Journal of Convex Analysis
Volume 24 (2017), No. 1, 1–17

Quasistatic Viscoplasticity with Polynomial Growth Condition and with Frictional Contact

Lukasz Gleń
Warsaw University of Technology,
Faculty of Mathematics and Information Sciences,
ul. Koszykowa 75, 00-662 Warsaw, Poland
lukasz.glen at poczta.fm

Received: June 3, 2013
Accepted: March 16, 2015

We consider a problem in the inelastic deformation theory. There is a body consisted of a viscoplastic material, which is deformed in a quasistatic process i.e. the movement varies slowly. Additionally we assume that the body has a contact with a rigid foundation: the body moves on the foundation with a friction modelled by a dissipative potential. Together with an inelastic constitutive function, it gives a problem that involves two monotone operators: one acting on the body, other acting on its boundary. We prove existence and uniqueness of a solution to this problem where inelastic constitutive function has a polynomial growth at infinity.

Keywords: Inelastic deformation theory, viscoplasticity, friction, frictional contact, sum of monotone operators

2000 Mathematics Subject Classification: 74C10

1. Introduction

Phenomena of viscoplasticity refers to the class of materials and their behaviour while an external forces deform them: partially irreversible (viscoplastic deformation or plastic flow) and partially elastic (reversible, material "remembers" its origin shape). For details on viscoplastic models we refer to [1] and [14]. As an example of a viscoplastic material we can take any metal in a "standard" temperature.

In the contact mechanics we describe how a body behaves when it touches a solid foundation. We assume that a body moves along the foundation if it is in touch. One kind of a contact is a frictional contact. Very well known models describing this kind of contact are Tresca's Friction Law and Coulomb Friction Law. For the details we refer to [10]. In this paper we consider viscoplastic and quasistatic problem with frictional contact. There are some results in this domain, for example those presented in [10]. These results are obtained with the assumption that inelastic constitutive function is Lipschitz. In this paper

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

we assume polynomial growth of this function at infinity. This assumption is closer to many applications and it is often considered, see for example [2], [9], [13]. But the case of rapid, i.e. faster than polynomial, growth of inelastic deformation with frictional contact is still to be solved.

In this paper we study quasistatic processes only. Quasistatic processes occur when external forces acting on the body vary slowly in time. Therefore the response of the material is equally slow and inertia can be neglected in the problem formulation.

One of the methods of solving problems in inelastic deformation theory is to use coercive approximations. This method consists of two steps. First we solve an approximated problem. This problem is obtained by adding a perturbation so that the problem becomes coercive. Second step is to pass to coercive limits. Such method was proposed in [6], [7] and [8]. Note that a notion of coercivity in the mentioned above papers differs from the one used in this paper.

We take other approach. Results presented in this paper base on standard theory of maximal monotone operators in Banach and Hilbert spaces. In this paper we use several theorems from the theory of monotone operators. For proofs of these results we refer to [4], [15], [16], [17]. For this reason, the method used in this paper can be considered as a geometric one, although the problems are stated as analytic.

Similar approach is adopted in [11] and [12]. Those results are even stronger because a nonreflexive Banach spaces are considered. In this paper we assume that an inelastic deformation is described by a function with a polynomial growth. This allows us to use reflexive Banach spaces. But nonlinear conditions of boundary (frictional contact) provide additional difficulties. Thus the problem, considered in this paper, is of a different class, compared to [11], [12], [6], [7] and [8].

At the beginning of this paper we formulate the problem. In the next parts we give some properties and we express the problem in an abstract form. At the end of this paper we prove main results. The first result is formulated in an abstract form, where we consider a sum of monotone operators. The second result presents how the problem from the beginning is solved using the abstract theorems.

2. Problem formulation

2.1. Constitutive equations

Suppose that the open and bounded set $\Omega \subset \mathbb{R}^n$ represents a solid material, with constant in time and space density. Let $u(t,x) \in \mathbb{R}^n$ denotes the displacement vector at the point $x \in \Omega$ and the time $t \geq 0$. We focus on the symmetric part of the gradient of u:

$$\epsilon = \epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \in \mathbb{R}^{n\times n}_{\mathrm{sym}},$$

which is called the linear strain tensor in the case of the small deformations. Recall that $\mathbb{R}^{n\times n}_{\mathrm{sym}} = \{v \in \mathbb{R}^{n\times n} \colon v = v^T\}$. Let ϵ^p denotes a plastic strain tensor which is also symmetric. We define the stress tensor $T(t,x) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ by using the following elastic constitutive equation:

$$T = \mathcal{D}(\epsilon - \epsilon^p), \tag{1}$$

where $\mathcal{D} \colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is a linear, symmetric and positive definite operator with constant coefficients. We consider a quasistatic problem i.e. we assume that the velocity u_t of the displacement vector is very small and it can be neglected. Recall that u_t denotes the time derivative of u. Hence the balance of forces is in the form:

$$\operatorname{div} T = f_0, \tag{2}$$

where f_0 are given external forces acting on the body Ω. We assume that the inelastic constitutive equation has the following form:

$$\epsilon^p_t = \mathcal{G}(T),$$

where $\mathcal{G}$ is the nonlinear operator with the domain $\operatorname{Dom}\mathcal{G} \subset \mathbb{R}^{n\times n}_{\mathrm{sym}}$. The function $\mathcal{G}$ is the given inelastic constitutive function and it arises from engineering applications. We apply (1) and consider the constitutive equation in the form:

$$\mathcal{D}^{-1}T_t - \epsilon(u_t) = -\mathcal{G}(T). \tag{3}$$

We study the model of the monotone type i.e. the model where $\mathcal{G}$ is a maximal monotone function. Further, we assume that the body Ω is made of a viscoplastic material, hence constitutive function $\mathcal{G}$ is continuous and everywhere defined i.e. $\operatorname{Dom}\mathcal{G} = \mathbb{R}^{n\times n}_{\mathrm{sym}}$. A case where $\operatorname{Dom}\mathcal{G} \subsetneq \mathbb{R}^{n\times n}_{\mathrm{sym}}$ refers to a different type of problem domain and models, for example a perfect plasticity. Methods presented in this paper are not applicable to this case.

2.2. The boundary conditions

Let $\Gamma = \partial\Omega$ be the boundary. We assume that the boundary is Lipschitz and it is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, relatively open in Γ and such that $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$ ($\overline{\Gamma_i}$ denotes the closure of Γ_i). We assume $|\Gamma_1| > 0$. Thanks to this assumption, the solutions to the problems considered in this paper are unique. Moreover, we assume that Γ_1 and Γ_3 are separated i.e. $\overline{\Gamma_1} \cap \overline{\Gamma_3} = \emptyset$.

On part Γ_1 we consider the Dirichlet condition:

$$u = f_1 \quad \text{on } \Gamma_1, \tag{4}$$

on part Γ_2 we consider the Neumann condition:

$$T\nu = f_2 \quad \text{on } \Gamma_2, \tag{5}$$

where ν is the exterior normal vector to the boundary.

Now we make the most important assumption in this paper. We consider a fictional contact problem. The body Ω has a contact with a rigid foundation through Γ_3. The contact is frictional, i.e. the body can move on a foundation with a friction. The frictional contact is given by the relation:

$$\phi(v) \geq -T\nu \cdot (v - u_t) + \phi(u_t) \quad \text{on } \Gamma_3,$$

holds for every $v \in \operatorname{Dom}\phi$, where ϕ is a proper, convex, lower semicontinuous function $\phi\colon \operatorname{Dom}\phi \to \mathbb{R}$ and $\operatorname{Dom}\phi \subset \mathbb{R}^n$. We do not assume that $\operatorname{Dom}\phi = \mathbb{R}^n$. The above relation is equivalent to

$$-T\nu \in \partial\phi(u_t) \quad \text{on } \Gamma_3, \tag{6}$$

where $\partial\phi$ denotes the subgradient of the function ϕ.

In the case of $|\Gamma_3| = 0$ the problem takes the well known form with linear boundary conditions. Thus we assume that $|\Gamma_3| > 0$, through the rest of this paper. The nonlinearity occurring at the boundary is the main problem of this paper. We study the existence and uniqueness of solutions to the system (2) – (6).

2.3. The solutions domain

Through this paper, we set a constant $r \geq 2$. Let r' be conjugate exponent of r i.e. $\frac{1}{r} + \frac{1}{r'} = 1$. Solutions are consisted in Banach spaces: $T \in L^r(\Omega)$ and $\epsilon(u) \in L^{r'}(\Omega)$. Moreover, solutions are defined on bounded time interval $[0, \mathcal{T}]$.

Let $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$. We write L^p instead of $L^p(\Omega)$ and $L^q(L^p)$ instead of $L^q(0, \mathcal{T}; L^p(\Omega))$ through the rest of this paper.

3. Transformation to the abstract problem

3.1. Weak formulation of the boundary conditions

The relation (6) is pointwise. Let us define $\bar{\phi}\colon W^{1-\frac{1}{r'},r'}(\Gamma_3) \to \mathbb{R}$ as follows:

$$\bar{\phi}(v) = \int\limits_{\Gamma_3} \phi(v(x))\, dx$$

for $v \in W^{1-\frac{1}{r'},r'}(\Gamma_3)$ such that $\phi(v) \in L^1(\Gamma_3)$. The function $\bar{\phi}$ is convex, proper and lower semicontinuous. On part Γ_3 of the boundary we consider the following relation:

$$\bar{\phi}(v) \geq - < T\nu, v - u_t >_{\Gamma_3} + \bar{\phi}(u_t), \tag{7}$$

holds for every $v \in W^{1-\frac{1}{r'},r'}(\Gamma_3)$, where $< \_,\_ >_{\Gamma_3}$ denotes the pairing in $W^{-\frac{1}{r},r}(\Gamma_3) \times W^{1-\frac{1}{r'},r'}(\Gamma_3)$. Note that $T\nu$ is considered as an element of $W^{-\frac{1}{r},r}(\Gamma_3)$ and u_t is considered as an element of $W^{1-\frac{1}{r'},r'}(\Gamma_3)$.

The space $W^{-\frac{1}{r},r}(\Gamma_3)$ is the dual space to $W^{1-\frac{1}{r'},r'}(\Gamma_3)$. Therefore pairing $< \_,\_ >_{\Gamma_3}$ makes sense and it holds $< T\nu, v >_{\Gamma_3}= T\nu(v)$. Moreover, the function $\bar{\phi}$ has a subgradient $\partial\bar{\phi}\colon W^{1-\frac{1}{r'},r'}(\Gamma_3) \to W^{-\frac{1}{r},r}(\Gamma_3)$. The following relation

$$-T\nu \in \partial\bar{\phi}(u_t)$$

holds if and only if $T\nu$ and u_t satisfy the relation (7). The operator $\partial\bar{\phi}$ is a monotone operator in the following sense: if $(v_1, T_1\nu)$ and $(v_2, T_2\nu)$ belong to Graph $\partial\bar{\phi}$ then holds

$$< T_1\nu - T_2\nu, v_1 - v_2 >_{\Gamma_3} \geq 0.$$

Moreover, from properties of $\bar{\phi}$ we see that $\partial\bar{\phi}$ is also maximal monotone.

If $-T\nu \in \partial\bar{\phi}(v)$ and $T\nu$ is defined pointwise, then inclusion $-T\nu \in \partial\phi(v)$ is satisfied pointwise for almost all $x \in \Gamma_3$.

We say that $T \in L^r$ and $\epsilon(u_t) \in L^{r'}$ satisfy (6) in the weak sense (or weakly) if $-T\nu \in \partial\bar{\phi}(u_t)$.

The relation (4) is understood as an equation in the space $W^{1-\frac{1}{r'},r'}(\Gamma_1)$, and the relation (5) is understood as an equation in the space $W^{-\frac{1}{r},r}(\Gamma_2)$. Moreover, a divergence operator in the relation (2) is considered as a weak divergence.

3.2. Transformation of the data functions

Let $v\colon [0,\mathcal{T}] \times \Omega \to \mathbb{R}^n$ be the solution to the following system of equations:

$$\begin{cases} \operatorname{div}(\epsilon(v)) = 0 & \\ v = f_1 & \text{on } \Gamma_1 \\ v = 0 & \text{on } \Gamma_3, \end{cases}$$

and $w\colon [0,\mathcal{T}] \times \Omega \to \mathbb{R}^n$ denotes the solution to the system:

$$\begin{cases} \operatorname{div}(\epsilon(w)) = f_0 & \\ \epsilon(w)\nu = f_2 & \text{on } \Gamma_2 \\ \epsilon(w)\nu = 0 & \text{on } \Gamma_3. \end{cases}$$

Remark 3.1. Notice that it suffices to assume that for almost every t holds

$$f_0(t) \in L^r(\Omega), \qquad f_1(t) \in W^{1-\frac{1}{r'},r'}(\Gamma_1), \qquad f_2(t) \in W^{-\frac{1}{r},r}(\Gamma_2),$$

to imply existence of the solutions such that:

$$v(t) \in W^{1,r'}(\Omega), \qquad w(t) \in W^{1,r}(\Omega),$$

holds for almost every t, where div denotes the weak divergence, the equation $v = f_1$ on Γ_1 holds in $W^{1-\frac{1}{r'},r'}(\Gamma_1)$ and the equation $\epsilon(w)\nu = f_2$ on Γ_2 holds in $W^{-\frac{1}{r},r}(\Gamma_2)$. The assumption $\overline{\Gamma_1} \cap \overline{\Gamma_3} = \emptyset$ is used. It implies that we can define the boundary conditions of v independently on Γ_1 and Γ_3 i.e. it holds $v(t)|_{\Gamma_1\cup\Gamma_3} \in W^{1-\frac{1}{r'},r'}(\Gamma_1 \cup \Gamma_3)$ for any $f_1(t) \in W^{1-\frac{1}{r'},r'}(\Gamma_1)$. Therefore, the solution v exists for any $f_1(t) \in W^{1-\frac{1}{r'},r'}(\Gamma_1)$.

Remark 3.2. The solutions v and w are not unique. The problems are not defined on some parts of boundary. Fortunately, the uniqueness is not needed. We could extend the problems definitions and hope to obtain unique solutions, but we would encounter difficulties in case of v.

We consider substitutions:

$$\begin{aligned} u &:= u + v, \\ T &:= T + \epsilon(w). \end{aligned}$$

Coupling together the relations (2) – (6) and using the above substitutions we obtain the problem:

$$\begin{cases} \mathcal{D}^{-1}T_t - \epsilon(u_t) & \\ \qquad = -\mathcal{G}\left(T + \epsilon(w)\right) - \mathcal{D}^{-1}\epsilon(w_t) + \epsilon(v_t) & \\ \operatorname{div} T = 0 & \\ u = 0 & \text{on } \Gamma_1 \\ T\nu = 0 & \text{on } \Gamma_2 \\ -T\nu \in \partial\phi(u_t) & \text{on } \Gamma_3, \quad \text{in the weak sense given previously.} \end{cases}$$

In general the problem has the following form:

$$\begin{cases} \mathcal{D}^{-1}T_t - \epsilon(u_t) = -\mathcal{G}(T + g) + f & \\ \operatorname{div} T = 0 & \\ u_t = 0 & \text{on } \Gamma_1 \\ T\nu = 0 & \text{on } \Gamma_2 \\ -T\nu \in \partial\phi(u_t) & \text{on } \Gamma_3, \quad \text{in the weak sense given previously,} \end{cases} \tag{8}$$

where the unknowns are the stress tensor T and the displacement vector u. Note that ϵ^p does not appear in this problem. We also ignore relation (1). This way T and $\epsilon(u)$ are considered as independent unknowns. If T and $\epsilon(u)$ are the solutions to (8) then ϵ^p can be calculated from (1).

3.3. Purging of the boundary conditions

We replace relation (6) with a relation extended to L^r as follows. Let $T \in L^r$, $\epsilon(u_t) \in L^{r'}$. For Γ_3 we consider more general condition:

$$-T\nu \in \psi(u_t),$$

where ψ is a maximal monotone operator $W^{1-\frac{1}{r'},r'}(\Gamma_3) \to W^{-\frac{1}{r},r}(\Gamma_3)$. Recall that the space $W^{-\frac{1}{r},r}(\Gamma_3)$ is the dual space to $W^{1-\frac{1}{r'},r'}(\Gamma_3)$, $T\nu$ is an element of $W^{-\frac{1}{r},r}(\Gamma_3)$ and u_t is an element of $W^{1-\frac{1}{r'},r'}(\Gamma_3)$. The monotonicity of the operator ψ is defined as follows: if $T_1\nu \in \psi(v_1)$ and $T_2\nu \in \psi(v_2)$ then holds

$$< T_1\nu - T_2\nu, v_1 - v_2 >_{\Gamma_3} \geq 0,$$

where $< \_, \_ >_{\Gamma_3}$ denotes the pairing in $W^{-\frac{1}{r},r}(\Gamma_3) \times W^{1-\frac{1}{r'},r'}(\Gamma_3)$. Note that we do not assume gradient structure of the operator as it is in (6) and that the operator ψ is considered as a multifunction.

We state the boundary condition as:

$$u_t \in \psi^{-1}(-T\nu).$$

We introduce function $v = -u_t$. The above relation and relations (8) lead to the following problem:

$$\begin{cases} \mathcal{D}^{-1}T_t + \epsilon(v) = -\mathcal{G}(T+g) + f & & \\ \operatorname{div} T = 0 & & \\ v = 0 & \text{on } \Gamma_1 & \\ T\nu = 0 & \text{on } \Gamma_2 & \\ v \in -\psi^{-1}(-T\nu) & \text{on } \Gamma_3, & \begin{array}{l}\text{in the sense}\\ \text{given above.}\end{array} \end{cases} \tag{9}$$

We define a multifunction $\Psi : L^r \to L^{r'}$ as follows: $\epsilon(v) \in \Psi(T)$ if and only if

$$\begin{cases} \operatorname{div} T = 0 & & \\ v = 0 & \text{on } \Gamma_1 & \\ T\nu = 0 & \text{on } \Gamma_2 & \\ v \in -\psi^{-1}(-T\nu) & \text{on } \Gamma_3, & \begin{array}{l}\text{in the sense}\\ \text{given above.}\end{array} \end{cases}$$

Fact. Ψ *is a maximal monotone operator.*

Proof. We prove monotonicity by the following calculation:

$$\int_\Omega (T_1 - T_2) \cdot (\epsilon(v_1) - \epsilon(v_2)) =< (T_1\nu - T_2\nu) \cdot (v_1 - v_2) >_{\Gamma_3} \geq 0.$$

To prove maximality it suffices to show that if a given $S \in L^r$, $W \in L^{r'}$ satisfies

$$< S - T, W - \epsilon(v) > \geq 0 \quad \forall (T, \epsilon(v)) \in \mathrm{Graph}\, \Psi$$

then $(S, W) \in \mathrm{Graph}\, \Psi$. Here $< \_, \_ >$ denotes pairing in $L^r \times L^{r'}$. Let us assume such S and W. Let $\epsilon(\bar{v}) \in L^{r'}$ be chosen such that

$$\bar{v} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_3.$$

If $(T, \epsilon(v)) \in \mathrm{Graph}\, \Psi$ then also $(T, \epsilon(v) + \epsilon(\bar{v})) \in \mathrm{Graph}\, \Psi$ and:

$$< S - T, W - \epsilon(v) - \epsilon(\bar{v}) > \geq 0.$$

Since $\epsilon(\bar{v})$ is chosen arbitrary, by taking $\lambda \epsilon(\bar{v})$ where $\lambda \in \mathbb{R}$, we conclude that

$$< S - T, \epsilon(\bar{v}) >= 0.$$

We have $\operatorname{div} T = 0$ and $T\nu = 0$ on Γ_2, thus

$$< T, \epsilon(\bar{v}) >= 0$$

and in consequence:

$$< S, \epsilon(\bar{v}) >= 0$$

Further, since $\epsilon(\bar{v})$ is chosen arbitrary, then we conclude:

$$\operatorname{div} S = 0, \quad S\nu = 0 \quad \text{on } \Gamma_2.$$

Similarly we show that there exists $\epsilon(w) \in L^{r'}$ such that $w = 0$ on Γ_1 and $W = \epsilon(w)$. Hence:

$$< S - T, W - \epsilon(v) >=< S\nu - T\nu, w - v >_{\Gamma_3} \geq 0$$

for all $(T, \epsilon(v)) \in \mathrm{Graph}\, \Psi$. Notice that for every $(\tau, \mu) \in \mathrm{Graph}\, \psi$, there exists $(T, \epsilon(v)) \in \mathrm{Graph}\, \Psi$ such that $T\nu|_{\Gamma_3} = \tau$ and $v|_{\Gamma_3} = \mu$. Therefore, from maximality of ψ, we obtain that $(S\nu|_{\Gamma_3}, w|_{\Gamma_3}) \in \mathrm{Graph}\, \psi$. All four parts of definition of Ψ are satisfied, thus $(S, W) \in \mathrm{Graph}\, \Psi$. □

This way the problem (9) is transformed to the single relation:

$$\mathcal{D}^{-1} T_t + \Psi(T) \ni -\mathcal{G}(T + g) + f \tag{10}$$

Note that the problems (9) and (10) are equivalent. Furthermore, the solutions to the problem (9) are also the solutions to the problem (8).

4. Properties

4.1. The Growth Condition

The growth condition is very important in this paper. It implies that an operator has a domain with nonempty interior.

Definition. We say that the operator $\mathcal{G}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfies the growth condition if there exists a constant $C > 0$ such that the inequality:

$$|\mathcal{G}(T)| \leq C + C|T|^{r-1}$$

holds for any $T \in \mathbb{R}^{n\times n}$.

Fact. *Let us assume that an operator $\mathcal{G}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfies the growth condition. Then the induced operator $\mathcal{G}\colon L^r \to L^{r'}$ satisfies the growth condition.*

Proof. For any $T \in L^r$ holds:

$$\begin{aligned}
\|\mathcal{G}(T)\|_{r'} &= \left(\int_\Omega |\mathcal{G}(T(\zeta))|^{r'}\, d\zeta\right)^{1/r'} \leq \left(\int_\Omega (C + C|T(\zeta)|^{r-1})^{r'}\, d\zeta\right)^{1/r'} \\
&\leq \left(\int_\Omega C^{(r-1)r'}\, d\zeta\right)^{r-1} + C^{r-1}\left(\int_\Omega |T(\zeta)|^{(r-1)r'}\, d\zeta\right)^{1/r'} \\
&\leq C_1 + C_1\left(\int_\Omega |T(\zeta)|^r\, d\zeta\right)^{(r-1)/r} = C_1 + C_1\|T\|_r^{r-1}. \qquad \square
\end{aligned}$$

Corollary. *Let us assume that an operator $\mathcal{G}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfies the growth condition. Then the induced operator $\mathcal{G}\colon L^r \to L^{r'}$ is everywhere defined and bounded (i.e. images of bounded sets are bounded).* $\square$

4.2. Hemicontinuity

In the following three sections, Hemicontinuity, Coercivity and Theorems of Monotonicity, we include definitions and theorems from theory of monotone operators which we use in this paper. They are included for the purpose of completeness and to avoid any confusions. For details of hemicontinuity and its applications we refer to [4]. We use it to conclude that a monotone operator is maximal.

Definition. Let X be Banach space and X^* denotes its dual space. Let $A\colon X \to X^*$ be a single valued, everywhere defined, maximal monotone operator. A is said to be hemicontinuous if

$$w - \lim_{t\to 0} A(x + ty) = Ax$$

for all $x, y \in X$. ($w - \lim$ denotes a weak limit).

Fact. *Let an operator $\mathcal{G}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfy the growth condition and be continuous. Then the induced operator $\mathcal{G}\colon L^r \to L^{r'}$ is continuous on directions.*

Proof. We need to prove that

$$\lim_{t\to 0} \mathcal{G}(T + tS) = \mathcal{G}(T)$$

holds for all $T, S \in L^r$. Notice that $\mathcal{G}(T+tS) \to \mathcal{G}(T)$ pointwise everywhere in $\mathbb{R}^{n\times n}$ and $\mathcal{G}(T+tS)$ is pointwise bounded in $\mathbb{R}^{n\times n}$ by some integrable function for bounded t. From the dominated convergence theorem we finish the proof. □

Corollary. *Let an operator $\mathcal{G}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfy the growth condition and be continuous. Then the induced operator $\mathcal{G}\colon L^r \to L^{r'}$ is hemicontinuous.* □

4.3. Coercivity

For the notion of coercivity of monotone operators we refer again to [4]. The most important application in this paper is that coercivity implies surjectivity.

Definition. Let X be Banach space and X^* denotes its dual space. Moreover, let $<\_,\_>_X$ denotes pairing in $X \times X^*$ and $\|\cdot\|_X$ is the norm in X. An operator $A\colon X \to X^*$ is said to be coercive if

$$\frac{< x, x^* >_X}{\|x\|_X} \to +\infty$$

whenever

$$\|x\|_X \to +\infty, \quad x \in \mathrm{Dom}(A), \quad x^* \in Ax.$$

Remark. If A is coercive, $g \in X$ and $f \in X^*$ then $x \mapsto A(x+g)+f$ is also coercive.

Fact. *Let an operator $\mathcal{G}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfy the following condition:*

$$\mathcal{G}(x)\cdot x \geq C|x|^r - C.$$

Then the induced operator $\mathcal{G}\colon L^r \to L^{r'}$ is coercive.

Proof. It is enough to calculate:

$$\frac{\int_\Omega \mathcal{G}(T(\zeta))\cdot T(\zeta)\, d\zeta}{\|T\|_{L^r}} \geq \frac{\int_\Omega C|T(\zeta)|^r - C\, d\zeta}{\|T\|_{L^r}} = \frac{C_1\|T\|^r_{L^r} - C_2}{\|T\|_{L^r}}$$
$$= C_1\|T\|^{r-1}_{L^r} - \frac{C_2}{\|T\|_{L^r}} \xrightarrow[\|T\|_{L^r}\to\infty]{} +\infty. \qquad \square$$

4.4. Theorems of Monotonicity

We list important theorems and corollaries used in proceeding parts of this paper. Once again, for details and proofs we refer to [4]. Let X be a reflexive Banach space and X^* denotes its dual space.

Theorem. *Let A and B be maximal monotone operators $X \to X^*$. Moreover, assume that the following condition:*

$$(\operatorname{int}\operatorname{Dom}(A)) \cap \operatorname{Dom}(B) \neq \emptyset$$

is satisfied. Then $A + B$ is a maximal monotone operator. □

Corollary. *Let A and B be maximal monotone operators $X \to X^*$ and* $\operatorname{Dom}(A) = X$*. Then $A + B$ is a maximal monotone operator.*

Theorem. *Let $A\colon X \to X^*$ be a maximal monotone operator. A is a surjection if and only if A^{-1} is locally bounded at every point in the image of A.* □

Corollary. *Coercive and maximal monotone operators are surjections.* □

Corollary. *Let A and B be maximal monotone operators $X \to X^*$,* $\operatorname{Dom}(A) =$ *X and A be coercive. Then $A + B$ is a maximal monotone and surjective operator.*

Theorem. *Let $A\colon X \to X^*$ be a monotone operator, everywhere defined and hemicontinuous. Then A is a single valued, maximal monotone operator.* □

Corollary. *Let an operator $\mathcal{G}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ satisfies the growth condition, be continuous and monotone. Then induced operator $\mathcal{G}\colon L^r \to L^{r'}$ is maximal monotone.* □

5. The main results

Theorem. *Let $\mathcal{G}\colon L^r \to L^{r'}$ be a coercive, maximal monotone operator satisfying the growth condition and $g \in L^r$ be given. Let $\Psi\colon L^r \to L^{r'}$ be any maximal monotone operator. Then $\mathcal{G}(T + g) + \Psi T$ is a maximal monotone operator as $L^2 \to L^2$ operator.*

Remark. Note that the operators $\mathcal{G}$ and Ψ are $L^r \to L^{r'}$ operators, but its sum $\mathcal{G}(T + g) + \Psi T$ (with perturbation g) is considered as $L^2 \to L^2$ operator. To make it clear we provide auxiliary definitions. Let $A\colon L^r \to L^{r'}$ be defined by the formula:

$$AT = \mathcal{G}(T + g) + \Psi T.$$

It is clear that A is a maximal monotone operator. Now we truncate it to obtain an operator from L^2 to L^2 (note that $r \geq 2$). Let $\bar{A}\colon L^2 \to L^2$ be defined as follows:

$$(T, S) \in \operatorname{Graph}\bar{A} \Leftrightarrow T \in L^r, \quad S \in L^2, \quad (T, S) \in \operatorname{Graph} A.$$

Proof of the Theorem. The proof of the monotonicity is straightforward. To prove maximality, it is enough to solve in L^2 the following inclusion:

$$T + \bar{A}T \ni h,$$

for any $h \in L^2$.

Firstly we solve it in L^r. Literally we solve

$$T + AT \ni h.$$

An expression T is not just an identity, it is a mapping $T \mapsto T$ treated as $L^r \to L^{r'}$ operator, i.e. it is an embedding $L^r \subset L^{r'}$. It is a continuous linear monotone operator. Thus $T + \Psi T$ is a maximal monotone operator. Further, $(T + \Psi T) + \mathcal{G}(T+g)$ is a maximal monotone operator and is a surjection because $\mathcal{G}(T + g)$ is a maximal monotone, everywhere defined, bounded and coercive. Note that $h \in L^2 \subset L^{r'}$. Hence, there exists $T \in L^r$ such that

$$T + (\mathcal{G}(T + g) + \Psi(T)) \ni h.$$

We show that the solution T is also a solution in L^2. Notice that:

$$\mathcal{G}(T + g) + \Psi(T) \ni h - T.$$

An element $h - T \in L^2$ thus T is an element of $\operatorname{Dom} \bar{A}$. □

Now it is matter of routine to solve (10).

Theorem. *Let $\mathcal{G} \colon L^r \to L^{r'}$ be a coercive, maximal monotone operator satisfying the growth condition. Let $\Psi \colon L^r \to L^{r'}$ be any maximal monotone operator. Moreover let $T_0 \in \operatorname{Dom} \Psi$, $f \in W^{1,1}(0, \mathcal{T}; L^2)$ and g be piecewise constant i.e. there exists $0 = t_0 \leq \ldots \leq t_m = T$ and $g_0, \ldots, g_{m-1} \in L^r$ such that $g(t) = g_i$ if $t_i \leq t \leq t_{i+1}$. Then the inclusion:*

$$D^{-1}T_t + \Psi T + \mathcal{G}(T + g) \ni f \tag{11}$$

has a unique solution $T \in W^{1,\infty}(0, \mathcal{T}; L^2)$ such that $T(0) = T_0$.

Proof. For $t = t_i$ it is satisfied that $T \in \operatorname{Dom}(\Psi)$, $T + g_{i-1} \in \operatorname{Dom}(\mathcal{G})$ and $T + g_i \in \operatorname{Dom}(\mathcal{G})$. Thus, it is enough to find solutions on time intervals $[t_i, t_{i+1}]$ and concatenate them.

Assume that g is constant in time. According to the previous theorem, the inclusion:

$$\mathcal{D}^{-1}T_t + \Psi T + \mathcal{G}(T + g) \ni f$$

is an evolution inclusion in Hilbert space L^2. Existence and uniqueness of solution follows from standard theories of monotone evolution equations in Hilbert spaces. □

To end this paper we give an answer to the problem stated at the beginning of this paper. It is a direct implication of the above theorem.

We recall the problem. Let $\mathcal{G}\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ be a continuous, everywhere defined maximal monotone operator satisfying the growth condition and the following condition: there exists a constant $C > 0$ such that

$$\mathcal{G}(x)\cdot x \geq C|x|^r - C \tag{12}$$

holds for every $x \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$. Let $\phi\colon \mathbb{R}^n \to \mathbb{R}$ be any proper, convex and lower semicontinuous function, not necessary everywhere defined. We consider the problem:

$$\begin{cases} \mathcal{D}^{-1}T_t - \epsilon(u_t) = -\mathcal{G}(T) & \\ \operatorname{div} T = f_0 & \\ u = f_1 & \text{on } \Gamma_1 \\ T\nu = f_2 & \text{on } \Gamma_2 \\ -T\nu \in \partial\phi(u_t) & \text{on } \Gamma_3 \end{cases} \tag{13}$$

where $T, \epsilon(u) \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ are unknowns.

We say that T and u are the solutions in the following sense. The solutions are considered on the given time interval $[0,\mathcal{T}]$. The equation of balance of forces (divergence) is satisfied in the weak sense. The condition on Γ_3 is satisfied in the weak sense and the boundary conditions are taken in the sense of traces as we defined previously.

We say that T_0 and u_0 are compatible initial values if they can be applied to (13) i.e. $\operatorname{div} T_0 = f_0(0)$, $u_0 = f_1(0)$ on Γ_1, $T_0\nu = f_2(0)$ on Γ_2 and $-T_0\nu$ belongs to the image of $\partial\phi$.

Theorem. *Let $\epsilon(u_0) \in L^{r'}$ and $T_0 \in L^r$ be compatible initial values. Let us assume that*

$$\begin{cases} f_0 \in L^r(\Omega) \quad \textit{is constant in time,} \\ f_1 \in H^1(0,\mathcal{T}; H^{1/2}(\Gamma_1)), \\ f_2 \in W^{-\frac{1}{r},r}(\Gamma_2) \quad \textit{is constant in time.} \end{cases}$$

Then there exists unique solution $(T,\epsilon(u))$ to the problem (13) such that: $T \in H^1(0,\mathcal{T}; L^2(\Omega)) \cap L^r(0,\mathcal{T}; L^r(\Omega))$ and $\epsilon(u) \in W^{1,r'}(0,\mathcal{T}; L^{r'}(\Omega))$.

Proof. Let $T \in W^{1,\infty}(L^2)$ be the solution to the problem (11) stated in the above theorem. It exists since $T_0 \in \operatorname{Dom}\Psi$. In addition we know that $T(t) \in L^r$ holds for almost every $t \in [0,\mathcal{T}]$ and there exists $\xi\colon [0,\mathcal{T}] \to L^{r'}$ such that $\xi(t) \in \Psi(T(t))$ and

$$\mathcal{D}^{-1}T_t + \mathcal{G}(T+g) + \xi = f$$

holds for almost every $t \in [0,\mathcal{T}]$. According to (12) there exists a constant $C > 0$ such that the condition:

$$\mathcal{G}(T)\cdot T \geq C\|T\|_r^r - C$$

holds for every $T \in L^r$, where $\cdot$ denotes the pairing in $L^r \times L^{r'}$.

Recall that $\xi(t) \in \Psi(T(t))$. Let $\xi_0 \in \Psi(T_0)$. Thus for almost every t holds:

$$\begin{aligned}
& \mathcal{G}(T(t)+g)\cdot(T(t)-T_0) \\
= & -\xi(t)\cdot(T(t)-T_0) - \mathcal{D}^{-1}T_t\cdot(T(t)-T_0) + f(t)\cdot(T(t)-T_0) \\
= & -(\xi(t)-\xi_0)\cdot(T(t)-T_0) - \xi_0\cdot(T(t)-T_0) \\
& \qquad - \mathcal{D}^{-1}T_t\cdot(T(t)-T_0) + f(t)\cdot(T(t)-T_0) \\
\le & -\xi_0\cdot(T(t)-T_0) - \mathcal{D}^{-1}T_t\cdot(T(t)-T_0) + f(t)\cdot(T(t)-T_0) \\
= & -\xi_0\cdot(T(t)+g) + \xi_0\cdot(T_0+g) + (f(t)-\mathcal{D}^{-1}T_t)\cdot(T(t)-T_0) \\
\le & \frac{\varepsilon}{r}\|T(t)+g\|_r^r + \frac{1}{\varepsilon r'}\|\xi_0\|_{r'}^{r'} + (f(t)-\mathcal{D}^{-1}T_t)\cdot(T(t)-T_0).
\end{aligned}$$

On the other hand:

$$\begin{aligned}
& \mathcal{G}(T(t)+g)\cdot(T(t)-T_0) \\
& \ge \mathcal{G}(T(t)+g)\cdot(T(t)+g) - \mathcal{G}(T(t)+g)\cdot(g+T_0) \\
& \ge C\|T(t)+g\|_r^r - C - \mathcal{G}(T(t)+g)\cdot(g+T_0) \\
& \ge C\|T(t)+g\|_r^r - C - \|\mathcal{G}(T(t)+g)\|_{r'}\cdot\|g+T_0\|_r \\
& \ge C\|T(t)+g\|_r^r - C - (\bar{C}\|T(t)+g\|_r^{r-1} + \bar{C})\cdot\|g+T_0\|_r \\
& \ge C\|T(t)+g\|_r^r - C - \frac{\varepsilon\bar{C}}{r'}\|T(t)+g\|_r^r - \frac{\bar{C}}{\varepsilon r}\cdot\|g+T_0\|_r^r - \bar{C}\cdot\|g+T_0\|_r \\
& = \left(C - \frac{\varepsilon\bar{C}}{r'}\right)\|T(t)+g\|_r^r - C - \frac{\bar{C}}{\varepsilon r}\cdot\|g+T_0\|_r^r - \bar{C}\cdot\|g+T_0\|_r,
\end{aligned}$$

where C is the constant from (12) and $\bar{C}$ is the constant from the growth condition. Thus:

$$\begin{aligned}
& \left(C - \frac{\varepsilon\bar{C}}{r'}\right)\|T(t)+g\|_r^r - C - \frac{\bar{C}}{\varepsilon r}\cdot\|g+T_0\|_r^r - \bar{C}\cdot\|g+T_0\|_r \\
\le & \frac{\varepsilon}{r}\|T(t)+g\|_r^r + \frac{1}{\varepsilon r'}\|\xi_0\|_{r'}^{r'} + (f(t)-\mathcal{D}^{-1}T_t)\cdot(T(t)-T_0).
\end{aligned}$$

We integrate it with respect to time:

$$\begin{aligned}
& \left(C - \frac{\varepsilon\bar{C}}{r'} - \frac{\varepsilon}{r}\right)\int_0^{\mathcal{T}} \|T(t)+g\|_r^r\,dt \\
\le & \left(C + \frac{\bar{C}}{\varepsilon r}\cdot\|g+T_0\|_r^r + \bar{C}\cdot\|g+T_0\|_r + \frac{1}{\varepsilon r'}\|\xi_0\|_{r'}^{r'}\right)\mathcal{T} + \\
& \qquad + \int_0^{\mathcal{T}} (f(t)-\mathcal{D}^{-1}T_t)\cdot(T(t)-T_0)\,dt < \infty\,.
\end{aligned}$$

The last inequality holds since $f(t) - \mathcal{D}^{-1}T_t, T(t) - T_0 \in L^2(L^2)$. We conclude that $T \in L^r(L^r)$, since ε is arbitrary and can be selected very small.

The growth condition and fact that $T \in L^r(L^r)$ implies that $\mathcal{G}(T+g) \in L^{r'}(L^{r'})$. The equation

$$\xi = f - T_t - \mathcal{G}(T+g)$$

implies that $\xi \in L^{r'}(L^{r'})$ since $T_t, f \in L^2(L^2)$.

Recall that $g = \epsilon(w)$, where w is the solution to the problem:

$$\begin{cases} \operatorname{div}(\epsilon(w)) = f_0 & \\ \epsilon(w)\nu = f_2 & \text{on } \Gamma_2 \text{ (in } W^{-\frac{1}{r},r}(\Gamma_2)) \\ \epsilon(w)\nu = 0 & \text{on } \Gamma_3. \end{cases}$$

The regularity of the given data functions yields that $g \in L^r$. Let $\bar{T} = T + g$. Notice that $\bar{T} \in H^1(L^2) \cap L^r(L^r)$ and the relations

$$\begin{cases} \bar{T}_t = T_t, & \\ \operatorname{div} \bar{T} = f_0 & \text{in the weak sense,} \\ \bar{T}\nu = f_2 & \text{on } \Gamma_2 \text{ (in } W^{-\frac{1}{r},r}(\Gamma_2)), \\ \bar{T}\nu = T\nu & \text{on } \Gamma_3 \text{ (in } W^{-\frac{1}{r},r}(\Gamma_3)), \end{cases}$$

are satisfied.

Recall that $f = \epsilon(v_t)$ since $w_t = 0$, where v is the solution to the problem:

$$\begin{cases} \operatorname{div}(\epsilon(v)) = 0 & \\ v = f_1 & \text{on } \Gamma_1 \text{ (in } H^{1/2}(\Gamma_1)) \\ v = 0 & \text{on } \Gamma_3. \end{cases}$$

Since $f_1 \in H^1(0, \mathcal{T}; H^{1/2}(\Gamma_1))$ then $\epsilon(v) \in H^1(L^2)$ and $f \in L^2(L^2)$. Let $\xi = -\epsilon(u_t)$. Notice that $\epsilon(u_t) \in L^{r'}(L^{r'})$. Let $\bar{u} = u + v$. Thus $\epsilon(\bar{u}_t) \in L^{r'}(L^{r'})$ and $\epsilon(\bar{u}) \in W^{1,r'}(L^{r'})$ and the relations:

$$\begin{aligned} \xi - f &= -\epsilon(\bar{u}_t), \\ \bar{u} = f_1 &\quad \text{on } \Gamma_1 \text{ (in } W^{1-\frac{1}{r'},r'}(\Gamma_1)), \\ \bar{u}_t = u_t &\quad \text{on } \Gamma_3 \text{ (in } W^{1-\frac{1}{r'},r'}(\Gamma_3)), \end{aligned}$$

are satisfied.

Now it is clear that the relations

$$\mathcal{D}^{-1}\bar{T}_t - \epsilon(\bar{u}_t) = \mathcal{G}(\bar{T}),$$

$$-\bar{T}\nu \in \partial\phi(\bar{u}_t) \quad \text{on } \Gamma_3 \quad \begin{array}{l}\text{in the weak sense}\\ \text{given previously}\end{array}$$

are satisfied. Thus $\bar{T}$ and $\bar{u}$ are desired solutions to (13).

The solution T is unique for given functions f and g. But the choice of the functions v and w is not unique for given functions f_0, f_1 and f_2. In consequence, the functions f and g are not chosen uniquely. That is why we need to prove uniqueness of $\bar{T}$ and $\bar{u}$.

Let $(\bar{T}_1, \epsilon(\bar{u}_1))$ and $(\bar{T}_2, \epsilon(\bar{u}_2))$ be the solutions. We have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathcal{D}^{-1/2}(\bar{T}_1 - \bar{T}_2)\|_2^2 = \int_\Omega \mathcal{D}^{-1}(\bar{T}_{1,t} - \bar{T}_{2,t}) \cdot (\bar{T}_1 - \bar{T}_2)\, dx \\
= &\int_\Omega (\epsilon(\bar{u}_{1,t}) - \epsilon(\bar{u}_{2,t})) \cdot (\bar{T}_1 - \bar{T}_2)\, dx - \int_\Omega (\mathcal{G}(\bar{T}_1) - \mathcal{G}(\bar{T}_2)) \cdot (\bar{T}_1 - \bar{T}_2)\, dx \\
\leq &\int_\Omega (\epsilon(\bar{u}_{1,t}) - \epsilon(\bar{u}_{2,t})) \cdot (\bar{T}_1 - \bar{T}_2)\, dx \\
= &-\int_\Omega (\bar{u}_{1,t} - \bar{u}_{2,t}) \cdot (\operatorname{div} \bar{T}_1 - \operatorname{div} \bar{T}_2)\, dx + < \bar{T}_1\nu - \bar{T}_2\nu, \bar{u}_{1,t} - \bar{u}_{2,t} >_\Gamma \\
= &< \bar{T}_1\nu - \bar{T}_2\nu, \bar{u}_{1,t} - \bar{u}_{2,t} >_{\Gamma_3} \leq 0.
\end{aligned}$$

We integrate the above with respect to time and we obtain:

$$\|\mathcal{D}^{-1/2}(\bar{T}_1 - \bar{T}_2)\|_2^2 \leq 0.$$

Hence $\bar{T}_1 = \bar{T}_2$. From the relation

$$\mathcal{D}^{-1}T_t - \epsilon(u_t) = -\mathcal{G}(T)$$

and the continuity of the function $\mathcal{G}$, it follows that $\epsilon(\bar{u}_1) = \epsilon(\bar{u}_2)$. □

Remark. In this paper, we assume that $\overline{\Gamma_1} \cap \overline{\Gamma_3} = \emptyset$. This assumption allows us to avoid difficulties related to solving the problem:

$$\begin{cases}
\operatorname{div}(\epsilon(v)) = 0 & \\
v = f_1 & \text{on } \Gamma_1 \quad (\text{in } H^{1/2}(\Gamma_1)) \\
v = 0 & \text{on } \Gamma_3.
\end{cases}$$

and other similar problems. Alternatively, we can assume that the function f_1 is constant in time and in consequence $u_t = 0$ on Γ_1. Similar approach is proposed in [10]. We can repeat the reasoning presented in this paper with the assumption $u_t = 0$ on Γ_1 and obtain analogous results.

References

[1] H.-D. Alber: Materials with Memory, Springer, Berlin (1998).

[2] H.-D. Alber, K. Chełmiński: Quasistatic problems in viscoplasticity theory II: models with nonlinear hardening, Math. Models Methods Appl. Sci. 17(2) (2007) 189–213.

[3] J.-P. Aubin, A. Cellina: Differential Inclusions, Springer, Berlin (1984).

[4] V. Barbu: Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, Leyden (1975).

[5] H. Brézis: Opérateurs Maximaus Monotones et Semigroupes de Contractions dans les Espaces de Hilbert, North-Holland, Amsterdam (1973).

[6] K. Chełmiński: Coercive approximation of viscoplasticity and plasticity, Asymptotic Anal. 26 (2001) 115–135.

[7] K. Chełmiński: Coercive limits for a subclass of monotone constitutive equations in the theory of inelastic material behaviour of metals, Rocz. Pol. Tow. Mat., Ser. III, Mat. Stosow. 40 (1997) 41–81.

[8] K. Chełmiński, P. Gwiazda: Convergence of coercive approximations for strictly monotone quasistatic models in inelastic deformation theory, Math. Methods Appl. Sci. 30 (2007) 1357–1374.

[9] K. Chełmiński: On monotone plastic constitutive equations with polynomial growth condition, Math. Methods Appl. Sci. 22 (1999) 547–562.

[10] W. Han, M. Sofonea: Quasistatic Contact Problems in Viscoelasticity and Viscoelasticity, Studies in Advanced Mathematics 30, AMS, Providence (2002).

[11] W. Pompe: Linear functionals on nonlinear spaces and applications to problems from viscoplasticity theory, Math. Methods Appl. Sci. 31(7) (2008) 775–792.

[12] W. Pompe: Nonautonomous initial-boundary value problems in the viscoplasticity, Asymptotic Anal. 66 (2010) 87–102.

[13] W. Pompe: Quasistatic and dynamic problems in viscoplasticity theory – nonlinearities with power growth, Math. Methods Appl. Sci. 27(11) (2004) 1347–1365.

[14] R. Temam: Mathematical Problems in Plasticity, Gauthier-Villars, Paris (1985).

[15] R. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[16] R. Rockafellar: Local boundedness of nonlinear monotone operators, Mich. Math. J. 16 (1970) 397–407.

[17] R. Rockafellar: On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 163 (1970) 75–88.