Journal of Convex Analysis
Volume 24 (2017), No. 1, 19–40

On the Maximal Extensions of Monotone Operators and Criteria for Maximality

Andrew Eberhard*
Mathematics Department, RMIT – GPO Box 2476V,
Melbourne, Vict. 3001, Australia
andrew.eberhard@ems.rmit.edu.au

Robert Wenczel
Mathematics Department, RMIT – GPO Box 2476V,
Melbourne, Vict. 3001, Australia
robert.wenczel@rmit.edu.au

Dedicated to the memory of Charles Pearce.

Received: October 21, 2013
Revised manuscript received: December 24, 2014
Accepted: May 13, 2015

Within a nonzero, real Banach space we study the problem of characterising a maximal extension of a monotone operator in terms of minimality properties of representative functions that are bounded by the Penot and Fitzpatrick functions. We single out a property of this space of representative functions that enable a very compact treatment of maximality and pre-maximality issues.

Keywords: Sum theorems, maximal extensions, monotone operators, representative functions

2000 Mathematics Subject Classification: 47H05, 46N10, 47H04, 49J53

Introduction

The last 20 years has seen the emergence of the Fitzpatrick function as a primary tool in the study of monotone operators [9]. This tool has resulted in the simplification of proofs of known results in reflexive space, and ushered in numerous new results on monotone operators in general real Banach spaces [2, 3, 4, 5, 6, 7, 8, 10, 12, 13]. The critical role of the space of *representative functions with bigger conjugates* has become evident in recent years [15] and will figure in this study.

Suppose X is a Banach space and X^* its dual. We may view $X \times X^*$ paired with $X^* \times X$ using the coupling $\langle (z, z^*), (x^*, x) \rangle = \langle z, x^* \rangle + \langle x, z^* \rangle$ and the

*The first author's research was funded by ARC grant DP120100567.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

norm $\|(z,z^*)\|^2 = \|z\|^2 + \|z^*\|^2$. Denote by $PC\,(X \times X^*)$ the set of all proper convex functions $f : X \times X^* \to \overline{\mathbb{R}} := (-\infty, +\infty]$. When we pair the topologies $s(X) \times \sigma(X^*, X)$ (product of the strong and weak*) with $\sigma(X^*, X) \times s(X)$ (product of the weak* and strong) the associated conjugation operation on a proper closed convex function $f \in \Gamma_{s\times w^*}(X \times X^*)$ yields $f^* \in \Gamma_{w^*\times s}(X^* \times X)$ given by

$$f^*(z^*, z) := \sup_{(x,x^*)} \{\langle (x, x^*), (z^*, z)\rangle - f(x, x^*)\}. \tag{1}$$

Definition 0.1. We call a proper convex function $\mathcal{H}_T \in PC\,(X \times X^*)$ a *representative function of a monotone mapping* T on X when $\mathcal{H}_T(y, y^*) \geq \langle y, y^*\rangle$ for all $(y, y^*) \in X \times X^*$ with $\mathcal{H}_T(y, y^*) = \langle y, y^*\rangle$ if $y^* \in T(y)$. If T is not specified we say a proper convex function f is *representative* when $f(y, y^*) \geq \langle y, y^*\rangle$ for all $(y, y^*) \in X \times X^*$ and then say that it *represents* (the monotone set) $M_f := \{(x, x^*) \in X \times X^* \mid f(x, x^*) = \langle x, x^*\rangle\}$.

The interest in representative functions stems from the fact that M_f is monotone whenever f is representative. Recall that a maximal monotone operator is one whose graph is not contained in a larger monotone set. Define the 'transpose' operator †: $(x^*, x) \leftrightarrow (x, x^*)$, and $c_T(\cdot,\cdot) := \delta_T(\cdot,\cdot) + \langle\cdot,\cdot\rangle$, where δ_T denotes the indicator function of the graph of T. Fitzpatrick [9] showed that $\mathcal{F}_T := c_T^{*\dagger}$ and $\mathcal{P}_T := \mathcal{F}_T^{*\dagger}$ induce functions that represent T, when T is maximal monotone. In [4] it is shown that $\mathcal{P}_T$ is representative when T is just monotone, while $\mathcal{F}_T$ may fail to be so when T is not maximal. It has been noted by a number of authors (see [4, 7, 13]) that when T is maximal, the largest representative function is $\mathcal{P}_T$ and the smallest is $\mathcal{F}_T$ ($\mathcal{P}_T \geq \mathcal{F}_T$ pointwise, see [11, Proposition 3]). Recall that when f is a representative function then f^* is also a representative function, [8]. Denote $[\mathcal{F}_T, \mathcal{P}_T] := \{f \in PC\,(X \times X^*) \mid \mathcal{F}_T \leq f \leq \mathcal{P}_T\}$ and $bR\,(T) := \left\{f \in [\mathcal{F}_T, \mathcal{P}_T] \mid f^{*\dagger} \geq f \geq \langle\cdot,\cdot\rangle\right\}$. When $\mathcal{F}_T$ is representative (i.e. T is "almost maximal", see [5, Proposition 2]) then $\mathcal{F}_T$ represents a maximal extension of T, given by $T^\mu := \{(x, x^*) \mid \mathcal{F}_T(x, x^*) \leq \langle x, x^*\rangle\} := M^{\leq}_{\mathcal{F}_T}$. The related class of monotone operators are those that have a unique maximal extension. This class was studied in [12] and there termed "pre-maximal" and correspond to operators T such that T^μ is maximal. Clearly these classes are closely related, with almost-maximal operators being a subset of pre-maximal operators.

When M is not maximal then the question may be asked as to what role the smallest representative function, with a bigger conjugate, might have? In this paper we study the relationship that minimality of representative functions, within $bR(T)$, has with maximality and/or pre-maximality of the monotone operator T. It is known that in a reflexive space the minimal such representative function must always be a Fitzpatrick function of some maximal monotone operator [11]. It is precisely this property that singles out those Banach spaces that admit a theory paralleling that in reflexive Banach space. We will show that the problem of characterising representative functions that describe maximal extensions is closely related to minimality, with respect to the pointwise partial ordering of representative functions.

Clearly, when g and f are representative and $g \geq f$ then $M_g \subseteq M_f$ with M_f monotone. Thus, one could conjecture: the search for a maximal extension of a monotone set M_g can be recast as a search for a "minimal" representative function. It is this study we undertake in this paper. We see that when T is not maximal and f represents a maximal extension of T, then this representative function lies inside the interval $[\mathcal{F}_T, \mathcal{P}_T]$. Suppose this function is f; then as $f \geq \langle \cdot, \cdot \rangle$, we have for all $g \in [\mathcal{F}_T, \mathcal{P}_T]$ with $g \geq f$ that g is also representative. Indeed, one might conjecture that the minimal representative functions in $R(T) := \{f \in [\mathcal{F}_T, \mathcal{P}_T] \mid f \geq \langle \cdot, \cdot \rangle\}$, under the partial order $\leq$, have a special role and characterise maximal extensions. This was first observed and studied in [11] in the context of a reflexive Banach space, where it was shown that such minimal elements are Fitzpatrick functions of maximal monotone operators. It is not immediately clear that within an arbitrary Banach space *all* such minimal elements of $[\mathcal{F}_T, \mathcal{P}_T]$ are Fitzpatrick functions. One can easily show that this deceptively simple property has a considerable number of consequences, in particular one has immediately that all minimal elements of $[\mathcal{F}_T, \mathcal{P}_T]$ represent maximal monotone operators. We shall call BMLS (**B**urachik, **M**artínez-**L**egaz, **S**vaiter) spaces to be those Banach spaces X for which (for all monotone operators T on X) all minimal elements of $[\mathcal{F}_T, \mathcal{P}_T]$ are Fitzpatrick functions (note: all reflexive spaces are BMLS spaces). Then we show the following are all equivalent to X being a BMLS space:

1. The space X has the property that for every monotone T, every minimal element f in $[R(T), \leq]$ represents a maximal monotone set M_f.
2. The space X has the property that for every monotone T, every $f \in bR(T)$ represent a maximal monotone set M_f.

Thus this property is a potent tool for establishing maximality theorems, see Theorem 3.2, 3.3 and 3.4. In a BMLS space (without any *a priori* assumption of reflexivity) one can show the standard certificates for maximality without reference to any of the usual duality machinery central to other approaches. Indeed the order-reversal property of conjugation is one of the main tools used. All reflexive spaces are BMLS spaces [11]. It is an open question if there exist any non-reflexive BMLS spaces or indeed if all real Banach spaces are BMLS spaces. We do show that the study of this property gives a new avenue to the study of maximality and some new results, even in the reflexive case.

1. Fixed-Points under Conjugation

We will occasionally require the existence of fixed-points under conjugation during this development. In a reflexive space, Bauschke and Wang [2] have provided an explicit representation of an autoconjugate representative function of a given maximal monotone operator. In [1] it is shown that this formula fails to give such a representation, for certain linear, skew-symmetric monotone operators on a non-reflexive space. Thus, outside of reflexive spaces, no explicit formula for an auto-conjugate representative function is known. Despite this, such auto-conjugate representative functions do often exist. The following is a minor variation on the results of [13]. The proof is provided in an appendix for completeness.

Proposition 1.1. *Let X be a general Banach space. Let $f \in PC(X \times X^*)$ be a proper convex function such that $f^{*\dagger} \geq f$, and, if f is not closed, we assume there is $r \in \Gamma(X \times X^*)$ such that $f^{*\dagger} \geq r \geq r^{*\dagger} \geq f$. Then there exists a closed proper convex function $q \in \Gamma(X \times X^*)$ such that $q^* = q^\dagger$ and $f^{*\dagger} \geq q \geq f$.*

Proof. See Appendix A. □

Remark 1.2. As we begin with $\mathcal{F}_T \leq \mathcal{P}_T$, both proper closed and convex, there exists an autoconjugate $q \in [\mathcal{F}_T, \mathcal{P}_T]$.

Note that q must be representative since by using the Fenchel inequality

$$q(x, x^*) = \frac{1}{2}\left(q(x, x^*) + q^*(x^*, x)\right) \geq \langle x, x^* \rangle.$$

Representability with convex functions with bigger conjugates is critical to the study of monotone operators, as was first demonstrated by Simons [15].

Proposition 1.3. *Let $f : X \times X^* \to \overline{\mathbb{R}}$ be a representative function for which $f^{*\dagger} \geq f$ with $f^* \in PC(X^* \times X)$. Assume also, if f is not closed, that there exists $r \in \Gamma(X \times X^*)$ for which $f^{*\dagger} \geq r \geq r^{*\dagger} \geq f$. Then there exists an auto–conjugate $q^* = q^\dagger$ with $f^{*\dagger} \geq q \geq f$ and so q represents M_f.*

Proof. As $f^* \geq f$ the existence of q is assured by Proposition 1.1. As $f^{*\dagger} \geq q \geq f$ it follows that

$$M_{f^{*\dagger}} \subseteq M_q \subseteq M_f = M_{f^{*\dagger}},$$

where the last equality follows from [15, Lemma 19.12]. Thus equality ensues. □

We note that there now exist some explicit constructions of auto–conjugates and a survey of these may be found in [2] and [3] which are set mainly in reflexive spaces.

2. On Maximal Extensions of T

We know that when $T : X \rightrightarrows X^*$ is not a *maximal* monotone operator then $\mathcal{F}_T$ is not, in general, representative. Thus we are presented with the problem of finding a maximal extension. We begin this study by collecting some known results regarding partial-orders and inclusions.

We commit the usual abuse of notation, by using T to denote the mapping: $X \rightrightarrows X^*$, *and* its graph $\{(x, x^*) \in X \times X^* \mid x^* \in T(x)\} \subseteq X \times X^*$.

Lemma 2.1. *If $f \in PC\,(X \times X^*)$ is representative, and either: (i) f is closed, or (ii) f satisfies $f \leq f^{*\dagger}$, then*

$$T \subseteq M_f \quad \Longrightarrow \quad \mathcal{F}_T \leq f \leq \mathcal{P}_T.$$

Proof. For any representative f, [9, Theorem 2.4] yields $T \subseteq M_f \subseteq M_{f^{*\dagger}}$, so that

$$f \leq c_T, \quad f^{*\dagger} \leq c_T \quad \text{and hence} \quad \mathcal{F}_T \leq f^{*\dagger}, \quad \mathcal{F}_T \leq f\,.$$

If f is closed, these yield $f = f^{**} \leq \mathcal{F}_T^{*\dagger} = \mathcal{P}_T$ giving $\mathcal{F}_T \leq f \leq \mathcal{P}_T$. If, instead, have $f \leq f^{*\dagger}$, then as $\mathcal{F}_T \leq f$, which implies $f^{*\dagger} \leq \mathcal{P}_T$ we obtain $\mathcal{F}_T \leq f \leq f^{*\dagger} \leq \mathcal{P}_T$. □

The representativeness of $\mathcal{F}_T$ (i.e. the "almost" maximality of T, see [5, Proposition 2]) has been recognised as an important step towards establishing maximality. When T is not almost–maximal we will show that we need to search for minimal elements inside $[[\mathcal{F}_T, \mathcal{P}_T]\,, \leq]$ to obtain a maximal extension of T.

Recall

$$M_f^{\leq} := \{(x, x^*) \in X \times X^* \mid f\,(x, x^*) \leq \langle x, x^* \rangle\}$$

and note that when $f \geq \langle \cdot, \cdot \rangle$ we have $M_f^{\leq} = M_f$.

Lemma 2.2 ([15, Lemma 19.12]). *Suppose $f : X \times X^* \to \overline{\mathbb{R}}$ is a proper convex function.*

1. *If $f^{*\dagger} \geq f$ then $M_{f^{*\dagger}}^{\leq} \subseteq M_f^{\leq}$. If $(x, x^*) \in M_{f^{*\dagger}}^{\leq}$ then $(x, x^*) \in \partial f^*(x^*, x) = (\partial f)^{-1}(x^*, x)$. Conversely when $(x^*, x) \in \partial f(x, x^*) = (\partial f^*)^{-1}(x, x^*)$ then $(x, x^*) \in M_f^{\leq}$. If, in addition, $f \geq \langle \cdot, \cdot \rangle$ then we have*

$$\begin{aligned} M_f = M_{f^{*\dagger}} &= \{(x, x^*) \mid (x^*, x) \in \partial f(x, x^*)\} \\ &= \{(x, x^*) \mid (x, x^*) \in \partial f^*(x^*, x)\}\,. \end{aligned} \tag{2}$$

2. *If $f \geq \langle \cdot, \cdot \rangle$ then M_f is a monotone set.*
3. *When $T : X \rightrightarrows X^*$ is monotone we have $\mathcal{F}_T\,(x, x^*) = \langle x, x^* \rangle$ for all $(x, x^*) \in T$.*

Proof. See the appendix for proof. □

If T is not maximal, $\mathcal{F}_T$ is not necessarily representative, while $\mathcal{P}_T$ does represent $M_{\mathcal{P}_T}$. When $\mathcal{F}_T$ is not representative we study

$$T^\mu = M_{\mathcal{F}_T}^{\leq} = \{(x, x^*) \mid \langle x - y, x^* - y^* \rangle \geq 0, \forall\,(y, y^*) \in T\}\,.$$

We now recall some results from [12]. One always has $T^{\mu\mu\mu} = T^\mu$ and if T is monotone then $T \subseteq T^\mu$. In [12] T^μ is shown to be a polarity and as a consequence $A \subseteq B$ implies $A^{\mu} \supseteq B^{\mu}$, $T \subseteq T^{\mu\mu}$ and $(A \cup B)^{\mu} = A^{\mu} \cap B^{\mu}$. We shall call $T^{\mu\mu}$ the monotonic closure of T.

Proposition 2.3 ([12, Proposition 22]). *The following are equivalent:*

1. $T : X \rightrightarrows X^*$ *is monotone,*
2. $T \subseteq T^{\mu}$,
3. $T^{\mu\mu} \subseteq T^{\mu}$,
4. $T^{\mu\mu}$ *is monotone.*

Moreover, we have T maximal monotone iff $T = T^{\mu}$ (or T monotone and $T \supseteq T^{\mu}$).

Proposition 2.4 ([12, Proposition 25]). *Let $T : X \rightrightarrows X^*$ be a monotone operator. Then $\mathcal{F}_{T^{\mu}}$ is representative and*

$$M_{\mathcal{F}_{T^{\mu}}} = M^{\leq}_{\mathcal{F}_{T^{\mu}}} = T^{\mu\mu}.$$

We say that an element f in a partially ordered set $[S, \leq]$ is *minimal* iff $g \in S$ with $g \leq f$ implies $g = f$. When T is maximal, $\mathcal{F}_T$ is minimal in the class of representative functions (with bigger conjugates). When T is almost maximal, $\mathcal{F}_T$ is still minimal in $[\mathcal{F}_T, \mathcal{P}_T]$ but this is not necessarily so when we have pre-maximality.

Remark 2.5. We will work with the subset of $[\mathcal{F}_T, \mathcal{P}_T]$ consisting of representative functions with bigger conjugates. These functions are potentially not closed but the desire to find minimal elements has the effect of seeking a closed function (indeed a Fitzpatrick function). Let f be representative and suppose that its closure g is also representative. Then clearly $g \leq f$ (pointwise) and so in the case when $g \neq f$ we see that f could not have been minimal. Thus minimal functions should be expected to be closed with respect to all topologies (or closure operations with respect to a convergence) that are compatible with the duality pairing, such as the strong topology or the closure with respect to bounded $s \times w^*$-convergent nets. Furthermore a minimal element with respect to the set of proper convex, representative functions will also be minimal with respect to smaller subclasses to which it also belongs (i.e. closed, proper convex representative functions). Our main tool for obtaining the existence of minimal elements is Zorn's Lemma. Thus the proofs are not constructive. The use of proper convex functions (rather than closed ones) aids the use of this technique. The analysis would be greatly simplified if one could claim the existence of $s \times w^*$-closed minimal elements but this cannot be claimed *a priori in a non-reflexive space.*

The next result extends the study of minimality of the Fitzpatrick function to cases when T is not maximal providing a new conceptual framework, even for the case of reflexive Banach spaces.

Lemma 2.6. *Let $T \subseteq X \times X^*$.*

A. Denote

$$b(T) := \big\{ f \in PC(X \times X^*) \mid f^{*\dagger} \geq f \\ \text{and } f^*(x^*, x) \leq \langle x, x^* \rangle \text{ for all } (x, x^*) \in T \big\}.$$

For all $f \in b(T)$ we have $M^{\leq}_{f^{*\dagger}}$ a monotone set. Thus $b(T) \neq \emptyset$ iff T is a monotone set, in which case $\mathcal{F}_T \in b(T)$, and $\mathcal{F}_T \leq f$ for any $f \in b(T)$.

Suppose f is minimal in $[b(T), \leq]$.

1. *Then $f = \mathcal{F}_{M^{\leq}_{f^{*\dagger}}}$ and $M^{\leq}_f = \left(M^{\leq}_{f^{*\dagger}}\right)^{\mu}$. Thus $\mathcal{F}_{M^{\leq}_f} = \mathcal{F}_{\left(M^{\leq}_{f^{*\dagger}}\right)^{\mu}}$ is representative.*
2. *Additionally, $M^{\leq}_f$ is monotone iff $\left(M^{\leq}_{f^{*\dagger}}\right)^{\mu\mu} = \left(M^{\leq}_f\right)^{\mu\mu}$, or equivalently, $\left(M^{\leq}_{f^{*\dagger}}\right)^{\mu} = \left(M^{\leq}_f\right)^{\mu}$.*
3. *If T is monotone then $f = \mathcal{F}_T$ is the unique minimal element of $[b(T), \leq]$.*

B. Suppose T is monotone. Recall

$$bR(T) := \left\{ f \in PC(X \times X^*) \mid f^{*\dagger} \geq f \geq \langle \cdot, \cdot \rangle,\ \mathcal{F}_T \leq f \leq \mathcal{P}_T \right\}.$$

1. *Then $bR(T) \subseteq b(T)$, and for all $f \in bR(T)$ we have $f \geq \mathcal{F}_{M_f}$.*
2. *Let f be minimal in $[bR(T), \leq]$. If $\mathcal{F}_{M_{f^{*\dagger}}} (= \mathcal{F}_{M_f})$ is representative, then $f = \mathcal{F}_{M_{f^{*\dagger}}} = \mathcal{F}_{M_f}$ and $M_f = (M_f)^{\mu}$ is maximal monotone. Moreover it follows that $\mathcal{F}_{(M_f)^{\mu}}$ has a bigger conjugate.*

Proof. *Part A:* Let $f \in b(T)$. Observing that $f^*(x^*, x) \leq \langle x, x^* \rangle$ for all $(x, x^*) \in T$ we have $M^{\leq}_{(f^*)^{\dagger}} \supseteq T$. By Lemma 2.2, $M^{\leq}_f \supseteq M^{\leq}_{(f^*)^{\dagger}} (\supseteq T$ by the definition of $b(T))$. Then

$$f \ \leq f^{*\dagger} \leq \langle \cdot, \cdot \rangle \quad \text{on } M^{\leq}_{(f^*)^{\dagger}} \supseteq T,$$

so

$$f \leq f^{*\dagger} \leq \langle \cdot, \cdot \rangle + \delta_{M^{\leq}_{(f^*)^{\dagger}}} \leq \langle \cdot, \cdot \rangle + \delta_T,$$

$$\text{implying} \quad f^{*\dagger} \geq f \geq \left(\langle \cdot, \cdot \rangle + \delta_{M^{\leq}_{(f^*)^{\dagger}}}\right)^{*\dagger} = \mathcal{F}_{M^{\leq}_{(f^*)^{\dagger}}} \geq \mathcal{F}_T. \tag{3}$$

Note that (3) gives $M^{\leq}_{(f^*)^{\dagger}} \subseteq M^{\leq}_f \subseteq M^{\leq}_{\mathcal{F}_{M^{\leq}_{(f^*)^{\dagger}}}} = \left(M^{\leq}_{(f^*)^{\dagger}}\right)^{\mu} \subseteq M^{\leq}_{\mathcal{F}_T} = T^{\mu}$ and $M^{\leq}_{(f^*)^{\dagger}} \subseteq \left(M^{\leq}_{(f^*)^{\dagger}}\right)^{\mu}$. The latter containment and Proposition 2.3 imply $M^{\leq}_{(f^*)^{\dagger}}$ is monotone and hence T is monotone. Thus $b(T) \neq \emptyset$ then T is a monotone set. When T is monotone then $\mathcal{F}_T \in b(T) \neq \emptyset$, and $\mathcal{F}_T \leq f$, as seen from (3).

Thus we have $\left(\mathcal{F}^*_{M^{\leq}_{(f^*)^\dagger}}\right)^\dagger = \mathcal{P}_{M^{\leq}_{(f^*)^\dagger}} = \langle\cdot,\cdot\rangle$ on $M^{\leq}_{(f^*)^\dagger} \supseteq T$, as $\mathcal{P}_{M^{\leq}_{(f^*)^\dagger}}$ is representative for the monotone set $M^{\leq}_{(f^*)^\dagger}$, see [4, Proposition 8.1.7]. As we have

$$\mathcal{F}_T \leq \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}} \leq f \leq f^{*\dagger} \leq \left(\mathcal{F}^*_{M^{\leq}_{(f^*)^\dagger}}\right)^\dagger = \mathcal{P}_{M^{\leq}_{(f^*)^\dagger}} \leq \mathcal{P}_T,$$

it follows that $\mathcal{F}_{M^{\leq}_{(f^*)^\dagger}} \in b\,(T)$. When f is minimal then $f = \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}$. Consequently

$$M^{\leq}_{f} = M^{\leq}_{\mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}} = \left(M^{\leq}_{(f^*)^\dagger}\right)^\mu \tag{4}$$

establishing *1*.

For *2* we note that if $\left(M^{\leq}_{(f^*)^\dagger}\right)^{\mu\mu} = \left(M^{\leq}_{f}\right)^{\mu\mu}$ it follows from (4) that

$$\left(M^{\leq}_{f}\right)^{\mu} = \left(M^{\leq}_{(f^*)^\dagger}\right)^{\mu\mu} = \left(M^{\leq}_{f}\right)^{\mu\mu}$$

and Proposition 2.3 implies $M^{\leq}_{f}$ is a monotone set. On the other hand, when $M^{\leq}_{f}$ is monotone Proposition 2.3 implies $M^{\leq}_{f} \subseteq \left(M^{\leq}_{f}\right)^{\mu}$. Using (4) gives

$$\left(M^{\leq}_{(f^*)^\dagger}\right)^{\mu} \subseteq \left(M^{\leq}_{f}\right)^{\mu}, \quad \text{and hence } \left(M^{\leq}_{(f^*)^\dagger}\right)^{\mu\mu} \supseteq \left(M^{\leq}_{f}\right)^{\mu\mu}.$$

For the reverse inclusion, the first part of Lemma 2.2 gives $M^{\leq}_{(f^*)^\dagger} \subseteq M^{\leq}_{f}$, so $\left(M^{\leq}_{(f^*)^\dagger}\right)^{\mu\mu} \subseteq \left(M^{\leq}_{f}\right)^{\mu\mu}$ by polarity. Thus equality ensues. Clearly, the relation $\left(M^{\leq}_{(f^*)^\dagger}\right)^{\mu\mu} = \left(M^{\leq}_{f}\right)^{\mu\mu}$ is equivalent to $\left(M^{\leq}_{(f^*)^\dagger}\right)^{\mu} = \left(M^{\leq}_{f}\right)^{\mu}$ (as $A^{\mu\mu\mu} = A^{\mu}$).

For the last part assume T a monotone operator. Then by Lemma 2.2 (part *3*) we have $(\mathcal{F}^*_T)^\dagger = \mathcal{P}_T = \langle\cdot,\cdot\rangle$ on T and so $\mathcal{F}_T \in b\,(T)$. Let $f \in b\,(T)$ be minimal. Then as $f \geq \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}} \geq \mathcal{F}_T$ we have $f = \mathcal{F}_T$.

Part B: By Lemma 2.2 we have $M_f = M_{f^{*\dagger}} = M^{\leq}_{(f^*)^\dagger}$, a monotone set, hence for all $f \in bR\,(T)$ we have $f \in b\left(M^{\leq}_{(f^*)^\dagger}\right) = b\,(M_f)$. Thus $f \geq \mathcal{F}_{M_f}$ (by part *3* applied to the monotone set M_f). Furthermore $f^{*\dagger} \leq \mathcal{P}_{M_f} = \langle\cdot,\cdot\rangle$ on $M_f \supseteq T$, giving $bR\,(T) \subseteq b\,(T)$.

From the inequalities (3) we have $f \geq \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}$ and so when $\mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}$ is representative and f minimal we have $f = \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}} \in bR\,(T)$. As $f = \mathcal{F}_{M^{\leq}_{(f^*)^\dagger}}$ implies $M^{\leq}_{f} = \left(M^{\leq}_{(f^*)^\dagger}\right)^{\mu} = (M_f)^\mu$, and as $f \in bR(T)$ we have $M^{\leq}_{f} = M_f$ and so Proposition 2.3 gives maximality of M_f. □

As noted earlier, if $T \subseteq A$ are monotone sets then $\mathcal{F}_T \leq \mathcal{F}_A$. To date the behaviour of an arbitrary representative function for A in relation to $\mathcal{F}_T$ has not been addressed.

Corollary 2.7. *Suppose $T \subseteq X \times X^*$ is monotone. Then $\mathcal{F}_T \in b(T)$, and $f \geq \mathcal{F}_T$ for all $f \in b(T)$. Thus $\mathcal{F}_T$ is the unique minimal element of $[b(T), \leq]$.*

Remark 2.8. If $g \in bR(T)$ does not represent a maximal monotone set M_g, then for an arbitrary $h \in b(T)$ for which $M^{\leq}_{h^{*\dagger}} \supseteq M_g$ we have $h \in b(M_g)$. Applying Corollary 2.7 (on the monotone set M_g) we deduce

$$\mathcal{F}_{M_g} \leq h, \tag{5}$$

as in Lemma 2.1. In particular this implies that all maximal extensions M of M_g and all bigger conjugate representations $h \in bR(M)$ with $M = M_h = M_{h^{*\dagger}} \supseteq M_g$ have (5) holding.

When dealing with pre-maximal monotone operators we have:

Proposition 2.9. ([12, Props. 22, 36]) *Let $T : X \rightrightarrows X^*$ be monotone. Denote by $\mathcal{M}(T)$ the set of all maximal monotone extensions of T. Then*

1. $T^\mu = \bigcup_{B \in \mathcal{M}(T)} B$ *and* $T^{\mu\mu} = \bigcap_{B \in \mathcal{M}(T)} B$.
2. *The following are equivalent:*
   (*a*) *T has a unique maximal monotone extension (i.e. T is pre-maximal),*
   (*b*) $T^\mu = T^{\mu\mu}$,
   (*c*) *T^μ is monotone,*
   (*d*) *T^μ is maximal monotone,*
   (*e*) *$T^{\mu\mu}$ is maximal monotone.*
   Moreover, it follows that T^μ (or equivalently $T^{\mu\mu}$) is the unique maximal extension of T.

The following is well known.

Proposition 2.10. *Let M be a maximal monotone extension of T. Then $\mathcal{F}_M$ is a minimal element of $[bR(T), \leq]$. Hence also, $\mathcal{F}_M$ is the unique minimal element of $bR(M)$.*

Proof. Clearly $\mathcal{F}_M$ is in $bR(M)$. For the minimality, suppose $g \leq \mathcal{F}_M$ where $g \in bR(T)$. Then M_g is a monotone extension of $M_{\mathcal{F}_M} = M$, so $M_g = M$ by maximality. From part B of Lemma 2.6 follows that $g \geq \mathcal{F}_{M_g} = \mathcal{F}_M$ and so $g = \mathcal{F}_M$ and $\mathcal{F}_M$ is thus minimal. □

The following is trivial but a surprising statement of the results of [15].

Lemma 2.11. *Let $T : X \rightrightarrows X^*$ be a monotone operator, let $k, h \in bR(T)$ for which $h \leq k$. Then $M_k = M_h \supseteq T$.*

Proof. Note that as $h \leq k$ and $h, k \in bR(T)$ we have $h \leq k \leq k^{*\dagger} \leq h^{*\dagger}$ and so $M_h \supseteq M_k \supseteq M_{k^{*\dagger}} \supseteq M_{h^{*\dagger}} = M_h$, where the last equality follows from Lemma 2.2, giving $M_h = M_k$. □

The issue as to whether $\mathcal{F}_{M_f} = f$ for any minimal element of $bR(T)$ is clearly of interest as it implies the identity $(M_f)^{\mu} = M_f$ and hence maximality of M_f. By $\operatorname{co} f$ we denote the convex function whose epigraph corresponds to $\operatorname{co}\operatorname{epi} f$ and by $\operatorname{co}\{f,g\}$ we denote the convex function $\operatorname{co}\{\min\{f,g\}\}$. Clearly $\operatorname{epi}\operatorname{co}\{\min\{f,g\}\} = \operatorname{co}\{\operatorname{epi} f \cup \operatorname{epi} g\}$.

The next result shows that the BMLS property of a general Banach space is equivalent to a number of seemingly natural statements about the partial order on $bR(T)$, which all hold in a reflexive space.

Proposition 2.12. *Suppose f is minimal in $bR(T)$. Then the following are all equivalent to the condition $f = \mathcal{F}_{M_f}$.*

1. *M_f is maximal monotone.*
2. *$\mathcal{F}_{M_f} \in bR(T)$.*
3. *For any maximal monotone extension $M \supseteq M_f$, the function $\max\{\operatorname{co}\{f,\mathcal{F}_M\}, \langle\cdot,\cdot\rangle\}$ is convex.*
4. *For any maximal monotone extension $M \supseteq M_f$, we have that $\mathcal{F}_{M_f} \leq \mathcal{F}_M \leq f$.*
5. *For all $k \in bR(T)$ (with $k \geq f$) and any maximal monotone extension $M \supseteq M_k$, we have $\mathcal{F}_{M_k} \leq \mathcal{F}_M \leq k$.*
6. *For any $g \in R(T)$ with $g \geq f$ (and hence $M_g \subseteq M_f$) the function $\operatorname{co}\left\{\mathcal{F}_{M_f}, g\right\}$ is representative.*

Proof. If M_f is maximal, then $\mathcal{F}_{M_f} \in bR(T)$. If $\mathcal{F}_{M_f} \in bR(T)$ then $\mathcal{F}_{M_f} \leq f$, and hence $f = \mathcal{F}_{M_f}$ by minimality of f. If $f = \mathcal{F}_{M_f}$ then $M_f = (M_f)^{\mu}$. This establishes that parts *1* and *2* are equivalent to the condition $f = \mathcal{F}_{M_f}$.

(*2* implies *3*): We consider $\max\{\operatorname{co}\left\{f,\mathcal{F}_{M_f}\right\}, \langle\cdot,\cdot\rangle\} = \max\{g, \langle\cdot,\cdot\rangle\}$ where $g := \operatorname{co}\left\{f,\mathcal{F}_{M_f}\right\}$. From the equivalences already verified above, M_f is maximal and $f = \mathcal{F}_{M_f}$, so $g = \operatorname{co}\left\{f,\mathcal{F}_{M_f}\right\} = \mathcal{F}_{M_f} \in bR(T)$. Thus we have $\max\{\operatorname{co}\left\{f,\mathcal{F}_{M_f}\right\}, \langle\cdot,\cdot\rangle\} = g$ because $g \geq \langle\cdot,\cdot\rangle$. Hence $\max\{\operatorname{co}\left\{f,\mathcal{F}_{M_f}\right\}, \langle\cdot,\cdot\rangle\}$ is convex. Now suppose $M \supseteq M_f$ is any maximal extension but as part *1* is equivalent to *2* we have $M = M_f$ as M_f is maximal and hence $\max\{\operatorname{co}\{f,\mathcal{F}_M\}, \langle\cdot,\cdot\rangle\} = \max\{\operatorname{co}\left\{f,\mathcal{F}_{M_f}\right\}, \langle\cdot,\cdot\rangle\} = g$ is convex, yielding part *3*.

(*3* implies *4*): Let $M \supseteq M_f$ be a maximal extension, and assume that $g := \max\{\operatorname{co}\{f,\mathcal{F}_M\}, \langle\cdot,\cdot\rangle\}$ is convex. Now, $g \geq \langle\cdot,\cdot\rangle$, and $g \leq f$, $g \leq \mathcal{F}_M$, so that $g \leq f \leq f^{*\dagger} \leq g^{*\dagger}$. Thus $g \in bR(M_g)$, so $M_g = M_{g^{*\dagger}}$ by Lemma 2.2. Further, $g \leq \mathcal{F}_M$ implies $M_g \supseteq M_{\mathcal{F}_M} = M$, so by maximality of the latter, it follows that $M_g = M$. Then, as $g^{*\dagger} = \langle\cdot,\cdot\rangle$ on $M_{g^{*\dagger}} = M$, we have that $g^{*\dagger} \leq \langle\cdot,\cdot\rangle + \delta_M$, so $g \geq \mathcal{F}_M$. This yields that $f \geq g \geq \mathcal{F}_M \geq \mathcal{F}_{M_f}$, as required.

(*4* implies *5*): Let $k \geq f$ and $M \supseteq M_k$ any maximal extension. Apply Lemma 2.11 to deduce that $M_k = M_f$ and hence M extends M_f. Applying the inequalities from part *4* we have $\mathcal{F}_{M_k} = \mathcal{F}_{M_f} \leq \mathcal{F}_M \leq f \leq k$, establishing part *5*.

(*5* implies *1*): Assume part *5* with $k = f$, so when $M \supseteq M_f$ is any maximal extension, we have $\mathcal{F}_{M_f} \leq \mathcal{F}_M \leq f$. As f is minimal in $bR(T)$ (and $\mathcal{F}_M \in bR(T)$ since M is maximal) it follows that $f = \mathcal{F}_M$ and so $M_f = M$, maximal monotone, and we have part *1.* Thus *1* to *5* are equivalent.

(*5* implies *6*): As $M_g \subseteq M_f$ then since part *5* implies part *1*, M_f is maximal so M_f is a maximal extension of M_g. Applying part *5* we have $\mathcal{F}_{M_g} \leq \mathcal{F}_{M_f} \leq g$, implying that $\operatorname{co}\left\{\mathcal{F}_{M_f}, g\right\} = \mathcal{F}_{M_f}$, the latter being representative.

(*6* implies *3*): Taking $g = f$ in part *6*, then $\operatorname{co}\left\{\mathcal{F}_{M_f}, g\right\} = \operatorname{co}\left\{\mathcal{F}_{M_f}, f\right\}$ is representative. Hence $\operatorname{co}\{\mathcal{F}_M, f\} \geq \langle\cdot,\cdot\rangle$, implying $\max\{\operatorname{co}\{f, \mathcal{F}_M\}, \langle\cdot,\cdot\rangle\} = \operatorname{co}\left\{\mathcal{F}_{M_f}, f\right\}$, which is convex. □

Remark 2.13. It is clear from the proof of Proposition 2.12 that part *5* could be replaced by the proposition that: For all $k \geq f$ then *there exists* a maximal monotone extension $M \supseteq M_k$ with $\mathcal{F}_{M_k} \leq \mathcal{F}_M \leq k$. Clearly part *5* implies this. Conversely we may apply this for $k = f$ in the same way we have argued that part *5* implies part *1.*

Definition 2.14. (The space of all representative functions for the monotone set T.)

$$R(T) := \{f \in PC(X \times X^*) \mid f \geq \langle\cdot,\cdot\rangle\,,\ T \subseteq M_f\}\,.$$

Note that $R(T)$ is nonempty (it contains $\mathcal{P}_T$), so that minimal elements in $[R(T), \leq]$ always exist.

The following was observed in [11, Theorem 5] for reflexive spaces but is also a corollary of the earlier work [8], again for the case of a reflexive Banach space.

Lemma 2.15. *Let $g \in R(T)$ be minimal in $[R(T), \leq]$, with T monotone. Then $g \in bR(T)$, $g \geq \mathcal{F}_{M_g}$, and g is minimal in $[bR(T), \leq]$.*

Proof. By a trivial adaptation of the proof of [11, Theorem 5], it follows that $g \leq g^{*\dagger}$, and as $g \geq \langle\cdot,\cdot\rangle$, Lemma 2.2 gives $M_g = M_{g^{*\dagger}}$, and since then $g^{*\dagger} = \langle\cdot,\cdot\rangle$ on M_g, we have that $g^{*\dagger} \leq \langle\cdot,\cdot\rangle + \delta_{M_g}$, so $g \geq g^{**} \geq \mathcal{F}_{M_g}$ ($\geq \mathcal{F}_T$ since $M_g \supseteq T$). Thus $g \in bR(T)$, and since $bR(T) \subseteq R(T)$, it follows that g is also minimal for $[bR(T), \leq]$. □

Since we do not demand closedness from the outset, the existence of minimal elements can be deduced via Zorn's Lemma.

Lemma 2.16. *For T monotone, and $f \in R(T)$, let*

$$L_f(T) := \{g \in bR(T) \mid g \leq f\}\,. \tag{6}$$

Then $L_f(T)$ is nonempty, there exists a minimal element of $[L_f(T), \leq]$, and this element is also minimal for $[bR(T), \leq]$.

Proof. For the nonemptiness, consider the set $R_f(T) := \{g \in R(T) \mid g \leq f\}$. Since $f \in R_f(T)$, the latter is nonempty. Any *totally-ordered* subset S of $[R_f(T), \leq]$ has a lower bound in $R_f(T)$, given by the pointwise infimum of the members of S. To see this, if $h := \inf_{g \in S} g$, then h is convex, since its epigraph is the union of a nested family of convex sets (nested in the sense that for any pair of functions from S, one epigraph must contain the other) and therefore itself convex. Also, since $\langle \cdot, \cdot \rangle \leq g \leq f$ and $T \subseteq M_g$ for all $g \in S \subseteq R_f(T)$ it follows that $\langle \cdot, \cdot \rangle \leq h \leq f$ and $T \subseteq M_g \subseteq M_h$, so $h \in R_f(T)$ as claimed. Zorn's Lemma then yields a minimal element $k \in R_f(T)$. Then by Lemma 2.15, $k \in bR(T)$, hence $k \in L_f(T)$, and the latter is nonempty.

For the remaining assertions we repeat this argument, but now on $L_f(T)$. Indeed, if S totally ordered in $[L_f(T), \leq]$, again defining $h := \inf_{g \in S} g$, yields $\langle \cdot, \cdot \rangle \leq h \leq f$ and $h \leq g \leq g^{*\dagger} \leq h^{*\dagger}$, giving $h \in bR(T)$ and so $h \in L_f(T)$. Thus every chain in $[L_f(T), \leq]$ has a lower bound in $L_f(T)$ and again Zorn's Lemma yields a minimal element we denote (again) by $h \in L_f(T)$. Now suppose $g \in bR(T)$ and $g \leq h$. From transitivity of the partial order follows $g \leq h \leq f$ and hence $g \in L_f(T)$. By minimality of h in $[L_f(T), \leq]$ we have $g = h$ and hence h is minimal in $[bR(T), \leq]$. □

Any representative function f for which there is a criterion for maximality of M_f (for example [10]) will yield $\mathcal{F}_{M_f}$ as the minimal element of $L_f(T)$. Consequently by Lemma 2.11 all $g \geq \mathcal{F}_{M_f}$ will represent the same maximal set M_f. Maximality now becomes endemic on spaces where any of the equivalent statements in Proposition 2.12 hold (for any monotone T) for any minimal element of $bR(T)$. Let us call these BMLS spaces i.e. BMLS spaces possess the property that every minimal element in $bR(T)$ is a Fitzpatrick function. From the work of Burachik, Martínez-Legaz and Svaiter [8, 11] we know that BMLS spaces include all reflexive spaces. The following result makes a fundamental connect to these spaces with the problem of representation of maximal monotone operators by bigger-conjugate representative functions.

Theorem 2.17. *The space X is a* BMLS space if and only if (for every monotone T) all $f \in bR(T)$ represent maximal monotone operators.

Proof. By Lemma 2.16, for any $g \in bR(T)$ we have $g \geq h$ for some minimal h in $[bR(T), \leq]$. When X is a BMLS space then $h = \mathcal{F}_{M_h}$ for a maximal M_h. By Lemma 2.11 it follows that $M_g = M_h$ and as M_h is maximal so is M_g. Conversely, suppose $bR(T)$ only represent maximal monotone operators. Then Proposition 2.12 part *1* holds and X is then a BMLS space. □

3. Detecting Maximal Extensions in a BMLS Space

We now develop practical criteria for detecting the maximality of a monotone operator by identifying it with its maximal extension. This approach allows us to totally avoid the use of duality theorems of the Fenchel type but instead use the simple fact that conjugation is order-reversing. Bringing this together we have the following extensions, to general real Banach spaces, of known results for reflexive Banach spaces.

Theorem 3.1 (Maximality via representative functions). *Let $T : X \rightrightarrows X^*$ be a monotone operator.*

1. *For every maximal extension $\widetilde{T}$ of T there exists a representative function p with $\mathcal{F}_T \leq p \leq p^{*\dagger} \leq \mathcal{P}_T$ such that $p = \mathcal{F}_{\widetilde{T}}$, and there does not exist any other representative function $g \in PC(X \times X^*)$ with $g \neq p$ and $g \leq p$ (i.e. p is a minimal element of $[bR(T), \leq]$). Moreover,*

$$M_p = \{(x, x^*) \mid (x^*, x) \in \partial p(x, x^*)\}. \tag{7}$$

2. *If $q \in bR(T)$ is such that $q = q^{*\dagger}$ (i.e. is auto-conjugate), then, whenever X is a* BMLS *space, M_q is a maximal monotone extension of T. Conversely (for general X) an auto-conjugate representative function exists for every maximal monotone extension $\widetilde{T}$ of T.*
3. *Suppose X is a* BMLS *space. If $f \in R(T)$ and both f, $f^{*\dagger}$ are representative, then $M_{f^{*\dagger}} = M_{f^{**}}$ is maximal monotone.*

Proof. To begin, we note the following: applying Lemma 2.1 we know that as $T \subseteq \widetilde{T}$ we have $\mathcal{F}_T \leq \mathcal{F}_{\widetilde{T}} \leq \mathcal{P}_{\widetilde{T}} \leq \mathcal{P}_T$, so $\mathcal{P}_{\widetilde{T}} \in \{r \in \Gamma(X \times X^*) \mid \mathcal{P}_T \geq r \geq r^{*\dagger} \geq \mathcal{F}_T\} \neq \emptyset$, and Proposition 1.1 yields q with $\mathcal{F}_T \leq q = q^{*\dagger} \leq \mathcal{P}_T$, so $q \geq \langle \cdot, \cdot \rangle$ by Fenchel inequality.

(Part *1*): By Proposition 2.10, $p := \mathcal{F}_{\widetilde{T}}$ is minimal in $bR(T)$. Lemma 2.2 then yields (7).

(Part *2*): First we show that (for any Banach space) such auto–conjugate representative functions always exist. Observe that if $f \in bR(T)$ is a representative of a maximal extension $\widetilde{T}$ of T then $f \geq \mathcal{F}_{M_f} = \mathcal{F}_{\widetilde{T}}$ and we have

$$\mathcal{F}_T \leq \mathcal{F}_{\widetilde{T}} \leq f \leq f^{*\dagger} \leq \mathcal{F}^{*\dagger}_{\widetilde{T}} = \mathcal{P}_{\widetilde{T}} \leq \mathcal{P}_T.$$

Applying Proposition 1.1 to the pair $\mathcal{F}_{\widetilde{T}}$ and $\mathcal{P}_{\widetilde{T}}$ there exists $\mathcal{F}_{\widetilde{T}} \leq q \leq \mathcal{P}_{\widetilde{T}}$ such that $q = q^{*\dagger}$. From Lemma 2.11 we get $M_q = \widetilde{T}$. Thus the autoconjugate $q \in bR(T)$ represents $\widetilde{T}$.

Now suppose instead that X is BMLS, and that $\mathcal{F}_T \leq q \leq \mathcal{P}_T$ with $q = q^{*\dagger}$. Then $q \in bR(T)$, and by Theorem 2.17, M_q is maximal, and hence a maximal extension of T.

(Part *3*): By [11, Theorem 5 and 6] one may deduce that for any $f \in R(T)$ there always exists a minimal element $h \leq \min\{f, f^{*\dagger}\}$ in $R(T)$ such that $h \leq h^{*\dagger}$ and $T \subseteq M_f \subseteq M_h$. Thus, by Lemma 2.15, $h \in bR(T)$ and is also minimal in $bR(T)$. The BMLS property now implies maximality of M_h, so that $\mathcal{F}_{M_h} \in bR(T)$, and then $h = \mathcal{F}_{M_h}$ by minimality of h. We see now that h is closed, so that $h = h^{**} \leq f^{**} \leq f$, implying that $f^{**} \in R(T)$ also. As $h \leq f^{*\dagger}$ we have $f^{**} \leq h^{*\dagger}$ and so $M_h = M_{h^{*\dagger}} \subseteq M_{f^{**}}$, implying $M_{f^{**}}$ is maximal and $M_h = M_{f^{**}}$. Similarly $h \leq f$ implies $f^{*\dagger} \leq h^{*\dagger}$ and so $M_h = M_{h^{*\dagger}} \subseteq M_{f^{*\dagger}}$. The maximality of M_h gives $M_h = M_{f^{*\dagger}}$. $\square$

In [15] Lemma 21.9 it is shown that in reflexive, real Banach spaces, an autoconjugate function (i.e. $q = q^{*\dagger}$) with $\mathcal{P}_T \geq q \geq \mathcal{F}_T$, represents a maximal monotone set which is an extension of T. This phenomenon was further studied by Bauschke and Wang in [2], where it is shown that in reflexive Banach spaces, any autoconjugate representative function represents a maximal monotone operator. Indeed this fact is also a straightforward corollary of the earlier result of [8]. We will extend [15, Lemma 21.9] and [2, Theorem 5.7] first to the case of a BMLS space.

Theorem 3.2 (Maximality via Auto-conjugates). *Let $T : X \rightrightarrows X^*$ be a monotone operator. Suppose X is BLMS . If there exists $q \in \Gamma(X \times X^*)$ with $q^{*\dagger} = q$ that represents T (i.e. $M_q = T$), then T is maximal and $M_q = M_{q^{*\dagger}} = \{(x, x^*) \mid (x^*, x) \in \partial q(x, x^*)\} = T$.*

Proof. Here $M_q = T$ is its own maximal extension and hence T is maximal. $\square$

The next result follows directly from Theorem 2.17 but it is interesting that it also follows directly from Theorem 3.2.

Theorem 3.3 (Maximality via bigger conjugates). *Let $T : X \rightrightarrows X^*$ be a monotone operator, with X a BMLS space. If there exists $f,\ f^{*\dagger} \in PC(X \times X^*)$ with $f^{*\dagger} \geq f \geq \langle \cdot, \cdot \rangle$ that represents T (i.e. $M_f = T$) then T is maximal and $M_f = M_{f^{*\dagger}} = \{(x, x^*) \mid (x^*, x) \in \partial f(x, x^*)\} = T$.*

Proof. By Proposition 1.3 there exists an auto–conjugate q with $f^{*\dagger} \geq q \geq f$ and $M_q = M_f = T$ by Lemma 2.11 . Hence T is maximal. $\square$

Theorem 3.4 (Maximality via representative functions). *Let $T : X \rightrightarrows X^*$ be a monotone operator, with X a BMLS space. If there exists $f,\ f^{*\dagger} \in PC(X \times X^*)$ with $f^{*\dagger} \geq \langle \cdot, \cdot \rangle$ and $f \geq \langle \cdot, \cdot \rangle$ that represents T (in that $M_{f^{*\dagger}} = T$) then T is maximal and $M_{f^{**}} = M_{f^{*\dagger}} = \{(x, x^*) \mid (x^*, x) \in \partial f^{**}(x, x^*)\} = T$.*

Proof. By Theorem 3.1 part *3* when $M_{f^{*\dagger}} = T$ then T is maximal. $\square$

One might ask whether it is possible to improve the last result to claim that T is maximal and $M_f = M_{f^{*\dagger}} = T$. A recent counterexample [6, Example 6.2] implies that no non-reflexive space can exhibit such a result. Here a representative f function has a conjugate $f^{*\dagger}$ which is representative for which M_f is not maximal. We note that for this example $f \gtrnsim f^{*\dagger} = f^{**}$ and hence f does not have bigger conjugate. In particular $f = \delta_{\{0\} \times e_\perp}$ (where $e \in X^{**} \backslash X$ and

$e_\perp := \{x^* \mid \langle x^*, e\rangle = 0\}$) and $f^{*\dagger} = \delta_{\{0\}\times X^*}$. Thus $f^{*\dagger} = \delta_{\{0\}\times X^*} = f^{**}$ represents a maximal skew-symmetric monotone operator, and does not provide a contradiction to the Theorem on maximality via representative functions. Any counterexample for this property would be of interest as we have framed this paper in such as way as to provide equivalences wherever possible. Thus a counterexample to one of these properties provides a counterexample to them all and would establish the existence of a Banach space which is not a BMLS space. Moreover, this would also implies some quite nonintuitive violations of partial orderings of representative functions, as compared with the graph-inclusions associated with the corresponding monotone operators (see Proposition 2.12).

4. On Pre-Maximality

Pre-maximality may also be studied via representative functions and minimality. We embark on this next. Maximality of M_f is related to the minimality of $\mathcal{F}_{M_f}$ in $bR(T)$, while pre-maximality is related to the minimality of $\mathcal{F}_{(M_f)^\mu}$ in $[R(T), \leq]$. These results are new even for the case of a reflexive Banach space.

Let $R(T)$ denote the space of representative functions of the monotone operator T, that is, $R(T) := \{f \in PC(X \times X^*) \mid f \geq \langle\cdot,\cdot\rangle,\ T \subseteq M_f\}$. Minimal elements of $[R(T), \leq]$ always exist. It would be of interest to know if a minimal element of the subset of $s \times w^*$-closed representative functions exists. We will return to this issue later.

Lemma 4.1. *Let $f \in R(T)$. Then*

$$\mathcal{F}_{M_f} \leq \mathcal{F}_{(M_f)^{\mu\mu}} \leq \mathcal{F}_{(M_f)^{\mu}} \leq \mathcal{P}_{(M_f)^{\mu\mu}} \leq \mathcal{P}_{M_f}. \tag{8}$$

Proof. The first two inequalities are a consequence of $M_f \subseteq (M_f)^{\mu\mu} \subseteq (M_f)^{\mu}$. Now note that for all $(x_i, x_i^*) \in (M_f)^{\mu\mu}$ we have

$$\langle x_i, x_i^*\rangle = \mathcal{F}_{(M_f)^{\mu}}(x_i, x_i^*)$$

since $\mathcal{F}_{(M_f)^{\mu}}$ is a representative function of $(M_f)^{\mu\mu}$. Thus by the convexity of $\mathcal{F}_{(M_f)^{\mu}}$,

$$\begin{aligned}\sum_i \lambda_i \langle x_i, x_i^*\rangle &= \sum_i \lambda_i \mathcal{F}_{(M_f)^{\mu}}(x_i, x_i^*) \geq \mathcal{F}_{(M_f)^{\mu}}\left(\sum_i \lambda_i (x_i, x_i^*)\right) \\ &= \mathcal{F}_{(M_f)^{\mu}}(x, x^*)\end{aligned}$$

for any $\sum_i \lambda_i (x_i, x_i^*) = (x, x^*)$.

Hence, for all $(x, x^*) \in \mathrm{co}\,(M_f)^{\mu\mu} = \mathrm{dom}(\mathrm{co}(\langle\cdot,\cdot\rangle + \delta_{(M_f)^{\mu\mu}}))$,

$$\inf\left\{ \sum_i \lambda_i \langle x_i, x_i^*\rangle \,\middle|\, (x_i, x_i^*) \in (M_f)^{\mu\mu}, \right.$$
$$\left. \sum_i \lambda_i (x_i, x_i^*) = (x, x^*), \ \sum_i \lambda_i = 1 \text{ and } \lambda_i \geq 0 \right\} \geq \mathcal{F}_{(M_f)^\mu}(x, x^*)$$

and on taking a closure we have $\mathcal{P}_{(M_f)^{\mu\mu}} \geq \mathcal{F}_{(M_f)^\mu}$. Finally $\mathcal{P}_{(M_f)^{\mu\mu}} \leq \mathcal{P}_{M_f}$ because $M_f \subseteq (M_f)^{\mu\mu}$. □

Lemma 4.2. *For $f \in R(T)$, we have $\mathcal{F}_{M_f} \leq f$, and there exists $g \in bR(T)$ minimal for $[R(T), \leq]$ for which $\mathcal{F}_{M_f} \leq \mathcal{F}_{M_g} \leq g \leq f$.*

Proof. By the Zorn Lemma, there is a minimal $g \in [R(T), \leq]$ for which $g \leq f$. By Lemma 2.15, $g \in bR(T)$ and $g \geq \mathcal{F}_{M_g}$. Since $g \leq f$, we have $T \subseteq M_f \subseteq M_g$, and hence that $\mathcal{F}_{M_f} \leq \mathcal{F}_{M_g} \leq g \leq f$. □

Lemma 4.3. *Let $f \in R(T)$, and let M any maximal monotone extension of M_f. Then $\mathcal{F}_M \in bR(T)$, and is minimal in $[R(T), \leq]$.*

Proof. Clearly $\mathcal{F}_M \in R(T)$, from the maximality. Suppose $h \in R(T)$ with $h \leq \mathcal{F}_M$. There is a minimal $g \in R(T)$ with $g \leq h$, and it satisfies $g \geq \mathcal{F}_{M_g}$ (by Lemma 2.15). Since $M = M_{\mathcal{F}_M} \subseteq M_h \subseteq M_g$ and hence $g \geq \mathcal{F}_{M_g} \geq \mathcal{F}_M \geq h \geq g$, it follows that $h = \mathcal{F}_M$, establishing the minimality of $\mathcal{F}_M$ in $[R(T), \leq]$. □

Note that when M_f is pre-maximal, then $(M_f)^\mu$ is a maximal extension of M_f and hence by Lemma 4.3 we have $\mathcal{F}_{(M_f)^\mu} \in bR(T)$ and minimal in $[R(T), \leq]$. Then $\mathcal{F}_{(M_f)^\mu} \geq \mathcal{F}_{M_f}$, where $\mathcal{F}_{M_f}$ and $\mathcal{F}_{(M_f)^\mu}$ represent the same maximal monotone set (in that $M_{\mathcal{F}_{(M_f)^\mu}} = (M_f)^{\mu\mu} = (M_f)^\mu = M^{\leq}_{\mathcal{F}_{M_f}}$) but we do not know that $\mathcal{F}_{M_f}$ is representative. Indeed in a BMLS space, representativity of $\mathcal{F}_{M_f}$ implies maximality of M_f (by Proposition 2.12).

We revisit pre-maximality and then link this to maximality (of operators) via minimality of representative functions.

Theorem 4.4. *For $f \in R(T)$, let*

$$bR_f(T) := \left\{ g \in R(T) \,\middle|\, g \text{ minimal in } [R(T), \leq], \text{ with } g \geq \mathcal{F}_{M_f} \right\}.$$

1. *Denote by h (or h_f to indicate the role of f) the function defined by*

$$h^* := \sup_{g \in bR_f(T)} g^* \quad \text{and} \quad h = h^{**}. \tag{9}$$

Then,

(a) $h^{*\dagger} \geq h$,
(b)

$$f \geq h = \overline{\text{co}}\Big(\inf_{g \in bR_f(T)} g^{**}\Big) \geq \mathcal{F}_{M_f}, \quad \textit{and} \tag{10}$$

(c) *we have*

$$M_{h^{*\dagger}} \subseteq (M_f)^{\mu\mu} \subseteq (M_f)^{\mu} \subseteq M_h^{\leq}, \tag{11}$$

with $M_{h^{*\dagger}} = (M_f)^{\mu\mu}$ *if* X *is a* BMLS *space.*

2. *When* $h = h_f$ *in* (9) *is representative, then* M_f *is pre-maximal, with* $h = \mathcal{F}_{(M_f)^\mu}$. *Conversely, whenever* M_f *is pre-maximal we have* $(M_f)^\mu = M_h^{\leq}$, *and when* X *is a* BMLS *space, have* $h = \mathcal{F}_{(M_f)^\mu}$ (*and hence* h *is representative*).

Proof. Observe that $bR_f(T) \neq \emptyset$, for if M is a maximal monotone extension of M_f, then $\mathcal{F}_M$ is minimal in $R(T)$ by Lemma 4.2, so $\mathcal{F}_M \in bR_f(T)$.

1: Note that for any maximal extension $M \supseteq M_f$ we have $\mathcal{F}_M \in bR_f(T)$ by Lemma 4.3, giving

$$h^{*\dagger} = \sup_{g \in bR_f(T)} g^{*\dagger} \geq \sup\{\mathcal{P}_M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\}. \tag{12}$$

Observe that if X has the BMLS property, then there is an *equality* in (12) (for if $g \in bR_f(T)$, then $g \in bR(T)$ and is minimal in $bR(T)$. If X is BMLS, then M_g is maximal, so $g = \mathcal{F}_{M_g}$ by Proposition 2.12, and since $\mathcal{F}_{M_g} \geq \mathcal{F}_{M_f}$ we get $M_f \subseteq M_g$ so $g^{*\dagger} \leq$ rhs of (12)) and so

$$\begin{aligned} M_{h^{*\dagger}} &= \bigcap\{M_{\mathcal{P}_M} \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \\ &= \bigcap\{M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} = (M_f)^{\mu\mu}. \end{aligned}$$

Otherwise an inclusion ($\subseteq$) holds. Now consider

$$h := h^{**} = \Big(\sup_{g \in bR_f(T)} g^*\Big)^* = \Big(\Big(\text{co} \inf_{g \in bR_f(T)} g\Big)^*\Big)^* = \overline{\text{co}}\Big(\inf_{g \in bR_f(T)} g^{**}\Big). \tag{13}$$

We have

$$\begin{aligned} M_h^{\leq} &\supseteq \bigcup_{g \in bR_f(T)} M_g = \bigcup_{g \in bR_f(T)} M_{g^{*\dagger}} \\ &\supseteq \bigcup\{M_{\mathcal{P}_M} \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \\ &= \bigcup\{M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} = (M_f)^{\mu}. \end{aligned} \tag{14}$$

As $(M_f)^{\mu\mu} \subseteq (M_f)^{\mu}$ for any monotone set M_f (recall $f \in R(T)$) this gives part *1*.

For the inequalities in (10), to see that $h \leq f$, note that by Lemma 4.2, there is $g \in bR_f(T)$ with $g \leq f$ so that $h \leq g^{**} \leq g \leq f$. Also, $h \geq \mathcal{F}_{M_f}$ since $g \geq \mathcal{F}_{M_f}$ (and hence $g^{**} \geq \mathcal{F}_{M_f}$) for each $g \in bR_f(T)$.

One may show $h^{*\dagger} \geq h$ by direct calculation; as $g^{*\dagger} \geq g$ for each $g \in bR_f(T)$ we have

$$h^{*\dagger} = \sup_{g \in bR_f(T)} g^{*\dagger} \geq \inf_{g \in bR_f(T)} g \geq \overline{\mathrm{co}}\Big(\inf_{g \in bR_f(T)} g^{**} \Big) = h.$$

2:($\Longrightarrow$) When h is representative we have by Lemma 2.2 that

$$M_h = M_{h^{*\dagger}} \subseteq (M_f)^{\mu\mu} \subseteq (M_f)^{\mu} \subseteq M_h^{\leq} = M_h,$$

forcing $M_h = (M_f)^{\mu\mu} = (M_f)^{\mu}$. By Proposition 2.9, M_f is pre-maximal, so $M_h = (M_f)^{\mu}$ is maximal monotone. By Lemma 4.3, $\mathcal{F}_{(M_f)^{\mu}} \in bR_f(T)$, so that $\mathcal{F}_{(M_f)^{\mu}} = (\mathcal{F}_{(M_f)^{\mu}})^{**} \geq h$. But as $\mathcal{F}_{(M_f)^{\mu}}$ is minimal in $[R(T), \leq]$ and h is representative (so that $h \in R(T)$) we have $h = \mathcal{F}_{(M_f)^{\mu}}$.

2:($\Longleftarrow$) Assume M_f is pre-maximal. We now argue that $\mathcal{F}_{(M_f)^{\mu}} \leq h^{*\dagger}$ and so $\mathcal{P}_{(M_f)^{\mu}} \geq h$. Using (12) we have

$$\begin{aligned} h^{*\dagger} &:= \sup_{g \in bR_f(T)} g^{*\dagger} \geq \sup \{\mathcal{P}_M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \\ &\geq \sup \{\mathcal{F}_M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \\ &= \sup \{\mathcal{F}_M \mid M \supseteq (M_f)^{\mu\mu} \text{ and } M \text{ maximal monotone}\} \\ &= \mathcal{F}_{\cup\left\{M \text{ max mon. } \mid M \supseteq (M_f)^{\mu\mu}\right\}} = \mathcal{F}_{\left((M_f)^{\mu\mu}\right)^{\mu}} = \mathcal{F}_{(M_f)^{\mu}}. \end{aligned}$$

Thus $\mathcal{P}_{(M_f)^{\mu}} \geq h \geq \mathcal{F}_{M_f}$ and as $(M_f)^{\mu}$ is maximal we have

$$(M_f)^{\mu} = (M_f)^{\mu\mu} \subseteq M_h^{\leq} \subseteq (M_f)^{\mu}, \tag{15}$$

giving $M_h^{\leq} = (M_f)^{\mu}$.

To show that h is representative, we assume now the BMLS condition. By Lemma 4.3, $\mathcal{F}_{(M_f)^{\mu}} \in bR(T)$ and is minimal in $[R(T), \leq]$, so that $\mathcal{F}_{(M_f)^{\mu}} \in bR_f(T)$ and hence $h \leq \mathcal{F}_{(M_f)^{\mu}}$ from (13). When BMLS holds, (12) consists of equalities, and gives

$$\begin{aligned} h^{*\dagger} &= \sup \{\mathcal{P}_M \mid M \supseteq M_f \text{ and } M \text{ maximal monotone}\} \\ &\leq \mathcal{P}_{\cap\{M \text{ max mon } \mid M \supseteq M_f\}} = \mathcal{P}_{(M_f)^{\mu\mu}} = \mathcal{P}_{(M_f)^{\mu}} \end{aligned}$$

which yields that $h \geq \mathcal{F}_{(M_f)^{\mu}}$. Hence $h = \mathcal{F}_{(M_f)^{\mu}}$ is now representative. $\square$

Corollary 4.5. *Let X be a* BMLS *space. Then $f \in R(T)$ has M_f pre-maximal if and only if:*

$$h_f = \mathcal{F}_{\left(M_f\right)^\mu} \text{ and this is the unique}$$
$$\text{minimal element of } bR(T) \text{ with } f \geq h \geq \mathcal{F}_{M_f}.$$

Proof. ($\Longrightarrow$) This follows from Theorem 4.4 part *2*, which ensures uniqueness, and Lemma 4.2, which ensures there exists a minimal element $g \in bR_f(T)$ with $f \geq g \geq \mathcal{F}_{M_f}$. Indeed by Theorem 4.4 part *2*, $h = \mathcal{F}_{\left(M_f\right)^\mu}$ is representative when M_f is pre-maximal. As $h = \mathcal{F}_{\left(M_f\right)^\mu} \in bR_f(T)$ then (10) implies $g \geq \mathcal{F}_{\left(M_f\right)^\mu}$ for all $g \in bR_f(T)$. But as g is minimal and $\mathcal{F}_{\left(M_f\right)^\mu} \in bR_f(T)$ we have $g = \mathcal{F}_{\left(M_f\right)^\mu}$ and uniqueness ensues.

($\Longleftarrow$) Immediate from part *2*($\Longrightarrow$) of the Theorem. $\square$

5. Appendix A

Proof of Proposition 1.1. Let $\mathcal{R}$ be the set of closed proper convex functions $r \in \Gamma(X \times X^*)$ for which $p \geq r \geq (r^*)^\dagger \geq f$ (where $p := (f^*)^\dagger$). This set is nonempty by definition, as $p \in \mathcal{R}$ when f is closed.

Now let $(r_i)_{i\in I} \in \mathcal{R}$ be a totally (downwardly) ordered family and let $r := \overline{(\inf_{i\in I} r_i)} \in \Gamma(X \times X^*)$. As $p \geq r_i$ we have $p \geq \inf_{i\in I} r_i$ and on taking closures

$$p \geq r = \overline{\left(\inf_{i\in I} r_i\right)} .$$

Also $r^* = \left(\overline{\inf_{i\in I} r_i}\right)^* = \sup_{i\in I} r_i^* \geq f^\dagger$. Now $r_i^* \leq r_i$ for all i, and as $\{r_i\}_{i\in I}$ is downwardly ordered we have $\{r_i^*\}_{i\in I}$ upwardly ordered, so $\sup_{i\in I} r_i^* = \lim_i r_i^* \leq \lim_i r_i^\dagger = \inf_{i\in I} r_i^\dagger$ and

$$r = \left(\inf_{i\in I} r_i\right)^{**} = \left(\sup_{i\in I} r_i^*\right)^* \geq \left(\inf_{i\in I} r_i^\dagger\right)^* = \left(\overline{\inf_{i\in I} r_i}\right)^{*\dagger} = (r^*)^\dagger .$$

Consequently $p \geq r \geq (r^*)^\dagger \geq f$ and $r \in \mathcal{R}$. Thus there exists a lower bound in $\mathcal{R}$ to every totally–ordered chain. By Zorn's Lemma $\mathcal{R}$ has a minimal element q. We now show $q^* = q^\dagger$. Let $t = \frac{1}{2}\left(q + (q^*)^\dagger\right) \leq q$. Then $t^* \geq q^* \geq f^\dagger$, and

$$\begin{aligned} t^*(x^*, x) &= \frac{1}{2}\left(q + (q^*)^\dagger\right)^*(2\,(x^*, x)) = \frac{1}{2}\overline{\left(q^* \Box\, q^\dagger\right)}\,(2\,(x^*, x)) \\ &\leq \frac{1}{2}\left(q^* \Box\, q^\dagger\right)(2\,(x^*, x)) \leq \frac{1}{2}\left(q(x^*, x) + (q^*)^\dagger(x^*, x)\right) = t^\dagger(x^*, x) . \end{aligned}$$

Thus $t \in \mathcal{R}$, and $t \leq q$, implying $t = q$ by minimality of q, and so $q = (q^*)^\dagger$, on $\operatorname{dom} g$.

We have by construction that $q^* \leq q^\dagger$ and that q is minimal with $(q^*)^\dagger = q$ on $\operatorname{dom} q$. Suppose for now that $\operatorname{dom} q \subsetneq \operatorname{dom}(q^*)^\dagger$.

Take $(x^*, x) \in \operatorname{dom} q^*$ but with $(x, x^*) \notin \operatorname{dom} q$. Let $\beta = \max\{q^*(x, x^*), \langle x, x^*\rangle\}$. Adjoin a point $((x, x^*), \beta)$ to the epigraph of the minimal q and convexify in order to define h i.e.

$$h := \operatorname{co}\{q, \delta_{\{(x,x^*)\}} + \beta\} \leq q.$$

Then by construction $h(x, x^*) = \beta$ and via a simple calculation $h^*(z^*, z) = \max\{q^*(z^*, z), \langle (x, x^*), (z, z^*)\rangle - \beta\}$. By construction $q^* \leq h^\dagger$ and so $h^* \leq q^\dagger$. Now

$$h^*(x^*, x) \leq \max\{q^*(x^*, x), 2\langle x, x^*\rangle - q^*(x^*, x)\} \leq \max\{q^*(x^*, x), \langle x^*, x\rangle\} = \beta$$

because $q^*(x^*, x) \geq t(x^*, x) \geq \langle x^*, x\rangle$. Hence we may assert that $(h^*)^\dagger \leq h := \operatorname{co}\{q, \delta_{\{(x,x^*)\}} + \beta\}$. Thus $h \in \mathcal{R}$ and $h \leq q$. But by minimality of q we must have $h = q$ and so $\operatorname{dom} q = \operatorname{dom} q^*$, contrary to assumption. $\square$

Proof of Lemma 2.2 part *1*. Clearly $M^{\leq}_{f^{*\dagger}} \subseteq M^{\leq}_{f}$ whenever $f^{*\dagger} \geq f$. The fact that $M^{\leq}_{f} \subseteq M_{f^{*\dagger}}$ follows via [15, Lemma 19.12] when one assumes $f \geq \langle \cdot, \cdot \rangle$, we show this for completeness. Let $(x, x^*) \in M^{\leq}_{f}$ and $\lambda \in [0, 1]$ and $\mu := 1 - \lambda \in [0, 1]$. Then

$$\begin{aligned}
&\lambda^2\langle y, y^*\rangle + \lambda\mu\langle (x, x^*), (y, y^*)\rangle + \mu^2\langle x, x^*\rangle \\
&= \frac{1}{2}\langle \lambda(y, y^*) + \mu(y, y^*), \lambda(y, y^*) + \mu(x, x^*)\rangle = \langle \lambda y + \mu x, \lambda y^* + \mu x^*\rangle \\
&\leq f(\lambda(y, y^*) + \mu(x, x^*)) \leq \lambda f(y, y^*) + \mu f(x, x^*) \leq \lambda f(y, y^*) + \mu\langle x, x^*\rangle
\end{aligned}$$

so

$$\begin{aligned}
\lambda\mu\langle (x, x^*), (y, y^*)\rangle - \lambda f(y, y^*) &\leq \left(\mu - \mu^2\right)\langle x, x^*\rangle - \lambda^2\langle y, y^*\rangle \\
&= \mu\lambda\langle x, x^*\rangle - \lambda^2\langle y, y^*\rangle.
\end{aligned}$$

Dividing by λ and letting $\lambda \to 0$ we have $\langle (x, x^*), (y, y^*)\rangle - f(y, y^*) \leq \langle x, x^*\rangle$ for all (y, y^*) giving $f^*(x^*, x) \leq \langle x, x^*\rangle$ and so $(x, x^*) \in M^{\leq}_{(f^*)^\dagger}$. Thus $M^{\leq}_{f} \subseteq M^{\leq}_{(f^*)^\dagger}$ with equality ensuing.

Now suppose $(x, x^*) \in M^{\leq}_{f^{*\dagger}} \subseteq M^{\leq}_{f}$. Then

$$\begin{aligned}
&\langle x, x^*\rangle \geq f(x, x^*) \quad \text{and} \quad \langle x, x^*\rangle \geq f^*(x^*, x) \\
\text{so} \quad &2\langle x, x^*\rangle \geq f(x, x^*) + f^*(x^*, x) \geq \langle (x, x^*), (x^*, x)\rangle = 2\langle x, x^*\rangle
\end{aligned}$$

using the Fenchel inequality, so $f(x, x^*) + f^*(x^*, x) = \langle (x, x^*), (x^*, x)\rangle$. Hence $(x^*, x) \in \partial f(x, x^*)$ (and by $\partial f^*(x^*, x) = (\partial f)^{-1}(x^*, x)$ we have the last identity in (2)). Conversely when $(x^*, x) \in \partial f(x, x^*)$ we have

$$f(x, x^*) + f^*(x^*, x) = \langle (x, x^*), (x, x^*)\rangle,$$

which combined with $f^{*\dagger} \geq f$ gives $2f(x,x^*) \leq 2\langle x,x^*\rangle$ and $(x,x^*) \in M_f^{\leq} = M_f$ when $f \geq \langle\cdot,\cdot\rangle$.

The assertion *2* is well known (see [15, Lemma 19.8]), and assertion *3* follows from the definition of the Fitzpatrick function

$$\mathcal{F}_T(x,x^*) = \langle x,x^*\rangle - \inf_{(y,y^*)\in T}\langle x-y,x^*-y^*\rangle = \langle x,x^*\rangle$$

because when $(x,x^*) \in T$ we have $\inf_{(y,y^*)\in T}\langle x-y,x^*-y^*\rangle = 0$ (as T is monotone implies $\langle x-y,x^*-y^*\rangle \geq 0$ for all $(y,y^*) \in T$). $\square$

Acknowledgements. The first author would like to acknowledge the important influence his PhD supervisor, mentor and friend Charles Pearce has had on his career. The work of this paper was completed after Charles Pearce's untimely passing on 8/6/2012. The last part of the proof to Proposition 1.1 was discussed with Constantin Zălinescu.

References

[1] F. J. Aragon Artacho, J. M. Borwein, V. Martín-Márquez, L. Yao: Applications of convex analysis within mathematics, preprint (2013).

[2] H. H. Bauschke, X. Wang: The kernel average for two convex functions and its applications to the extension and representation of monotone operators, Trans. Amer. Math. Soc. 361 (2009) 5947–5965.

[3] H. H. Bauschke, X. Wang, L. Yao: Autoconjugate representatives for linear monotone operators, Math. Program. 123(B) (2010) 5–24.

[4] J. M. Borwein: Maximal monotonicity via convex analysis, J. Convex Analysis 13(3–4) (2006) 561–586.

[5] J. M. Borwein: Maximality of sums of two maximal monotone operators in general Banach space, Proc. Amer. Math. Soc 135 (2007) 3917–3924.

[6] J. M. Borwein, L. Yao: Recent progress on monotone operator theory, arXiv:1210.3401v3 [math.FA] (2013).

[7] R. Burachik, B. F. Svaiter: Maximal monotone operators, convex functions and a special family of enlargements, Set-Valued Anal. 10(4) (2002) 297–316.

[8] R. S. Burachik, B. F. Svaiter: Maximal monotonicity, conjugation, and the duality product, Proc. Amer. Math. Soc. 132(8) (2003) 2379–2383.

[9] S. Fitzpatrick: Representing monotone operators by convex functions, in: Functional Analysis and Optimization, Workshop / Miniconference (Canberra, 1988), Proc. Cent. Math. Anal. Aust. Natl. Univ. 20, Australian National University, Canberra (1988) 59–65.

[10] M. Marques Alves, B. F. Svaiter: Maximal monotonicity, conjugation and duality product in non-reflexive spaces, J. Convex Analysis 17(2) (2010) 533–563.

[11] J.-E. Martínez-Legaz, B. F. Svaiter: Minimal convex functions bounded below by the duality product, Proc. Amer. Math. Soc. 136(3) (2008) 873–878.

[12] J.-E. Martínez-Legaz, B. F. Svaiter: Monotone operators representable by l.s.c. convex functions, Set-Valued Anal. 13 (2005) 21–46.

[13] J.-P. Penot: The relevance of convex analysis for the study of monotonicity, Nonlinear Anal. 58(7–8) (2004) 855–871.

[14] R. T. Rockafellar: On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 159 (1970) 81–99.

[15] S. Simons: From Hahn-Banach to Monotonicity, Springer, Berlin (2007).