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# Finer Properties of Ultramaximally Monotone Operators on Banach Spaces

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*Dedicated to the memory of Jean Jacques Moreau.*

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In this paper, we study properties of ultramaximally monotone operators. We characterize the interior and the closure of the range of an ultramaximally monotone operator. We establish the Brézis–Haraux condition in the setting of a general Banach space. Moreover, we show that every ultramaximally monotone operator is of type (NA), which generalizes Bauschke and Simons' result.

*Keywords:* Brézis-Haraux condition, Fenchel conjugate, Fitzpatrick function, linear relation, maximally monotone operator, monotone operator, operator of type (D), operator of type (NI), operator of type (NA), rectangular, set-valued operator, subdifferential operator, ultramaximally monotone operator

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## 1. Introduction

Throughout this paper, we assume that $X$ is a real Banach space with norm $\|\cdot\|$, that $X^*$ is the continuous dual of $X$, and that $X$ and $X^*$ are paired by $\langle\cdot,\cdot\rangle$. Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as a relation, point-to-set mapping or multifunction) from $X$ to $X^*$, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A := \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of $A$. Recall that $A$ is *monotone* if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (x,x^*) \in \operatorname{gra} A \; \forall (y,y^*) \in \operatorname{gra} A, \tag{1}$$

and *maximally monotone* if $A$ is monotone and $A$ has no proper monotone extension (in the sense of graph inclusion). Let $A : X \rightrightarrows X^*$ be monotone and $(x,x^*) \in X \times X^*$. We say $(x,x^*)$ is *monotonically related to* $\operatorname{gra} A$ if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (y,y^*) \in \operatorname{gra} A.$$

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The following is the key concept in our paper.

**Definition 1.1 (Ultramaximally monotone operators; see [4]).** Let $A : X \rightrightarrows X^*$ be monotone. We say $A$ is *ultramaximally monotone* if $A$ is maximally monotone with respect to $X^{**} \times X^*$.

Our paper is motivated by Bauschke and Simons' paper [4], where the authors study the properties of an unbounded linear ultramaximally monotone operator. In this paper, we continue to study the properties of a general ultramaximally monotone operator.

Monotone operators have proven to be important objects in modern Optimization and Analysis; see, e.g., [7, 8, 12, 1, 26, 35] and the books [3, 10, 16, 17, 23, 29, 30, 27, 36, 37, 38] and the references therein. We adopt standard notation used in these books: $\operatorname{dom} A := \{x \in X \mid Ax \neq \varnothing\}$ is the *domain* of $A$, and the *range* of $A$ is written as $\operatorname{ran} A := A(X)$. Given a subset $C$ of $X$, $\operatorname{int} C$ is the *interior* of $C$ and $\overline{C}$ is the norm *closure* of $C$. We say $A$ is a *linear relation* if $\operatorname{gra} A$ is a linear subspace.

The *indicator function* of $C$, written as $\iota_C$, is defined at $x \in X$ by

$$\iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{otherwise.} \end{cases}$$

If $D \subseteq X$, we set $C - D := \{x - y \mid x \in C, y \in D\}$. For every $x \in X$, the normal cone operator of $C$ at $x$ is defined by $N_C(x) := \{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^* \rangle \leq 0\}$, if $x \in C$; and $N_C(x) = \varnothing$, if $x \notin C$. For $x, y \in X$, we set $[x, y] := \{tx + (1-t)y \mid 0 \leq t \leq 1\}$. Given $f \colon X \to \left]-\infty, +\infty\right]$, we set $\operatorname{dom} f := f^{-1}(\mathbb{R})$ and $f^* \colon X^* \to [-\infty, +\infty] : x^* \mapsto \sup_{x \in X} \big(\langle x, x^* \rangle - f(x)\big)$ is the *Fenchel conjugate* of $f$. We say $f$ is *proper* if $\operatorname{dom} f \neq \varnothing$. Let $f$ be proper. Then $\partial f \colon X \rightrightarrows X^* \colon x \mapsto \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^* \rangle + f(x) \leq f(y)\}$ is the *subdifferential operator* of $f$. We denote by $J_X$ the *duality map* from $X$ to $X^*$, i.e., the subdifferential of the function $\frac{1}{2}\|\cdot\|^2$. For convenience, we denote by $J := J_X$.

Let $A : X \rightrightarrows X^*$ and $F \colon X \times X^* \to \left]-\infty, +\infty\right]$. We say $F$ is a *representer* for $\operatorname{gra} A$ if

$$\operatorname{gra} A = \big\{(x, x^*) \in X \times X^* \mid F(x, x^*) = \langle x, x^* \rangle\big\}. \tag{2}$$

Given two real Banach spaces $X, Y$ and $F_1, F_2 \colon X \times Y \to \left]-\infty, +\infty\right]$, the *partial inf-convolution* $F_1 \Box_2 F_2$ is the function defined on $X \times Y$ by

$$F_1 \Box_2 F_2 \colon (x, y) \mapsto \inf_{v \in Y} \big\{F_1(x, y - v) + F_2(x, v)\big\}.$$

We set $P_X : X \times Y \to X \colon (x, y) \mapsto x$, and $P_Y : X \times Y \to Y \colon (x, y) \mapsto y$. We denote by $\longrightarrow$ and $\rightharpoonup_{\mathrm{w}^*}$ the norm convergence and weak$^*$ convergence of nets, respectively.

We now recall two fundamental properties of maximally monotone operators.

**Definition 1.2.** Let $A : X \rightrightarrows X^*$ be maximally monotone.

1. We say $A$ is *of dense type or type (D)* (1976, [20]; see also [24] and [31, Theorem 9.5]) if for every $(x^{**}, x^*) \in X^{**} \times X^*$ with
$$\inf_{(a,a^*)\in \operatorname{gra} A} \langle a - x^{**}, a^* - x^*\rangle \geq 0,$$
there exist a bounded net $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ in $\operatorname{gra} A$ such that $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ weak*× strong converges to $(x^{**}, x^*)$.
2. We say $A$ is *of type negative infimum (NI)* (1996, [28]) if
$$\inf_{(a,a^*)\in \operatorname{gra} A} \langle x^{**} - a, x^* - a^*\rangle \leq 0, \quad \forall (x^{**}, x^*) \in X^{**} \times X^*.$$

These two properties coincide by Simons, Marques Alves and Svaite (see [28, Lemma 15] or [30, Theorem 36.3(a)], and [22, Theorem 4.4]). By the definition of ultrmaximally monotone operators, every ultramaximally monotone operator is of type (D) (i.e., type (NI)). For convenience, we write type (NI) to represent type (D) as well. Note that not every operator of type (NI) is ultramaximally monotone. For instance, suppose that $X$ is a nonreflexive space. Let $A : X \rightrightarrows X^*$ be defined by $\operatorname{gra} A := X \times \{0\}$. Then $\operatorname{gra} A = \partial\iota_X$ and $A$ is of type (NI) by Fact 2.8. But $\operatorname{gra} A \subsetneq X^{**} \times \{0\}$. Hence $A$ is not ultramaximally monotone.

However, there always exists an ultramximally monotone operator in $X$, for instance, the operator with the graph: $\{0\} \times X^*$.

The remainder of this paper is organized as follows. In Section 2, we collect auxiliary results for future reference and for the reader's convenience. Our main results (Theorem 3.3, Theorem 3.4, Theorem 3.9 and Theorem 3.12) are presented in Section 3.

## 2. Auxiliary results

In this section, we introduce some basic facts. We start with the Attouch-Brézis' Fenchel duality theorem.

**Fact 2.1 (Attouch-Brézis; see [2, Theorem 1.1] or [30, Remark 15.2]).** *Let $f, g : X \to \left]-\infty, +\infty\right]$ be proper lower semicontinuous and convex. Assume that $\bigcup_{\lambda>0} \lambda [\operatorname{dom} f - \operatorname{dom} g]$ is a closed subspace of $X$. Then $\partial(f+g) = \partial f + \partial g$ and*
$$(f + g)^*(z^*) = \min_{y^*\in X^*} [f^*(y^*) + g^*(z^* - y^*)], \quad \forall z^* \in X^*.$$

**Fact 2.2 (Simons and Zălinescu; see [33, Theorem 4.2] or [30, Theorem 16.4(a)]).** *Let $X, Y$ be real Banach spaces and $F_1, F_2 \colon X\times Y \to \left]-\infty, +\infty\right]$ be proper lower semicontinuous and convex. Assume that for every $(x, y) \in X \times Y$,*
$$(F_1 \Box_2 F_2)(x, y) > -\infty$$

*and that* $\bigcup_{\lambda>0} \lambda\, [P_X \operatorname{dom} F_1 - P_X \operatorname{dom} F_2]$ *is a closed subspace of* $X$. *Then for every* $(x^*, y^*) \in X^* \times Y^*$,

$$(F_1 \Box_2 F_2)^*(x^*, y^*) = \min_{u^* \in X^*} \left\{ F_1^*(x^* - u^*, y^*) + F_2^*(u^*, y^*) \right\}.$$

The Fitzpatrick function below is a key tool in Monotone Operator Theory, which has been applied comprehensively.

**Fact 2.3 (Fitzpatrick; see [18, Corollary 3.9]).** *Let* $A\colon X \rightrightarrows X^*$ *be maximally monotone, and set*

$$F_A\colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \sup_{(a,a^*) \in \operatorname{gra} A} \big(\langle x, a^*\rangle + \langle a, x^*\rangle - \langle a, a^*\rangle\big), \quad (3)$$

*the* Fitzpatrick function *associated with* $A$. *Then for every* $(x, x^*) \in X \times X^*$, *the inequality* $\langle x, x^*\rangle \le F_A(x, x^*)$ *is true, and equality holds if and only if* $(x, x^*) \in \operatorname{gra} A$.

**Fact 2.4 (Fitzpatrick; see [18, Proposition 4.2 and Theorem 4.3]).** *Let* $A : X \rightrightarrows X^*$ *be a monotone operator with* $\operatorname{dom} A \neq \varnothing$. *Then* $F_A^* = \langle \cdot, \cdot \rangle$ *on* $\operatorname{gra} A^{-1}$, $F_A^*(x^*, x) \ge F_A(x, x^*)$, $\forall (x, x^*) \in X \times X^*$, *and*

$$\big\{ x \in X \mid \exists x^* \in X^* \text{ such that } F_A^*(x^*, x) < +\infty \big\} \subseteq \overline{\operatorname{conv}(\operatorname{dom} A)}.$$

The following three results are the fundamental characterizations of the domain of a maximally monotone operator.

**Fact 2.5 (Simons, see [30, Theorem 27.1]).** *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone operator. Then*

$$\operatorname{int} \operatorname{dom} A = \operatorname{int} [\operatorname{conv} \operatorname{dom} A] = \operatorname{int} \big[P_X \operatorname{dom} F_A)\big].$$

**Fact 2.6 (See [11, Theorem 3.6] or [12]).** *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone operator. Then*

$$\overline{\operatorname{conv} [\operatorname{dom} A]} = \overline{P_X [\operatorname{dom} F_A]}.$$

Let $A : X \rightrightarrows X^*$ be maximally monotone. We say $A$ is *of type (FPV)* if for every open convex set $U \subseteq X$ such that $U \cap \operatorname{dom} A \neq \varnothing$, the implication

$$x \in U \text{ and } (x, x^*) \text{ is monotonically related to } \operatorname{gra} A \cap U \times X^* \Rightarrow (x, x^*) \in \operatorname{gra} A$$

holds. We do not know if every maximally monotone operator is necessarily of type (FPV) in a general Banach space [30, 10], but it is true in reflexive spaces. Every operator of type (NI) is of type (FPV) (see [31, Theorem 9.10(b)]).

**Fact 2.7 (Simons; see [30, Theorem 44.2]).** *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone operator that is of type (FPV). Then*

$$\overline{\operatorname{dom} A} = \overline{\operatorname{conv}\big(\operatorname{dom} A\big)} = \overline{P_X\big(\operatorname{dom} F_A\big)}.$$

Now we introduce some properties of operators of type (NI) (i.e., type (D)).

**Fact 2.8 (Gossez; see [19, Theorem 3.1] or [30, Theorem 48.4(b)]).** *Let $f : X \to ]-\infty, +\infty]$ be proper lower semicontinuous and convex. Then $\partial f$ is maximally monotone of type (NI).*

The following result provides a sufficient condition for the sum operator to be of type (NI).

**Fact 2.9 (See [21, Theorem 1.2] and [34, Corollary 3.5] or [32, Corollary 18]).** *Let $A, B : X \rightrightarrows X^*$ be maximally monotone of type (NI). Assume that $\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B]$ is a closed subspace. Then $A+B$ is maximally monotone of type (NI).*

**Fact 2.10 (Marques Alves and Svaiter, see [22, Theorem 4.4]).** *Let $A : X \rightrightarrows X$ be maximally monotone of type (NI), and $F : X \times X^* \to ]-\infty, +\infty]$ be proper (norm) lower semicontinuous and convex. Let $B : X^* \rightrightarrows X^{**}$ be defined by*

$$\operatorname{gra} B := \big\{(x^*, x^{**}) \in X^* \times X^{**} \mid \langle x^* - a^*, x^{**} - a\rangle \geq 0,\, \forall (a, a^*) \in \operatorname{gra} A\big\}.$$

*Assume that $F$ is a representer for $\operatorname{gra} A$. Then $F^*$ is a representer for $\operatorname{gra} B$ .*

**Corollary 2.11.** *Let $A : X \rightrightarrows X$ be maximally monotone of type (NI), and $F : X \times X^* \to ]-\infty, +\infty]$ be proper (norm) lower semicontinuous and convex. Assume that $F$ is a representer for $\operatorname{gra} A$ and that*

$$\Big\{(x^{**}, x^*) \in X^{**} \times X^* \mid F^*(x^*, x^{**}) = \langle x^*, x^{**}\rangle\Big\} \subseteq \operatorname{gra} A. \tag{4}$$

*Then $A$ is ultramaximally monotone.*

**Proof.** Let $B : X^* \rightrightarrows X^{**}$ be defined as in Fact 2.10. Let $(y^{**}, y^*) \in X^{**} \times X^*$ be monotonically related to $\operatorname{gra} A$. Thus $(y^{**}, y^*) \in \operatorname{gra} B^{-1}$. On the other hand, by Fact 2.10 and (4), we have $\operatorname{gra} B^{-1} \subseteq \operatorname{gra} A$. Since $(y^{**}, y^*) \in \operatorname{gra} B^{-1}$, $(y^{**}, y^*) \in \operatorname{gra} A$. Hence we have $A$ is ultramaximally monotone. □

## 3. Properties of ultramaximally monotone operators

In Theorem 3.3, we provide a sufficient condition for the sum operator to be ultramaximally monotone. We first need the following two technical results.

**Lemma 3.1.** *Let $A, B\colon X \rightrightarrows X^*$ be maximally monotone, and suppose that $\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B]$ is a closed subspace of $X$. Set*

$$E := \big\{x \in X \mid \exists x^* \in X^* \text{ such that } F_A^*(x^*, x) < +\infty\big\}$$

*and*

$$F := \big\{x \in X \mid \exists x^* \in X^* \text{ such that } F_B^*(x^*, x) < +\infty\big\}.$$

*Then*

$$\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] = \bigcup_{\lambda>0} \lambda\,[E - F]\,.$$

*Moreover, if $A$ and $B$ are of type (FPV), then we have*

$$\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] = \bigcup_{\lambda>0} \lambda\,[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B]\,.$$

**Proof.** Using Fact 2.4, we see that

$$\begin{aligned}
&\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] \subseteq \bigcup_{\lambda>0} \lambda\,[E - F] \\
&\subseteq \bigcup_{\lambda>0} \lambda\left[\overline{\operatorname{conv}(\operatorname{dom} A)} - \overline{\operatorname{conv}(\operatorname{dom} B)}\right] \\
&\subseteq \bigcup_{\lambda>0} \lambda\left[\overline{\operatorname{conv}(\operatorname{dom} A) - \operatorname{conv}(\operatorname{dom} B)}\right] = \bigcup_{\lambda>0} \lambda\overline{[\operatorname{conv}(\operatorname{dom} A - \operatorname{dom} B)]} \\
&\subseteq \overline{\bigcup_{\lambda>0} \lambda\,[\operatorname{conv}(\operatorname{dom} A - \operatorname{dom} B)]} \\
&= \bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] \quad \text{(using the assumption).}
\end{aligned}$$

Hence $\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] = \bigcup_{\lambda>0} \lambda\,[E - F]\,.$

Now assume that $A, B$ are of type (FPV). Then by Fact 2.7, we have

$$\begin{aligned}
&\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] \subseteq \bigcup_{\lambda>0} \lambda\,[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B] \\
&\subseteq \bigcup_{\lambda>0} \lambda\left[\overline{\operatorname{dom} A} - \overline{\operatorname{dom} B}\right] \subseteq \bigcup_{\lambda>0} \lambda\left[\overline{\operatorname{dom} A - \operatorname{dom} B}\right] \\
&\subseteq \overline{\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B]} = \bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] \quad \text{(using the assumption).}
\end{aligned}$$

Thus, $\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] = \bigcup_{\lambda>0} \lambda\,[P_X \operatorname{dom} F_A - P_X \operatorname{dom} F_B]$. $\square$

Proposition 3.2 below was first established by Bauschke, Wang and Yao in [5, Proposition 5.9] when $X$ is a reflexive space. We now provide a nonreflexive version.

**Proposition 3.2.** *Let $A, B\colon X \rightrightarrows X^*$ be maximally monotone and suppose that $\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B]$ is a closed subspace of $X$. Then $F_A \Box_2 F_B$ is proper, norm$\times$weak* lower semicontinuous and convex, and the partial infimal convolution is exact everywhere.*

**Proof.** Define $F_1, F_2 \colon X \times X^* \to \left]-\infty, +\infty\right]$ by

$$F_1 : (x, x^*) \mapsto F_A^*(x^*, x), \qquad F_2 : (x, x^*) \mapsto F_B^*(x^*, x).$$

Since $F_A$ and $F_B$ are norm-weak* lower semicontinuous,

$$F_1^*(x^*, x) = F_A(x, x^*), \qquad F_2^*(x^*, x) = F_B(x, x^*), \quad \forall (x, x^*) \in X \times X^*. \tag{5}$$

Take $(x, x^*) \in X \times X^*$. By Fact 2.4 and Fact 2.3,

$$\big(F_1 \Box_2 F_2\big)(x, x^*) \geq \langle x, x^* \rangle > -\infty.$$

In view of Lemma 3.1,

$$\bigcup_{\lambda>0} \lambda \left[P_X \operatorname{dom} F_1 - P_X \operatorname{dom} F_2\right] = \bigcup_{\lambda>0} \lambda \left[\operatorname{dom} A - \operatorname{dom} B\right] \quad \text{is a closed subspace.}$$

By Fact 2.2 and (5),

$$\begin{aligned} \big(F_1 \Box_2 F_2\big)^*(x^*, x) &= \min_{y^* \in X^*} \left[F_1^*(x^* - y^*, x) + F_2^*(y^*, x)\right] \\ &= \min_{y^* \in X^*} \left[F_A(x, x^* - y^*) + F_B(x, y^*)\right] = \big(F_A \Box_2 F_B\big)(x, x^*). \end{aligned}$$

Hence $F_A \Box_2 F_B$ is proper, norm×weak* lower semicontinuous and convex, and the partial infimal convolution is exact. □

Now we come to our first main result.

**Theorem 3.3.** *Let $A, B : X \rightrightarrows X^*$ be maximally monotone. Assume that $\bigcup_{\lambda>0} \lambda \left[\operatorname{dom} A - \operatorname{dom} B\right]$ is a closed subspace. Suppose that $A$ is ultramaximally monotone, and that $B$ is of type (NI). Then $A + B$ is ultramaximally monotone.*

**Proof.** Clearly, $A + B$ is monotone and $A$ is of type (NI). Fact 2.9 shows that $A + B$ is maximally monotone of type (NI). Define $K : X \times X^* \to \left]-\infty, +\infty\right]$ by $K := F_A \Box_2 F_B$. Proposition 3.2 and Fact 2.3 imply that $K$ is a representer for $\operatorname{gra}(A + B)$.

Let $(x^*, x^{**}) \in X^* \times X^{**}$ be such that $K^*(x^*, x^{**}) = \langle x^*, x^{**} \rangle$. [31, Theorem 9.10(b)] implies that $A$ and $B$ are of type (FPV). Then by Lemma 3.1, Fact 2.3 and Fact 2.2, there exists $y^* \in X^*$ such that

$$K^*(x^*, x^{**}) = F_A^*(y^*, x^{**}) + F_B^*(x^* - y^*, x^{**}).$$

Then [21, Theorem 1.2] shows that

$$F_A^*(y^*, x^{**}) = \langle y^*, x^{**} \rangle \quad \text{and} \quad F_B^*(x^* - y^*, x^{**}) = \langle x^* - y^*, x^{**} \rangle. \tag{6}$$

Since $A$ is ultramaximally monotone, by Fact 2.10 and (6), we have

$$x^{**} \in X \quad \text{and} \quad y^* \in A x^{**}. \tag{7}$$

Then combining (6), Fact 2.3 and Fact 2.4, $x^* - y^* \in Bx^{**}$ and hence $(x^{**}, x^*) \in \operatorname{gra}(A+B)$. Hence

$$\big\{(x^{**}, x^*) \in X^{**} \times X^* | K^*(x^*, x^{**}) = \langle x^*, x^{**}\rangle\big\} \subseteq \operatorname{gra}(A+B). \tag{8}$$

Since $K$ is a representer for $\operatorname{gra}(A+B)$ and $A+B$ is of type (NI), Corollary 2.11 shows that $A+B$ is ultramaximally monotone. $\square$

Let $A : X \rightrightarrows X^*$ be monotone. For convenience, we defined $\Phi_A$ on $X^{**} \times X^*$ by

$$\Phi_A \colon (x^{**}, x^*) \mapsto \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x^{**}, a^*\rangle + \langle a, x^*\rangle - \langle a, a^*\rangle\big).$$

Then we have $\Phi_A|_{X\times X^*} = F_A$.

Now we present some characterizations of the interior and the closure of the range of an ultramaximally monotone operator, which generalizes Simons' results in a reflexive space (see [30, Theorem 31.2 and Lemma 31.1]).

**Theorem 3.4.** *Let $A : X \rightrightarrows X^*$ be ultramaximally monotone. Then*

$$\operatorname{int} \operatorname{ran} A = \operatorname{int} [\operatorname{conv} \operatorname{ran} A] = \operatorname{int} [P_{X^*} \operatorname{dom} F_A] = \operatorname{int} [P_{X^*} \operatorname{dom} \Phi_A] \tag{9}$$

$$\overline{\operatorname{ran} A} = \overline{\operatorname{conv} \operatorname{ran} A} = \overline{P_{X^*} \operatorname{dom} F_A} = \overline{P_{X^*} \operatorname{dom} \Phi_A}. \tag{10}$$

**Proof.** We first show (9). Fact 2.3 implies that

$$\operatorname{ran} A \subseteq \operatorname{conv} [\operatorname{ran} A] \subseteq P_{X^*} [\operatorname{dom} F_A] \subseteq P_{X^*} [\operatorname{dom} \Phi_A]. \tag{11}$$

Define $B : X^* \rightrightarrows X^{**}$ by $\operatorname{gra} B := \operatorname{gra} A^{-1}$. By the assumption, $B$ is maximally monotone. By Fact 2.5, we have

$$\operatorname{int} \operatorname{dom} B = \operatorname{int} [\operatorname{conv} \operatorname{dom} B] = \operatorname{int} [P_{X^*} \operatorname{dom} F_B]. \tag{12}$$

By the definition of $B$, we have

$$\operatorname{dom} B = \operatorname{ran} A \quad \text{and} \quad \Phi_A(x^{**}, x^*) = F_B(x^*, x^{**}), \quad \forall (x^{**}, x^*) \in X^{**} \times X^*. \tag{13}$$

Then (12) shows that

$$\operatorname{int} \operatorname{ran} A = \operatorname{int} [\operatorname{conv} \operatorname{ran} A] = \operatorname{int} P_{X^*} [\operatorname{dom} \Phi_A].$$

Thus combining (11), we have $\operatorname{int} \operatorname{ran} A = \operatorname{int} [\operatorname{conv} \operatorname{ran} A] = \operatorname{int} [P_{X^*} \operatorname{dom} F_A] = \operatorname{int} [P_{X^*} \operatorname{dom} \Phi_A]$.

Now we show (10). Fact 2.6 implies that

$$\overline{\operatorname{conv} \operatorname{dom} B} = \overline{P_{X^*} \operatorname{dom} F_B}.$$

Thus (13) implies that

$$\overline{\operatorname{conv} \operatorname{ran} A} = \overline{P_{X^*} \operatorname{dom} \Phi_A}.$$

Thus combining (11), we have

$$\overline{\operatorname{conv}\operatorname{ran} A} = \overline{P_{X^*}\operatorname{dom} F_A} = \overline{P_{X^*}\operatorname{dom}\Phi_A}. \tag{14}$$

Since $A$ is of type (NI), [30, Theorem 43.2] and (14) show that $\overline{\operatorname{ran} A} = \overline{\operatorname{conv}\operatorname{ran} A}$ $= \overline{P_{X^*}\operatorname{dom} F_A} = \overline{P_{X^*}\operatorname{dom}\Phi_A}$ and hence (10) holds. □

**Remark 3.5.** We cannot significantly weaken the conditions in Theorem 3.4. For instance, we cannot replace "ultramaximally monotone" by "of type (NI)". In a nonreflexive space there always exists a continuous, coercive, and convex function $f$ such that $\operatorname{int}[\operatorname{ran}\partial f]$ is not convex (where $\partial f$ is of type (NI) by Fact 2.8, see also [9, Theorem 3.1] and [30, page 169] for more information), but (9) grantees that $\operatorname{int}[\operatorname{ran}\partial f]$ is convex.

The following result is very useful, which allows us to show that every ultramaximally monotone operator is of type (NA) (see Theorem 3.12 below).

**Corollary 3.6.** *Let $A : X \rightrightarrows X^*$ be ultramaximally monotone. Then $A + J$ is ultramaximally monotone and* $\operatorname{ran}(A + J) = X^*$.

**Proof.** By Fact 2.8, $J$ is of type (NI). Then applying Theorem 3.3, we have $A + J$ is ultramaximally monotone. Now we show that $\operatorname{ran}(A + J) = X^*$.

By [30, Eq. (23.9), page 101], $\operatorname{dom} F_J = X \times X^*$. Then $X^* = P_{X^*}[\operatorname{dom}(F_A \Box_2 F_J)]$. Thus, [30, Lemma 23.9] implies that $X^* = P_{X^*}[\operatorname{dom} F_{A+J}]$. Since $A + J$ is ultramaximally monotone, Theorem 3.4 implies that

$$\operatorname{int}\operatorname{ran}(A + J) = \operatorname{int}[P_{X^*}\operatorname{dom} F_{A+J}] = \operatorname{int} X^* = X^*.$$

Hence $\operatorname{ran}(A + J) = X^*$. □

**Corollary 3.7.** *Let $A : X \rightrightarrows X^*$ be ultramaximally monotone with* $0 \in \operatorname{dom} A$. *Assume that* $\lim_{\|x\|\to\infty} \inf \frac{\langle x, Ax\rangle}{\|x\|} = +\infty$. *Then* $\operatorname{ran} A = X^*$.

**Proof.** We first show that

$$\{0\} \times X^* \subseteq \operatorname{dom} F_A. \tag{15}$$

Since $0 \in \operatorname{dom} A$, there exists $x_0^* \in X^*$ such that $(0, x_0^*) \in \operatorname{gra} A$. Let $x^* \in X^*$. By the assumption, there exists $\rho > 0$ such that

$$\langle a, a^*\rangle \geq \|x^*\| \cdot \|a\|, \quad \forall \|a\| \geq \rho,\ (a, a^*) \in \operatorname{gra} A. \tag{16}$$

Let $(a, a^*) \in \operatorname{gra} A$.

*Case 1:* $\|a\| < \rho$. By $(0, x_0^*) \in \operatorname{gra} A$, we have $\langle a, a^* - x_0^*\rangle = \langle a - 0, a^* - x_0^*\rangle \geq 0$. We have

$$\langle a, x^*\rangle - \langle a, a^*\rangle \leq \langle a, x^*\rangle - \langle a, x_0^*\rangle \leq \rho\|x^* - x_0^*\|.$$

*Case 2:* $\|a\| \geq \rho$. Using (16), we have

$$\langle a, x^* \rangle - \langle a, a^* \rangle = \langle a, x^* \rangle - \|x^*\| \cdot \|a\| \leq 0.$$

Thus combining the above two cases,

$$F_A(0, x^*) = \sup_{(a,a^*)\in \operatorname{gra} A} \{\langle a, x^* \rangle - \langle a, a^* \rangle\} \leq \rho\|x^* - x_0^*\|.$$

Thus $(0, x^*) \in \operatorname{dom} F_A$. Hence (15) holds and then $P_{X^*} \operatorname{dom} F_A = X^*$. Thus Theorem 3.4 implies that

$$\operatorname{int} \operatorname{ran} A = \operatorname{int} [P_{X^*} \operatorname{dom} F_A] = \operatorname{int} X^* = X^*.$$

Thus $\operatorname{ran} A = X^*$. $\square$

Corollary 3.7 was first proved by Browder in a reflexive space (see [15, Theorem 3]).

**Corollary 3.8.** *Let $A : X \rightrightarrows X^*$ be a set-valued mapping. Let $\delta > 0$ and $\alpha > 1$. Assume that*

$$\langle x - y, x^* - y^* \rangle \geq \delta\|x - y\|^\alpha, \quad \forall (x, x^*), (y, y^*) \in \operatorname{gra} A.$$

*Then $A$ is ultramaximally monotone if and only if* $\operatorname{ran} A = X^*$.

**Proof.** Clearly, we have $A$ is monotone.

"$\Rightarrow$": Let $(x_0, x_0^*) \in \operatorname{gra} A$. Define $B : X \rightrightarrows X^*$ by $\operatorname{gra} B := \operatorname{gra} A - \{(x_0, x_0^*)\}$. Then by the assumption, we have $B$ is ultramaximally monotone with $(0, 0) \in \operatorname{gra} B$ and

$$\langle b, b^* \rangle \geq \delta\|b\|^\alpha, \quad \forall (b, b^*) \in \operatorname{gra} B. \tag{17}$$

Thus, $\lim_{\|x\|\to\infty} \inf \frac{\langle x, Bx\rangle}{\|x\|} = +\infty$. Then by Corollary 3.7, $\operatorname{ran} B = X^*$ and hence $\operatorname{ran} A = X^*$ by the definition of $B$.

"$\Leftarrow$": We first show that $A^{-1}$ is single-valued on $X^*$. Let $x, y \in A^{-1}x^*$. Then $(x, x^*), (y, x^*) \in \operatorname{gra} A$. Then by the assumption, $0 = \langle x-y, x^*-x^* \rangle \geq \delta\|x-y\|^\alpha$. Thus, $x = y$ and hence $A^{-1}$ is single-valued on $X^*$ since $\operatorname{ran} A = X^*$. By the assumption again, we have

$$\begin{aligned} \|A^{-1}x^* - A^{-1}y^*\| \cdot \|x^* - y^*\| &\geq \langle A^{-1}x^* - A^{-1}y^*, x^* - y^* \rangle \\ &\geq \delta\|A^{-1}x^* - A^{-1}y^*\|^\alpha, \quad \forall x^*, y^* \in X^*. \end{aligned}$$

Thus

$$\|x^* - y^*\| \geq \delta\|A^{-1}x^* - A^{-1}y^*\|^{\alpha-1}, \quad \forall x^*, y^* \in X^*.$$

Since $\alpha > 1$, $A^{-1}$ is continuous and then $A^{-1}$ is maximally monotone from $X^*$ to $X^{**}$. Hence $A$ is ultramaximally monotone. $\square$

Let $A : X \rightrightarrows X^*$ be such that $\operatorname{dom} A \neq \varnothing$. We say that $A$ is *rectangular* if $\operatorname{dom} A \times \operatorname{ran} A \subseteq \operatorname{dom} F_A$ (see [14, 30, 38, 3, 6] for more information on rectangular operators).

The proof of Theorem 3.9 closely follows the lines of that of [30, Corollary 31.6].

**Theorem 3.9 (The Brézis-Haraux condition in general Banach space).** *Let $A, B : X \rightrightarrows X^*$ be monotone with $\operatorname{dom} A \cap \operatorname{dom} B \neq \varnothing$. Assume that $A+B$ is ultramaximally monotone. Suppose that one of the following conditions holds:*

(*i*) $A$ *and* $B$ *are rectangular.*

(*ii*) $\operatorname{dom} A \subseteq \operatorname{dom} B$ *and* $B$ *is rectangular.*

*Then*

$$\operatorname{int} \operatorname{ran}(A+B) = \operatorname{int} [\operatorname{ran} A + \operatorname{ran} B] \quad \text{and} \quad \overline{\operatorname{ran} A + \operatorname{ran} B} = \overline{\operatorname{ran}(A+B)}.$$

**Proof.** We first show that

$$\operatorname{ran} A + \operatorname{ran} B \subseteq P_{X^*} \operatorname{dom} F_{A+B}. \tag{18}$$

Let $x^* \in \operatorname{ran} A + \operatorname{ran} B$, and $x_0 \in \operatorname{dom} A \cap \operatorname{dom} B$. Thus there exists $x_1^* \in \operatorname{ran} A, x_2^* \in \operatorname{ran} B$ such that $x^* = x_1^* + x_2^*$. Then we have $(x_0, x_1^*) \in \operatorname{dom} A \times \operatorname{ran} A$ and $(x_0, x_2^*) \in \operatorname{dom} B \times \operatorname{ran} B$.

Now we consider two cases.

*Case 1*: (*i*) holds. We have $(x_0, x_1^*) \in \operatorname{dom} F_A$ and $(x_0, x_2^*) \in \operatorname{dom} F_B$. Since $F_A \Box_2 F_B(x_0, x^*) \leq F_A(x_0, x_1^*) + F_B(x_0, x_2^*) < +\infty$, $(x_0, x^*) \in \operatorname{dom} F_A \Box_2 F_B$. Then [30, Lemma 23.9] implies that $(x_0, x^*) \in \operatorname{dom} F_{A+B}$ and thus $x^* \in P_{X^*} \operatorname{dom} F_{A+B}$. Hence (18) holds.

*Case 2*: (*ii*) holds. Since $x_1^* \in \operatorname{ran} A$, there exists $x_1 \in X$ such that $(x_1, x_1^*) \in \operatorname{gra} A$. By the assumption, $(x_1, x_2^*) \in \operatorname{dom} B \times \operatorname{ran} B \subseteq \operatorname{dom} F_B$. Similar to the corresponding lines in Case 1, we have $x^* \in P_{X^*} \operatorname{dom} F_{A+B}$ and thus (18) holds.

Combining all the above cases, (18) holds.

By Theorem 3.4 and (18)

$$\operatorname{int} [\operatorname{ran} A + \operatorname{ran} B] \subseteq \operatorname{int} [P_{X^*} \operatorname{dom} F_{A+B}] = \operatorname{int} \operatorname{ran}(A+B) \subseteq \operatorname{int} [\operatorname{ran} A + \operatorname{ran} B].$$

Hence $\operatorname{int} [\operatorname{ran} A + \operatorname{ran} B] = \operatorname{int} \operatorname{ran}(A+B)$.

By Theorem 3.4 and (18) again,

$$\operatorname{ran} A + \operatorname{ran} B \subseteq P_{X^*} \operatorname{dom} F_{A+B} \subseteq \overline{\operatorname{ran}(A+B)} \subseteq \overline{\operatorname{ran} A + \operatorname{ran} B}.$$

Hence $\overline{\operatorname{ran} A + \operatorname{ran} B} = \overline{\operatorname{ran}(A+B)}$. □

**Remark 3.10.** Brézis and Haraux proved Theorem 3.9 in the setting of a Hilbert space (see [14, Theorems 3&4, pp. 173–174]). Reich extended the above result to a reflexive space (see [25, Theorem 2.2]).

Let $A : X \rightrightarrows X^*$ be monotone. We say $A$ is *of type (NA)* (where (NA) stands for “negative alignment") [4] if for every $(x, x^*) \in X \times X^* \setminus \operatorname{gra} A$, there exists $(a, a^*) \in \operatorname{gra} A$ such that $a \neq x, a^* \neq x^*$ and

$$\langle x - a, x^* - a^* \rangle = -\|x - a\| \cdot \|x^* - a^*\|.$$

**Proposition 3.11.** *Let $A : X \rightrightarrows X^*$ be ultramaximally monotone and $(x, x^*) \in X \times X^*$. Then there exists $(a, a^*) \in \operatorname{gra} A$ such that $\|x^* - a^*\| = \|x - a\|$ and*

$$\|x - a\|^2 + \|x^* - a^*\|^2 + 2\langle x - a, x^* - a^* \rangle = 0.$$

**Proof.** Define $B : X \rightrightarrows X^*$ by $B := A(\cdot + x)$. By the assumption, we have $B$ is ultramaximally monotone. Then Corollary 3.6 implies that $x^* \in \operatorname{ran}(B + J)$. Thus there exists $b \in X$ such that $x^* \in Bb + Jb = A(b + x) + Jb$. Let $a := b + x$. Thus $b := a - x$. We have $x^* \in Aa + J(a - x)$. Let $a^* \in Aa$ be such that $x^* \in a^* + J(a - x)$. Hence we have $x^* - a^* \in J(a - x)$ and then

$$\langle x^* - a^*, a - x \rangle = \|x^* - a^*\| \cdot \|a - x\| \quad \text{and} \quad \|x^* - a^*\| = \|a - x\|.$$

Then

$$\begin{aligned} &\|x - a\|^2 + \|x^* - a^*\|^2 + 2\langle x - a, x^* - a^* \rangle \\ &= 2\|x - a\|^2 - 2\|x^* - a^*\| \cdot \|a - x\| = 0. \end{aligned}$$

□

Now we show that every ultramaximally monotone operator is of type (NA).

**Theorem 3.12.** *Let $A : X \rightrightarrows X^*$ be ultramaximally monotone. Then $A$ is of type (NA).*

**Proof.** Let $(x, x^*) \in X \times X^* \setminus \operatorname{gra} A$. Proposition 3.11 implies that there exists $(a, a^*) \in \operatorname{gra} A$ such that $\|x^* - a^*\| = \|x - a\|$ and $2\|x - a\| \cdot \|x^* - a^*\| + 2\langle x - a, x^* - a^* \rangle = 0$. Thus

$$\langle x - a, x^* - a^* \rangle = -\|x - a\| \cdot \|x^* - a^*\|.$$

Since $(x, x^*) \notin \operatorname{gra} A$ and $\|x^* - a^*\| = \|x - a\|$, we have $\|x^* - a^*\| = \|x - a\| \neq 0$. Hence $x \neq a$ and $x^* \neq a^*$. □

**Remark 3.13.** In [30, Remark 29.4], Simons shows that not every operator of type (NI) is necessarily of type (NA) (see the operator $A := \partial \iota_X$ when $X$ is non-reflexive). Hence we cannot significantly weaken the conditions in Theorem 3.12.

**Corollary 3.14 (Bauschke and Simons, see [4, Theorem 3.5]).** *Let $A : X \rightrightarrows X^*$ be an ultramaximally monotone linear relation. Assume that $A$ is at most single-valued. Then $A$ is of type (NA).*

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