underline{\mathcal{I}}_D \cap \mathcal{Y} \neq \emptyset$, where $\mathcal{Y}$ is the set of minimizers of (D).*

Proof. From Remark 3.4 we know that $\mathcal{Y} = \bigcup_{x \in \mathcal{X}} \partial h(x)$, where the subdifferentials $\partial h(x)$ are polyhedral elementary convex sets. Hence, we have $\operatorname{ext}(\partial h(x)) \subseteq \underline{\mathcal{I}}_D \cap \mathcal{Y}$ for all $x \in \mathcal{X}$. □

Note that one can formulate an analogous discretization result concerning the set $\overline{\mathcal{I}}$ and Problem (P).

4.3. Assignment between Primal and Dual Elements

In the following a description of the relationship between primal and dual elementary convex sets is developed by applying results from the field of geometric duality [13].

Since the function g, as given in the location problem (P), and its conjugate g^* are polyhedral and convex functions, their epigraphs $\operatorname{epi} g \subseteq \mathbb{R}^{n+1}$ and $\operatorname{epi} g^* \subseteq \mathbb{R}^{n+1}$ are polyhedral convex sets.

We consider the cone

$$K := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} |\ x = 0, r \geq 0\}. \tag{18}$$

A proper face F of $\operatorname{epi} g$ is called K-minimal if all of its points are minimal with respect to the cone K, i.e.,

$$\forall y \in F : \ (\{y\} - (K \backslash (-K))) \cap \operatorname{epi} g = \emptyset.$$

According to [13, Propositon 3.2], each K-minimal face $\overline{F}$ of $\operatorname{epi} g$ is given by

$$\overline{F} = \{(x, g(x)) |\ x \in \partial g^*(y)\} := \overline{F}_y,$$

for some $y \in \operatorname{dom} g^*$, and each K-minimal face $\overline{F}^*$ of $\operatorname{epi} g^*$ is given by

$$\overline{F}^* = \{(y, g^*(y)) |\ y \in \partial g(x)\} := \overline{F}^*_x,$$

for some $x \in \operatorname{dom} g$. According to [13, Theorem 3.3 and Subsection 4.2], the mapping $\psi^* : 2^{\mathbb{R}^{n+1}} \to 2^{\mathbb{R}^{n+1}}$ defined by

$$\psi^*(\overline{F}) = \bigcap_{(x,g(x))\in\overline{F}} \overline{F}^*_x$$

provides an inclusion-reversing one-to-one map between the K-minimal faces of $\operatorname{epi} g$ and the K-minimal faces of $\operatorname{epi} g^*$, such that

$$\dim \overline{F} + \dim \psi^*(\overline{F}) = n.$$

Remark 4.12. Let $y \in \operatorname{dom} g^*$. The projection of a K-minimal face $\overline{F}_y = \{(x, g(x))|\, x \in \partial g^*(y)\}$ of $\operatorname{epi} g$ onto $\mathbb{R}^n$ is a polyhedral convex set in $\mathbb{R}^n$, which coincides with the elementary convex set $\partial g^*(y)$. On the other hand, each elementary convex set $\partial g^*(y)$ generates a face $\overline{F}_y$. Hence, there is a one-to-one mapping between the K-minimal faces $\overline{F}_y$ of $\operatorname{epi} g^*$ and the primal elementary convex sets $\partial g^*(y)$, and it holds $\dim \overline{F}_y = \dim \partial g^*(y)$.

Analogously, there is a one-to-one mapping between the K-minimal faces $\overline{F}^*_x$ of $\operatorname{epi} g$ and the dual elementary convex sets $\partial g(x)$, and it holds $\dim \overline{F}^*_x = \dim \partial g(x)$.

Consequently, the following duality result holds:

Theorem 4.13. *Let* $\overline{\mathfrak{C}} := \{\partial g^*(y)|\, y \in \operatorname{dom} g^*\} \subseteq 2^{\mathbb{R}^n}$ *and* $\overline{\mathfrak{D}} := \{\partial g(x)|\, x \in \mathbb{R}^n\} \subseteq 2^{\mathbb{R}^n}$. *The function* $\Phi^* : \overline{\mathfrak{C}} \to \overline{\mathfrak{D}}$ *defined by*

$$\Phi(\partial g^*(y)) := \bigcap_{x\in\partial g^*(y)} \partial g(x)$$

provides an inclusion-reversing one-to-one map between the primal and dual elementary convex sets, such that

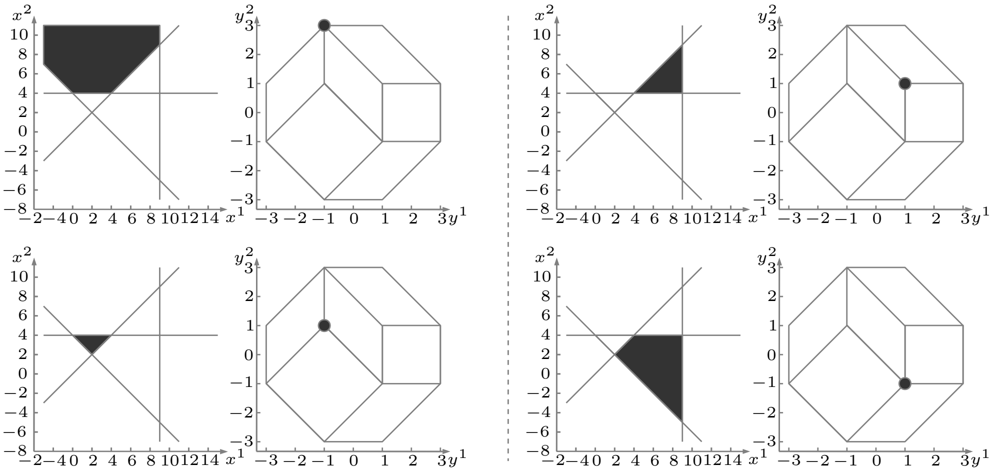
$$\dim \partial g^*(y) + \dim \Phi^*(\partial g^*(y)) = n.$$

Note that Theorem 4.13 analogously holds for the function h and its conjugate h^*.

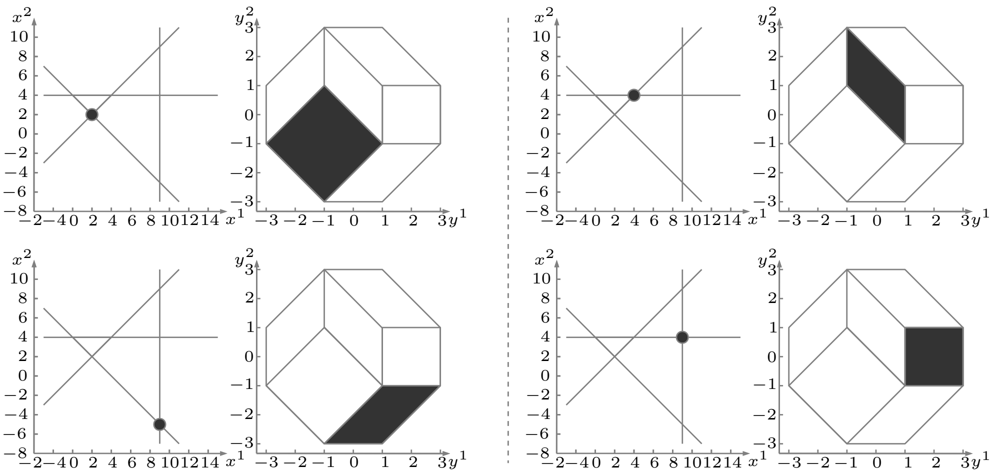
The assignment between primal and dual elementary convex sets in case of the 2-dimensional space $\mathbb{R}^2$ is illustrated in Figure 4.1. The figure refers to the attracting points of Example 3.6 with weights $\overline{w}_1 = 2$ and $\overline{w}_2 = 1$ and illustrates the primal and dual construction grids $\overline{G}$ and $\overline{G}_D$ w.r.t. attraction. The sum of the dimensions in each dual pair of assigned elementary convex sets is $\dim \partial g^*(y) + \dim \overline{\Psi}(\partial g^*(y)) = 2$.

5. An Algorithm for Solving Location Problems with Obnoxious Facilities

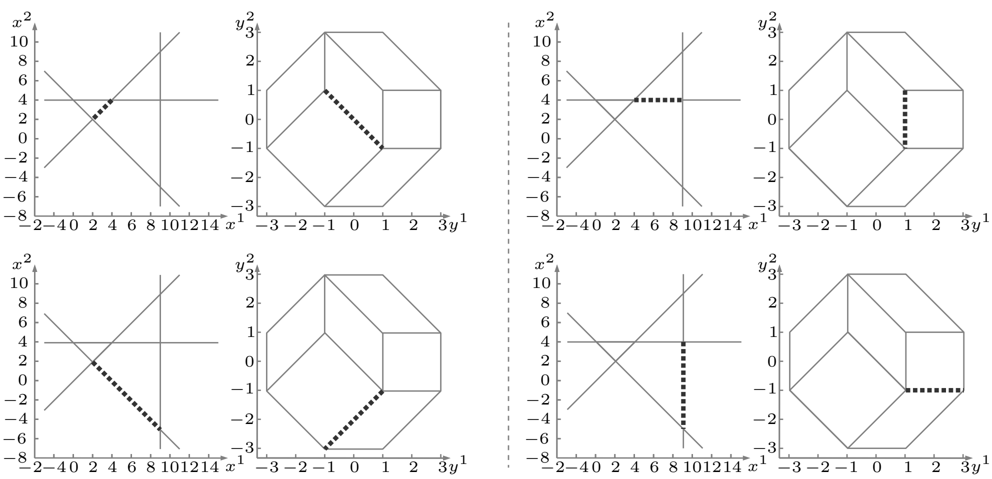
By means of Theorem 4.11 we obtain that the set $\underline{\mathcal{I}}_D$ of dual grid points in $\underline{G}_D$ is a finite dominating set for the optimal dual solutions. Although we are considering a scalar optimization problem, it is possible to apply Benson's algorithm



(a) Assignment between primal elementary convex cells and their corresponding dual grid points.



(b) Assignment between primal grid points and their corresponding dual elementary convex cells.



(c) Assignment between edges of primal elementary convex sets and their corresponding dual edges.

Figure 4.1: The primal construction grid $\overline{G}$ and the corresponding dual construction grid $\overline{G}_D$ with respect to the attracting facilities $\overline{a}^1$ and $\overline{a}^2$ of Example 3.6 and weights $\overline{w}_1 = 2$ and $\overline{w}_2 = 1$.

for linear vector optimization problems in order to determine the set $\underline{\mathcal{I}}_D$. This will be explained in a different paper.

In Algorithm 5.1 below we first check the existence of a finite optimal solution based on Theorem 4.2. Then, in steps 2 and 3, we determine the set of minimizing dual grid points w.r.t. repulsion, i.e., the finite set $\underline{\mathcal{I}}_D \cap \mathcal{Y}$. Applying Corollary 4.10 to the elements $y \in \underline{\mathcal{I}}_D \cap \mathcal{Y}$, we can determine the corresponding optimal solutions of the primal problem (P) in step 4.

Remarks on the implementation of Algorithm 5.1 in `Matlab` are presented in [32].

Algorithm 5.1. *(Solving the Location Problem (P))*

Input: $\overline{a}^m, \overline{w}_m, \overline{B}^*_m$, *$(m = 1, \ldots, \overline{M})$; $\underline{a}^m, \underline{w}_m, \underline{B}^*_m$, $(m = 1, \ldots, \underline{M})$.*

Output: *The sets $\mathcal{X}$ and $\mathcal{Y}$ of optimal solutions of (P) and(D), the optimal objective value α.*

1. *Check finiteness of the optimal objective value, using the condition*

$$\operatorname{dom} h^* = \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}^*_m \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}^*_m = \operatorname{dom} g^*$$

given in Theorem 4.2. If the condition is satisfied go on with 2., otherwise set $\alpha := -\infty$, $\mathcal{Y} := \operatorname{dom} h^* \setminus \operatorname{dom} g^*$ and $\mathcal{X} := \emptyset$ and `STOP`.

2. *Determine the set $\underline{\mathcal{I}}_D$ as well as $h^*(y)$ for each $y \in \underline{\mathcal{I}}_D$.*
3. *For each $y \in \underline{\mathcal{I}}_D$ determine the objective value $g^*(y)$ by solving the linear optimization problem*

$$g^*(y) = \inf \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{a}^m, \overline{y}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}^*_M}(\overline{y}^m) \right] \middle| \sum_{m=1}^{\overline{M}} \overline{y}^m = y \right\}$$

and determine the set $\widehat{\mathcal{Y}}$ of tuple $(\overline{y}^1, \ldots, \overline{y}^{\overline{M}})$ at which the infimum is attained.

4. *Determine the optimal objective value*

$$\alpha := \min_{y \in \underline{\mathcal{I}}_D} \{h^*(y) - g^*(y)\},$$

the dual set of optimal grid points

$$\mathcal{Y} := \operatorname*{argmin}_{y \in \underline{\mathcal{I}}_D} \{h^*(y) - g^*(y)\},$$

and the corresponding primal set of optimal solutions

$$\mathcal{X} := \bigcup_{y \in \mathcal{Y}} \partial g^*(y) = \bigcup_{y \in \mathcal{Y}} \bigcap_{m=1,\ldots,\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m) \right].$$

Obviously, the time complexity of Algorithm 5.1 strongly depends on the number of repulsion points and the resulting dual grid w.r.t. repulsion. The more grid points exist, the more linear optimization problems need to be solved. The complexity of each sub-problem depends on the number of attraction points. Further considerations on duality based algorithms for practical usage, including results concerning complexity and efficiency, will be presented in a forthcoming article.

5.1. The Special Case of no Repulsion

If the number of repulsion points equals zero, i.e., $\underline{M} = 0$, then we obviously have $h \equiv 0$ and thus $h^*(y) = \mathbb{I}_{\{0\}}(y)$, i.e., $\operatorname{dom} h^* = \{0\} = \underline{\mathcal{I}}_D = \mathcal{Y}$. The dual problem (D) simplifies to

$$-g^*(0) = -\inf \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{a}^m, \overline{y}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m) \right] \,\middle|\, \sum_{m=1}^{\overline{M}} \overline{y}^m = 0 \right\}.$$

The primal optimal set is then given by $\mathcal{X} := \partial g^*(0) = \bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m) \right]$ and the optimal objective value is finite, i.e., $\alpha \in \mathbb{R}$.

In [9, Lemma 3.1] the authors give a statement on the existence of such a tuple of elements $\overline{y}^1, \ldots, \overline{y}^M$ and a sufficient condition for a tuple to deliver optimal solutions for this special case of no repulsion. Nevertheless, we do not know from Lemma 3.1 in [9], how to determine such a tuple. Using the duality based results of Section 4 of our paper, we are able to determine such a tuple on the one hand, and also to generalize the assertions of Lemma 3.1 in [9] so that an arbitrary number of repulsion points may be considered, on the other hand.

6. Conclusion

In this paper we have presented a new approach for solving the non-convex location problem (P) with obnoxious facilities. The duality theory by Toland and Singer for D.C. problems has been applied in order to formulate a dual problem (D) and to study geometrical properties, to derive statements on the finiteness and attainment of the optimal objective value and to give duality assertions. Based on results from the field of geometric duality, the relationship between primal and dual elements has been described.

Based on the discretization results presented in Theorem 4.11, we have formulated an algorithm which determines exact solutions of the given location problem (P) with obnoxious facilities by reducing the non-convex optimization problem to a finite number of linear programs.

Acknowledgements. The authors wish to thank Frank Heyde and Andreas Löhne for fruitful discussions and valuable contributions.

We express our thanks to the referee for careful reading and for valuable comments.

References

[1] A. D. Aleksandrov: On the surfaces representable as difference of convex functions, Izv. Akad. Nauk Kazakh. SSR 3 (1949) 3–20 (in Russian).

[2] A. D. Aleksandrov: Surfaces representable as difference of convex functions, Dokl. Akad. Nauk SSSR, n. Ser. 72 (1950) 613–616 (in Russian).

[3] J. Borwein, A. Lewis: Convex Analysis and Nonlinear Optimization, 2nd Ed., Springer, New York (2006).

[4] P. C. Chen, P. Hansen, B. Jaumard, H. Tuy: Weber's problem with attraction and repulsion, J. Region. Sci. 32 (1992) 467–486.

[5] R. L. Church, R. S. Garfinkel: Locating an obnoxious facility on a network, Transp. Sci. 12(2) (1978) 107–118.

[6] Z. Drezner (ed.): Facility Location: A Survey of Applications and Methods, Springer Series in Operations Research, Springer, New York (1995).

[7] Z. Drezner, H. W. Hamacher (eds.): Facility Location: Application and Theory, Contributions to Management Science, Physica, Heidelberg (2002).

[8] Z. Drezner, G. O. Wesolowsky: The Weber problem on the plane with some negative weights, INFOR 29(2) (1991) 87–99.

[9] R. Durier, C. Michelot: Geometrical properties of the Fermat-Weber problem, Eur. J. Oper. Res. 20(3) (1985) 332–343.

[10] A. J. Goldman: Concepts of optimal location for partially noxious facilities, Bull. Oper. Res. Soc. Amer. 23(1) (1975) B–31.

[11] H. W. Hamacher: Mathematische Lösungsverfahren für planare Standortprobleme, Vieweg, Wiesbaden (1995).

[12] P. Hartman: On functions representable as a difference of convex functions, Pac. J. Math. 9(3) (1959) 707–713.

[13] F. Heyde: Geometric duality for convex vector optimization problems, J. Convex Analysis 20(3) (2013) 813–832.

[14] J. B. Hiriart-Urruty, C. Lemaréchal: Convex Analysis and Minimization Algorithms II. Advanced Theory and Bundle Methods, Grundlehren der Mathematischen Wissenschaften 306, Springer, Berlin (1993).

[15] R. Horst, N. V. Thoai: DC programming: overview, J. Optimization Theory Appl. 103(1) (1999) 1–43.

[16] R. F. Love, J. G. Morris, G. O. Wesolowsky (eds.): Facilities Location: Models and Methods, North-Holland, Amsterdam (1988).

[17] C. D. Maranas, C. Floudas: A global optimization method for Weber's problem with attraction and repulsion, in: Large Scale Optimization (Gainesville, 1993), W. W. Hager, D. W. Hearn, P. M. Pardalos (eds.), Kluwer, Dordrecht (1994) 259–285.

[18] J. E. Martínez-Legaz: Generalized conjugation and related topics, in: Generalized Convexity and Fractional Programming with Economic Applications (Pisa, 1988), A. Cambini et al. (ed.), Lecture Notes in Econom. and Math. Systems 345, Springer, Berlin (1990) 168–197.

[19] J. E. Martínez-Legaz, A. Seeger: A formula on the approximate subdifferential of the difference of convex functions, Bull. Aust. Math. Soc. 45(1) (1992) 37–41.

[20] J. E. Martínez-Legaz, I. Singer: An extension of d.c. duality theory, with an appendix on $*$-subdifferentials, Optimization 42(1) (1997) 9–37.

[21] J. E. Martínez-Legaz, M. Volle: Duality in D.C. programming: the case of several D.C. constraints, J. Math. Anal. Appl. 237(2) (1999) 657–671.

[22] J. J. Moreau: Fonctionnelles convexes, Séminaire Jean Leray 2 (1966–67) 1–108.

[23] S. Nickel, E. M. Dudenhöffer: Weber's problem with attraction and repulsion under polyhedral gauges, J. Glob. Optim. 11(4) (1997) 409–432.

[24] R. T. Rockafellar: Convex Analysis, Princeton Landmarks in Mathematics, Princeton University Press, Princeton (1997).

[25] D. Romero-Morales, E. Carrizosa, E. Conde: Semi-obnoxious location models: A global optimization approach, Eur. J. Oper. Res. 102(2) (1997) 295–301.

[26] I. Singer: A Fenchel-Rockafellar type duality theorem for maximization, Bull. Aust. Math. Soc. 20(2) (1979) 193–198.

[27] I. Singer: Duality for Nonconvex Approximation and Optimization, CMS Books in Mathematics 24, Springer, New York (2006).

[28] J. F. Toland: Duality in nonconvex optimization, J. Math. Anal. Appl. 66(2) (1978) 399–415.

[29] H. Tuy: A general d.c. approach to location problems, in: State of the Art in Global Optimization: Computational Methods and Applications (Princeton, 1995), C. A. Floudas, P. M. Pardalos (eds.), Nonconvex Optimization and its Applications 7, Kluwer, Dordrecht (1996) 413–432.

[30] H. Tuy: Convex Analysis and Global Optimization, Nonconvex Optimization and its Applications 22, Kluwer, Dordrecht (1998).

[31] H. Tuy, F. Al-Khayyal, F. Zhou: A d.c. optimization method for single facility location problems, J. Glob. Optim. 7(2) (1995) 209–227.

[32] A. Wagner: A new Duality-Based Approach for the Problem of Locating a Semi-Obnoxious Facility, PhD Thesis, Martin-Luther-University Halle-Wittenberg, Halle/Saale (2014).

[33] A. Weber: Über den Standort der Industrien. Erster Teil: Reine Theorie des Standortes, Mohr, Tübingen (1909).